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Quasi-similarity of contractions having a 2×1 characteristic function

Sergio Bermudo, Carmen H. Mancera, Pedro J. Paúl and Vasily Vasyunin

Abstract

Let $T_1 \in \mathcal{B}(\mathcal{H}_1)$ be a completely non-unitary contraction having a non-zero characteristic function Θ_1 which is a 2×1 column vector of functions in H^∞. As it is well-known, such a function Θ_1 can be written as $\Theta_1 = w_1 m_1 \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}$ where $w_1, m_1, a_1, b_1 \in H^\infty$ are such that w_1 is an outer function with $|w_1| \leq 1$, m_1 is an inner function, $|a_1|^2 + |b_1|^2 = 1$, and $a_1 \wedge b_1 = 1$ (here $\wedge$ stands for the greatest common inner divisor). Now consider a second completely non-unitary contraction $T_2 \in \mathcal{B}(\mathcal{H}_2)$ having also a 2×1 characteristic function $\Theta_2 = w_2 m_2 \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}$. We prove that T_1 is quasi-similar to T_2 if, and only if, the following conditions hold:

1. $m_1 = m_2$,
2. $\{z \in \mathbb{T} : |w_1(z)| < 1\} = \{z \in \mathbb{T} : |w_2(z)| < 1\}$ a.e., and
3. the ideal generated by a_1 and b_1 in the Smirnov class $\mathcal{N}^+$ equals the corresponding ideal generated by a_2 and b_2.

1. Statement of the main theorem

Can one characterize the quasi-similarity of contractions in terms of their characteristic functions? Quasi-similarity is an equivalence relation between Hilbert space bounded operators which, being weaker than similarity, still preserves many interesting features as the eigenvalues, the spectral multiplicity or the non-triviality of the lattice of invariant subspaces (see [1], [3], [6] and references therein).

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Two Hilbert space bounded operators $T_1 : \mathcal{H}_1 \to \mathcal{H}_1$ and $T_2 : \mathcal{H}_2 \to \mathcal{H}_2$ are said to be *quasi-similar* if there exist two bounded operators $X : \mathcal{H}_1 \to \mathcal{H}_2$ and $V : \mathcal{H}_2 \to \mathcal{H}_1$ such that

$$\begin{aligned} XT_1 = T_2X, \quad \operatorname{clos}\{X\mathcal{H}_1\} = \mathcal{H}_2, \quad \ker(X) = \{0\};\\ T_1V = VT_2, \quad \operatorname{clos}\{V\mathcal{H}_2\} = \mathcal{H}_1, \quad \ker(V) = \{0\}.\end{aligned}$$

Such operators X and V are called *deformations* or *quasi-affinities.*

There have been several very deep and interesting approaches towards the description of quasi-similarity in terms of the characteristic functions of the operators involved. Namely, the Jordan model for C_0-contractions, completed by Bercovici, Sz.-Nagy and Foiaş and, independently, Müller, after pioneering work by Sz.-Nagy and Foiaş (see [6] and [1]); the Jordan model for weak contractions due to Wu [7], [8]; and the classification, up to quasi-similarity, of C_{10}-contractions with finite defects and Fredholm index equal to -1 due to Makarov and Vasyunin [2].

In particular, the theorem of Wu's tells us that the quasi-similariry of completely non-unitary contractions T_1 and T_2 with scalar (i.e., 1×1) characteristic functions $\Theta_1, \Theta_2 \in H^\infty$ can be expressed in terms of their inner-outer factorizations, say $\Theta_1 = m_1w_1$ and $\Theta_2 = m_2w_2$, as follows: *T_1 is quasi-similar to T_2 if, and only if, $m_1 = m_2$ and $\{z \in \mathbb{T} : |w_1(z)| < 1\} = \{z \in \mathbb{T} : |w_2(z)| < 1\}$ a.e.* The purpose of this paper is to study, with the help of the coordinate-free function model developed by Nikolski and Vasyunin [5] (see also [3, Ch. 1]), the quasi-similarity of contractions having characteristic functions which are 2×1 matrices of elements in H^∞. As we shall see, this case seems to be already somewhat difficult to manage, but we hope that it will provide hints to tackle a more general case when the characteristic functions are $(n + 1) \times n$ matrices.

So let $T \in \mathcal{B}(\mathcal{H})$ be a completely non-unitary contraction having a non-zero characteristic function Θ which is a 2×1 column vector of functions in H^∞. As it is well-known, such a function Θ can be factorized as $\Theta = wm\left[\begin{smallmatrix} a \\ b\end{smallmatrix}\right]$, where $w, m, a, b \in H^\infty$ are such that (i) w is an outer function with $|w| \le 1$, (ii) m is the greatest common inner divisor of the components of Θ (this inner function m is unique up to a constant multiple of modulus one), (iii) $|a|^2 + |b|^2 = 1$, and (iv) a and b are relatively prime inner functions, that is $a \wedge b = 1$ where $\wedge$ stands for the greatest common inner divisor. Associated to these functions we can consider the set

$$\Omega \stackrel{\text{def}}{=} \{z \in \mathbb{T} : |w(z)| < 1\}$$

and the ideal $\mathcal{N}^+\{a, b\}$ generated by a and b in the Smirnov class $\mathcal{N}^+ \stackrel{\text{def}}{=} \{f/g : f, g \in H^\infty \text{ and } g \text{ is outer}\}$, that is,

$$\mathcal{N}^+\{a, b\} \stackrel{\text{def}}{=} \left\{\nu a + \mu b : \nu, \mu \in \mathcal{N}^+\right\}.$$

We fix this notation (with subindices when appropriate) throughout the paper.

The main result of this paper is the following.

Main Theorem *Let* T_i $(i = 1, 2)$ *be completely non-unitary contractions having non-zero* 2×1 *characteristic functions* $\Theta_i = w_i m_i \begin{bmatrix} a_i \\ b_i \end{bmatrix}$.

Then T_1 *is quasi-similar to* T_2 *if, and only if, the following conditions hold*:

1. $m_1 = m_2$,
2. $\Omega_1 = \Omega_2$ *a.e., and*
3. $\mathcal{N}^+\{a_1, b_1\} = \mathcal{N}^+\{a_2, b_2\}$.

Remarks. We would like to underline at this point that for characteristic functions $\Theta_i = w_i \begin{bmatrix} a_i \\ b_i \end{bmatrix}$ without scalar inner factor m_i, the assertion of the Theorem follows from [2] and from the fact that such a contraction is quasi-similar to the direct sum of its outer and inner parts (Proposition 6.1 below gives a slightly more general result).

However, the presence of scalar inner factors makes the situation more complicated in spite of the fact that the Main Theorem tells us that every part in the canonical factorization of one operator has to be quasi-similar to the corresponding part of the second operator. As a matter of fact, one could try, a priori, to prove this result by, as in the proof of Wu for the 1×1 case mentioned above, showing firstly that the operators T_1 with characteristic function $wm \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ and the operator T_2 with characteristic function

$$\begin{bmatrix} w & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] \end{bmatrix}$$

are quasi-similar. On the contrary, even in a simpler case, this does not hold; indeed, in Proposition 6.2 below we shall prove that the operators T_1 with characteristic function $m \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ and the operator T_2 with characteristic function

$$\begin{bmatrix} m & 0 \\ 0 & \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] \end{bmatrix}$$

are quasi-similar if, and only if, there exist three functions $f_1, f_2, f_3 \in H^\infty$ such that $mf_1 + af_2 + bf_3$ is an outer function; a condition that not always holds.

Notations. In what follows, $\operatorname{clos}\{\cdot\}$ stands for the closure of the linear span of the set within the brackets. In particular, if T is a bounded operator defined in a Hilbert space $\mathcal{H}$ and $\mathcal{M}$ is a linear subspace of $\mathcal{H}$, we shall frequently use that $\operatorname{clos}\big\{T \operatorname{clos}\{\mathcal{M}\}\big\} = \operatorname{clos}\{T\mathcal{M}\}$. Whenever we write L^2 or $L^2(\mathcal{H})$, our underlying measure space is assumed to be the unit circle $\mathbb{T}$ of the complex plane endowed with the Lebesgue measure; in particular, for two sets Ω_1 and Ω_2 we shall write $\Omega_1 = \Omega_2$ whenever these sets coincide up to a set of Lebesgue measure zero.

Otherwise, our terminology and notations are standard. A label $(m.n)$ refers to the n-th formula of section m.

2. The coordinate-free function model

Since we shall make an intensive use of the properties and the notation of the coordinate-free function model for completely non-unitary contractions given in [5] (see also [3, Ch. 1]), we shall describe it briefly for the convenience of the reader.

Given a completely non-unitary contraction $T \in \mathcal{B}(\mathcal{H})$, let

$$D_T \stackrel{\text{def}}{=} (I - T^*T)^{1/2}$$

be its defect operator and

$$\mathcal{D}_T \stackrel{\text{def}}{=} \operatorname{clos}\{D_T\mathcal{H}\}$$

be its defect subspace, and take two auxiliary Hilbert spaces $\mathcal{E}$ and $\mathcal{E}_*$ such that

$$\dim(\mathcal{E}) = \dim(\mathcal{D}_T) \qquad \text{and} \qquad \dim(\mathcal{E}_*) = \dim(\mathcal{D}_{T^*}).$$

Now, let $U \in \mathcal{B}(\mathcal{K})$ be the minimal unitary dilation of T. Then U has a triangular matrix with respect to the canonical decomposition $\mathcal{K} = \mathcal{G}_* \oplus \mathcal{H} \oplus \mathcal{G}$, where $\mathcal{G}$ and $\mathcal{G}_*$ are the so-called outgoing and incoming subspace, respectively, and there exists a functional operator

$$\Pi = (\pi_*, \pi) : L^2(\mathcal{E}_*) \oplus L^2(\mathcal{E}) \to \mathcal{K}$$

where π and π_* are isometries intertwining U and the operator M_z of multiplication by the independent variable and Π has dense range in $\mathcal{K}$. Among other properties, the operator defined by

$$\Theta \stackrel{\text{def}}{=} (\pi_*)^*\pi \in \mathcal{B}\big(L^2(\mathcal{E}), L^2(\mathcal{E}_*)\big)$$

is the operator of multiplication by a contractive-valued analytic function $z \to \Theta(z) \in \mathcal{B}(\mathcal{E}, \mathcal{E}_*)$, that is, $(\Theta f)(z) = \Theta(z)f(z)$, and this analytic function is equivalent to the characteristic function Θ_T of T defined by

$$\Theta_T(z) \stackrel{\text{def}}{=} \big(-T + zD_{T^*}(I - zT^*)^{-1}D_T\big)|\mathcal{D}_T.$$

Moreover, T is unitarily equivalent to the model operator defined as the compression of U to the subspace $\mathcal{H}_\Theta$, the orthogonal complement of $\big(\pi H^2(\mathcal{E}) \oplus \pi_* H^2_-(\mathcal{E}_*)\big)$ in $\mathcal{K}$.

To describe the intertwining lifting theorem that we shall use, we need to introduce some more operators appearing in this model.

