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Projections of hypersurfaces in the hyperbolic space to hyperhorospheres and hyperplanes

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Abstract

We study in this paper orthogonal projections in a hyperbolic space to hyperhorospheres and hyperplanes. We deal in more details with the case of embedded surfaces M in $H^3_+(-1)$. We study the generic singularities of the projections of M to horospheres and planes. We give geometric characterizations of these singularities and prove duality results concerning the bifurcation sets of the families of projections. We also prove Koenderink type theorems that give the curvature of the surface in terms of the curvatures of the profile and the normal section of the surface.

1. Introduction

Projections of surfaces in the Euclidean and projective 3-spaces are well studied (see for example [1, 3, 4, 5, 6, 7, 9, 23, 24, 27, 28, 29, 30]). We initiate in this paper an analogous study for embedded surfaces in the hyperbolic space $H^3_+(-1)$. Projections in the Euclidean space $\mathbb{R}^n$ are linear maps. By such projections, a point in $\mathbb{R}^n$ is taken along a line (a geodesic) until it hits an orthogonal hyperplane of projection (which is an $(n-1)$-dimensional flat object). There are two notions of flat objects in the hyperbolic space $H^n_+(-1)$. One is given by the everywhere vanishing of de Sitter Gaussian curvature and the other by the everywhere vanishing of the hyperbolic Gaussian curvature (see Section 2). It is shown in [17] that a totally umbilic hypersurface has everywhere zero hyperbolic Gaussian curvature if and only if it is part of a hyperhorosphere, and it has everywhere zero de Sitter Gaussian curvature

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if and only if it is part of a hyperplane ([19]). So we consider in this paper orthogonal projections to hyperhorospheres and to hyperplanes. By such projections, a point in $H^n_+(-1)$ is taken along the unique geodesic to the point where such geodesic meets orthogonally the chosen hyperhorosphere or hyperplane of projection.

We deal in Section 3 with projection to hyperhorospheres and in Section 4 with projections to hyperplanes. In both cases we start by finding the expressions of the families of orthogonal projections in $H^n_+(-1)$ to hyperhorospheres and hyperplanes (Theorems 3.1 and 4.1). We then restrict to the cases of embedded surfaces M in $H^3_+(-1)$. We give geometric characterisations of the generic singularities of the orthogonal projections of M to horospheres and planes (Theorems 3.5 and 4.4). We observe that the singularities of these projections measure the contact of the surface with geodesics in $H^3_+(-1)$. We prove duality results (Theorems 3.2 and 4.2) concerning the bifurcation sets of the families of projections, analogous to those of Shcherback in [29]. Here, we use the duality concepts introduced by the first author in [11, 12]; see §2 for details. We also prove Koenderink type theorems that give the curvature of the surface in terms of the curvature of the profile and of the normal section of the surface (Theorems 3.6 and 4.5).

2. Preliminaries

We start by recalling some basic concepts in hyperbolic geometry (see for example [26] for details). The *Minkowski* $(n+1)$-*space* $(\mathbb{R}^{n+1}_1, \langle , \rangle)$ is the $(n+1)$-dimensional vector space $\mathbb{R}^{n+1}$ endowed by the *pseudo scalar product* $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_0y_0 + \sum_{i=1}^n x_iy_i$, for $\boldsymbol{x} = (x_0, \dots, x_n)$ and $\boldsymbol{y} = (y_0, \dots, y_n)$ in $\mathbb{R}^{n+1}_1$. We say that a vector $\boldsymbol{x}$ in $\mathbb{R}^{n+1}_1 \setminus \{\boldsymbol{0}\}$ is *spacelike*, *lightlike* or *timelike* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0$, $= 0$ or < 0 respectively. The norm of a vector $\boldsymbol{x} \in \mathbb{R}^{n+1}_1$ is defined by $\|\boldsymbol{x}\| = \sqrt{|\langle \boldsymbol{x}, \boldsymbol{x} \rangle|}$.

Given a vector $\boldsymbol{v} \in \mathbb{R}^{n+1}_1$ and a real number c, the hyperplane with pseudo normal $\boldsymbol{v}$ is defined by

$$HP(\boldsymbol{v}, c) = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 \mid \langle \boldsymbol{x}, \boldsymbol{v} \rangle = c\}.$$

We say that $HP(\boldsymbol{v}, c)$ is a *spacelike*, *timelike* or *lightlike hyperplane* if $\boldsymbol{v}$ is timelike, spacelike or lightlike respectively. For $\boldsymbol{v} = \boldsymbol{e}_0 = (1, 0, \dots, 0)$, we have $HP(\boldsymbol{e}_0, 0) = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 \mid x_0 = 0\}$. This space is identified with the Euclidean n-space and is denoted by $\mathbb{R}^n_0$.

We have the following three types of pseudo-spheres in $\mathbb{R}^{n+1}_1$:

$$\begin{aligned}
&\textit{Hyperbolic } n\textit{-space}: & H^n(-1) &= \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1\},\\
&\textit{de Sitter } n\textit{-space}: & S^n_1 &= \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1\},\\
&(\textit{open}) \textit{ lightcone}: & LC^* &= \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 \setminus \{\boldsymbol{0}\} \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0\}.
\end{aligned}$$

We also define the *lightcone hypersphere*

$$S_+^{n-1} = \{\boldsymbol{x} = (x_0, \dots, x_n) \mid \ \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0,\ x_0 = 1\}.$$

For $\boldsymbol{x} \in LC^*$, we have $x_0 \neq 0$ so

$$\widetilde{\boldsymbol{x}} = \left(1, \frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right) \in S_+^{n-1}.$$

The hyperbolic space has two connected components $H_+^n(-1) = \{\boldsymbol{x} \in H^n(-1) \mid x_0 \geq 1\}$ and $H_-^n(-1) = \{\boldsymbol{x} \in H^n(-1) \mid x_0 \leq -1\}$. We only consider embedded surfaces in $H_+^n(-1)$ as the study is similar for those embedded in $H_-^n(-1)$.

The wedge product of n vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n \in \mathbb{R}_1^{n+1}$ is given by

$$\boldsymbol{a}_1 \wedge \boldsymbol{a}_2 \wedge \cdots \wedge \boldsymbol{a}_n = \begin{vmatrix} -\boldsymbol{e}_0 & \boldsymbol{e}_1 & \cdots & \boldsymbol{e}_n \\ a_0^1 & a_1^1 & \cdots & a_n^1 \\ a_0^2 & a_1^2 & \cdots & a_n^2 \\ \vdots & \vdots & \cdots & \vdots \\ a_0^n & a_1^n & \cdots & a_n^n \end{vmatrix},$$

where $\{\boldsymbol{e}_0, \boldsymbol{e}_1, \dots, \boldsymbol{e}_n\}$ is the canonical basis of $\mathbb{R}_1^{n+1}$ and $\boldsymbol{a}_i = (a_0^i, a_1^i, \dots, a_n^i)$, $i = 1, \dots, n$. One can check that

$$\langle \boldsymbol{a}, \boldsymbol{a}_1 \wedge \boldsymbol{a}_2 \wedge \cdots \wedge \boldsymbol{a}_n \rangle = \det(\boldsymbol{a}, \boldsymbol{a}_1, \dots, \boldsymbol{a}_n),$$

so the vector $\boldsymbol{a}_1 \wedge \boldsymbol{a}_2 \wedge \cdots \wedge \boldsymbol{a}_n$ is pseudo orthogonal to all the vectors $\boldsymbol{a}_i$, $i = 1, \dots, n$.

The extrinsic geometry of hypersurfaces in the hyperbolic space is studied in [11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21]. Let M be a hypersurface embedded in $H_+^n(-1)$. Given a local chart $\boldsymbol{i} : U \to M$, where U is an open subset of $\mathbb{R}^{n-1}$, we denote by $\boldsymbol{x} : U \to H_+^n(-1)$ such embedding, identify $\boldsymbol{x}(U)$ with U through the embedding $\boldsymbol{x}$ and write $M = \boldsymbol{x}(U)$. Since $\langle \boldsymbol{x}, \boldsymbol{x} \rangle \equiv -1$, we have $\langle \boldsymbol{x}_{u_i}, \boldsymbol{x} \rangle \equiv 0$, for $i = 1, \dots, n-1$, where $u = (u_1, \dots, u_{n-1}) \in U$. We define the spacelike unit normal vector $\boldsymbol{e}(u)$ to M at $\boldsymbol{x}(u)$ by

$$\boldsymbol{e}(u) = \frac{\boldsymbol{x}(u) \wedge \boldsymbol{x}_{u_1}(u) \wedge \ldots \wedge \boldsymbol{x}_{u_{n-1}}(u)}{\|\boldsymbol{x}(u) \wedge \boldsymbol{x}_{u_1}(u) \wedge \ldots \wedge \boldsymbol{x}_{u_{n-1}}(u)\|}.$$

It follows that the vector $\boldsymbol{x} \pm \boldsymbol{e}$ is a lightlike vector. Let

$$\mathbb{E} : U \to S_1^n \quad \text{and} \quad \mathbb{L}^{\pm} : U \to LC^*$$

be the maps defined by $\mathbb{E}(u) = \boldsymbol{e}(u)$ and $\mathbb{L}^{\pm}(u) = \boldsymbol{x}(u) \pm \boldsymbol{e}(u)$. These are called, respectively, the *de Sitter Gauss map* and the *lightcone Gauss map* (or *hyperbolic Gauss indicatrix*) of M ([17]). For any $p = \boldsymbol{x}(u_0) \in M$ and $\boldsymbol{v} \in T_pM$, one can show that $D_v\mathbb{E} \in T_pM$, where D_v denotes the covariant derivative with respect to the tangent vector $\boldsymbol{v}$. Since the derivative $d\boldsymbol{x}(u_0)$ can be identified with the identity mapping 1_{T_pM} on the tangent space T_pM, we have $d\mathbb{L}^{\pm}(u_0) = 1_{T_pM} \pm d\mathbb{E}(u_0)$, under the identification of U and M via the embedding $\boldsymbol{x}$. The linear transformation $A_p = -d\mathbb{E}(u_0)$ is called the *de Sitter shape operator*. Its eigenvalues κ_i, $i = 1, \ldots, n-1$, are called the *de Sitter principal curvature* and the corresponding eigenvectors $\boldsymbol{p}_i$, $i = 1, \ldots, n-1$, are called the *de Sitter principal directions*. The linear transformation $S_p^{\pm} = -d\mathbb{L}^{\pm}(u_0)$ is labelled the *lightcone* (*or hyperbolic*) *shape operator* of M at p. It has the same eigenvectors as A_p but its eigenvalues are distinct from those of A_p. In fact the eigenvalues $\bar{\kappa}_i^{\pm}$ of $S_p^{\pm}$ satisfy $\bar{\kappa}_i^{\pm} = -1 \pm \kappa_i$, $i = 1, \ldots, n-1$.

