Rev. Mat. Iberoamericana **22** (2006), no. 1, 93–110

An extension of the Krein-Šmulian Theorem

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Abstract

Let X be a Banach space, $u \in X^{**}$ and K, Z two subsets of X^{**}. Denote by $d(u, Z)$ and $d(K, Z)$ the distances to Z from the point u and from the subset K respectively. The Krein-Šmulian Theorem asserts that the closed convex hull of a weakly compact subset of a Banach space is weakly compact; in other words, every w*-compact subset $K \subset X^{**}$ such that $d(K, X) = 0$ satisfies $d(\overline{\text{co}}^{w^*}(K), X) = 0$.

We extend this result in the following way: if $Z \subset X$ is a closed subspace of X and $K \subset X^{**}$ is a w*−compact subset of X^{**}, then

$$d(\overline{\text{co}}^{w^*}(K), Z) \leq 5d(K, Z).$$

Moreover, if $Z \cap K$ is w*-dense in K, then $d(\overline{\text{co}}^{w^*}(K), Z) \leq 2d(K, Z)$. However, the equality $d(K, X) = d(\overline{\text{co}}^{w^*}(K), X)$ holds in many cases, for instance, if $\ell_1 \not\subseteq X^*$, if X has w*-angelic dual unit ball (for example, if X is WCG or WLD), if $X = \ell_1(I)$, if K is fragmented by the norm of X^{**}, etc. We also construct under CH a w*-compact subset $K \subset B(X^{**})$ such that $K \cap X$ is w*-dense in K, $d(K, X) = \frac{1}{2}$ and $d(\overline{\text{co}}^{w^*}(K), X) = 1$.

1. Introduction

If X is a Banach space, let $B(X)$ and $S(X)$ be the closed unit ball and unit sphere of X, respectively, and X^* its topological dual. If $u \in X^{**}$ and K, Z are two subsets of X^{**}, let $d(u, Z) = \inf\{\|u - z\| : z \in Z\}$ be the distance to Z from u, $d(K, Z) = \sup\{d(k, Z) : k \in K\}$ the distance to Z from K, $\text{co}(K)$ the convex hull of K, $\overline{\text{co}}(K)$ the norm-closure of $\text{co}(K)$ and $\overline{\text{co}}^{w^*}(K)$ the w*-closure of $\text{co}(K)$.

2000 Mathematics Subject Classification: 46B20, 46B26.
Keywords: Krein-Šmulian Theorem, Banach spaces, compact sets.

This paper is devoted to investigate the connection between the distances $d(\overline{\text{co}}^{w^*}(K), Z)$ and $d(K, Z)$, when $Z \subset X^{**}$ is a subspace of X (in particular, when $Z = X$) and K is a w*-compact subset of X^{**}. There exist some facts that suggest that the distance $d(\overline{\text{co}}^{w^*}(K), Z)$ is controlled by the distance $d(K, Z)$. Indeed, on the one hand, we have the classical Theorem of Krein-Šmulian (see [5, p. 51]). Using the terminology of distances, this Theorem asserts the following: if X is a Banach space, every w*-compact subset $K \subset X^{**}$ with $d(K, X) = 0$ (that is, $K \subset X$ is a weakly compact subset of X) satisfies $d(\overline{\text{co}}^{w^*}(K), X) = 0$ (that is, the closed convex hull $\overline{\text{co}}(K)$ of K in X is weakly compact).

On the other hand, if the dual X^* of the Banach space X does not contain a copy of ℓ_1, it is very easy to prove that $d(K, Z) = d(\overline{\text{co}}^{w^*}(K), Z)$ for every w*-compact subset $K \subset X^{**}$ of X^{**} and every subspace $Z \subset X^{**}$. Indeed, in this case $\overline{\text{co}}(K) = \overline{\text{co}}^{w^*}(K)$ (see [9]). So, as $d(\text{co}(K), Z) = d(K, Z)$ (this follows from the fact that the function $\varphi(u) := d(u, Z)$, $\forall u \in X^{**}$, is convex when $Z \subset X^{**}$ is a convex subset of X^{**}), we easily obtain that $d(K, Z) = d(\overline{\text{co}}^{w^*}(K), Z)$.

In view of these facts, one is inclined to conjecture that $d(K, X) = d(\overline{\text{co}}^{w^*}(K), X)$ for every w*-compact subset $K \subset X^{**}$ and every Banach space X. Unfortunately, assuming the Continuum Hypothesis (for short, CH), this is not true because of the following result we will prove here.

Theorem 1 *Under CH, if $X = \ell^{c}_{\infty}(\omega^+)$ (= subspace of the elements $f \in \ell_\infty(\omega^+)$ with countable support), there exists a w*-compact subset $H \subset B(X^{**})$ such that $d(H, X) = 1/2$, $H \cap X$ is w*-dense in H and $d(\overline{co}^{w^*}(H), X) = 1$.*

However, there exist many Banach spaces X for which the equality $d(K, X) = d(\overline{\text{co}}^{w^*}(K), X)$ holds, for every w*-compact subset $K \subset X^{**}$, for example, the class of Banach spaces with property J.