Consider the function $\Delta \stackrel{\text{def}}{=} (I - \Theta^*\Theta)^{1/2}$ defined a.e. on the unit circle. Then Δ is the positive part of the polar decomposition $\pi - \pi_*\Theta = \tau\Delta$ that also provides us with an isometry τ acting from the so-called residual subspace $L^2(\Delta\mathcal{E}) \stackrel{\text{def}}{=} \operatorname{clos}\{\Delta L^2(\mathcal{E})\}$ to $\mathcal{K}$. Similarly, for $\Delta_* \stackrel{\text{def}}{=} (I - \Theta\Theta^*)^{1/2}$ there is an isometry τ_* defined in $L^2(\Delta_*\mathcal{E})$. These operators satisfy a number of relationships [5, p. 237] and some of them will be used time and again in the sequel, namely

$$\begin{aligned} &\tau\tau^* + \pi_*\pi_*^* = I,\ \tau^*\pi = \Delta, \quad \tau^*\pi_* = 0,\ \tau^*\tau_* = -\Theta^*,\ \pi = \pi_*\Theta + \tau\Delta, \\ &\tau_*\tau_*^* + \pi\pi^* = I,\ \tau_*^*\pi_* = \Delta_*,\ \tau_*^*\pi = 0, \quad \tau_*^*\tau = -\Theta, \quad \pi_* = \pi\Theta^* + \tau_*\Delta_*. \end{aligned} \tag{2.1}$$

We will also need the following equalities:

$$\begin{aligned} \mathcal{G} &= \pi H^2(\mathcal{E}) & \mathcal{H} \oplus \mathcal{G} &= \pi_* H^2(\mathcal{E}_*) \oplus \tau L^2(\Delta\mathcal{E}) \\ \mathcal{G}_* &= \pi_* H^2_-(\mathcal{E}_*) & \mathcal{H} \oplus \mathcal{G}_* &= \pi H^2_-(\mathcal{E}) \oplus \tau_* L^2(\Delta_*\mathcal{E}_*). \end{aligned} \tag{2.2}$$

Now let $T_1 \in \mathcal{B}(\mathcal{H}_1)$ and $T_2 \in \mathcal{B}(\mathcal{H}_2)$ be arbitrary completely non-unitary contractions. Let $X \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ be a bounded operator intertwining T_1 and T_2, that is, $T_2X = XT_1$. Then the liftings $Y \in \mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)$ of X intertwining the minimal unitary dilations of T_1 and T_2 and preserving the outgoing-incoming structure, in the sense that $Y\mathcal{G}_1 \subset \mathcal{G}_2$ and $Y^*\mathcal{G}_{*2} \subset \mathcal{G}_{*1}$, can be parametrized in either of the following forms [5, p. 252–258]

$$\begin{aligned} Y &= \pi_{*2}A_*\pi_{*1}^* + \tau_2\Delta_2A\pi_1^* + \tau_2A_0\tau_{*1}^* \\ &= \pi_2A\pi_1^* + \pi_{*2}A_*\Delta_{*1}\tau_{*1}^* + \tau_2A_0\tau_{*1}^*, \end{aligned}$$

where the parameters $z \mapsto A(z) \in \mathcal{B}(\mathcal{E}_1, \mathcal{E}_2)$ and $z \mapsto A_*(z) \in \mathcal{B}(\mathcal{E}_{*1}, \mathcal{E}_{*2})$ are operator-valued, bounded analytic functions such that $A_*\Theta_1 = \Theta_2A$, and $z \mapsto A_0(z) \in \mathcal{B}(\Delta_{*1}\mathcal{E}_{*1}, \Delta_2\mathcal{E}_2)$ is an operator-valued, bounded measurable function, which can be regarded as a function in $\mathcal{B}(\mathcal{E}_{*1}, \Delta_2\mathcal{E}_2)$ equal to zero on $\ker(\Delta_{*1})$. This parametrization theorem will be essential in our computations.

3. Lifting quasi-affinities

The lemmas that we give in this section tell us how to relate the conditions that define a quasi-affinity to the parameters of any of its liftings. These lemmas are formulated in the general case.

Lemma 3.1 *Let $X : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded operator such that $XT_1 = T_2X$ and let*

$$Y = \pi_{*2}A_*\pi_{*1}^* + \tau_2\Delta_2 A\pi_1^* + \tau_2 A_0\tau_{*1}^*$$

be a lifting of X intertwining the minimal unitary dilations of T_1 and T_2. Then $\operatorname{clos}\{X\mathcal{H}_1\} = \mathcal{H}_2$ if, and only if,

$$\operatorname{clos}\left\{\begin{bmatrix} A_* & \Theta_2 & 0 \\ \Delta_2 A\Theta_1^* + A_0\Delta_{*1} & \Delta_2 & \Delta_2 A\Delta_1 - A_0\Theta_1 \end{bmatrix}\begin{bmatrix} H^2(\mathcal{E}_{*1}) \\ H^2(\mathcal{E}_2) \\ L^2(\Delta_1\mathcal{E}_1)\end{bmatrix}\right\} = \begin{bmatrix} H^2(\mathcal{E}_{*2}) \\ L^2(\Delta_2\mathcal{E}_2)\end{bmatrix}. \tag{3.1}$$

Moreover, in this case the operator $\begin{bmatrix} A_* & \Theta_2\end{bmatrix}$ defined on $H^2(\mathcal{E}_{*1}) \oplus H^2(\mathcal{E}_2)$ is outer, that is, its range is dense in $H^2(\mathcal{E}_{*2})$.

Proof. Since $X\mathcal{H}_1 = P(\mathcal{H}_2)Y\mathcal{H}_1$ and $\mathcal{G}_2 \perp \mathcal{H}_2$, we have that

$$\operatorname{clos}\{X\mathcal{H}_1\} = \mathcal{H}_2 \qquad \Longleftrightarrow \qquad \operatorname{clos}\{P(\mathcal{H}_2)Y\mathcal{H}_1 \oplus \mathcal{G}_2\} = \mathcal{H}_2 \oplus \mathcal{G}_2.$$

It follows from $Y^*\mathcal{G}_{*2} \subset \mathcal{G}_{*1}$ that $Y(\mathcal{H}_1 \oplus \mathcal{G}_1) \subset \mathcal{H}_2 \oplus \mathcal{G}_2$ and, using this, it follows easily the equality

$$P(\mathcal{H}_2)Y\mathcal{H}_1 \oplus \mathcal{G}_2 = Y\mathcal{H}_1 + \mathcal{G}_2.$$

Hence the equivalence above can be written as

$$\operatorname{clos}\{X\mathcal{H}_1\} = \mathcal{H}_2 \qquad \Longleftrightarrow \qquad \operatorname{clos}\{Y\mathcal{H}_1 + \mathcal{G}_2\} = \mathcal{H}_2 \oplus \mathcal{G}_2.$$

Since $Y\mathcal{G}_1 \subset \mathcal{G}_2$, we have

$$Y(\mathcal{H}_1 \oplus \mathcal{G}_1) + \mathcal{G}_2 = Y\mathcal{H}_1 + \mathcal{G}_2,$$

and therefore

$$\operatorname{clos}\{X\mathcal{H}_1\} = \mathcal{H}_2 \qquad \Longleftrightarrow \qquad \operatorname{clos}\{Y(\mathcal{H}_1 \oplus \mathcal{G}_1) + \mathcal{G}_2\} = \mathcal{H}_2 \oplus \mathcal{G}_2.$$

Now let us express the left hand side of the latter equality in terms of the parameters of the lifting $Y = \pi_{*2}A_*\pi_{*1}^* + \tau_2\Delta_2 A\pi_1^* + \tau_2 A_0\tau_{*1}^*$.

Using (2.1) and (2.2) we obtain

$$
\begin{aligned}
&\operatorname{clos}\{Y(\mathcal{H}_1 \oplus \mathcal{G}_1) + \mathcal{G}_2\} \\
&= \operatorname{clos}\left\{ \begin{array}{c} (\pi_{*2} A_* \pi_{*1}^* + \tau_2 \Delta_2 A \pi_1^* + \tau_2 A_0 \tau_{*1}^*)\,(\pi_{*1} H^2(\mathcal{E}_{*1}) \oplus \tau_1 L^2(\Delta_1 \mathcal{E}_1)) \\ +\pi_2 H^2(\mathcal{E}_2) \end{array} \right\} \\
&= \operatorname{clos}\left\{ \begin{array}{c} \big(\pi_{*2} A_* + \tau_2 \Delta_2 A \Theta_1^* + \tau_2 A_0 \Delta_{*1}\big) H^2(\mathcal{E}_{*1}) \\ +\big(\pi_{*2}\Theta_2 + \tau_2 \Delta_2\big) H^2(\mathcal{E}_2) \\ +\big(\tau_2 \Delta_2 A \Delta_1 - \tau_2 A_0 \Theta_1\big) L^2(\Delta_1 \mathcal{E}_1) \end{array} \right\} \\
&= \operatorname{clos}\left\{ \begin{bmatrix} \pi_{*2} A_* & \pi_{*2}\Theta_2 & 0 \\ \tau_2(\Delta_2 A \Theta_1^* + A_0 \Delta_{*1}) & \tau_2 \Delta_2 & \tau_2\big(\Delta_2 A \Delta_1 - A_0 \Theta_1\big) \end{bmatrix} \begin{bmatrix} H^2(\mathcal{E}_{*1}) \\ H^2(\mathcal{E}_2) \\ L^2(\Delta_1 \mathcal{E}_1) \end{bmatrix} \right\} \\
&= \begin{bmatrix} \pi_{*2} & 0 \\ 0 & \tau_2 \end{bmatrix} \operatorname{clos}\left\{ \begin{bmatrix} A_* & \Theta_2 & 0 \\ \Delta_2 A \Theta_1^* + A_0 \Delta_{*1} & \Delta_2 & \Delta_2 A \Delta_1 - A_0 \Theta_1 \end{bmatrix} \begin{bmatrix} H^2(\mathcal{E}_{*1}) \\ H^2(\mathcal{E}_2) \\ L^2(\Delta_1 \mathcal{E}_1) \end{bmatrix} \right\}
\end{aligned}
$$

Since π_{*2} and τ_2 are isometries and, according to (2.2), we have $\mathcal{H}_2 \oplus \mathcal{G}_2 = \pi_{*2} H^2(\mathcal{E}_{*2}) \oplus \tau_2 L^2(\Delta_2 \mathcal{E}_2)$, it follows from the above chain of equalities that the identity $\operatorname{clos}\{Y(\mathcal{H}_1 \oplus \mathcal{G}_1) + \mathcal{G}_2\} = \mathcal{H}_2 \oplus \mathcal{G}_2$ is equivalent to (3.1). (Note that, in fact, the removing from the formula the operators π_{*2} and τ_2 is equivalent to the choice of the standard Szőkefalvi-Nagy–Foiaş functional model with $\pi_* = \left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right]$ and $\tau = \left[\begin{smallmatrix} 0 \\ I \end{smallmatrix}\right]$.)

This finishes the proof of the equivalence.

To prove that if $\operatorname{clos}\{X\mathcal{H}_1\} = \mathcal{H}_2$ then $\begin{bmatrix} A_* & \Theta_2 \end{bmatrix}$ is outer, it is sufficient to look on the first line of (3.1)

$$
\operatorname{clos}\left\{ A_* H^2(\mathcal{E}_{*1}) + \Theta_2 H^2(\mathcal{E}_2) \right\} = H^2(\mathcal{E}_{*2})
$$

and this means that $\begin{bmatrix} A_* & \Theta_2 \end{bmatrix}$ is outer. ∎

Our next result is a converse of the second part of Lemma 3.1.

Lemma 3.2 *Let $X : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded operator such that $XT_1 = T_2 X$ and let $Y = \pi_{*2} A_* \pi_{*1}^* + \tau_2 \Delta_2 A \pi_1^* + \tau_2 A_0 \tau_{*1}^*$ be a lifting of X intertwining the minimal unitary dilations of T_1 and T_2. If*

$$
\operatorname{clos}\left\{ \big(\Delta_2 A \Delta_1 - A_0 \Theta_1\big) L^2(\Delta_1 \mathcal{E}_1) \right\} = L^2(\Delta_2 \mathcal{E}_2), \tag{3.2}
$$

then the claim $\operatorname{clos}\{X\mathcal{H}_1\} = \mathcal{H}_2$ is equivalent to the assertion that the function $\begin{bmatrix} A_* & \Theta_2 \end{bmatrix}$ is outer.

Proof. A straightforward computation shows that, under conjecture (3.2), the criterion (3.1) of the fact that the operator X has a dense range is equivalent to the equality

$$\operatorname{clos}\left\{A_* H^2(\mathcal{E}_{*1}) + \Theta_2 H^2(\mathcal{E}_2)\right\} = H^2(\mathcal{E}_{*2}),$$

what just means that the function $\begin{bmatrix} A_* & \Theta_2 \end{bmatrix}$ is outer. ∎

Taking into account that $\ker(X) = \{0\}$ if, and only if, $\operatorname{clos}\{X^* \mathcal{H}_2\} = \mathcal{H}_1$ and that X^* is a compression of Y^*, the following lemmas can be proved analogously.

Lemma 3.3 *Let $X : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded operator such that $XT_1 = T_2 X$ and let $Y = \pi_2 A \pi_1^* + \pi_{*2} A_* \Delta_{*1} \tau_{*1}^* + \tau_2 A_0 \tau_{*1}^*$ be a lifting of X intertwining the minimal unitary dilations of T_1 and T_2. Then $\ker(X) = \{0\}$ if, and only if,*

$$\operatorname{clos}\left\{\begin{bmatrix} A^* & \Theta_1^* & 0 \\ \Delta_{*1} A_*^* \Theta_2 + A_0^* \Delta_2 & \Delta_{*1} & \Delta_{*1} A_*^* \Delta_{*2} - A_0^* \Theta_2^* \end{bmatrix} \begin{bmatrix} H^2_-(\mathcal{E}_2) \\ H^2_-(\mathcal{E}_{*1}) \\ L^2(\Delta_{*2} \mathcal{E}_{*2}) \end{bmatrix}\right\} = \begin{bmatrix} H^2_-(\mathcal{E}_1) \\ L^2(\Delta_{*1} \mathcal{E}_{*1}) \end{bmatrix}.$$

Moreover, in this case the operator $\left[\begin{smallmatrix} A \\ \Theta_1 \end{smallmatrix}\right]$ defined on $H^2(\mathcal{E}_1)$ is $*$-outer, that is, the range of its adjoint $\begin{bmatrix} A^* & \Theta_1^* \end{bmatrix}$ defined on $H^2_-(\mathcal{E}_2) \oplus H^2_-(\mathcal{E}_{*1})$ is dense in $H^2_-(\mathcal{E}_1)$.