We call $K_e(p) = \prod_{i=1}^n \kappa_i(p)$ (resp. $K_h(p) = \prod_{i=1}^n \bar{\kappa}_i(p)$) the *de Sitter* (resp. *hyperbolic*) *Gauss-Kronecker curvature* of M at p. The curvature K_e is also called the *extrinsic Gaussian curvature*. The set of points where $K_e(p) = 0$ (resp. $K_h(p) = 0$) is labelled the *de Sitter* (resp. *horospherical*) *parabolic set* of M. The restriction of the pseudo-scalar product to the hyperbolic space is a scalar product, so $H_+^n(-1)$ is a Rimaniann manifold. When $n = 3$, we have the sectional curvature K_I of M which is also called the *intrinsic Gaussian curvature*. It is known that $K_e = K_I + 1$ (see (2.2) in [8]).

A hypersurface given by the intersection of $H_+^n(-1)$ with a spacelike, timelike or lightlike hyperplane is called respectively *hypersphere*, *equidistant hypersurface* or *hyperhorosphere*. The intersection of the surface with timelike hyperplane through the origin is called simply a *hyperplane*. As pointed out in the introduction, the hyperhorospheres (resp. hyperplanes) are the only hypersurfaces with everywhere zero lightcone (resp. de Sitter) Gaussian curvature. We deal in Section 3 with projections to hyperhorospheres and in Section 4 with projections to hyperplanes.

We need the notion of curvature of a curve in $H_+^3(-1)$. Let $\gamma : I \to H_+^3(-1)$ be a regular curve. Since $H_+^3(-1)$ is a Riemannian manifold, we can parametrise γ by arc-length and assume that $\gamma(s)$ is unit speed. Let $t(s) = \gamma'(s)$, with $||t(s)|| = 1$. When $\langle t'(s), t'(s)\rangle \neq -1$, we have a unit normal vector $n(s) = \frac{t'(s)-\gamma(s)}{||t'(s)-\gamma(s)||}$. Let $e(s) = \gamma(s) \wedge t(s) \wedge n(s)$, then we have a pseudo orthogonal frame $\{\gamma(s), t(s), n(s), e(s)\}$ in $\mathbb{R}_1^4$ along γ. Frenet-Serret type formulae, similar to those for a space curve in $\mathbb{R}^3$, can be proved for the curve γ ([20]). The *curvature* of γ at $\gamma(s)$ is defined to be

$$\kappa_h(s) = ||t'(s) - \gamma(s)||.$$

In particular, $t'(s) = \kappa_h(s)n(s) + \gamma(s)$. The condition $\langle t'(s), t'(s)\rangle \neq -1$ above is in fact equivalent to $\kappa_h(s) \neq 0$. See [20] for more results on curves in the hyperbolic plane.

We require some properties of contact manifolds and Legendrian submanifolds for the duality results in this paper (for more details see for example [2]). Let N be a $(2n+1)$-dimensional smooth manifold and K be a field of tangent hyperplanes on N. Such a field is locally defined by a 1-form α. The tangent hyperplane field K is said to be *non-degenerate* if $\alpha \wedge (d\alpha)^n \neq 0$ at any point on N. The pair (N, K) is a *contact manifold* if K is a non-degenerate hyperplane field. In this case K is called a *contact structure* and α a *contact form*.

A submanifold $\boldsymbol{i} : L \subset N$ of a contact manifold (N, K) is said to be *Legendrian* if $\dim L = n$ and $d\boldsymbol{i}_x(T_x L) \subset K_{\boldsymbol{i}(x)}$ at any $x \in L$. A smooth fibre bundle $\pi : E \to M$ is called a *Legendrian fibration* if its total space E is furnished with a contact structure and the fibres of π are Legendrian submanifolds. Let $\pi : E \to M$ be a Legendrian fibration. For a Legendrian submanifold $\boldsymbol{i} : L \subset E$, $\pi \circ \boldsymbol{i} : L \to M$ is called a *Legendrian map.* The image of the Legendrian map $\pi \circ \boldsymbol{i}$ is called a *wavefront set* of $\boldsymbol{i}$ and is denoted by $W(\boldsymbol{i})$.

There are two families of maps defined on an embedded manifold M in the Euclidean space $\mathbb{R}^n$. These are the family of height functions given by

$$\begin{array}{rccc} H: & M \times S^{n-1} & \to & \mathbb{R} \times S^{n-1} \\ & (q, v) & \mapsto & q.v \end{array}$$

and the family of orthogonal projections given by

$$\begin{array}{rccc} P: & M \times S^{n-1} & \to & TS^{n-1} \\ & (q, v) & \mapsto & (q, q - (q.v)v) \end{array}$$

where S^{n-1} denotes the unit sphere and "." the scalar product in $\mathbb{R}^n$. The local bifurcation set of H (resp. P) is the set of $u \in S^{n-1}$ for which there exists $p \in M$ such that H_u (resp. P_u) has a non-stable singularity at p. When $n = 3$, a result in [7] shows that the dual of the A_2-stratum of the bifurcation set of the family of height functions on $M \subset \mathbb{R}^3$ is the lips/beaks stratum of the family of orthogonal projections on M. The duality in [7] refers to the double Legendrian fibration $S^2 \xleftarrow{\pi_1} \Delta \xrightarrow{\pi_2} S^2$, where S^2 is the unit sphere in $\mathbb{R}^3$ and $\Delta = \{(u, v) \in S^2 \times S^2 \mid u.v = 0\}$. The contact structure on Δ is given by the 1-form $\theta = v.du|\Delta$. (There are also other duality results in [29] regarding the strata of the bifurcation set of the family of projections of surfaces in the projective space $\mathbb{R}P^3$. Details of these are given in §3.2.)

We prove in §3.2 and §4.2 analogous results to those in [7] and [29]. The duality concepts we use here are those introduced in [11, 12, 22], where five Legendrian double fibrations are considered on the subsets Δ_i, $i = 1, \ldots, 5$

below, of the product of two of the pseudo spheres $H^n(-1)$, S_1^n and LC^*. The geometric ideas behind the choice of the subsets Δ_i and the Legendrian double fibrations are as follows (for more details see [11, 12, 22]).

To any hypersurface $\boldsymbol{x} : U \to H^n(-1)$ is associated the de Sitter Gauss map $\mathbb{E} : U \to S_1^n$. It is easy to show that the pair $(\boldsymbol{x}, \mathbb{E}) : U \to H^n(-1) \times S_1^n$ is a Legedrian embedding into the set $\Delta_1 = \{(\boldsymbol{v}, \boldsymbol{w}) \in H^n(-1) \times S_1^n \mid \langle \boldsymbol{v}, \boldsymbol{w} \rangle = 0\}$. (The contact structure on Δ_1 is given below.) This means that $M = \boldsymbol{x}(U)$ and $M^* = \mathbb{E}(U)$ are dual. We call this duality the Δ_1-*duality*. This is a direct analogue of the spherical duality in the Euclidean space.

Consider now the lightcone Gauss map $\mathbb{L}^\pm : U \to H^n(-1) \times LC^*$ which satisfies $\langle \boldsymbol{x}(u), \mathbb{L}^\pm(u) \rangle = -1$. The pair $(\boldsymbol{x}, \mathbb{L}^\pm) : U \to H^n(-1) \times LC^*$ determines a Legedrian embedding into the set $\Delta_2 = \{(\boldsymbol{v}, \boldsymbol{w}) \in H^n(-1) \times LC^* \mid \langle \boldsymbol{v}, \boldsymbol{w} \rangle = -1\}$, so $M = \boldsymbol{x}(U)$ and $M^* = \mathbb{L}^\pm(U)$ are dual. We call this duality the Δ_2-*duality*.

Similarly, we have $\langle \mathbb{E}(u) \pm \boldsymbol{x}(u), \mathbb{E}(u) \rangle = 1$ and $\langle \mathbb{L}^+(u), \mathbb{L}^-(u) \rangle = -2$ and these lead to the concepts of Δ_3-*duality* and Δ_4-*duality* respectively.

For spacelike hypersurfaces embedded in one of the pseudo-spheres in the Minkowski space (i.e. surfaces whose tangent spaces at all points are spacelike), we need to consider only the above four Δ_i-dualities, $i = 1, \dots, 4$. However, if we consider timelike hypersurfaces in S_1^n, (i.e. surfaces whose tangent spaces at all points are timelike) we need the concept of Δ_5-*duality* below which is also a direct analogue to the spherical duality in the Euclidean space. To summarise, we have the following five Legendrian double fibrations.

(1) (a) $H^n(-1) \times S_1^n \supset \Delta_1 = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w} \rangle = 0\}$,
(b) $\pi_{11} : \Delta_1 \to H^n(-1)$, $\pi_{12} : \Delta_1 \to S_1^n$,
(c) $\theta_{11} = \langle d\boldsymbol{v}, \boldsymbol{w} \rangle | \Delta_1$, $\theta_{12} = \langle \boldsymbol{v}, d\boldsymbol{w} \rangle | \Delta_1$.

(2) (a) $H^n(-1) \times LC^* \supset \Delta_2 = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w} \rangle = -1 \}$,
(b) $\pi_{21} : \Delta_2 \to H^n(-1)$,$\pi_{22} : \Delta_2 \to LC^*$,
(c) $\theta_{21} = \langle d\boldsymbol{v}, \boldsymbol{w} \rangle | \Delta_2$, $\theta_{22} = \langle \boldsymbol{v}, d\boldsymbol{w} \rangle | \Delta_2$.

(3) (a) $LC^* \times S_1^n \supset \Delta_3 = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w} \rangle = 1 \}$,
(b) $\pi_{31} : \Delta_3 \to LC^*$,$\pi_{32} : \Delta_3 \to S_1^n$,
(c) $\theta_{31} = \langle d\boldsymbol{v}, \boldsymbol{w} \rangle | \Delta_3$, $\theta_{32} = \langle \boldsymbol{v}, d\boldsymbol{w} \rangle | \Delta_3$.