Definition 2 *A Banach space X has property J (for short, $X \in J$) if for every $z \in B(X^{**}) \setminus X$ and for every number $b \in \mathbb{R}$ with $0 < b < d(z, X)$, there exists a sequence $\{x^*_n\}_{n\geq 1} \subset \mathfrak{S}(B(X^*), z, b) := \{u \in B(X^*) : z(u) \geq b\}$ such that $x^*_n \xrightarrow{w^*} 0$.*

For this class of Banach spaces with property J we prove the following result.

Theorem 3 *Let X be a Banach space such that $X \in J$. Then for every w*-compact subset $K \subset X^{**}$ we have $d(K, X) = d(\overline{co}^{w^*}(K), X)$.*

In the following corollary we state that many Banach spaces have property J and, so, satisfy Theorem 3. Recall that, for a Banach X, the dual unit ball $(B(X^*), w^*)$ is *angelic* in the w*-topology if, for every subset $A \subset B(X^*)$ and every $z \in \overline{A}^{w^*}$, there exists a sequence $\{a_n\}_{n\geq 1} \subset A$ such that $a_n \xrightarrow{w^*} z$.

Corollary 4 *If X is a Banach space such that $(B(X^*), w^*)$ is angelic (for example, if X is weakly compactly generated (for short, WCG) or weakly Lindelöf determined (for short, WLD)), then $X \in J$ and, so, for every w^*-compact subset $K \subset X^{**}$ we have $d(K, X) = d(\overline{co}^{w^*}(K), X)$.*

Although the equality $d(K, X) = d(\overline{\text{co}}^{w^*}(K), X)$ does not hold in general, we can ask whether there exists a universal constant $1 \leq M < \infty$ such that $d(\overline{\text{co}}^{w^*}(K), X) \leq M d(K, X)$ for every Banach space X and every w*-compact subset $K \subset X^{**}$.

The answer to this question is affirmative. We prove the following result, which extends the Krein-Šmulian Theorem.

Theorem 5 *If X is a Banach space, $Z \subset X$ a closed subspace of X and $K \subset X^{**}$ a w^*-compact subset, then $d(\overline{co}^{w^*}(K), Z) \leq 5d(K, Z)$.*

When $K \cap Z$ is w*-dense in K, the argument used in Theorem 5 gives the following result.

Theorem 6 *Let X be a Banach space, $Z \subset X$ a closed subspace and $K \subset X^{**}$ a w^*-compact subset. If $Z \cap K$ is w^*-dense in K, then $d(\overline{co}^{w^*}(K), Z) \leq 2d(K, Z)$.*

Finally, we also obtain the following result.

Theorem 7 *Let I be an infinite set and $X = \ell_1(I)$. Then for every w^*-compact subset $K \subset X^{**}$ we have $d(\overline{co}^{w^*}(K), X) = d(K, X)$.*

A version of the problem we study here was considered (independently) by M. Fabian, P. Hájek, V. Montesinos and V. Zizler in [7]. They study the class of w*-compact subsets $K \subset X^{**}$ such that $K \cap X$ is w*-dense in K. Instead of distances, they deal with the notion of *ϵ-weakly relatively compact* subsets of X (for short, ϵ-WRK) introduced in [8]. A bounded subset H of X is said to be ϵ-WRK, for some $\epsilon > 0$, if $\overline{H}^{w^*} \subset X + \epsilon B(X^{**})$, that is, if $d(\overline{H}^{w^*}, X) \leq \epsilon$. Using arguments based on the techniques of double limit due to Grothendieck and Pták, they prove that the constant $M = 2$ holds for this category of w*-compact subsets $K \subset X^{**}$ such that $K \cap X$ is w*-dense in K. More precisely, they prove the following beautiful result.

Theorem ([7]) *Let X be a Banach space and $H \subset X$ a bounded subset of X. Assume that H is ϵ-WRK for some $\epsilon > 0$. Then the convex hull $co(H)$ is 2ϵ-WRK. Moreover, if $(B(X^*), w^*)$ is angelic, or X^* does not contain a copy of ℓ_1, then $co(H)$ is ϵ-WRK.*

Observe that the Theorem of Krein-Šmulian follows from this result when $\epsilon = 0$.

2. Proofs of the results

Let us introduce some notation and terminology (see [1], [4], [6], [11]). $|A|$ denotes the cardinality of a set A, ω^+ the first uncountable ordinal, $\aleph_1$ the first uncountable cardinal and CH the continuum hypothesis. A Hausdorff compact space K is said to have property (M) if every Radon Borel measure μ on K has separable support $\operatorname{supp}(\mu)$. If K is a convex compact subset of some locally convex linear space X and μ is a Radon Borel probability measure on K, $r(\mu)$ denotes the *barycentre* of μ. Recall (see [3]) that $r(\mu) \in K$ and that $r(\mu)$ satisfies $x^*(r(\mu)) = \int_K x^*(k)d\mu$ for every $x^* \in X^*$.