Lemma 3.4 *Let $X : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded operator such that $XT_1 = T_2 X$ and let $Y = \pi_2 A \pi_1^* + \pi_{*2} A_* \Delta_{*1} \tau_{*1}^* + \tau_2 A_0 \tau_{*1}^*$ be a lifting of X intertwining the minimal unitary dilations of T_1 and T_2. If*

$$\operatorname{clos}\left\{\left(\Delta_{*1} A_*^* \Delta_{*2} - A_0^* \Theta_2^*\right) L^2(\Delta_{*2} \mathcal{E}_{*2})\right\} = L^2(\Delta_{*1} \mathcal{E}_{*1}),$$

then the claim $\ker(X) = \{0\}$ is equivalent to the assertion that the function $\left[\begin{smallmatrix} A \\ \Theta_1 \end{smallmatrix}\right]$ is $*$-outer.

4. Features of the model for the case at hand

Let us introduce at this point the following notation: given the characteristic function $\Theta = wm \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$, we define

$$\theta \stackrel{\text{def}}{=} \begin{bmatrix} a \\ b \end{bmatrix} \quad \text{and} \quad \eta \stackrel{\text{def}}{=} \begin{bmatrix} \bar{b} \\ -\bar{a} \end{bmatrix}, \text{ so that } \theta^* = \begin{bmatrix} \bar{a} & \bar{b} \end{bmatrix} \text{ and } \eta^* = \begin{bmatrix} b & -a \end{bmatrix}.$$

Note also, for later use, that $\theta^* \theta = \eta^* \eta = 1$ and $\theta^* \eta = \eta^* \theta = 0$.

Let us now describe some of the embeddings and subspaces of the coordinate-free functional model of our operator T with characteristic function $\Theta = wm\theta$.

If we consider the scalar outer function w as an 1×1 characteristic function, then we have $\Delta_w = \sqrt{1-|w|^2}$ and it is well-known that its residual subspace is $L^2(\Delta_w\mathbb{C}) = \operatorname{clos}\{\Delta_w L^2\} = \chi_\Omega L^2$, where $\Omega \stackrel{\text{def}}{=} \{z \in \mathbb{T} : |w(z)| < 1\}$.

Since Θ is a 2×1 vector with entries in H^∞, we can take as auxiliary spaces $\mathcal{E} = \mathbb{C}$ and $\mathcal{E}_* = \mathbb{C}^2$.

Proposition 4.1 *For $\Theta = wm\theta$ and the auxiliary spaces $\mathcal{E} = \mathbb{C}$ and $\mathcal{E}_* = \mathbb{C}^2$, the corresponding functions Δ and Δ_* in the function model are*

$$\Delta = \Delta_w \qquad \text{and} \qquad \Delta_* = \eta\eta^* + \Delta_w\theta\theta^*,$$

and the corresponding residual subspaces are

$$L^2(\Delta\mathcal{E}) = L^2(\Delta_w\mathbb{C}) = \chi_\Omega L^2 \qquad \text{and} \qquad L^2(\Delta_*\mathcal{E}_*) = \eta L^2 \oplus \theta\chi_\Omega L^2.$$

Proof. On the one hand, we have

$$\Delta^2 = 1 - \Theta^*\Theta = 1 - \overline{w}\,\overline{m}\theta^* wm\theta = 1 - |w|^2,$$

therefore $\Delta = \Delta_w$ and, consequently,

$$L^2(\Delta\mathcal{E}) = L^2(\Delta\mathbb{C}) = L^2(\Delta_w\mathbb{C}) = \chi_\Omega L^2.$$

On the other hand,

$$\Delta_*^2 \stackrel{\text{def}}{=} I - \Theta\Theta^* = I - wm\theta\overline{w}\,\overline{m}\theta^* = I - |w|^2\theta\theta^*.$$

Since

$$\Delta_*^2\eta = (I - |w|^2\theta\theta^*)\eta = \eta \qquad \text{and} \qquad \Delta_*^2\theta = (I - |w|^2\theta\theta^*)\theta = (1-|w|^2)\theta,$$

we have that the eigenvalues of Δ_*^2 are 1 and $1-|w|^2$ with respective orthonormal eigenvectors η and θ. Therefore,

$$\Delta_* = \eta\eta^* + \sqrt{1-|w|^2}\,\theta\theta^* = \eta\eta^* + \Delta_w\theta\theta^*.$$

Now, the multiplication by the matrix $\begin{bmatrix}\theta & \eta\end{bmatrix}$ is a unitary operator on $\mathbb{C}^2$, hence

$$L^2(\mathbb{C}^2) = \begin{bmatrix}\theta & \eta\end{bmatrix} L^2(\mathbb{C}^2) = \theta L^2 \oplus \eta L^2.$$

Therefore,

$$\begin{aligned}\operatorname{clos}\left\{\Delta_* L^2(\mathbb{C}^2)\right\} &= \operatorname{clos}\left\{\left(\eta\eta^* + \Delta_w\theta\theta^*\right)\left(\theta L^2 \oplus \eta L^2\right)\right\}\\ &= \operatorname{clos}\left\{\eta L^2 \oplus \theta\Delta_w L^2\right\} = \eta L^2 \oplus \theta\chi_\Omega L^2,\end{aligned}$$

So that, for the $*$-residual subspace $L^2(\Delta_*\mathcal{E}_*)$ we finally have

$$L^2(\Delta_*\mathcal{E}_*) = \eta L^2 \oplus \theta\chi_\Omega L^2.$$

■

5. Proof of the main theorem

Main Theorem 5.1 *Let $T_i \in \mathcal{B}(\mathcal{H}_i)$ $(i = 1, 2)$ be completely non-unitary contractions having non-zero 2×1 characteristic functions $\Theta_i = w_i m_i \begin{bmatrix} a_i \\ b_i \end{bmatrix}$.*

Then T_1 is quasi-similar to T_2 if, and only if, the following conditions hold:

1. $m_1 = m_2$,
2. $\Omega_1 = \Omega_2$ *a.e., and*
3. $\mathcal{N}^+\{a_1, b_1\} = \mathcal{N}^+\{a_2, b_2\}$.

The proof of the Main Theorem has been decomposed in a series of lemmas in order to make it more transparent the role of each condition in the network of implications. First we prove two easy lemmas of general character.

Lemma 5.2 *Let $a, b \in H^\infty$ such that $|a|^2 + |b|^2 = 1$ and $a \wedge b = 1$. Then for any pair of functions $f, g \in H^\infty$ satisfying $af + bg = 0$ there exists a function $\varphi \in H^\infty$ such that $f = \varphi b$ and $g = -\varphi a$.*

Proof. We take two functions $f, g \in H^\infty$ satisfying $af + bg = 0$. Firstly, we suppose that they also satisfy $|f|^2 + |g|^2 = 1$. Taking modulus in the equality $af = -bg$ we obtain $|a|^2|f|^2 = |b|^2|g|^2 = |b|^2(1 - |f|^2)$, therefore $|b|^2 = (|a|^2+|b|^2)|f|^2 = |f|^2$, in consequence $|b| = |f|$ and $|a| = |g|$. Using the properties of inner-outer factorizations we have the equality of outer parts ($b^{\mathrm{e}} = f^{\mathrm{e}}$, $a^{\mathrm{e}} = g^{\mathrm{e}}$), and therefore, for inner parts we get $a^{\mathrm{i}} f^{\mathrm{i}} = -b^{\mathrm{i}} g^{\mathrm{i}}$. Since a and b do not have a common inner divisor, b^{i} divides f^{i}, that is, there exists an inner function φ such that $f^{\mathrm{i}} = \varphi b^{\mathrm{i}}$ and, consequently $g^{\mathrm{i}} = -\varphi a^{\mathrm{i}}$. Hence we conclude that $f = \varphi b$ and $g = -\varphi a$.

If we have now two functions $f, g \in H^\infty$ satisfying $af + bg = 0$, we consider an outer function $w \in H^\infty$ such that $|w|^2 = |f|^2 + |g|^2$. Then we use the last result with f/w and g/w to obtain an inner function ψ such that $f = w\psi b$ and $g = -w\psi a$. ∎

Lemma 5.3 *Let a, b, and m be three functions in $\mathcal{N}^+$. The following properties hold*:

1. *If $a \wedge b = 1$, then there exists $t_0 \in [0, 1]$ such that $(a + t_0 b) \wedge m = 1$.*
2. *If $b \wedge m = 1$, then there exists $t_0 \in [0, 1]$ such that $(a + t_0 b) \wedge m = 1$.*

Proof. We consider the function $a + tb$ for every $t \in [0, 1]$. Firstly, if we denote $m_t = a + tb$, let us see that $m_{t_1} \wedge m_{t_2} = 1$ for every $t_1 \neq t_2$. We suppose that there exists a function φ which divides m_{t_1} and m_{t_2}, then φ

divides $(t_1 - t_2)b$ and, therefore, it divides b. As φ divides $a + t_1 b$ and b, it also divides a, in consequence, $\varphi = 1$. By using the nonexistence of an uncountable number of pairwise relatively prime inner divisors of an inner function [1, 2.14], we obtain that there exists a real number $t_0 \in [0,1]$ such that $(a_1 + t_0 b_1) \wedge m = 1$. Analogously, we can prove the property (2). ∎

Notation. We denote by $\mathcal{N}^+_{2\times 2}$ the set of all 2×2 matrices with entries in the Smirnov class $\mathcal{N}^+$ and by $\det^{\mathrm{i}}(\Lambda)$ the inner part of $\det(\Lambda)$; as usual, we assume $0^{\mathrm{i}} = 0$.

Lemma 5.4 *Let* $\theta_i = \begin{bmatrix} a_i \\ b_i \end{bmatrix}$ *be such that* $a_i, b_i \in H^\infty$, $|a_i|^2 + |b_i|^2 = 1$ *and* $a_i \wedge b_i = 1$ *for* $i = 1, 2$, *and let* m *be an inner function. If* $a_1, b_1 \in \mathcal{N}^+\{a_2, b_2\}$, *then there exists a matrix* $\Lambda \in \mathcal{N}^+_{2\times 2}$ *such that*

$$\Lambda\theta_2 = \theta_1 \qquad \textit{and} \qquad \det{}^{i}(\Lambda) \wedge m = 1.$$

Proof. Since $a_1, b_1 \in \mathcal{N}^+\{a_2, b_2\}$, there exist $\Lambda_{ij} \in \mathcal{N}^+$ $(i, j = 1, 2)$ such that

$$\Lambda_{11}a_2 + \Lambda_{12}b_2 = a_1 \qquad \text{and} \qquad \Lambda_{21}a_2 + \Lambda_{22}b_2 = b_1,$$

that is, $\Lambda \overset{\text{def}}{=} [\Lambda_{ij}]$ is in $\mathcal{N}^+_{2\times 2}$ and $\Lambda\theta_2 = \theta_1$. If $\det^{\mathrm{i}}(\Lambda) \wedge m = 1$ then we are done, so assume that $\det^{\mathrm{i}}(\Lambda) \wedge m \neq 1$. We shall prove that there exists a matrix $Q \in H^\infty_{2\times 2}$ such that $Q\theta_2 = 0$ and $\det^{\mathrm{i}}(\Lambda + Q) \wedge m = 1$; this matrix $\Lambda + Q$ does the job.

Since $|a_2|^2 + |b_2|^2 = 1$ and $a_2 \wedge b_2 = 1$, if $Q \in H^\infty_{2\times 2}$ is such that $Q\theta_2 = 0$, from Lemma 5.2 we know that $Q = \phi\eta_2^*$, where the entries of $\phi = \begin{bmatrix} c \\ d \end{bmatrix}$ are in H^∞ (we have used the notation introduced in the previous section: $\eta_2^* = \begin{bmatrix} b_2 & -a_2 \end{bmatrix}$). So we have to find $c, d \in H^\infty$ such that the matrix $Q = \phi\eta_2^*$ satisfies $\det^{\mathrm{i}}(\Lambda + Q) \wedge m = 1$.