(4) (a) $LC^* \times LC^* \supset \Delta_4 = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w} \rangle = -2 \}$,
(b) $\pi_{41} : \Delta_4 \to LC^*$,$\pi_{42} : \Delta_4 \to LC^*$,
(c) $\theta_{41} = \langle d\boldsymbol{v}, \boldsymbol{w} \rangle | \Delta_4$, $\theta_{42} = \langle \boldsymbol{v}, d\boldsymbol{w} \rangle | \Delta_4$.

(5) (a) $S_1^n \times S_1^n \supset \Delta_5 = \{(\boldsymbol{v}, \boldsymbol{w}) \mid \langle \boldsymbol{v}, \boldsymbol{w} \rangle = 0\}$,
(b) $\pi_{51} : \Delta_5 \to S_1^n$,$\pi_{52} : \Delta_5 \to S_1^n$,
(c) $\theta_{51} = \langle d\boldsymbol{v}, \boldsymbol{w} \rangle | \Delta_5$, $\theta_{52} = \langle \boldsymbol{v}, d\boldsymbol{w} \rangle | \Delta_5$.

Above, $\pi_{i1}(\boldsymbol{v},\boldsymbol{w}) = \boldsymbol{v}$ and $\pi_{i2}(\boldsymbol{v},\boldsymbol{w}) = \boldsymbol{w}$ for $i = 1,\dots,5$, $\langle d\boldsymbol{v},\boldsymbol{w}\rangle = -w_0dv_0 + \sum_{i=1}^n w_idv_i$ and $\langle \boldsymbol{v}, d\boldsymbol{w}\rangle = -v_0dw_0 + \sum_{i=1}^n v_idw_i$. The 1-forms θ_{i1}^{-1} and θ_{i2}^{-1}, $i = 1,\dots,5$, define the same tangent hyperplane field over Δ_i which is denoted by K_i.

We have the following duality theorem on the above spaces.

Theorem 2.1 ([11, 12, 22]) *The pairs (Δ_i, K_i), $i = 1,\dots,5$, are contact manifolds and π_{i1} and π_{i2} are Legendrian fibrations.*

We have the following general remarks, some of which follow from the discussion proceeding Theorem 2.1.

Remark 2.2 1. Given a Legendrian submanifold $\boldsymbol{i} : L \to \Delta_i$, $i = 1,\dots,5$, Theorem 2.1 states that $\pi_{i1}(\boldsymbol{i}(L))$ is the Δ_i-dual of $\pi_{i2}(\boldsymbol{i}(L))$ and vice-versa.

2. We have the following geometric properties for a Legendrian submanifold $L \subset \Delta_i$, $i = 1,\dots,5$. Take the case $i = 1$. If $\pi_{11}(\boldsymbol{i}(L))$ is smooth at a point $\pi_{11}(\boldsymbol{i}(\boldsymbol{u}))$, then $\pi_{12}(\boldsymbol{i}(\boldsymbol{u}))$ is the normal vector to the hypersurface $\pi_{11}(\boldsymbol{i}(L)) \subset H_+^n(-1)$ at $\pi_{11}(\boldsymbol{i}(\boldsymbol{u}))$. Conversely, if $\pi_{12}(\boldsymbol{i}(L))$ is smooth at a point $\pi_{12}(\boldsymbol{i}(\boldsymbol{u}))$, then $\pi_{11}(\boldsymbol{i}(\boldsymbol{u}))$ is the normal vector to the hypersurface $\pi_{12}(\boldsymbol{i}(L)) \subset S_1^n$. The same holds for the Δ_i-dualities, $i = 2,\dots,5$, where we take the normal to a hypersurface $M \subset LC^*$ at $p \in M$ as the direction given by the intersection of the normal plane to T_pM in $\mathbb{R}_1^{n+1}$ with T_pLC^*.

3. The Δ_4-duality is included for completion only and is not used in this paper.

4. Since the normal of a hypersurface in $H^n(-1)$ is always spacelike, we have no good duality relationship in $H^n(-1) \times H^n(-1)$.

3. Projections to hyperhorospheres

Our construction of the family of orthogonal projections works in $H_+^n(-1)$ for $n \geq 3$. So we shall first deal with the general case and then restrict to $n = 3$ for a detailed study of the singularities of the members of the family. Let $HP(\boldsymbol{v}, c)$ be a lightlike hyperplane (so $\boldsymbol{v} \in LC^*$ and $c \in \mathbb{R}$). Given a point $p \in H_+^n(-1)$, there is a unique geodesic in $H_+^n(-1)$ which intersects orthogonally the hyperhorosphere $HP(\boldsymbol{v}, c) \cap H_+^n(-1)$ at some point $q(p,\boldsymbol{v})$. We call the point $q(p,\boldsymbol{v})$ the orthogonal projection of p in the direction of $\boldsymbol{v}$ to the hyperhorosphere $HP(\boldsymbol{v}, c) \cap H_+^n(-1)$. By varying c, we obtain orthogonal projections to parallel hyperhorospheres. As the geometry we are investigating here is the same in all these parallel hyperhorospheres, we fix c to be $\langle \boldsymbol{e}_0, \boldsymbol{v}\rangle$, with $\boldsymbol{e}_0 = (1, 0, \dots, 0) \in H_+^n(-1)$. That is, we consider orthogonal projections to the hyperhorospheres that passe through the point $\boldsymbol{e}_0$. We observe that $HP(\boldsymbol{v}, \langle \boldsymbol{e}_0, \boldsymbol{v}\rangle) = HP(\frac{1}{v_0}\boldsymbol{v}, -1)$, so the hyperhorospheres we are considering are in fact parametrised by the sphere S^{n-1}_+. We define the fibre bundle

$$\mathcal{L} := \{(\boldsymbol{v}, q) \in S^{n-1}_+ \times H^n_+(-1) \mid \langle \boldsymbol{v}, q\rangle = -1\}.$$

By varying $\boldsymbol{v}$, we obtain a family of orthogonal projections to hyperhorospheres parametrised by vectors in S^{n-1}_+.

Theorem 3.1 *The family of orthogonal projections in $H^n_+(-1)$ to hyperhorospheres is given by*

$$\begin{array}{rccc} P_{HS}: & H^n_+(-1) \times S^{n-1}_+ & \to & \mathcal{L} \\ & (p, \boldsymbol{v}) & \mapsto & (\boldsymbol{v}, q(p, \boldsymbol{v})) \end{array}$$

where $q(p, \boldsymbol{v})$ has the following expression

$$q(p, \boldsymbol{v}) = -\frac{1}{\langle p, \boldsymbol{v}\rangle} p - \frac{1 - \langle p, \boldsymbol{v}\rangle^2}{2\langle p, \boldsymbol{v}\rangle^2} \boldsymbol{v}.$$

Proof. Let $p \in H^n_+(-1)$ and $\boldsymbol{v} \in S^{n-1}_+$. Consider the two parallel hyperhorospheres $HP(\boldsymbol{v}, -1) \cap H^n_+(-1)$ and $HP(\boldsymbol{v}, \langle p, \boldsymbol{v}\rangle) \cap H^n_+(-1)$, the first contains the point $\boldsymbol{e}_0$ and the second the point p. A geodesic orthogonal to one of these hyperhorospheres is also orthogonal to the other, and the length of the segment of such geodesics between a point on one hyperhorosphere and another point on the other hyperhorosphere is the same for all such geodesics. The geodesic in $H^n_+(-1)$ through $\boldsymbol{e}_0$ and orthogonal to $HP(\boldsymbol{v}, -1) \cap H^n_+(-1)$ is parametrised by

$$c(t) = \cosh(t)\boldsymbol{e}_0 + \sinh(t)\boldsymbol{u}, \tag{3.1}$$

where $\boldsymbol{u}$ is orthogonal to $HP(\boldsymbol{v}, -1) \cap H^n_+(-1)$ at $\boldsymbol{e}_0$ and satisfies $\langle \boldsymbol{u}, \boldsymbol{u}\rangle = 1$. A short calculation shows that

$$\boldsymbol{u} = \boldsymbol{e}_0 - \boldsymbol{v}.$$

We are seeking the expressions of $\cosh(t_0)$ and $\sinh(t_0)$ in (3.1) when $c(t_0)$ is on the hyperhorosphere $HP(\boldsymbol{v}, \langle p, \boldsymbol{v}\rangle) \cap H^n_+(-1)$. For such t_0 we have

$$\begin{array}{rcl} \langle p, \boldsymbol{v}\rangle & = & \langle c(t_0), \boldsymbol{v}\rangle \\ & = & -\cosh(t_0) + \langle \boldsymbol{u}, \boldsymbol{v}\rangle \sinh(t_0) \\ & = & -\cosh(t_0) + \langle \boldsymbol{e}_0 - \boldsymbol{v}, \boldsymbol{v}\rangle \sinh(t_0) \\ & = & -(\cosh(t_0) + \sinh(t_0)). \end{array}$$

Therefore

$$\cosh(t_0) + \sinh(t_0) = -\langle p, \boldsymbol{v}\rangle .$$

Combining the above relation with the identity $\cosh^2(t_0) - \sinh^2(t_0) = 1$ yields

$$\begin{aligned}\cosh(t_0) &= -\frac{\langle p, \boldsymbol{v}\rangle^2 + 1}{2\langle p, \boldsymbol{v}\rangle},\\ \sinh(t_0) &= -\frac{\langle p, \boldsymbol{v}\rangle^2 - 1}{2\langle p, \boldsymbol{v}\rangle}.\end{aligned}$$

Now the point $q(p, \boldsymbol{v})$, which is the orthogonal projection of p to the hyperhorosphere $HP(\boldsymbol{v}, -1) \cap H^n_+(-1)$ is given by

$$q(p, \boldsymbol{v}) = \cosh(-t_0)p + \sinh(-t_0)\boldsymbol{w},$$

with $\boldsymbol{w} = p + 1/\langle p, \boldsymbol{v}\rangle \boldsymbol{v}$. Substituting the expressions for $\cosh(t_0)$ and $\sinh(t_0)$ yields the expression of $q(p, \boldsymbol{v})$ in the statement of the theorem. ■

The projection P_{HS} can be interpreted as follows in the Poincaré ball model of $H^n_+(-1)$. Given a point $\boldsymbol{v}$ on the ideal boundary, the hyperhorospheres defined by $\boldsymbol{v}$ are the hyperspheres in the ball that are tangent to the boundary at $\boldsymbol{v}$. If we fix one of them, then the projection $q(p, \boldsymbol{v})$ is represented by the intersection of the geodesic linking $\boldsymbol{v}$ and p with the fixed hyperhorosphere. One can also define a projection to the ideal boundary by considering the point of intersection of the geodesic linking $\boldsymbol{v}$ and p with the ideal boundary. By varying $\boldsymbol{v}$, we obtain a family of projections to the ideal boundary. Under the identification between S^{n-1}_+ in the Minkowski model and the ideal boundary in the Poincaré ball model via the canonical stereographic projection, we also have the corresponding projection onto S^{n-1}_+ that we denote by $\overline{P_{LS}}$.