A straightforward computation shows that

$$\det(\Lambda + Q) = \det(\Lambda) + (cb_1 - da_1),$$

hence

$$\det{}^{\mathrm{i}}(\Lambda + Q) = \big[\det(\Lambda) + (cb_1 - da_1)\big]^{\mathrm{i}}.$$

Since $a_1 \wedge b_1 = 1$, we can use the first part of Lemma 5.3 to find a real number $t \in [0,1]$ such that $(a_1 + tb_1) \wedge m = 1$. Then, using the second part of the same lemma, we can find another real number $s \in [0,1]$ such that

$$\big[\det(\Lambda) + s(a_1 + tb_1)\big] \wedge m = 1.$$

Then it suffices to take $c = st$ and $d = -s$. ∎

Lemma 5.5 *Let $T_i \in \mathcal{B}(\mathcal{H}_i)$ $(i = 1, 2)$ be completely non-unitary contractions with 2×1 characteristic functions $\Theta_i = w_i m_i \begin{bmatrix} a_i \\ b_i \end{bmatrix}$.*

Then there exists an operator $X\colon \mathcal{H}_1 \to \mathcal{H}_2$ such that

$$XT_1 = T_2X, \quad \textit{and} \quad \operatorname{clos}\{X\mathcal{H}_1\} = \mathcal{H}_2$$

if, and only if, the following conditions hold:

1. *m_2 divides m_1,*
2. *$\Omega_2 \subseteq \Omega_1$ a.e., and*
3. *$\mathcal{N}^+\{a_1, b_1\} \subseteq \mathcal{N}^+\{a_2, b_2\}$.*

Proof. We suppose that there exists an operator $X\colon \mathcal{H}_1 \to \mathcal{H}_2$ such that $XT_1 = T_2X$ and $\operatorname{clos}\{X\mathcal{H}_1\} = \mathcal{H}_2$. Let

$$Y = \pi_{*2}A_*\pi_{*1}^* + \tau_2\Delta_2A\pi_1^* + \tau_2A_0\tau_{*1}^* = \pi_2A\pi_1^* + \pi_{*2}A_*\Delta_{*1}\tau_{*1}^* + \tau_2A_0\tau_{*1}^*$$

be a lifting of X intertwining the minimal unitary dilations of T_1 and T_2. Then A_0 is a row vector with two entries in L^∞, A is a function in H^∞ (formally, a 1×1 matrix) and A_* is in $H^\infty_{2\times 2}$, say

$$A_* = \begin{bmatrix} a_{*11} & a_{*12} \\ a_{*21} & a_{*22} \end{bmatrix},$$

satisfying $A_*\Theta_1 = \Theta_2A$. Multiply this equality by η_2^*, then

$$\eta_2^*A_*\Theta_1 = \eta_2^*\Theta_2A = w_2m_2\eta_2^*\theta_2A = 0$$

because $\eta_2^*\theta_2 = 0$. Since $\Theta_1 = m_1w_1\theta_1$, it follows that $\eta_2^*A_*\theta_1 = 0$. It follows from Lemma 5.2 that there exist functions f_1 and f_2 in H^∞ such that

$$\eta_2^*A_* = f_1\eta_1^* = f_1\begin{bmatrix} b_1 & -a_1 \end{bmatrix} \qquad \text{and} \qquad A_*\theta_1 = f_2\theta_2.$$

Taking into account that $\begin{bmatrix} \theta & \eta \end{bmatrix}$ is a unitary matrix of determinant 1 we have

$$\det(A_*) = \det\left(\begin{bmatrix} \theta_2^* \\ \eta_2^* \end{bmatrix} A_* \begin{bmatrix} \theta_1 & \eta_1 \end{bmatrix}\right) = \det\begin{bmatrix} f_2 & \theta_2^*A_*\eta_1 \\ 0 & f_1 \end{bmatrix} = f_1f_2.$$

Now, since $\operatorname{clos}\{X\mathcal{H}_1\} = \mathcal{H}_2$, Lemma 3.1 tells us that

$$\begin{bmatrix} A_* & \Theta_2 \end{bmatrix} = \begin{bmatrix} a_{*11} & a_{*12} & m_2w_2a_2 \\ a_{*21} & a_{*22} & m_2w_2b_2 \end{bmatrix}$$

is an outer operator.

This implies that the three determinants of the 2×2 minors that can be extracted form this matrix are relatively prime, i.e., have no common proper inner divisors (see [4]). That is, the three functions

$$\begin{aligned} \det(A_*) &= a_{*11}a_{*22} - a_{*21}a_{*12} = f_1 f_2, \\ m_2 w_2 (a_{*11} b_2 - a_{*21} a_2) &= m_2 w_2 \eta_2^* \begin{bmatrix} a_{*11} \\ a_{*21} \end{bmatrix} = m_2 w_2 \eta_2^* A_* \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= m_2 w_2 f_1 \eta_1^* \begin{bmatrix} 1 \\ 0 \end{bmatrix} = m_2 w_2 f_1 b_1, \\ m_2 w_2 (a_{*12} b_2 - a_{*22} a_2) &= m_2 w_2 \eta_2^* \begin{bmatrix} a_{*12} \\ a_{*22} \end{bmatrix} = m_2 w_2 \eta_2^* A_* \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= m_2 w_2 f_1 \eta_1^* \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -m_2 w_2 f_1 a_1, \end{aligned}$$

have no common proper inner divisors, hence f_1 must be an outer function and $f_2 \wedge m_2 = 1$.

Since

$$m_2 w_2 \theta_2 A = \Theta_2 A = A_* \Theta_1 = m_1 w_1 A_* \theta_1 = m_1 w_1 f_2 \theta_2,$$

we have $m_2 w_2 A = m_1 w_1 f_2$ and, therefore m_2 divides m_1.

To prove that $\Omega_2 \subset \Omega_1$ a.e or, equivalently, that $\chi_{\Omega_2}(1 - \chi_{\Omega_1}) = 0$ a.e., we will use Lemma 3.1. Since $\operatorname{clos}\{X\mathcal{H}_1\} = \mathcal{H}_2$ and, by applying Proposition 4.1, $\mathcal{E}_i = \mathbb{C}$, $\mathcal{E}_{*i} = \mathbb{C}^2$, $\Delta_i = \Delta_{w_i}$ and $L^2(\Delta_i \mathcal{E}_i) = \chi_{\Omega_i} L^2$ for $i = 1, 2$; Lemma 3.1 tells us that

$$\operatorname{clos}\left\{ \begin{bmatrix} A_* & \Theta_2 & 0 \\ \Delta_{w_2} A \Theta_1^* + A_0 \Delta_{*1} & \Delta_{w_2} & \Delta_{w_2} A \Delta_{w_1} - A_0 \Theta_1 \end{bmatrix} \begin{bmatrix} H^2(\mathbb{C}^2) \\ H^2 \\ \chi_{\Omega_1} L^2 \end{bmatrix} \right\} = \begin{bmatrix} H^2(\mathbb{C}^2) \\ \chi_{\Omega_2} L^2 \end{bmatrix}. \tag{5.1}$$

Now, since the orthogonal projection from the space $\chi_{\Omega_2} L^2$ onto $\chi_{\Omega_2}(1 - \chi_{\Omega_1}) L^2$ is the multiplication by $1 - \chi_{\Omega_1}$, the orthogonal projection from the space $H^2(\mathbb{C}^2) \oplus \chi_{\Omega_2} L^2$ onto its subspace $H^2(\mathbb{C}^2) \oplus \chi_{\Omega_2}(1 - \chi_{\Omega_1}) L^2$ is the operator $I \oplus (1 - \chi_{\Omega_1})$. Therefore, if we apply this orthogonal projection to the equality (5.1), taking into account that

$$\big(\Delta_{w_2} A \Delta_{w_1} - A_0 \Theta_1\big) \chi_{\Omega_1} L^2 \subset \chi_{\Omega_1} \chi_{\Omega_2} L^2 \perp \chi_{\Omega_2}(1 - \chi_{\Omega_1}) L^2,$$

we obtain

$$\operatorname{clos}\left\{ \begin{bmatrix} I & 0 \\ 0 & 1 - \chi_{\Omega_1} \end{bmatrix} \begin{bmatrix} A_* & \Theta_2 \\ \Delta_{w_2} A \Theta_1^* + A_0 \Delta_{*1} & \Delta_{w_2} \end{bmatrix} \begin{bmatrix} H^2(\mathbb{C}^2) \\ H^2 \end{bmatrix} \right\} = \begin{bmatrix} H^2(\mathbb{C}^2) \\ (1 - \chi_{\Omega_1}) \chi_{\Omega_2} L^2 \end{bmatrix}$$

or, equivalently

$$\operatorname{clos}\left\{\begin{bmatrix} A_* & \Theta_2 \\ (1-\chi_{\Omega_1})\big(\Delta_{w_2}A\Theta_1^*+A_0\Delta_{*1}\big) & (1-\chi_{\Omega_1})\Delta_{w_2}\end{bmatrix}\begin{bmatrix} H^2(\mathbb{C}^2) \\ H^2\end{bmatrix}\right\}=\begin{bmatrix} H^2(\mathbb{C}^2) \\ (1-\chi_{\Omega_1})\chi_{\Omega_2}L^2\end{bmatrix}.$$

We claim now that the matrix

$$\begin{bmatrix} A_* & \Theta_2 \\ (1-\chi_{\Omega_1})\big(\Delta_{w_2}A\Theta_1^*+A_0\Delta_{*1}\big) & (1-\chi_{\Omega_1})\Delta_{w_2}\end{bmatrix}$$

has rank 2 a.e., hence the preceding equality implies that for almost every $z \in \mathbb{T}$ the evaluation at z of each of the vectors in $\begin{bmatrix} H^2(\mathbb{C}^2) \\ (1-\chi_{\Omega_1})\chi_{\Omega_2}L^2\end{bmatrix}$ is two-dimensional, and this can only happen if $(1-\chi_{\Omega_1})\chi_{\Omega_2} = 0$ a.e.

To prove our claim, start by noting that, plainly, $\theta_1\theta_1^* + \eta_1\eta_1^* = I$ so that our matrix can be expressed as

$$\begin{bmatrix} A_* & \Theta_2 \\ (1-\chi_{\Omega_1})\big(\Delta_{w_2}A\Theta_1^*+A_0\Delta_{*1}\big) & (1-\chi_{\Omega_1})\Delta_{w_2}\end{bmatrix}$$

$$= \begin{bmatrix} A_*\theta_1\theta_1^* & \Theta_2 \\ (1-\chi_{\Omega_1})\Delta_{w_2}A\Theta_1^* & (1-\chi_{\Omega_1})\Delta_{w_2}\end{bmatrix} + \begin{bmatrix} A_*\eta_1\eta_1^* & 0 \\ (1-\chi_{\Omega_1})A_0\Delta_{*1} & 0\end{bmatrix}.$$

Let us see that these two matrices have rank 1 a.e. Concerning the first, consider the function defined a.e. by $\xi = Aw_1^{-1}\overline{m}_1\theta_1^*$. It will be enough to prove that the factorization

$$\begin{bmatrix} A_*\theta_1\theta_1^* & \Theta_2 \\ (1-\chi_{\Omega_1})\Delta_{w_2}A\Theta_1^* & (1-\chi_{\Omega_1})\Delta_{w_2}\end{bmatrix} = \begin{bmatrix} \Theta_2 \\ (1-\chi_{\Omega_1})\Delta_{w_2}\end{bmatrix}\begin{bmatrix} \xi & 1\end{bmatrix}$$

holds; note that both factors on the right hand have rank 1. Indeed, using that $\Theta_2 A = A_*\Theta_1$ and that m_1 is inner, we have

$$\Theta_2\xi = \Theta_2 A w_1^{-1}\overline{m}_1\theta_1^* = A_*\Theta_1 w_1^{-1}\overline{m}_1\theta_1^* = A_* m_1 w_1\theta_1 w_1^{-1}\overline{m}_1\theta_1^* = A_*\theta_1\theta_1^*,$$

and, using that $|w_1(z)| = 1$ for all $z \notin \Omega_1$, also

$$\begin{aligned}(1-\chi_{\Omega_1})\Delta_{w_2}\xi &= (1-\chi_{\Omega_1})\Delta_{w_2}Aw_1^{-1}\overline{m}_1\theta_1^* \\ &= (1-\chi_{\Omega_1})\Delta_{w_2}A\overline{w}_1\overline{m}_1\theta_1^* = (1-\chi_{\Omega_1})\Delta_{w_2}A\Theta_1^*.\end{aligned}$$

Concerning the second matrix, since $\Delta_{*1} = \eta_1\eta_1^* + \Delta_{w_1}\theta_1\theta_1^*$, by Proposition 4.1, and, on the other hand, $(1-\chi_{\Omega_1})\Delta_{w_1} = 0$ by the definition of Ω_1, it follows

$$(1-\chi_{\Omega_1})A_0\Delta_{*1} = (1-\chi_{\Omega_1})A_0(\eta_1\eta_1^* + \Delta_{w_1}\theta_1\theta_1^*) = (1-\chi_{\Omega_1})A_0\eta_1\eta_1^*$$

so that the following factorization holds

$$\begin{bmatrix} A_*\eta_1\eta_1^* & 0 \\ (1-\chi_{\Omega_1})A_0\Delta_{*1} & 0 \end{bmatrix} = \begin{bmatrix} A_*\eta_1 \\ (1-\chi_{\Omega_1})A_0\eta_1 \end{bmatrix} \begin{bmatrix} \eta_1^* & 0 \end{bmatrix}$$

where both factors have rank 1.

This finishes the proof that $\chi_{\Omega_2}(1-\chi_{\Omega_1}) = 0$ a.e. or, as we mentioned above, that $\Omega_2 \subseteq \Omega_1$ a.e.

So, finally, take an arbitrary element $\mu a_1 + \nu b_1 \in \mathcal{N}^+\{a_1, b_1\}$, where $\nu, \mu \in \mathcal{N}^+$. Then

$$\mu a_1 + \nu b_1 = \begin{bmatrix} b_1 & -a_1 \end{bmatrix} \begin{bmatrix} \nu \\ -\mu \end{bmatrix} = \eta_1^* \begin{bmatrix} \nu \\ -\mu \end{bmatrix} = \frac{1}{f_1}\eta_2^* A_* \begin{bmatrix} \nu \\ -\mu \end{bmatrix}$$

which is in $\mathcal{N}^+\{a_2, b_2\}$ since the entries of A_* are in H^∞ and $\nu/f_1, \mu/f_1 \in \mathcal{N}^+$ because f_1 is outer.

This finishes the proof that the conditions are necessary.

Now, assume that $m_1 = hm_2$ for some inner function h, $\Omega_2 \subseteq \Omega_1$ a.e., and $\mathcal{N}^+\{a_1, b_1\} \subseteq \mathcal{N}^+\{a_2, b_2\}$. We have to prove that there exists an operator $X : \mathcal{H}_1 \to \mathcal{H}_2$ such that

$$XT_1 = T_2X, \quad \text{and} \quad \operatorname{clos}\{X\mathcal{H}_1\} = \mathcal{H}_2.$$

We will prove the existence of such an operator X by using an adequate parametrization to produce a suitable lifting $Y = \pi_{*2}A_*\pi_{*1}^* + \tau_2\Delta_2 A\pi_1^* + \tau_2 A_0\tau_{*1}^*$ of X.

Start by taking $A_0 = 0$. Using Lemma 3.2 with the descriptions of the residual subspaces given in Proposition 4.1, it suffices to find a function $A \in H^\infty$ and a matrix $A_* = \begin{bmatrix} a_{*11} & a_{*12} \\ a_{*21} & a_{*22} \end{bmatrix}$ in $H^\infty_{2\times 2}$ such that the following three conditions hold:

$$\Theta_2 A = A_*\Theta_1, \quad \operatorname{clos}\left\{\Delta_{w_2} A \Delta_{w_1}\chi_{\Omega_1} L^2\right\} = \chi_{\Omega_2} L^2 \quad \text{and} \quad \begin{bmatrix} A_* & \Theta_2 \end{bmatrix} \text{ is outer.}$$

Since $a_1, b_1 \in \mathcal{N}^+\{a_2, b_2\}$, Lemma 5.4 tells us that there exists a matrix $\Lambda \in \mathcal{N}^+_{2\times 2}$ such that $\Lambda\theta_2 = \theta_1$ and $\det^{\mathrm{i}}(\Lambda) \wedge m_2 = 1$. So fix an outer function ϕ such that $\phi\Lambda \in H^\infty_{2\times 2}$ and call $\lambda \stackrel{\text{def}}{=} \det(\Lambda)$. Then, we take $A \stackrel{\text{def}}{=} hw_1\phi^2\lambda \in H^\infty$; note that $\lambda \wedge m_2 = 1$. Now, for $\Lambda = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{bmatrix}$ define

$$\Lambda^{\mathrm{ad}} \stackrel{\text{def}}{=} \begin{bmatrix} \Lambda_{22} & -\Lambda_{12} \\ -\Lambda_{21} & \Lambda_{11} \end{bmatrix}.$$

Then $\Lambda\Lambda^{\mathrm{ad}} = \Lambda^{\mathrm{ad}}\Lambda = \det(\Lambda)I = \lambda I$. We take $A_* \stackrel{\text{def}}{=} w_2\phi^2\Lambda^{\mathrm{ad}}$.

Proof that $\Theta_2 A = A_*\Theta_1$. Since $\Lambda\theta_2 = \theta_1$, multiplying by Λ^{ad} we get

$$\lambda\theta_2 = \Lambda^{\mathrm{ad}}\Lambda\theta_2 = \Lambda^{\mathrm{ad}}\theta_1$$

and, therefore,

$$\begin{aligned}\Theta_2 A &= (w_2 m_2\theta_2)(hw_1\phi^2\lambda) = w_2\phi^2 w_1 m_1(\lambda\theta_2)\\ &= w_2\phi^2 w_1 m_1(\Lambda^{\mathrm{ad}}\theta_1) = (w_2\phi^2\Lambda^{\mathrm{ad}})(w_1 m_1\theta_1) = A_*\Theta_1.\end{aligned}$$

Proof that $\operatorname{clos}\{\Delta_{w_2} A\Delta_{w_1}\chi_{\Omega_1}L^2\} = \chi_{\Omega_2}L^2$. This is easy: simply note that A never vanishes, that Δ_{w_i} does not vanish on Ω_i ($i = 1, 2$), and that $\Omega_2 \subseteq \Omega_1$.

Proof that $\begin{bmatrix} A_* & \Theta_2\end{bmatrix}$ is outer . To prove that $\begin{bmatrix} A_* & \Theta_2\end{bmatrix}$ is outer, we need to check that the three determinants of the 2×2 minors of this matrix have no common inner divisor. But, as we saw in the proof of the necessity part, it follows from the equality $\Theta_2 A = A_*\Theta_1$ that these three determinants are $\det(A_*)$, $m_2 w_2 f_1 b_1$, and $-m_2 w_2 f_1 a_1$. It is easy to see that $f_1 = \eta_2^* A_*\eta_1 = w_2\phi^2$. Hence, these three functions

$$\det(A_*) = w_2^2\phi^4\lambda, \qquad m_2 w_2^2\phi^2 b_1, \quad \text{and} \qquad - m_2 w_2^2\phi^2 a_1.$$

have no common inner divisor because w_2 and ϕ are outer, $a_1 \wedge b_1 = 1$ and $m_2 \wedge \lambda = 1$. This proves that $\begin{bmatrix} A_* & \Theta_2\end{bmatrix}$ is outer and the proof of the lemma is completed. ■

Lemma 5.6 *Let $T_i \in \mathcal{B}(\mathcal{H}_i)$ $(i = 1, 2)$ be completely non-unitary contractions having 2×1 characteristic functions $\Theta_i = w_i m_i \begin{bmatrix} a_i \\ b_i\end{bmatrix}$.*

Then there exists an operator $X : \mathcal{H}_1 \to \mathcal{H}_2$ such that

$$XT_1 = T_2 X, \quad \textit{and} \quad \ker(X) = \{0\}$$

if, and only if, the following conditions hold:

1. *m_1 divides m_2,*
2. *$\Omega_1 \subseteq \Omega_2$ a.e.*

Proof. We suppose that there exists an operator $X : \mathcal{H}_1 \to \mathcal{H}_2$ such that $XT_1 = T_2 X$ and $\ker(X) = \{0\}$. Let

$$Y = \pi_{*2}A_*\pi_{*1}^* + \tau_2\Delta_2 A\pi_1^* + \tau_2 A_0\tau_{*1}^* = \pi_2 A\pi_1^* + \pi_{*2}A_*\Delta_{*1}\tau_{*1}^* + \tau_2 A_0\tau_{*1}^*$$

be a lifting of X intertwining the minimal unitary dilations of T_1 and T_2. Then A_0 is a row vector with two entries in L^∞, A is a function in H^∞ and A_* is in $H^\infty_{2\times 2}$, say

$$A_* = \begin{bmatrix} a_{*11} & a_{*12}\\ a_{*21} & a_{*22}\end{bmatrix},$$

satisfying $\Theta_2 A = A_*\Theta_1$.

The last equality yields

$$m_2 w_2 A a_2 = m_1 w_1 (a_{*11} a_1 + a_{*12} b_1) \quad \text{and} \quad m_2 w_2 A b_2 = m_1 w_1 (a_{*21} a_1 + a_{*22} b_1).$$

Therefore, m_1 divides both $m_2 A a_2$ and $m_2 A b_2$, and $a_2 \wedge b_2 = 1$ implies that m_1 divides $m_2 A$. Now, since $\ker(X) = \{0\}$, Lemma 3.3 tells us that the operator $\begin{bmatrix} A \\ \Theta_1 \end{bmatrix}$ is $*$-outer and this implies that its entries A, $m_1 w_1 a_1$ and $m_1 w_1 b_1$ must have no common inner divisor. Hence $m_1 \wedge A = 1$ and since, as we have just seen, m_1 divides $m_2 A$, it follows that m_1 divides m_2.

To prove that $\Omega_1 \subseteq \Omega_2$ a.e. we will argue in the same way that in the last lemma, we will prove that $\chi_{\Omega_1}(1 - \chi_{\Omega_2}) = 0$ a.e.. For that, as in that lemma, we know that there exist functions f_1 and f_2 in H^∞ such that

$$\eta_2^* A_* = f_1 \eta_1^* = f_1 \begin{bmatrix} b_1 & -a_1 \end{bmatrix}, \qquad A_* \theta_1 = f_2 \theta_2 \qquad \text{and} \qquad \det(A_*) = f_1 f_2.$$

Since $\ker(X) = \{0\}$ and, by applying Proposition 4.1, $\mathcal{E}_i = \mathbb{C}$, $\mathcal{E}_{*i} = \mathbb{C}^2$, $\Delta_i = \Delta_{w_i}$, $\Delta_{*i} = \eta_i \eta_i^* + \Delta_{w_i} \theta_i \theta_i^*$ and $L^2(\Delta_{*i} \mathcal{E}_{*i}) = \eta_i L^2 \oplus \theta_i \chi_{\Omega_i} L^2$ for $i = 1, 2$, Lemma 3.3 tells us that

$$\operatorname{clos} \left\{ \begin{bmatrix} \overline{A} & \Theta_1^* & 0 \\ \Delta_{*1} A_*^* \Theta_2 + A_0^* \Delta_{w_2} & \Delta_{*1} & \Delta_{*1} A_*^* \Delta_{*2} - A_0^* \Theta_2^* \end{bmatrix} \begin{bmatrix} H_-^2 \\ H_-^2(\mathbb{C}^2) \\ \eta_2 L^2 \oplus \theta_2 \chi_{\Omega_2} L^2 \end{bmatrix} \right\}$$
$$= \begin{bmatrix} H_-^2 \\ \eta_1 L^2 \oplus \theta_1 \chi_{\Omega_1} L^2 \end{bmatrix}.$$

Now, since the orthogonal projection from the space $\eta_1 L^2 \oplus \theta_1 \chi_{\Omega_1} L^2$ onto $\theta_1 (1 - \chi_{\Omega_2}) \chi_{\Omega_1} L^2$ is the operator $\theta_1 (1 - \chi_{\Omega_2}) \theta_1^*$, let us see that

$$\theta_1 (1 - \chi_{\Omega_2}) \theta_1^* (\Delta_{*1} A_*^* \Delta_{*2} - A_0^* \Theta_2^*)(\eta_2 L^2 \oplus \theta_2 \chi_{\Omega_2} L^2) = \{0\}.$$