Theorem 3.2 *There is a bundle isomorphism taking P_{HS} to $\overline{P_{LS}}$.*

Proof. For any $\boldsymbol{v} \in S^{n-1}_+$, the tangent space of S^{n-1}_+ at $\boldsymbol{v}$ can be canonically identified with the space

$$T_{\boldsymbol{v}} S^{n-1}_+ = \{\boldsymbol{w} \in \mathbb{R}^n_0 \mid \langle \boldsymbol{v}, \boldsymbol{w}\rangle = 0\ \}.$$

We define the stereographic projection $\Pi_{\boldsymbol{v}} : S^{n-1}_+ \setminus \{\boldsymbol{v}\} \to T_{\boldsymbol{v}} S^{n-1}_+$ by

$$\Pi_{\boldsymbol{v}}(\boldsymbol{u}) = \boldsymbol{v} + \frac{\boldsymbol{v} - \boldsymbol{u}}{\langle \boldsymbol{u}, \boldsymbol{v}\rangle} - \boldsymbol{e}_0.$$

We consider the induced metric on $S^{n-1}_+ \setminus \{\boldsymbol{v}\}$ via the stereographic projection from the Euclidean space $T_{\boldsymbol{v}} S^{n-1}_+$, so that $\Pi_{\boldsymbol{v}}$ is an isometric diffeomorphism. We also define a projection $P^{\boldsymbol{v}}_{LS} : H^n_+(-1) \to S^{n-1}_+ \setminus \{\boldsymbol{v}\}$ as follows. Given a point $p \in H^n_+(-1)$, the line joining p and $\boldsymbol{v}$ meets the lightcone at another point q. Then $P_{LS}^{\boldsymbol{v}}(p)$ is defined to be the point $\widetilde{q} \in S_+^{n-1} \setminus \{\boldsymbol{v}\}$. One can show that

$$P_{LS}^{\boldsymbol{v}}(p) = \widetilde{2p + \frac{\boldsymbol{v}}{\langle p, \boldsymbol{v} \rangle}}.$$

We remark that the restriction

$$P_{LS}^{\boldsymbol{v}}|_{HP(\boldsymbol{v},-1)\cap H_+^n(-1)} : HP(\boldsymbol{v},-1) \cap H_+^n(-1) \to S_+^{n-1} \setminus \{\boldsymbol{v}\}$$

is an isometric diffeomorphism. Therefore,

$$\Pi_{\boldsymbol{v}} \circ P_{LS}^{\boldsymbol{v}}|_{HP(\boldsymbol{v},-1)\cap H_+^n(-1)} : HP(\boldsymbol{v},-1) \cap H_+^n(-1) \to T_{\boldsymbol{v}} S_+^{n-1}$$

is an isometric diffeomorphism. Varying $\boldsymbol{v}$ in S_+^{n-1} yields a family of mappings $P_{LS} : H_+^n(-1) \times S_+^{n-1} \to S_+^{n-1} \times S_+^{n-1}$ given by $P_{LS}(p, \boldsymbol{v}) = (\boldsymbol{v}, P_{LS}^{\boldsymbol{v}}(p))$.

The tangent bundle of the lightcone hypersphere is

$$TS_+^{n-1} = \{(\boldsymbol{v}, \boldsymbol{w}) \in S_+^{n-1} \times \mathbb{R}_0^n \mid \langle \boldsymbol{v}, \boldsymbol{w} \rangle = 0 \}.$$

Therefore we have a family of projections to the tangent bundle of S_+^{n-1}

$$\overline{P_{LS}} : H_+^n(-1) \times S_+^{n-1} \to TS_+^{n-1}$$

defined by $\overline{P_{LS}}(p, \boldsymbol{v}) = (1_{S_+^{n-1}} \times \Pi_{\boldsymbol{v}}) \circ P_{LS}(p, \boldsymbol{v}) = (\boldsymbol{v}, \Pi_{\boldsymbol{v}} \circ P_{LS}^{\boldsymbol{v}}(p))$. A straightforward calculation shows that $P_{LS}^{\boldsymbol{v}}(q(p, \boldsymbol{v})) = P_{LS}^{\boldsymbol{v}}(p)$, where $q(p, \boldsymbol{v})$ is as in Theorem 3.1.

Let $\Phi : \mathcal{L} \to TS_+^{n-1}$ be the mapping defined by $\Phi(\boldsymbol{v}, q) = (\boldsymbol{v}, \Pi_{\boldsymbol{v}} \circ P_{LS}^{\boldsymbol{v}}(q))$. Since $\Pi_{\boldsymbol{v}} \circ P_{LS}^{\boldsymbol{v}}|_{HP(\boldsymbol{v},-1)\cap H_+^n(-1)}$ is an isometric diffeomorphism, Φ is a bundle isomorphism and $\Phi \circ P_{HS} = \overline{P_{LS}}$.

On the Poincaré ball model of $H_+^n(-1)$, the ideal boundary can be identified with S_+^{n-1} through the canonical stereographic projection. Therefore, the bundle $\mathcal{L}$ can be identified with the tangent bundle of the ideal boundary. ∎

In this paper, the family of orthogonal projections of a given submanifold M in $H_+^n(-1)$ to hyperhorospheres refers to the restriction of the family P_{HS} to M. We still denote this restriction by P_{HS}. We have the following result where the term generic is defined in terms of transversality to submanifolds of multi-jet spaces (see for example [10]).

Theorem 3.3 *For a residual set of embeddings $\boldsymbol{x} : M \to H_+^n(-1)$, the family P_{HS} is a generic family of mappings.*

Proof. The theorem follows from Montaldi's result in [25] and the fact that $P_{HS|H_+^n(-1)}$ is a stable map. ∎

We denote by $P_{HS}^{\boldsymbol{v}}$ the map $H_+^n(-1) \to H_+^n(-1)$, given by $P_{HS}^{\boldsymbol{v}}(p) = q(p, \boldsymbol{v})$, with $q(p, \boldsymbol{v})$ as in Theorem 3.1.

3.1. Projections of surfaces in $H^3(-1)$ to horospheres

We now study projections of embedded surfaces in $H^3_+(-1)$ to horospheres. For a given $\boldsymbol{v} \in S^2_+$ and a point $p_0 \in M$, one can choose local coordinates so that $P^{\boldsymbol{v}}_{HS}$ restricted to M can be considered locally as a map-germ $\mathbb{R}^2, 0 \to \mathbb{R}^2, 0$. These map-germs are extensively studied. We refer to [27] for the list of the $\mathcal{A}$-orbits with $\mathcal{A}_e$-codimension ≤ 6, where $\mathcal{A}$ denotes the Mather group of smooth changes of coordinates in the source and target. In Table 1, we reproduce from [27] the list of local singularities of $\mathcal{A}_e$-codimension ≤ 3. Some of these singularities are also called as follows: 4_2 (lips/beaks), 4_3 (goose), 5 (swallowtail), 6 (butterfly), 11_5 (gulls). The multi-local singularities of $\mathcal{A}_e$-codimension ≤ 2 are as follows:

codimension 0:	double fold.
codimension 1:	triple fold; double tangent fold; fold plus cusp.
codimension 2:	quadruple fold; double cusp; double fold plus cusp; double tangent; fold plus fold; 3-point contact folds; cusp plus tangent fold; swallowtail plus fold; lips/beaks plus fold.

Table 1: $\mathcal{A}_e$-codimension ≤ 3 local singularities of map-germs $\mathbb{R}^2, 0 \to \mathbb{R}^2, 0$ ([27]).

Name	Normal form	$\mathcal{A}_e$-codimension
Immersion	(x, y)	0
Fold	(x, y^2)	0
Cusp	$(x, xy + y^3)$	0
4_k	$(x, y^3 \pm x^k y), k = 2, 3, 4$	$k - 1$
5	$(x, xy + y^4)$	1
6	$(x, xy + y^5 \pm y^7)$	2
7	$(x, xy + y^5)$	3
11_{2k+1}	$(x, xy^2 + y^4 + y^{2k+1}), k = 2, 3$	k
12	$(x, xy^2 + y^5 + y^6)$	3
16	$(x, x^2 y + y^4 \pm y^5)$	3

It follows from Theorem 3.3 that for generic embeddings of the surface only singularities of $\mathcal{A}_e$-codimension $\leq \dim(S^2_+) = 2$ can occur in the members of the family of orthogonal projections. So we have the following result.

Proposition 3.4 *For a residual set of embeddings $\boldsymbol{x} : M \to H^3_+(-1)$, the projections $P^{\boldsymbol{v}}_{HS} : M \to H^3_+(-1)$ in the family P_{HS} have local singularities $\mathcal{A}$-equivalent to one in Table 1 whose $\mathcal{A}_e$-codimension ≤ 2. Moreover, these singularities are versally unfolded by the family P_{HS}.*

The members of P_{HS} can also have multi-local local singularities $\mathcal{A}$-equivalent to one listed above with $\mathcal{A}_e$-codimension ≤ 2, and these singularities are also versally unfolded by the family P_{HS}. In this paper, we deal mainly with the geometry of the local singularities.

As A_p and S_p are self-adjoint operators on M we can define the notion of asymptotic directions at p. We say that $u \in T_pM$ is a *de Sitter* (*resp. horospherical*) *asymptotic direction* if and only if $\langle A_p.u, u\rangle = 0$ (resp. $\langle S_p.u, u\rangle = 0$). There are 0/1/2 de Sitter (resp. horospherical) asymptotic directions at every point where $K_e(p)$ (resp. $K_h(p)$) $0 > / = / < 0$.