Using that

$$(1 - \chi_{\Omega_2}) \chi_{\Omega_2} = 0, \qquad \Delta_{*i} \eta_i = \eta_i, \qquad \theta_i^* \eta_i = 0 \qquad \text{and} \qquad A_*^* \eta_2 = \overline{f}_1 \eta_1,$$

we obtain

$$\begin{aligned} \theta_1 (1 - \chi_{\Omega_2}) \theta_1^* (\Delta_{*1} A_*^* \Delta_{*2} - A_0^* \Theta_2^*)(\eta_2 L^2 \oplus \theta_2 \chi_{\Omega_2} L^2) \\ &= \theta_1 (1 - \chi_{\Omega_2}) \theta_1^* (\Delta_{*1} A_*^* \Delta_{*2} - A_0^* \Theta_2^*) \eta_2 L^2 \\ &= \theta_1 (1 - \chi_{\Omega_2}) \theta_1^* \Delta_{*1} A_*^* \Delta_{*2} \eta_2 L^2 \\ &= \theta_1 (1 - \chi_{\Omega_2}) \theta_1^* \Delta_{*1} \overline{f}_1 \eta_1 L^2 \\ &= \theta_1 (1 - \chi_{\Omega_2}) \theta_1^* \eta_1 \overline{f}_1 L^2 = \{0\}. \end{aligned}$$

Therefore

$$\operatorname{clos}\left\{\begin{bmatrix} 1 & 0 \\ 0 & \theta_1(1-\chi_{\Omega_2})\theta_1^* \end{bmatrix}\begin{bmatrix} \overline{A} & \Theta_1^* \\ \Delta_{*1}A_*^*\Theta_2 + A_0^*\Delta_{w_2} & \Delta_{*1} \end{bmatrix}\begin{bmatrix} H_-^2 \\ H_-^2(\mathbb{C}^2) \end{bmatrix}\right\}$$

$$= \operatorname{clos}\left\{\begin{bmatrix} \overline{A} & \Theta_1^* \\ \theta_1(1-\chi_{\Omega_2})\theta_1^*(\Delta_{*1}A_*^*\Theta_2+A_0^*\Delta_{w_2}) & \theta_1(1-\chi_{\Omega_2})\theta_1^*\Delta_{*1} \end{bmatrix}\begin{bmatrix} H_-^2 \\ H_-^2(\mathbb{C}^2) \end{bmatrix}\right\}$$

$$= \begin{bmatrix} H_-^2 \\ \theta_1(1-\chi_{\Omega_2})\chi_{\Omega_1}L^2 \end{bmatrix}.$$

It remains to prove that the matrix

$$\begin{bmatrix} \overline{A} & \Theta_1^* \\ \theta_1(1-\chi_{\Omega_2})\theta_1^*(\Delta_{*1}A_*^*\Theta_2 + A_0^*\Delta_{w_2}) & \theta_1(1-\chi_{\Omega_2})\theta_1^*\Delta_{*1} \end{bmatrix}$$

has rank 1. Since

$$\theta_1^*\Delta_{*1} = \Delta_{w_1}\theta_1^*, \quad \theta_1^*A_*^* = \overline{f}_2\theta_2^* \quad \text{and} \quad (1-\chi_{\Omega_2})\Delta_{w_2} = 0,$$

we have

$$\theta_1(1-\chi_{\Omega_2})\theta_1^*(\Delta_{*1}A_*^*\Theta_2 + A_0^*\Delta_{w_2}) = \theta_1(1-\chi_{\Omega_2})\Delta_{w_1}m_2w_2\overline{f}_2.$$

Now, from the equality $\Theta_2 A = A_*\Theta_1$ it follows that

$$m_2w_2\theta_2A = A_*m_1w_1\theta_1 = m_1w_1f_2\theta_2,$$

consequently, $\overline{A} = m_2\overline{w}_2^{-1}\overline{m}_1\overline{w}_1\overline{f}_2$. Taking into account that $(1-\chi_{\Omega_2})w_2 = (1-\chi_{\Omega_2})\overline{w}_2^{-1}$ we can write the matrix above as

$$\begin{bmatrix} \overline{m}_1\overline{w}_1 \\ \theta_1(1-\chi_{\Omega_2})\Delta_{w_1} \end{bmatrix}\begin{bmatrix} m_2\overline{w}_2^{-1}\overline{f}_2 & \theta_1^* \end{bmatrix},$$

which is of rank 1. It follows from this fact that $(1-\chi_{\Omega_2})\chi_{\Omega_1} = 0$ a.e. and, therefore, the inclusion $\Omega_1 \subseteq \Omega_2$ a.e. This finishes the proof that the conditions are necessary.

Now, assume that $m_2 = hm_1$ for some inner function h and $\Omega_1 \subseteq \Omega_2$ a.e. We have to prove that there exists an operators $X : \mathcal{H}_1 \to \mathcal{H}_2$ such that

$$XT_1 = T_2X, \quad \text{and} \quad \ker(X) = \{0\}.$$

We will prove the existence of such an operator X by using an adequate parametrization to produce a suitable lifting $Y = \pi_{*2}A_*\pi_{*1}^* + \tau_2\Delta_2A\pi_1^* + \tau_2A_0\tau_{*1}^*$ of X.

Start by taking $A_0 = 0$. Using Lemma 3.4 together with the descriptions of the residual subspaces given in Proposition 4.1, it suffices to find a function $A \in H^\infty$ and a matrix $A_* = \begin{bmatrix} a_{*11} & a_{*12} \\ a_{*21} & a_{*22} \end{bmatrix}$ in $H^\infty_{2\times 2}$ such that the following three conditions hold:

1. $\Theta_2 A = A_* \Theta_1$
2. $\begin{bmatrix} A \\ \Theta_1 \end{bmatrix}$ is $*$-outer
3. $\operatorname{clos}\left\{\Delta_{*1} A_*^* \Delta_{*2}(\eta_2 L^2 \oplus \theta_2 \chi_{\Omega_2} L^2)\right\} = \eta_1 L^2 \oplus \theta_1 \chi_{\Omega_1} L^2$.

Since $a_1 \wedge b_1 = 1$, Lemma 5.3 tells us that there exists a number $t \in [0,1]$ such that $(a_1 + t b_1) \wedge m_1 = 1$. We consider the matrices

$$B = \begin{bmatrix} (1+b_1)a_2 & (t-a_1)a_2 \\ b_2 & t b_2 \end{bmatrix} \qquad \text{and} \qquad B^{\mathrm{ad}} = \begin{bmatrix} t b_2 & -(t-a_1)a_2 \\ -b_2 & (1+b_1)a_2 \end{bmatrix}.$$

It is not difficult to check that

$$B\theta_1 = (a_1 + t b_1)\theta_2, \qquad \det B = a_2 b_2 (a_1 + t b_1) \qquad \text{and} \qquad B^{\mathrm{ad}}\theta_2 = a_2 b_2 \theta_1.$$

Then, we take $A \stackrel{\mathrm{def}}{=} w_1(a_1 + t b_1)$ and $A_* \stackrel{\mathrm{def}}{=} h w_2 B$.

Proof that $\boldsymbol{\Theta_2 A = A_* \Theta_1}$. Since $B\theta_1 = (a_1 + t b_1)\theta_2$ we have

$$\begin{aligned} \Theta_2 A &= (w_2 m_2 \theta_2)(w_1(a_1 + t b_1)) = w_2 w_1 h m_1((a_1 + t b_1)\theta_2) \\ &= w_2 w_1 h m_1 B \theta_1 = (h w_2 B)(w_1 m_1 \theta_1) = A_* \Theta_1. \end{aligned}$$

Proof that $\boldsymbol{\begin{bmatrix} A \\ \Theta_1 \end{bmatrix}}$ is $*$-outer. Using that w_1 is outer and $(a_1 + t b_1) \wedge m_1 = 1$, we see that the components of

$$\begin{bmatrix} A \\ \Theta_1 \end{bmatrix} = \begin{bmatrix} w_1(a_1 + t b_1) \\ m_1 w_1 a_1 \\ m_1 w_1 b_1 \end{bmatrix}$$

have no common inner divisor, in consequence, $\begin{bmatrix} A \\ \Theta_1 \end{bmatrix}$ is $*$-outer.

Proof that $\boldsymbol{\operatorname{clos}\left\{\Delta_{*1} A_*^* \Delta_{*2}(\eta_2 L^2 \oplus \theta_2 \chi_{\Omega_2} L^2)\right\} = \eta_1 L^2 \oplus \theta_1 \chi_{\Omega_1} L^2}$. The inclusion

$$\operatorname{clos}\left\{\Delta_{*1} A_*^* \Delta_{*2}(\eta_2 L^2 \oplus \theta_2 \chi_{\Omega_2} L^2)\right\} \subseteq \eta_1 L^2 \oplus \theta_1 \chi_{\Omega_1} L^2$$

is clear because

$$\eta_1 L^2 \oplus \theta_1 \chi_{\Omega_1} L^2 = \operatorname{clos}\left\{\Delta_{*1} L^2(\mathbb{C}^2)\right\}.$$

A symmetrical relation $\eta_2 L^2 \oplus \theta_2 \chi_{\Omega_2} L^2 = \operatorname{clos}\{\Delta_{*2} L^2(\mathbb{C}^2)\}$ yields

$$\operatorname{clos}\left\{\Delta_{*2}(\eta_2 L^2 \oplus \theta_2 \chi_{\Omega_2} L^2)\right\} = \eta_2 L^2 \oplus \theta_2 \chi_{\Omega_2} L^2,$$

whence,

$$\begin{aligned}\operatorname{clos}\left\{\Delta_{*1} A_*^* \Delta_{*2}(\eta_2 L^2 \oplus \theta_2 \chi_{\Omega_2} L^2)\right\} &= \operatorname{clos}\left\{\Delta_{*1} A_*^* (\eta_2 L^2 \oplus \theta_2 \chi_{\Omega_2} L^2)\right\} \\ &= \operatorname{clos}\left\{(\eta_1\eta_1^* + \Delta_{w_1}\theta_1\theta_1^*) A_*^* (\eta_2 L^2 \oplus \theta_2 \chi_{\Omega_2} L^2)\right\}.\end{aligned}$$

Now, on one hand, we have

$$\begin{aligned}\operatorname{clos}\left\{\Delta_{w_1}\theta_1\theta_1^* A_*^* \theta_2 \chi_{\Omega_2} L^2\right\} &= \operatorname{clos}\left\{\Delta_{w_1}\theta_1 (A_*\theta_1)^* \theta_2 \chi_{\Omega_2} L^2\right\} \\ &= \operatorname{clos}\left\{\Delta_{w_1}\theta_1 (h w_2 B\theta_1)^* \theta_2 \chi_{\Omega_2} L^2\right\} \\ &= \operatorname{clos}\left\{\Delta_{w_1}\theta_1 (h w_2 (a_1 + t b_1)\theta_2)^* \theta_2 \chi_{\Omega_2} L^2\right\} \\ &= \operatorname{clos}\left\{\Delta_{w_1}\theta_1 \overline{h}\,\overline{w}_2 (\overline{a}_1 + t\overline{b}_1) \chi_{\Omega_2} L^2\right\} \\ &= \operatorname{clos}\left\{\theta_1 \chi_{\Omega_1} L^2\right\} = \theta_1 \chi_{\Omega_1} L^2.\end{aligned} \tag{5.2}$$

We used here the following facts: the functions h, w_2, and $a_1 + tb_1$ are different from zero almost everywhere; θ_1 is an isometry; Δ_{w_1} is different from zero on Ω_1 and vanishes outside Ω_1; and $\Omega_1 \subseteq \Omega_2$.

On the other hand, we have

$$\eta_2^* B = \begin{bmatrix} b_2 & -a_2 \end{bmatrix} \begin{bmatrix} (1+b_1)a_2 & (t-a_1)a_2 \\ b_2 & t b_2 \end{bmatrix} = a_2 b_2 \begin{bmatrix} b_1 & -a_1 \end{bmatrix} = a_2 b_2 \eta_1^*,$$

therefore,

$$\begin{aligned}\operatorname{clos}\left\{\Delta_{*1} A_*^* \eta_2 L^2\right\} &= \operatorname{clos}\left\{\Delta_{*1} (\eta_2^* A_*)^* L^2\right\} = \operatorname{clos}\left\{\Delta_{*1} (h w_2 \eta_2^* B)^* L^2\right\} \\ &= \operatorname{clos}\left\{\Delta_{*1} (h w_2 a_2 b_2 \eta_1^*)^* L^2\right\} = \operatorname{clos}\left\{\overline{h}\,\overline{w}_2 \overline{a}_2 \overline{b}_2 \Delta_{*1} \eta_1 L^2\right\} \\ &= \operatorname{clos}\left\{\overline{h}\,\overline{w}_2 \overline{a}_2 \overline{b}_2 \eta_1 L^2\right\} = \eta_1 L^2,\end{aligned} \tag{5.3}$$

using that η_1 is an isometry.

Recall that we need to prove that

$$\eta_1 L^2 \oplus \theta_1 \chi_{\Omega_1} L^2 \subset \operatorname{clos}\left\{(\eta_1\eta_1^* + \Delta_{w_1}\theta_1\theta_1^*) A_*^* (\eta_2 L^2 \oplus \theta_2 \chi_{\Omega_2} L^2)\right\}.$$

If we take an arbitrary element $u = \eta_1 u_1 + \theta_1 \chi_{\Omega_1} u_2$ in $\eta_1 L^2 \oplus \theta_1 \chi_{\Omega_1} L^2$, by (5.2) and (5.3) there exist two elements $h_1, h_2 \in L^2$ such that

$$\|\Delta_{w_1}\theta_1\theta_1^* A_*^* \theta_2 \chi_{\Omega_2} h_1 - \theta_1 \chi_{\Omega_1} u_2\| < \frac{\varepsilon}{2}$$

and

$$\|\Delta_{*1}A_*^*\eta_2 h_2 - (\eta_1 u_1 - \eta_1\eta_1^* A_*^*\theta_2\chi_{\Omega_2}h_1)\| < \frac{\varepsilon}{2},$$

therefore

$$\|\Delta_{*1}A_*^*\eta_2 h_2 + (\eta_1\eta_1^* + \Delta_{w_1}\theta_1\theta_1^*)A_*^*\theta_2\chi_{\Omega_2}h_1 - (\eta_1 u_1 + \theta_1\chi_{\Omega_1}u_2)\| < \varepsilon.$$

As the element $\Delta_{*1}A_*^*\eta_2 h_2 + (\eta_1\eta_1^* + \Delta_{w_1}\theta_1\theta_1^*)A_*^*\theta_2\chi_{\Omega_2}h_1 = \Delta_{*1}A_*^*(\eta_2 h_2 + \theta_2\chi_{\Omega_2}h_1)$ belongs to the space $\Delta_{*1}A_*^*(\eta_2 L^2 \oplus \theta_2\chi_{\Omega_2}L^2)$, we get the required inclusion and the lemma is proved. ■

Lemma 5.7 *Let $T_i \in \mathcal{B}(\mathcal{H}_i)$ $(i = 1, 2)$ be completely non-unitary contractions having 2×1 characteristic functions $\Theta_i = w_i m_i \begin{bmatrix} a_i \\ b_i \end{bmatrix}$.*

Then there exists an operator $X : \mathcal{H}_1 \to \mathcal{H}_2$ such that

$$XT_1 = T_2X, \quad \textit{and} \quad \operatorname{clos}\{X\mathcal{H}_1\} = \mathcal{H}_2, \quad \ker(X) = \{0\}$$

if, and only if, the following conditions hold:

1. $m_1 = m_2 = m$,
2. $\Omega_1 = \Omega_2 = \Omega$ *a.e., and*
3. $\mathcal{N}^+\{a_1, b_1\} \subseteq \mathcal{N}^+\{a_2, b_2\}$.

Proof. The proof that the conditions are necessary follows from Lemmas 5.5 and 5.6. To prove that the conditions are sufficient it is necessary, from Lemmas 3.2 and 3.4, to find the parameters A_0, A, and A_* of a lifting of X satisfying the conditions

1. $\Theta_2 A = A_*\Theta_1$
2. $\begin{bmatrix} A_* & \Theta_2 \end{bmatrix}$ is outer
3. $\operatorname{clos}\{(\Delta_{w_2}A\Delta_{w_1} - A_0\Theta_1)\chi_\Omega L^2\} = \chi_\Omega L^2$
4. $\begin{bmatrix} A \\ \Theta_1 \end{bmatrix}$ is $*$-outer
5. $\operatorname{clos}\{(\Delta_{*1}A_*^*\Delta_{*2} - A_0^*\Theta_2^*)(\eta_2 L^2 \oplus \theta_2\chi_\Omega L^2)\} = \eta_1 L^2 \oplus \theta_1\chi_\Omega L^2$.

We take the same parameters that we took in the proof of Lemma 5.5, that is, $A_0 = 0$, $A = w_1\phi^2\lambda$ and $A_* = w_2\phi^2\Lambda^{\mathrm{ad}}$, where $\Lambda \in \mathcal{N}_{2\times 2}^+$ satisfies $\Lambda\theta_2 = \theta_1$ and $\det^{\mathrm{i}}(\Lambda) \wedge m = 1$, ϕ is an outer function such that $\phi\Lambda \in H_{2\times 2}^\infty$ and the equality $\Lambda^{\mathrm{ad}}\theta_1 = \lambda\theta_2$ is satisfied with $\lambda = \det(\Lambda)$. From that proof we know the conditions (1), (2), and (3) are fulfilled.

Proof that $\begin{bmatrix} A \\ \Theta_1 \end{bmatrix}$ is $*$-outer. Using that w_1 is outer, we see that the components of

$$\begin{bmatrix} A \\ \Theta_1 \end{bmatrix} = \begin{bmatrix} w_1\phi^2\lambda \\ mw_1a_1 \\ mw_1b_1 \end{bmatrix}$$

have no common inner divisor, in consequence, $\begin{bmatrix} A \\ \Theta_1 \end{bmatrix}$ is $*$-outer.

Proof that $\operatorname{clos}\left\{\Delta_{*1}A_*^*\Delta_{*2}(\eta_2L^2 \oplus \theta_2\chi_\Omega L^2)\right\} = \eta_1L^2 \oplus \theta_1\chi_\Omega L^2$.

As in the proof of Lemma 5.6, it is sufficient to check two identities

$$\operatorname{clos}\left\{\Delta_{w_1}\theta_1\theta_1^*A_*^*\theta_2\chi_\Omega L^2\right\} = \theta_1\chi_\Omega L^2$$

and

$$\operatorname{clos}\left\{\Delta_{*1}A_*^*\eta_2L^2\right\} = \eta_1L^2.$$

For the first expression we have

$$\begin{aligned}
\operatorname{clos}\left\{\Delta_{w_1}\theta_1\theta_1^*A_*^*\theta_2\chi_\Omega L^2\right\} &= \operatorname{clos}\left\{\Delta_{w_1}\theta_1(A_*\theta_1)^*\theta_2\chi_\Omega L^2\right\} \\
&= \operatorname{clos}\left\{\Delta_{w_1}\theta_1(w_2\phi^2\Lambda^{\mathrm{ad}}\theta_1)^*\theta_2\chi_\Omega L^2\right\} \\
&= \operatorname{clos}\left\{\Delta_{w_1}\theta_1(w_2\phi^2\Lambda^{\mathrm{ad}}\Lambda\theta_2)^*\theta_2\chi_\Omega L^2\right\} \\
&= \operatorname{clos}\left\{\Delta_{w_1}\theta_1(w_2\phi^2\lambda\theta_2)^*\theta_2\chi_\Omega L^2\right\} \\
&= \operatorname{clos}\left\{\Delta_{w_1}\theta_1\overline{w}_2\overline{\phi}^2\overline{\lambda}\chi_\Omega L^2\right\} \\
&= \operatorname{clos}\left\{\theta_1\chi_\Omega L^2\right\} = \theta_1\chi_\Omega L^2,
\end{aligned}$$

using that w_2 and $\phi^2\lambda$ are not equal to zero almost everywhere, θ_1 is an isometry and Δ_{w_1} vanishes only outside Ω.

On the other hand, as $\Lambda\theta_2 = \theta_1$, we have

$$\eta_2^*\Lambda^{\mathrm{ad}} = \begin{bmatrix} b_2 & -a_2 \end{bmatrix}\begin{bmatrix} \Lambda_{22} & -\Lambda_{12} \\ -\Lambda_{21} & \Lambda_{11} \end{bmatrix} = \begin{bmatrix} b_1 & -a_1 \end{bmatrix} = \eta_1^*,$$

therefore, for the second expression we get

$$\begin{aligned}
\operatorname{clos}\left\{\Delta_{*1}A_*^*\eta_2L^2\right\} &= \operatorname{clos}\left\{\Delta_{*1}(\eta_2^*A_*)^*L^2\right\} = \operatorname{clos}\left\{\Delta_{*1}(w_2\phi^2\eta_2^*\Lambda^{\mathrm{ad}})^*L^2\right\} \\
&= \operatorname{clos}\left\{\Delta_{*1}(w_2\phi^2\eta_1^*)^*L^2\right\} = \operatorname{clos}\left\{\overline{w}_2\overline{\phi}^2\Delta_{*1}\eta_1L^2\right\} \\
&= \operatorname{clos}\left\{\overline{w}_2\overline{\phi}^2\eta_1L^2\right\} = \eta_1L^2,
\end{aligned}$$

and the lemma is proved. ■

Finally, Lemma 5.7 directly implies the Main Theorem.

6. Concluding remarks

First of all we would like to observe that it is possible to separate the inner and outer factors of $\Theta = mw\theta$, i.e., to consider the quasi-similarity of the operators with scalar outer and 2×1 inner characeristic function separately. To see this, let us check that the contractions with the characteristic functions

$$\Theta_1 = mw \begin{bmatrix} a \\ b \end{bmatrix} = mw\theta \quad \text{and} \quad \Theta_2 = \begin{bmatrix} w & 0 \\ 0 & \left[\begin{smallmatrix} ma \\ mb \end{smallmatrix}\right] \end{bmatrix} = \begin{bmatrix} w & 0 \\ 0 & m\theta \end{bmatrix}, \tag{6.1}$$

are quasi-similar.

For these characteristic functions we can take the auxiliary spaces as $\mathcal{E} = \mathbb{C}$, $\mathcal{E}_{*1} = \mathcal{E}_2 = \mathbb{C}^2$, and $\mathcal{E}_{*2} = \mathbb{C}^3$. Then

$$\begin{aligned} \Delta_1 &= \Delta_w, & \Delta_{*1} &= \eta\eta^* + \Delta_w \theta\theta^*; \\ L^2(\Delta_1 \mathcal{E}_1) &= L^2(\Delta_w \mathbb{C}) = \chi_\Omega L^2, & L^2(\Delta_{*1}\mathcal{E}_{*1}) &= \eta L^2 \oplus \theta \chi_\Omega L^2; \\ \Delta_2 &= \begin{bmatrix} \Delta_w & 0 \\ 0 & 0 \end{bmatrix}, & \Delta_{*2} &= \begin{bmatrix} \Delta_w & 0 \\ 0 & \eta\eta^* \end{bmatrix}; \\ L^2(\Delta_2 \mathcal{E}_2) &= \begin{bmatrix} \chi_\Omega L^2 \\ 0 \end{bmatrix}, & L^2(\Delta_{*2}\mathcal{E}_{*2}) &= \begin{bmatrix} \chi_\Omega L^2 \\ \eta L^2 \end{bmatrix}. \end{aligned}$$

Proposition 6.1 *The operators T_1 and T_2 with respective characteristic functions given in* (6.1) *are quasi-similar.*

Proof. To prove the assertion we construct two bounded operators $X : \mathcal{H}_2 \to \mathcal{H}_1$ and $X' : \mathcal{H}_1 \to \mathcal{H}_2$ such that

$$\begin{aligned} X'T_1 &= T_2 X', \quad \operatorname{clos}\{X'\mathcal{H}_1\} = \mathcal{H}_2, \quad \ker(X') = \{0\}; \\ T_1 X &= X T_2, \quad \operatorname{clos}\{X\mathcal{H}_2\} = \mathcal{H}_1, \quad \ker(X) = \{0\}. \end{aligned}$$

To do this we present two suitable liftings $Y = \pi_{*1} A_* \pi_{*2}^* + \tau_1 \Delta_1 A \pi_2^* + \tau_1 A_0 \tau_{*2}^*$ and $Y' = \pi_{*2} A'_* \pi_{*1}^* + \tau_2 \Delta_2 A' \pi_1^* + \tau_2 A'_0 \tau_{*1}^*$ of X and X' respectively. We take $A_0 = 0$ and $A'_0 = 0$. According Lemmas 3.2 and 3.4 it is sufficient to find four matrix-valued functions $A' \in H^\infty_{2\times 1}, A'_* \in H^\infty_{3\times 2}$, $A \in H^\infty_{1\times 2}$, and $A_* \in H^\infty_{2\times 3}$ satisfying the following ten conditions

1. $A'_* \Theta_1 = \Theta_2 A'$,
2. $\left[\begin{smallmatrix} A' \\ \Theta_1 \end{smallmatrix}\right]$ is $*$-outer,
3. $\begin{bmatrix} A'_* & \Theta_2 \end{bmatrix}$ is outer,
4. $\operatorname{clos}\{\Delta_2 A' \Delta_1 L^2(\Delta_1 \mathcal{E}_1)\} = L^2(\Delta_2 \mathcal{E}_2)$,
5. $\operatorname{clos}\{\Delta_{*1} A_*'^{*} \Delta_{*2} L^2(\Delta_{*2}\mathcal{E}_{*2})\} = L^2(\Delta_{*1}\mathcal{E}_{*1})$, (6.2)

6. $\Theta_1 A = A_* \Theta_2$,
7. $\begin{bmatrix} A \\ \Theta_2 \end{bmatrix}$ is $*$-outer,
8. $\begin{bmatrix} A_* & \Theta_1 \end{bmatrix}$ is outer,
9. $\operatorname{clos}\{\Delta_1 A \Delta_2 L^2(\Delta_2 \mathcal{E}_2)\} = L^2(\Delta_1 \mathcal{E}_1)$,
10. $\operatorname{clos}\{\Delta_{*2}(A_*)^* \Delta_{*1} L^2(\Delta_{*1}\mathcal{E}_{*1})\} = L^2(\Delta_{*2}\mathcal{E}_{*2})$.