Given $\boldsymbol{v} \in S^2_+$ and a point q on the horosphere $HP(\boldsymbol{v}, \langle q, \boldsymbol{v}\rangle) \cap H^3_+(-1)$, we denote by $\boldsymbol{v}^*$ the projection of $\boldsymbol{v}$ in the direction of q (considered as a vector in $\mathbb{R}^4_1$) to the tangent space of the horosphere at q. We have $\boldsymbol{v}^* = \boldsymbol{v} + \langle q, \boldsymbol{v}\rangle\, q$, and the map $\boldsymbol{v} \mapsto \boldsymbol{v}^*/||\boldsymbol{v}^*|| = -(\boldsymbol{v}/\langle q, \boldsymbol{v}\rangle + q)$ from S^2_+ to $T_qH^3_+(-1) \cap S^3_1$ is one-to-one. Also, given two parallel horospheres defined by $\boldsymbol{v} \in S^2_+$ and a geodesic orthogonal to both of them at p and q respectively, then the vector $\boldsymbol{v}^*$ associated to $\boldsymbol{v}$ is the same at p and q. The types of singularities in the following theorem are those in Table 1.

Theorem 3.5 *Let M be an embedded surface in $H^3_+(-1)$ and $\boldsymbol{v} \in S^2_+$.*

(1) *The projection $P^{\boldsymbol{v}}_{HS}$ is singular at a point $p \in M$ if and only if $\boldsymbol{v}^* \in T_pM$.*

(2) *The singularity of $P^{\boldsymbol{v}}_{HS}$ at p is of type cusp or worse if and only if $\boldsymbol{v}^*$ is a de Sitter asymptotic direction at p. In particular, p is a de Sitter hyperbolic or parabolic point.*

(3) *The singularities of $P^{\boldsymbol{v}}_{HS}$ of type 5 (swallowtail) occur generically on a curve in the de Sitter hyperbolic region, labelled the* horosphere flecnodal curve. *This curve can be characterised as the locus of points where the de Sitter asymptotic curves have geodesic inflections.*

(4) *The singularities of $P^{\boldsymbol{v}}_{HS}$ at p is of type 4_2 or 4_3 if and only if p is a de Sitter parabolic point but not a swallowtail point of the de Sitter Gauss map and $\boldsymbol{v}^*$ is the unique de Sitter asymptotic direction there. Singularities of type 11_5 occur at swallowtail points of the de Sitter Gauss map.*

Proof. We shall take the surface M in hyperbolic Monge form (H-Monge form, see [17]) at the point in consideration. In fact, by hyperbolic motions, we can suppose that the point of interest is $\boldsymbol{e}_0 = (1, 0, 0, 0)$ and the surface is given in H-Monge form

$$\boldsymbol{x}(x, y) = \left(\sqrt{f^2(x, y) + x^2 + y^2 + 1}, f(x, y), x, y\right),$$

with (x, y) in some neighbourhood of the origin. Here f is a smooth function with $f(0,0) = 0$ and $f_x(0,0) = f_y(0,0) = 0$. So a unit normal to M at $\boldsymbol{e}_0$ is given by $\boldsymbol{n}(0,0) = (0,1,0,0)$. We shall write the Taylor expansion of f at the origin in the form

$$f(x,y) = a_{20}x^2 + a_{21}xy + a_{22}y^2 + \sum_{i=0}^{3} a_{3i}x^{3-i}y^i + \sum_{i=0}^{4} a_{4i}x^{4-i}y^i + \text{h.o.t.}$$

Let $\boldsymbol{v} = (1, v_1, v_2, v_3) \in S^2_+$, so that at $\boldsymbol{e}_0$ we have $\boldsymbol{v}^* = (0, v_1, v_2, v_3)$. Then

$$\begin{aligned} \partial P^{\boldsymbol{v}}_{HS}/\partial x(0,0) &= (0, v_1v_2, 1 + v_2^2, v_2v_3) \\ \partial P^{\boldsymbol{v}}_{HS}/\partial y(0,0) &= (0, v_1v_3, v_2v_3, 1 + v_3^2) \end{aligned}$$

and these two vectors are linearly dependent if and only if $v_1 = 0$, if and only if $\boldsymbol{v}^* \in T_{\boldsymbol{e}_0}M$, which proves (1).

For the remaining cases we take, without loss of generality, $\boldsymbol{v} = (1,0,0,1)$. The restriction of the projection $\pi(x_0, x_1, x_2, x_3) \mapsto (0, x_1, x_2, 0)$ to the horosphere is a submersion at $\boldsymbol{e}_0$. As the singularities of $P^{\boldsymbol{v}}_{HS}$ and those of $\pi \circ P^{\boldsymbol{v}}_{HS}$ are $\mathcal{A}$-equivalent, we study $\pi \circ P^{\boldsymbol{v}}_{HS}$ instead. We have

$$\pi \circ P^{\boldsymbol{v}}_{HS}(x,y) = \Big(\frac{f(x,y)}{\sqrt{f^2(x,y) + x^2 + y^2 + 1}}, \frac{x}{\sqrt{f^2(x,y) + x^2 + y^2 + 1}}\Big).$$

We can now analyse the appropriate k-jets of $\pi \circ P^{\boldsymbol{v}}_{HS}$ and interpret geometrically the conditions for it to be $\mathcal{A}$-equivalent to a given singularity. For example, we have a fold singularity if and only if $a_{20} \neq 0$, if and only if $\boldsymbol{v}^* = (0,0,0,1)$ is not a de Sitter asymptotic direction at $\boldsymbol{e}_0$. The singularity is of type cusp if and only if $a_{20} = 0$ and $a_{21}a_{33} \neq 0$, and is of type swallowtail if and only if $a_{20} = a_{33} = 0$ and $a_{21}a_{44} \neq 0$.

The equation of the asymptotic curves in the parameter space is given by $ldx^2 + 2mdxdy + ndy^2 = 0$, where l, m, n are the coefficients of the de Sitter second fundamental form. Suppose that the projection in the direction of $\boldsymbol{v} = (1,0,0,1)$ has a singularity worse than fold at $\boldsymbol{e}_0$ and assume that this point is not a de Sitter parabolic point, i.e. $a_{20} = 0$ and $a_{21} \neq 0$. Then the de Sitter asymptotic curve tangent to $\boldsymbol{v}^*$ is parametrised by

$$\gamma(t) = \Big(1 + \frac{1}{2}t^2, -\frac{1}{2}a_{33}t^2, -\frac{3}{2}\frac{a_{33}}{a_{21}}t^2, t\Big) + \text{h.o.t.}$$

The geodesic curvature of this asymptotic curve at $\boldsymbol{e}_0$ is $-3a_{33}/a_{21}$ and its curvature, as a curve in $H^3_+(-1)$ (see §2 for definition), is given by $|a_{33}|\sqrt{1 + 9/a_{21}^2}$. Both these curvatures vanish at $\boldsymbol{e}_0$ if and only if $a_{33} = 0$, if and only if the singularity of the projection is of type swallowtail or worse.

The analysis for remaining cases is similar to the one above. ∎

We call the image of the critical set of $P^{\boldsymbol{v}}_{HS}$ the *contour* (or *profile*) of M in the direction $\boldsymbol{v}$. This is generically a curve on a horosphere. We shall suppose here that it is a smooth curve. (The bifurcations of the contour as $\boldsymbol{v}$ varies in S^2_+ are similar to those of the contour of a surface in the Euclidean space $\mathbb{R}^3$ and can be found in [1].) Let p be a point on M. We call the intersection of M with the 3-dimensional space generated by the vectors p, $\boldsymbol{v}$ and $\boldsymbol{e}(p)$ the *normal section* of M at p along $\boldsymbol{v}$. Koenderink showed in [24] that for embedded surfaces in $\mathbb{R}^3$, the Gaussian curvature of the surface at a given point is the product of the curvature of the contour with the curvature of the normal section in the direction of projection. We have the following result for projections of surfaces in $H^3_+(-1)$ to horospheres, where the curvature of a curve in $H^3_+(-1)$ is as given in §2.

Theorem 3.6 (Koenderink type theorem) *Let κ_c be the curvature of the contour and κ_n the curvature of the normal section in the projection direction. Then the de Sitter Gaussian curvature of the surface is given by*

$$K_e = \kappa_n \sqrt{\kappa_c^2 - 1}.$$

Proof. We consider the H-Monge form setting of the proof of Theorem 3.5 and take $\boldsymbol{v} = (1, 0, 0, 1)$. We assume that the singularity of the projection is a fold at $\boldsymbol{e}_0$, so $a_{22} \neq 0$. Then the 2-jet of the profile is given by

$$(1 + \frac{1}{2}t^2, \frac{4a_{20}a_{22} - a_{21}^2}{4a_{22}}t^2, t - \frac{a_{21}}{2a_{22}}t^2, \frac{1}{2}t^2),$$

so, following the formula in §2, its curvature at $\boldsymbol{e}_0$ is given by

$$\kappa_c^2 = \frac{(4a_{20}a_{22} - a_{21}^2)^2}{4a_{22}^2} + 1.$$

The normal section of the surface along $\boldsymbol{v}$ is given by

$$(\sqrt{f(0, y)^2 + y^2 + 1}, f(0, y), 0, y)$$

and its curvature at $\boldsymbol{e}_0$ is given by $\kappa_n = 2a_{22}$. Given the fact that the de Sitter Gaussian curvature $K_e = 4a_{20}a_{22} - a_{21}^2$ at $\boldsymbol{e}_0$, it follows that

$$\kappa_c^2 = \frac{K_e^2}{\kappa_n^2} + 1.$$

We remark that $K_I \equiv 0$ (i.e. flat in the intrinsic sense) for a horosphere, so that $K_e \equiv 1$. This explains why we have $+1$ in the last formula. ∎

3.2. Duality

We prove in this section duality result similar to those in [29] for central projections of surfaces in $\mathbb{R}P^3$. Following the notation in [29], let S be a two-dimensional surface in $\mathbb{R}P^3$ and q a point in $\mathbb{R}P^3$. The pencil of lines through q form a two dimensional projective space Q and one obtains a bundle $\mathbb{R}P^3 \setminus q \to Q$. The projection of the surface S from the point q is the diagram $S \hookrightarrow \mathbb{R}P^3 \setminus q \to Q$. For a generic surface, a germ of a projection is equivalent to one of 14 non-equivalent types of projections [30]. Three of these types occur when one projects from a point in an open set of $\mathbb{R}P^3$ and the rest when projecting from points on the bifurcation set of the family of projections parametrised by points in $\mathbb{R}P^3$. One component of the bifurcation set is the ruled surface A_2^{par} swept out by the asymptotic lines with origins at the parabolic points of S. Another stratum of the bifurcation set involving local singularities is the ruled surface A_3 swept out by the asymptotic lines of S which are tangent to S of order at least three (the origin of such lines form a smooth curve on S). The projection can have multi-local singularities. Three other ruled surfaces are considered in [29]. These are the A_1^3 whose lines are tangent to S at three points or more, $A_1 \times A_2$ whose lines are tangent to S at three points or more, so that each line is asymptotic tangent at one of the points, and the surface $A_1||A_1$ whose lines are tangent to S at two points, so that for each line, the projective planes tangent to S at the points coincide. The following result is proved in [29], where the dual surface S^* is the wavefront of $S \hookrightarrow PT^*\mathbb{R}P^3$. (The projectivised cotangent bundle $PT^*\mathbb{R}P^3$ is given the canonical contact structure, see [2] for more details.)

Theorem 3.7 ([29])

(1) A_2^{par} *is the front of the cuspidal edge of the surface* S^*.

(2) $A_1||A_1$ *is the front of the self-intersection line of the surface* S^*.

(3) *The surfaces* A_3, A_1^3, $A_1 \times A_2$ *are self-dual, i.e. the surface dual to these surfaces are the corresponding objects of the surface* S^*.

There are Euclidean analogues in [7] of the results in [29] (see also [3, 4, 6] for related results). It is shown for example in [7] that the dual of the A_2-stratum of the bifurcation set of the family of height functions on a smooth surface in $\mathbb{R}^3$ is dual to the lips/beaks stratum of the family of orthogonal projections of the surface. (As pointed out in §2, duality in [7] refers to the double Legendrian fibration $S^2 \xleftarrow{\pi_1} \Delta \xrightarrow{\pi_2} S^2$, where S^2 is the unit sphere in $\mathbb{R}^3$ and $\Delta = \{(u,v) \in S^2 \times S^2 \mid u.v = 0\}$. The contact structure on Δ is given by the 1-form $\theta = v.du|\Delta$.)

Let M be an embedded surface in $H^3_+(-1)$. The situation here is different from that in [29]. We shall use the duality concepts in [11, 12, 22] (see Section 2), so the Δ_1-dual of the surface M does not live in the dual space of the ambient space $H^3_+(-1)$ of the surface M. Also, the bifurcation set of the family of projections P_{HS} is not a subset of $H^3_+(-1)$. However, we still obtain results similar to those in [29].

We denote by A_2^{par} the ruled surface in $H^3_+(-1)$ swept out by the geodesics in $H^3_+(-1)$ with origins at the de Sitter parabolic points of M and whose tangent directions at these points are along the unique de Sitter asymptotic directions. We also denote by $A_1||A_1$ the ruled surface swept out by the geodesics in $H^3_+(-1)$ that are tangent to M at two points where the normals to M at such points are parallel. (So the projection $P^{\boldsymbol{v}}_{HS}$, with $\boldsymbol{v}$ well chosen, has a multi-local singularity of type double tangent fold or worse.)

Theorem 3.8 *Let M^* be the Δ_1-dual of the surface M embedded in $H^3_+(-1)$. Then,*

(1) *The Δ_1-dual of the surface A_2^{par} is the cuspidaledge of M^*.*

(2) *The Δ_1-dual of the surface $A_1||A_1$ is the self-intersection line of M^*.*

Proof. (1) We suppose that the de Sitter parabolic set $K_e^{-1}(0)$ is a regular curve. This property holds for generic embeddings of surfaces in $H^3_+(-1)$. Let $p(t)$, $t \in I$, be a parametrisation of the de Sitter parabolic set of M and $\boldsymbol{u}_i(t)$, $i = 1, 2$, denote the unit principal directions of M at $p(t)$. Suppose, without loss of generality, that the unique asymptotic direction at $p(t)$ is along $\boldsymbol{u}_1(t)$. Then we have the following local parametrisation of A_2^{par}:

$$\boldsymbol{y}(s,t) = \cosh(s)p(t) + \sinh(s)\boldsymbol{u}_1(t).$$

The normal to the surface A_2^{par} (in $H^3_+(-1)$) is along

$$\boldsymbol{y} \wedge \boldsymbol{y}_s \wedge \boldsymbol{y}_t = \cosh(s)p(t) \wedge \boldsymbol{u}_1(t) \wedge p'(t) + \sinh(s)p(t) \wedge \boldsymbol{u}_1(t) \wedge \boldsymbol{u}_1'(t).$$

At a generic point p on the de Sitter parabolic set (i.e. away from swallowtail of the de Sitter Gauss map), the de Sitter asymptotic direction is transverse to the parabolic set, so $p(t) \wedge \boldsymbol{u}_1(t) \wedge p'(t)$ is along $\boldsymbol{e}(p(t))$. It follows from Lemma 3.11 below that $p(t) \wedge \boldsymbol{u}_1(t) \wedge \boldsymbol{u}_1'(t)$ is also along $\boldsymbol{e}(p(t))$. Therefore $\boldsymbol{y} \wedge \boldsymbol{y}_s \wedge \boldsymbol{y}_t$ is along $\boldsymbol{e}(p(t))$. So the normal to the ruled surface A_2^{par} is constant along the rulings and is given by the normal vector $\boldsymbol{e}(p(t))$ to M at $p(t)$. This means that A_2^{par} is a de Sitter developable surface. Therefore, the Δ_1-wavefront of A_2^{par} is $\{\boldsymbol{e}(p),\ p \text{ a de Sitter parabolic point}\}$. This is precisely the singular set (i.e. the cuspidaledge) of the Δ_1-dual surface of M.

(2) Suppose a multi-local singularity (double tangent fold) occurs at two points p_1 and p_2 on M. The surface $A_1||A_1$ is then a ruled surface generated by geodesics along a curve C_1 on M through p_1 (or a curve C_2 on M through p_2). The normals to the surface at points on C_1 and C_2 that are on the same ruling of $A_1||A_1$ are parallel. Let $q(t)$ be a local parametrisation of the curve C_1 and $\boldsymbol{u}(t)$ be the unit tangent direction to the ruling in $A_1||A_1$ through $q(t)$. Then a parametrisation of $A_1||A_1$ is given by

$$w(s,t) = \cosh(s)q(t) + \sinh(s)\boldsymbol{u}(t).$$

The normal to this surface is along $\cosh(s)V_1(t) + \sinh(s)V_2(t)$ with $V_1(t) = q(t) \wedge \boldsymbol{u}(t) \wedge q'(t)$ and $V_2(t) = q(t) \wedge \boldsymbol{u}(t) \wedge \boldsymbol{u}'(t)$. These normals are parallel at two points on any ruling, one point being on the curve C_1 and the other on C_2. Therefore $V_1(t)$ and $V_2(t)$ are parallel, so the normal to the surface $A_1||A_1$ is constant along the rulings of this surface. As these are along the normal to the surface at $q(t)$, it follows that the Δ_1-wavefront of $A_1||A_1$ is $\{\boldsymbol{e}(p),\ p \in C_1\} = \{\boldsymbol{e}(p),\ p \in C_2\}$. This is precisely the self-intersection line of M^*, the Δ_1-dual surface of M. ■

With the notation in the proof of Theorem 3.8, the cuspidaledge of M^* (the Δ_1-dual of M) is parametrised by $\mathbb{E}(p(t))$ (recall that $M^* = \mathbb{E}(M)$ by definition). Theorem 3.8 asserts that $L(s,t) = (\boldsymbol{y}(s,t), \mathbb{E}(p(t)))$ is a Legendrian embedding into Δ_1. This can be checked directly using the parametrisation $L(s,t)$.

We consider now other dualities pointed out in Section 2. We define a diffeomorphism $\Psi_1 : H^3_+(-1) \times S^2_+ \to \Delta_1$ by

$$\Psi_1(q, \boldsymbol{v}) = (q, -\frac{\boldsymbol{v}}{\langle q, \boldsymbol{v}\rangle} - q).$$

The inverse mapping $\Psi_1^{-1} : \Delta_1 \to H^3_+(-1) \times S^2_+$ is given by

$$\Psi_1^{-1}(q, \boldsymbol{w}) = (q, \widetilde{q + \boldsymbol{w}}),$$

so $\widetilde{p(t) + \boldsymbol{u}_1}(t)$ gives a parametrisation of the stratum $Bif(P_{HS}, lips/beaks)$ in S^2_+. Let

$$\Sigma(4_2) = \{(q, \boldsymbol{v}) \in H^3_+(-1) \times S^2_+ \mid P^{\boldsymbol{v}}_{HS} \text{ has a singularity at } q \text{ of type } 4_2\},$$

so that $\pi(\Sigma(4_2)) = Bif(P_{HS}, lips/beaks)$, where $\pi : H^3_+(-1) \times S^2_+ \to S^2_+$ is the canonical projection. Therefore we have

$$\Psi_1\left(\overline{\Sigma(4_2)}\right) = \{(q, \boldsymbol{w}) \mid \boldsymbol{w} \text{ is the unique asymptotic direction at } q \in K_e^{-1}(0)\ \}.$$

Moreover, we define a surface in the lightcone by

$$\begin{aligned}\boldsymbol{z}(s,t) &= \boldsymbol{y}(s,t)+\mathbb{E}(p(t))\\ &= \cosh(s)p(t)+\sinh(s)\boldsymbol{u}_1(t)+\mathbb{E}(p(t))\end{aligned}$$

with notation as in the proof of Theorem 3.8. We now define the mappings $\Phi_{12}:\Delta_1\to\Delta_2$ and $\Phi_{13}:\Delta_1\to\Delta_3$ by $\Phi_{12}(q,\boldsymbol{w})=(q,q+\boldsymbol{w})$ and $\Phi_{13}(q,\boldsymbol{w})=(q+\boldsymbol{w},\boldsymbol{w})$. These mappings are contact diffeomorphisms. Since $\boldsymbol{y}(s,t)$ and $\mathbb{E}(p(t))$ are Δ_1-dual, it follows that $\boldsymbol{y}(s,t)$ and $\boldsymbol{z}(s,t)$ are Δ_2-dual and $\boldsymbol{z}(s,t)$ and $\mathbb{E}(p(t))$ are Δ_3-dual. We have therefore shown the following result.