It easy to check by direct calculation that all these conditions are verified if we take

$$A' = \begin{bmatrix} ma \\ w \end{bmatrix}, \qquad A'_* = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$A = \begin{bmatrix} 1 & 1 \end{bmatrix}, \qquad A_* = \begin{bmatrix} ma & w & 0 \\ mb & 0 & w \end{bmatrix} = \begin{bmatrix} m\theta & wI_{2\times 2} \end{bmatrix}.$$

Since this verification contains no specific difficulties we omit them. ■

We will now see that, as we said in our remarks following the statement of the Main Theorem, the situation is different if we try to split the inner parts of the characteristic function and get quasi-similarity for the operators with the characteristic functions

$$\Theta_1 = m \begin{bmatrix} a \\ b \end{bmatrix} = m\theta \qquad \text{and} \qquad \Theta_2 = \begin{bmatrix} m & 0 \\ 0 & \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] \end{bmatrix} = \begin{bmatrix} m & 0 \\ 0 & \theta \end{bmatrix}, \tag{6.3}$$

Proposition 6.2 *The operators T_1 and T_2 with respective characteristic functions given in* (6.3) *are quasi-similar if, and only if, $\mathcal{N}^+\{m, a, b\} = \mathcal{N}^+$, i.e., if there exist three functions $f_1, f_2, f_3 \in H^\infty$ such that $mf_1 + af_2 + bf_3$ is an outer function.*

Proof. First we show that the necessity of the stated condition follows from the density of the range of the intertwining operator $X : \mathcal{H}_2 \to \mathcal{H}_1$, $T_1 X = XT_2$. Let $Y = \pi_{*1} A_* \pi_{*2}^* + \tau_1 \Delta_1 A \pi_2^* + \tau_1 A_0 \tau_{*2}^* = \pi_1 A \pi_2^* + \pi_{*1} A_* \Delta_{*2} \tau_{*2}^* + \tau_1 A_0 \tau_{*2}^*$ be the lifting of X with the parameter A being a matrix-valued function from $H^\infty_{1\times 2}$ and A_* from $H^\infty_{2\times 3}$, say

$$A = \begin{bmatrix} a_1 & a_2 \end{bmatrix} \qquad \text{and} \qquad A_* = \begin{bmatrix} A_{*1} & A_{*2} \end{bmatrix},$$

where A_{*1} is the first column of A_* and A_{*2} is the square matrix consisting of second and third columns of A_*. Then the intertwining relation $\Theta_1 A = A_* \Theta_2$ yields

$$A_{*1} = a_1 \theta \qquad \text{and} \qquad A_{*2}\theta = m a_2 \theta.$$

The second relation implies that $\eta^* A_{*2}\theta = 0$ and therefore (see Lemma 5.2) there exists an H^∞-function φ such that $\eta^* A_{*2} = \varphi\eta^*$. Check that φ is a common divisor of all minors of the matrix $\begin{bmatrix} A_* & \Theta_1 \end{bmatrix} = \begin{bmatrix} a_1\theta & A_{*2} & m\theta \end{bmatrix}$. The situation is the same as in the proof of Lemma 5.5, where all minors of the corresponding matrix were divisible by f_1. Indeed, the determinant of A_{*2} is divisible by φ (moreover, $\det A_{*2} = \varphi m a_2$) because it is its eigenvalue (and the second eigenvalue is ma_2), and minors containing one collumn from A_{*2} and the second column being θ are just the entries of the row $\eta^* A_{*2} = \varphi\eta^*$, therefore, they are also divisible by φ. Since $\begin{bmatrix} A_* & \Theta_1 \end{bmatrix}$ has to be outer (Lemma 3.1), the function φ is outer as well.

Using again Lemma 5.2, the identity $(A_{*2} - ma_2 I)\theta = 0$ guarantees the existence of a vector $\psi = \left[\begin{smallmatrix} \psi_1 \\ \psi_2 \end{smallmatrix}\right]$ with H^∞-entries such that $A_{*2} - ma_2 I = \psi\eta^*$. Therefore,

$$A_{*2} = ma_2 I + \psi\eta^* = \begin{bmatrix} ma_2 + b\psi_1 & -a\psi_1 \\ b\psi_2 & ma_2 - a\psi_2 \end{bmatrix},$$

whence

$$\varphi m a_2 = \det A_{*2} = (ma_2)^2 + ma_2(b\psi_1 - a\psi_2),$$

which gives us the required property: the function $ma_2 + b\psi_1 - a\psi_2 = \varphi$ is outer.

To construct an intertwining operator $X' : \mathcal{H}_1 \to \mathcal{H}_2$ we need no additional properties of the functions m, a, and b. We take a number t from the first assertion of Lemma 5.3, i.e., such that $m \wedge (a + tb) = 1$, and put

$$A' = \begin{bmatrix} a + tb \\ m(2 + b + a) \end{bmatrix}, \qquad A'_* = \begin{bmatrix} 1 & t \\ 2 + a & a \\ b & 2 + b \end{bmatrix}.$$

To show that the operator X' given as the compression of its lifting Y' with these parameters and $A'_0 = 0$, is as required it is sufficient to check the first five conditions of (6.2). The first can be verified by direct calculation, the second is evident, because A' itself is already $*$-outer. So, let us check condition (3) of (6.2). The minors of the matrix

$$\begin{bmatrix} A'_* & \Theta_2 \end{bmatrix} = \begin{bmatrix} 1 & t & m & 0 \\ 2 + a & a & 0 & a \\ b & 2 + b & 0 & b \end{bmatrix}$$

are the functions

$$2m(2 + a + b), \quad 2(a + tb), \quad -2mb, \quad \text{and} \quad 2ma$$

which are mutually prime due to the choice of t.

Condition (4) is trivial because Δ_i are zero operators. Finally, to see that (5) is fulfilled we need to write down the expressions for Δ_{*i}:

$$\Delta_{*1} = \eta\eta^*, \qquad \Delta_{*2} = \begin{bmatrix} 0 & 0 \\ 0 & \eta\eta^* \end{bmatrix}.$$

And since

$$A_*'^{\,*} \begin{bmatrix} 0 \\ \eta \end{bmatrix} = 2\eta,$$

we have

$$\operatorname{clos}\{\Delta_{*1} A_*'^{\,*} L^2(\Delta_{*2}\mathcal{E}_{*2})\} = \operatorname{clos}\left\{\eta\eta^* A_*'^{\,*} \begin{bmatrix} 0 \\ \eta L^2 \end{bmatrix}\right\} = 2\eta L^2 = L^2(\Delta_{*1}\mathcal{E}_{*1}).$$

To construct intertwining operator X we shall use three H^∞-functions f_i such that $\varphi = mf_1 + af_2 + bf_3$ is outer. We check the conditions (6)–(10) of (6.2) for

$$A = \begin{bmatrix} 1 & -f_1 \end{bmatrix}$$

and

$$A_* = \begin{bmatrix} a & -mf_1 - bf_3 & af_3 \\ b & f_2 b & -mwf_1 - af_2 \end{bmatrix} = \begin{bmatrix} \theta & \begin{bmatrix} -f_3 \\ f_2 \end{bmatrix} \eta^* - mf_1 I_{2\times 2} \end{bmatrix}.$$

Again, condition (6) can be checked by direct calculation. The matrix

$$\begin{bmatrix} A \\ \Theta_2 \end{bmatrix} = \begin{bmatrix} 1 & -f_1 \\ mw & 0 \\ 0 & a \\ 0 & b \end{bmatrix}$$

is $*$-outer because

$$\det \begin{bmatrix} 1 & -f_1 \\ 0 & a \end{bmatrix} = a \qquad \text{and} \qquad \det \begin{bmatrix} 1 & -f_1 \\ 0 & b \end{bmatrix} = b$$

are mutually prime, i.e., condition (7) is fulfilled. The matrix

$$\begin{bmatrix} A_* & \Theta_1 \end{bmatrix} = \begin{bmatrix} a & -mf_1 - bf_3 & af_3 & ma \\ b & bf_2 & -mf_1 - af_2 & mb \end{bmatrix}$$

is outer because

$$\det \begin{bmatrix} a & -mf_1 - bf_3 \\ b & bf_2 \end{bmatrix} = b\varphi \qquad \text{and} \qquad \det \begin{bmatrix} a & af_3 \\ b & -mf_1 - af_2 \end{bmatrix} = -a\varphi$$

are mutually prime, i.e., condition (8) is fulfilled.

Condition (9) is trivial because all spaces there are zero spaces. And finally condition (10):

$$\begin{aligned}
&\operatorname{clos}\{\Delta_{*2}(A'_*)^*\Delta_{*1}L^2(\Delta_{*1}\mathcal{E}_{*1})\} \\
&\qquad= \operatorname{clos}\left\{\begin{bmatrix}0 & 0\\ 0 & \eta\eta^*\end{bmatrix}\begin{bmatrix}\theta^* \\ \eta\begin{bmatrix}-\overline{f_3} & \overline{f_2}\end{bmatrix} - \overline{m}\overline{f_1}I_{2\times 2}\end{bmatrix}\eta L^2\right\} \\
&\qquad= \operatorname{clos}\left\{\begin{bmatrix}0 & 0\\ 0 & \eta\end{bmatrix}\begin{bmatrix}0 \\ \begin{bmatrix}-\overline{f_3} & \overline{f_2}\end{bmatrix}\eta - \overline{m}\overline{f_1}\end{bmatrix} L^2\right\} \\
&\qquad= \operatorname{clos}\left\{\begin{bmatrix}0 & 0\\ 0 & \eta\end{bmatrix}\begin{bmatrix}0 \\ -\overline{\varphi}L^2\end{bmatrix}\right\} = \begin{bmatrix}0 \\ \eta L^2\end{bmatrix} = L^2(\Delta_{*2}\mathcal{E}_{*2}).
\end{aligned}$$

■

Conjecture

Concluding the paper we would like to conjecture that the same result could be true for general contractions with $(n+1)\times n$ characteristic function of rank n. Namely, if we factor this function as a product of an inner $*$-outer $(n+1)\times n$ function and a square $n\times n$ characteristic function of a weak contraction, then we have quasi-similar classification for both operators. And in spite of the fact that initial operator is not quasi-similar in general to the direct sum of these its parts, nevertheless the quasi-similarity of two such operators occurs if, and only if, these parts of one operator are quasi-similar to the corresponding parts of the other separately.

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Sergio Bermudo
Dpto. Economía, Métodos Cuantitativos e Historia Económica
Universidad Pablo de Olavide
Ctra. Utrera Km. 1, 41013-Sevilla, Spain
sbernav@upo.es

Carmen H. Mancera
ETS Ingenieros
Universidad de Sevilla
Camino de los Descubrimientos s/n, 41092-Sevilla, Spain
mancera@ma2.us.es

Pedro J. Paúl
ETS Ingenieros
Universidad de Sevilla
Camino de los Descubrimientos s/n, 41092-Sevilla, Spain
piti@us.es

Vasily Vasyunin
Petersburg Department of Steklov Institute of Mathematics
Russian Academy of Sciences
Fontanka 27, 191023 St. Petersburg, Russia
vasyunin@pdmi.ras.ru

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