Theorem 3.9 *Let M^* be the Δ_1-dual of the surface M embedded in $H^3_+(-1)$. Then the Δ_2-dual of A_2^{par} is the Δ_3-dual of the cuspidaledge of M^*.*

Remark 3.10 In Shcherback's Theorem 3.7, the surfaces A_3, A_1^3 and $A_1\times A_2$ are self-dual. In our case, we need the analogues of these surfaces for M^*. As M^* is not in $H^3_+(-1)$, we need to define the concept of projections for surfaces embedded in the de Sitter and lightcone pseudo-spheres. This will be dealt with in a forthcoming paper.

In the proof of Theorem 3.8 we used the following result.

Lemma 3.11 *Let M be a generic surface in $H^3_+(-1)$. Then the derivative of the de Sitter (resp. lightcone) asymptotic direction along the de Sitter (resp. lightcone) parabolic curve is tangent to the surface M.*

Proof. We consider the de Sitter case and the lightcone case follows in a similar way. We can suppose that the surface is parametrised by $\phi(x,y)$, where $x=const.$ and $y=const.$ represent the lines of curvature of M. Let $p(t)$ be a local parametrisation of the de Sitter parabolic curve. Then the unique de Sitter asymptotic direction on the parabolic set is also a principal direction. Suppose without loss of generality that this principal direction is $\boldsymbol{u}_1(t)$. Then $\boldsymbol{u}_1(t)=\lambda(t)\phi_x(p(t))=\lambda(t)\phi_x(x(t),y(t))$, where $\lambda(t)=1/||\phi_x(x(t),y(t))||$. Therefore $\boldsymbol{u}_1'(t)=\lambda(t)(x'(t)\phi_{xx}(p(t))+y'(t)\phi_{xy}(p(t)))+\lambda'(t)\phi_x(p(t))$. The coefficients of the de Sitter second fundamental form are given by $l=\langle\phi_{xx},\boldsymbol{e}\rangle=\kappa_1/E$, $m=\langle\phi_{xy},\boldsymbol{e}\rangle=0$ and $n=\langle\phi_{yy},\boldsymbol{e}\rangle=\kappa_2/G$ (where E,F,G are the coefficients of the first fundamental form). So

$$\begin{aligned}\langle\boldsymbol{u}_1'(t),\boldsymbol{e}(t)\rangle &= \lambda(t)\left(\langle\phi_{xx}(p(t)),\boldsymbol{e}(t)\rangle\, x'(t)+\langle\phi_{xy}(p(t)),\boldsymbol{e}(t)\rangle\, y'(t)\right)\\ &= \lambda(t)\kappa_1(t)/E\\ &= 0,\end{aligned}$$

and hence $\boldsymbol{u}_1'(t)\in T_{p(t)}M$. ∎

4. Projections to hyperplanes

We begin, as in Section 3, by considering the general case of orthogonal projections in $H^n_+(-1)$, for $n \geq 3$, to hyperplanes. Let $HP(\boldsymbol{v}, 0)$ be a timelike hyperplane (so $\boldsymbol{v} \in S^n_1$, that is, $\langle \boldsymbol{v}, \boldsymbol{v} \rangle = 1$). Given a point $p \in H^n_+(-1)$, there is a unique geodesic in $H^n_+(-1)$ which intersects orthogonally the hyperplane $HP(\boldsymbol{v}, 0) \cap H^n_+(-1)$ at some point $r(p, \boldsymbol{v})$. We call the point $r(p, \boldsymbol{v})$ the orthogonal projection of p in the direction of $\boldsymbol{v}$ to the hyperplane $HP(\boldsymbol{v}, 0) \cap H^n_+(-1)$. The space $HP(\boldsymbol{v}, 0)$ can be identified with the tangent space of S^n_1 at $\boldsymbol{v}$.

Theorem 4.1 *The family of orthogonal projections in $H^n_+(-1)$ to hyperplanes is given by*

$$\begin{array}{cccc} P_P: & H^n_+(-1) \times S^n_1 & \to & TS^n_1 \\ & (p, \boldsymbol{v}) & \mapsto & (\boldsymbol{v}, r(p, \boldsymbol{v})) \end{array}$$

where $r(p, \boldsymbol{v})$ has the following expression

$$r(p, \boldsymbol{v}) = \frac{1}{\sqrt{1 + \langle \boldsymbol{v}, p \rangle^2}} \big(p - \langle p, \boldsymbol{v} \rangle \boldsymbol{v} \big).$$

Proof. Let $p \in H^n_+(-1)$ and $\boldsymbol{v} \in S^n_1$. We consider the equidistant hypersurface $HP(\boldsymbol{v}, \langle p, \boldsymbol{v} \rangle) \cap H^n_+(-1)$ through p and the geodesic

$$c(t) = \cosh(t) p + \sinh(t) \boldsymbol{u} \tag{4.1}$$

orthogonal to $HP(\boldsymbol{v}, \langle p, \boldsymbol{v} \rangle) \cap H^n_+(-1)$ at p and to $HP(\boldsymbol{v}, 0) \cap H^n_+(-1)$ at $r(p, \boldsymbol{v})$. The vector $\boldsymbol{u}$ is given by

$$\boldsymbol{u} = \frac{1}{\sqrt{1 + \langle p, \boldsymbol{v} \rangle^2}} \big(\boldsymbol{v} + \langle p, \boldsymbol{v} \rangle p \big).$$

We are seeking the expressions of $\cosh(t_0)$ and $\sinh(t_0)$ in (4.1) when $c(t_0)$ is on the hyperplane $HP(\boldsymbol{v}, 0)$. For such t_0 we have

$$\begin{array}{rcl} \langle c(t_0), \boldsymbol{v} \rangle & = & \langle p, \boldsymbol{v} \rangle \cosh(t_0) + \langle \boldsymbol{u}, \boldsymbol{v} \rangle \sinh(t_0) \\ & = & \langle p, \boldsymbol{v} \rangle \cosh(t_0) + \sqrt{1 + \langle p, \boldsymbol{v} \rangle^2} \sinh(t_0) \\ & = & 0 \end{array}$$

Therefore

$$\sinh(t_0) = -\frac{\langle p, \boldsymbol{v} \rangle}{\sqrt{1 + \langle p, \boldsymbol{v} \rangle^2}} \cosh(t_0).$$

Combining the above relation with the identity $\cosh^2(t_0) - \sinh^2(t_0) = 1$ yields

$$\begin{array}{rcl}\cosh(t_0) & = & \sqrt{1 + \langle p, \boldsymbol{v}\rangle^2} \\ \sinh(t_0) & = & -\langle p, \boldsymbol{v}\rangle .\end{array}$$

The point $r(p, \boldsymbol{v})$ is given by $r(p, \boldsymbol{v}) = \cosh(t_0)p + \sinh(t_0)\boldsymbol{u}$. Substituting the expressions of $\cosh(t_0)$, $\sinh(t_0)$ and $\boldsymbol{u}$ yields the expression of $r(p, \boldsymbol{v})$ in the statement of the theorem. ■

The family of orthogonal projections of a given submanifold M in $H^n_+(-1)$ to hyperplanes is the restriction of the family P_P to M. We still denote this restriction by P_P.

Theorem 4.2 *For a residual set of embeddings $\boldsymbol{x} : M \to H^n_+(-1)$, the family P_P is a generic family of mappings.*

Proof. The theorem follows from Montaldi's result in [25] and the fact that $P_{P|H^n_+(-1)}$ is a stable map. ■

We denote by $P_P^{\boldsymbol{v}}$ the map $H^n_+(-1) \to H^n_+(-1)$, given by $P_P^{\boldsymbol{v}}(p) = r(p, \boldsymbol{v})$, with $r(p, \boldsymbol{v})$ as in Theorem 4.1

4.1. Projections of surfaces in $H^3(-1)$ to planes

We consider now embedded surfaces in $H^3_+(-1)$. For a given $\boldsymbol{v} \in S^3_1$ and a point $p_0 \in M$, one can choose local coordinates so that $P_P^{\boldsymbol{v}}$ restricted to M can be considered locally as a map-germ $\mathbb{R}^2, 0 \to \mathbb{R}^2, 0$. It follows from Theorem 4.2 that for generic embeddings of the surface, only singularities of $\mathcal{A}_e$-codimension $\leq \dim(S^3_1) = 3$ can occur in the members of the family of orthogonal projections. So we have the following result.

Proposition 4.3 *For a residual set of embeddings $\boldsymbol{x} : M \to H^3_+(-1)$, the projections $P_P^{\boldsymbol{v}} : M \to H^3_+(-1)$ in the family P_P have local singularities $\mathcal{A}$-equivalent to one in Table 1. Moreover, these singularities are versally unfolded by the family P_P.*

(The projection $P_P^{\boldsymbol{v}}$ can also have multi-local singularities of $\mathcal{A}_e$-codimension ≤ 3 and these singularities are versally unfolded by the family P_P; see §3.1 for the codimension ≤ 2 singularities.)

Given $\boldsymbol{v} \in S^3_1$ and a point q on the equidistant surface $HP(\boldsymbol{v}, \langle q, \boldsymbol{v}\rangle) \cap H^3_+(-1)$, we denote by $\boldsymbol{v}^*$ the projection of $\boldsymbol{v}$ in the direction of q to $T_q(HP(\boldsymbol{v}, \langle q, \boldsymbol{v}\rangle) \cap H^3_+(-1))$. Observe that when q is on $HP(\boldsymbol{v}, 0) \cap H^3_+(-1)$, then $\boldsymbol{v}^* = \boldsymbol{v}$. The map $\boldsymbol{v} \mapsto \boldsymbol{v}^*/||\boldsymbol{v}^*||$ from $S_1^3 \to T_qH_+^3(-1) \cap S_1^3$ is a submersion. In this case, the pre-image of a unit direction in $T_qH_+^3(-1)$ is a curve on S_1^3. The geodesic through a point $q \in HP(\boldsymbol{v}, 0) \cap H_+^3(-1)$ with tangent $\boldsymbol{v}$ at q intersects orthogonally any equidistant surface at some point p and its tangent there is the parallel transport of $\boldsymbol{v}$ to p, which is the vector $\boldsymbol{v}^*/||\boldsymbol{v}^*||$.

Theorem 4.4 *Let M be an embedded surface in $H_+^3(-1)$ and $\boldsymbol{v} \in S_1^3$.*

(1) *The projection $P_P^{\boldsymbol{v}}$ is singular at a point $p \in M$ if and only if the parallel transport $\boldsymbol{v}^*$ of $\boldsymbol{v}$ to the point p is in T_pM.*

(2) *The singularity of $P_P^{\boldsymbol{v}}$ at p is of type cusp or worse if and only if $\boldsymbol{v}^*$ is a de Sitter asymptotic direction at p. In particular, p is a de Sitter hyperbolic or parabolic point.*

(3) *The singularity of $P_P^{\boldsymbol{v}}$ at p is of type* 5 *(swallowtail) or worse if and only if $\boldsymbol{v}^*$ is a de Sitter asymptotic direction and p is a point on the horosphere flecnodal curve (see Theorem* 3.5(3)*).*

(4) *The singularity of $P_P^{\boldsymbol{v}}$ at p is of type* 6 *if and only if $\boldsymbol{v}^*$ is a de Sitter asymptotic direction and p is a point on the horosphere flecnodal curve where the asymptotic curve has a higher geodesic inflection. There is a unique direction $\boldsymbol{v} \in S_1^3$ where the singularity becomes of type* 7*.*

(5) *The singularities of $P_P^{\boldsymbol{v}}$ at p is of type 4_k, $k = 2, 3, 4$, if and only if p is a de Sitter parabolic point but not a swallowtail point of the de Sitter Gauss map and $\boldsymbol{v}^*$ is the unique de Sitter asymptotic direction there. There is a unique direction $\boldsymbol{v} \in S_1^3$ where the singularity becomes of type 4_3, and isolated points on the parabolic set where it becomes of type 4_4. At a swallowtail point of the de Sitter Gauss map, the singularity is of type 11_5 in general and for single directions $\boldsymbol{v} \in S_1^3$, it becomes of type 11_7 or of type* 12*.*

Proof. The proof follows by similar calculations to those in the proof of Theorem 3.5. We take the surface in H-Monge form at $\boldsymbol{e}_0$. When the projection is singular, we set $\boldsymbol{v} = (v_0, 0, 0, v_3)$ and consider the singularities of the modified projection $\pi \circ P_P^{\boldsymbol{v}}$ given by

$$\pi \circ P_P^{\boldsymbol{v}}(x, y) = \left(\frac{f(x, y)}{\lambda_{\boldsymbol{v}}(x, y)}, \frac{x}{\lambda_{\boldsymbol{v}}(x, y)}\right),$$

with $\lambda_{\boldsymbol{v}}(x, y) = (1 + (-v_0\sqrt{f^2(x, y) + x^2 + y^2 + 1} + v_3 y)^2)^{1/2}$ and π is as in the proof of Theorem 3.5. The results can then be obtained by analysing the map-germ $\pi \circ P_P^{\boldsymbol{v}}$. ■

Theorem 4.5 (Koenderink type theorem) *Let κ_c be the curvature of the contour and κ_n the curvature of the normal section in the projection direction. In general, the product of the hyperbolic curvature of the profile and of the normal section depends on the plane of projection. However, if the point on the surface is also on the plane of projection (alternatively, if $\boldsymbol{v} \in T_pM$) then*

$$K_e = \kappa_n \kappa_c.$$

Proof. We consider the H-Monge form setting of the proof of Theorem 3.5 and take $\boldsymbol{v} = (v_0, 0, 0, v_3) \in S_1^3$. We assume that the singularity of the projection is a fold at $\boldsymbol{e}_0$, so $a_{22} \neq 0$. Then the 2-jet of the profile is given by

$$\frac{1}{\sqrt{1+v_0^2}}\Big((\frac{3}{2}+v_0^2)t^2, \frac{4a_{20}a_{22}-a_{21}^2}{4a_{22}}t^2, t - \frac{v_0v_3a_{21}}{2(1+v_0^2)a_{22}}t^2, v_0v_3 + \frac{v_0v_3}{2(1+v_0^2)}t^2\Big).$$

A calculation shows that its curvature at $\boldsymbol{e}_0$ is given by

$$\kappa_c^2 = (1+v_0^2)\frac{K^2}{\kappa_n^2} + v_0^6\frac{a_{21}^2}{a_{22}^2}.$$

The above expression depends on $\boldsymbol{v}$. If $v_0 = 0$ (equivalently, if $\boldsymbol{v}^* = \boldsymbol{v}$ which means that $\boldsymbol{e}_0$ is on the hyperplane $HP(\boldsymbol{v}, 0) \cap H_+^3(-1)$ so $\boldsymbol{v} \in T_{\boldsymbol{e}_0}M$) then $K_e^2 = (\kappa_c\kappa_n)^2$. ∎

Remark 4.6 The locus of points on $M \subset H_+^3(-1)$ where degenerate singularities occur for $P_{HS}^{\boldsymbol{v}}$ and $P_P^{\boldsymbol{v}}$ coincide (de Sitter parabolic set and the horosphere flecnodal curve for the local singularities in Theorems 3.5 and 4.4). This is not surprising as both maps measure the contact of M with geodesics in $H_+^3(-1)$. The families P_{HS} and P_P have parameter spaces with different dimensions, so more singularities occur in the family P_P than in P_{HS}. Also, the target spaces of the projections are different. This influences the curvature of the profile and we get two different Koenderink type theorems.

4.2. Duality

We consider here the Δ_5-dual (see [11] and Section 2) of some components of the bifurcation set of the family P_P of orthogonal projections of an embedded surface M in $H_+^3(-1)$ to planes. Here the concepts of asymptotic directions and parabolic points are those associated to the de Sitter shape operator.

Let $p(t)$, $t \in I$, be a parametrisation of the parabolic set of M and $\boldsymbol{u}_i(t)$, $i = 1, 2$, denote the unit principal directions of M at $p(t)$. Suppose, without loss of generality, that the unique asymptotic direction at $p(t)$ is along $\boldsymbol{u}_1(t)$.

Theorem 4.7 *Let M^* be the Δ_1-dual of the surface M embedded in $H^3_+(-1)$. Then,*

(1) *The local stratum $Bif(P_P, lips/beaks)$ of the bifurcation set of P_P, which consits of vectors $\boldsymbol{v} \in S^3_1$ for which the projection $P^{\boldsymbol{v}}_P$ has a lips/beaks singularity, is a ruled surface parametrised by $\cosh(s)\boldsymbol{u}_1(t) + \sinh(s)p(t)$, with $t \in I$ and $s \in \mathbb{R}$. The Δ_5-dual of $Bif(P_P, lips/beaks)$ is the cuspidaledge of M^*.*

(2) *The multi-local stratum $Bif(P_P, DTF)$ of the bifurcation set of P_P, which consits of vectors $\boldsymbol{v} \in S^3_1$ for which the projection $P^{\boldsymbol{v}}_P$ has a multi-local singularity of type double tangent fold, is a ruled surface. The Δ_5-dual of this ruled surface is the self-intersection line of M^*.*

Proof. (1) It follows from Theorem 4.4(5) that the lips/beaks stratum $Bif(P_P, lips/beaks)$ of the family P_P is given by the set of $\boldsymbol{v} \in S^3_1$ such that $\boldsymbol{v}^*$ is an asymptotic direction at a parabolic point p, where $\boldsymbol{v}^*$ denotes the parallel transport of $\boldsymbol{v}$ to p. So $\boldsymbol{v}^* = \boldsymbol{u}_1(t)$ when $\boldsymbol{v} \in Bif(P_P, lips/beaks)$. We have then

$$\boldsymbol{u}_1(t) = \boldsymbol{v}^* = \frac{1}{\sqrt{1 + \langle p(t), \boldsymbol{v}\rangle^2}} (\boldsymbol{v} + \langle p(t), \boldsymbol{v}\rangle\, p(t))$$

and hence

$$\boldsymbol{v} = \sqrt{1 + \langle p(t), \boldsymbol{v}\rangle^2}\boldsymbol{u}_1(t) - \langle p(t), \boldsymbol{v}\rangle\, p(t).$$

If we set $\sinh(s) = \langle p(t), \boldsymbol{v}\rangle$ we get

$$Bif(P_P, lips/beaks) = \{\cosh(s)\boldsymbol{u}_1(t) + \sinh(s)p(t), t \in I, s \in \mathbb{R}\}.$$

For the duality result, following Remark 2.2, we need to find the unit normal vector to $Bif(P_P, lips/beaks)$. Following the same argument in the proof of Theorem 3.8(1) and using Lemma 3.11, we find that the normal vector is constant along the rulings of the surface $Bif(P_P, lips/beaks)$ and is along $\boldsymbol{e}(t)$, and the result follows.

(2) Let $q(t)$ and $\boldsymbol{u}(t)$ be as in the proof of Theorem 3.8(2). Then $\boldsymbol{u}(t) = \boldsymbol{v}^*$, so

$$\boldsymbol{v} = \sqrt{1 + \langle q(t), \boldsymbol{v}\rangle^2}\boldsymbol{u}(t) - \langle q(t), \boldsymbol{v}\rangle\, q(t).$$

If we set $\sinh(s) = \langle q(t), \boldsymbol{v}\rangle$ we get

$$Bif(P_P, DTF) = \{\cosh(s)\boldsymbol{u}(t) + \sinh(s)q(t), t \in I, s \in \mathbb{R}\}.$$

The normal to this surface is along $\cosh(s)V_1(t) + \sinh(s)V_2(t)$ with $V_1(t) = q(t) \wedge \boldsymbol{u}(t) \wedge q'(t)$ and $V_2(t) = q(t) \wedge \boldsymbol{u}(t) \wedge \boldsymbol{u}'(t)$. The same argument in the proof of Theorem 3.8(2) shows that $V_1(t)$ and $V_2(t)$ are parallel, so the normal to $Bif(P_P, DTF)$ is constant along the rulings of this surface. On the curve $\boldsymbol{u}(t)$, the normal to $Bif(P_P, DTF)$ is along the normal to the surface M at $q(t)$, so the Δ_5-wavefront of $Bif(P_P, DTF)$ is $\{\boldsymbol{e}(p),\ p \in C_1\} = \{\boldsymbol{e}(p),\ p \in C_2\}$. This is precisely the self-intersection line of M^*, the Δ_1-dual surface of M. ■

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