If X is a Banach space, let $X^\perp = \{z \in X^{***} : \langle z, x\rangle = 0,\ \forall x \in X\}$ denote the subspace of X^{***} orthogonal to X. If $Y \subset X$ is a subspace of X, let $Y^\perp(X^*) = \{z \in X^* : \langle z, y\rangle = 0,\ \forall y \in Y\}$ be the subspace of X^* orthogonal to Y, $Y^\perp(X^{***}) = \{z \in X^{***} : \langle z, y\rangle = 0,\ \forall y \in Y\}$, etc. So, $X^\perp = X^\perp(X^{***})$. Recall that, if $u \in X$ (resp., $u \in X^{**}$), then $d(u, Y) = \sup\{\langle z, u\rangle : z \in B(Y^\perp(X^*))\}$ (resp., $d(u, Y) = \sup\{\langle z, u\rangle : z \in B(Y^\perp(X^{***}))\}$). If $A \subset X$ is a subset of X, $[A]$ denotes the linear span of A.

Let I be an infinite set with the discrete topology. Then:

(0) We use the symbol $\ell_\infty(I)$ to denote the Banach space of all $f = (f(i))_{i\in I} \in \mathbb{R}^I$ with supremum norm finite $\|f\| := \sup\{|f(i)| : i \in I\} < \infty$. The symbol $c_0(I)$ means its subspace consisting from $f = (f(i))_{i\in I} \in \ell_\infty(I)$ such that the set $\{i \in I : |f(i)| > \epsilon\}$ is finite for all $\epsilon > 0$.

(1) If $f \in \ell_\infty(I)$, $\operatorname{supp}(f) = \{i \in I : f(i) \neq 0\}$ will be the *support* of f and $\check{f}$ the *Stone-Čech extension* of f to βI, where βI is the *Stone-Čech compactification* of I.

(2) Let $cI = \cup\{\overline{A}^{\beta I} : A \subset I,\ A \text{ countable}\}$ and $\ell^c_\infty(I) = \{f \in \ell_\infty(I) : \operatorname{supp}(f) \text{ countable}\}$. Observe that cI is an open subset of βI and that, if $f \in \ell_\infty(I)$, then $f \in \ell^c_\infty(I)$ if and only if $\check{f}_{\restriction \beta I\setminus cI} = 0$.

(3) Let $\Sigma(\{0,1\}^I) = \{x \in \{0,1\}^I :\ \operatorname{supp}(x) \text{ countable}\}$ and $\Sigma([-1,1]^I) = \{x \in [-1,1]^I :\ \operatorname{supp}(x) \text{ countable}\}$.

(4) Recall that a compact space is said to be a *Corson space* if it is homeomorphic to some compact subset of $\Sigma([-1,1]^I)$.

Proof of Theorem 1. We use a modification of the Argyros-Mercourakis-Negrepontis Corson compact space without property (M) [1, p. 219]. In the following we adopt the notation and terminology of [1, p. 219]. Let Ω be the space of Erdös, that is, the Stone space of the quotient algebra M_λ/N_λ, where λ is the Lebesgue measure on $[0,1]$, M_λ is the algebra of λ-measurable subsets of $[0,1]$ and N_λ is the ideal of λ-null subsets of $[0,1]$. Ω is a compact extremely disconnected space (because M_λ/N_λ is complete) and there exists a strictly positive regular Borel normal probability measure $\tilde{\lambda}$ on Ω, determined by the condition $\tilde{\lambda}(V) = \lambda(U)$, V being any clopen subset of Ω and U a λ-measurable subset of $[0,1]$ such that $V = U + N_\lambda$.

Now we proceed as in [1, 3.11 Lemma] with small changes. Write $[0,1] = \{x_\xi : \xi < \omega^+\}$ and let $\{K_\xi : \xi < \omega^+\}$ be the well-ordered class of compact subsets of $[0,1]$ with strictly positive Lebesgue measure. For each $\xi < \omega^+$ we choose a compact subset $U_\xi \subset [0,1]$ such that:

(a) $U_\xi \subset \{x_\rho : \xi < \rho < \omega^+\} \cap K_\xi$.

(b) If $\lambda(K_\xi) = 1$, then U_ξ satisfies the condition $\lambda(U_\xi) > 0$. If $\lambda(K_\xi) < 1$, U_ξ satisfies the condition $\lambda(K_\xi) - (1 - \lambda(K_\xi)) < \lambda(U_\xi) \leq \lambda(K_\xi)$.

Let V_ξ be the clopen subset of Ω corresponding to U_ξ. Then $\{V_\xi : \xi < \omega^+\}$ is a pseudobase of Ω that witnesses the failure of the property *caliber* ω^+, that is, if $A \subset \omega^+$ and $|A| = \aleph_1$, then $\cap_{\xi \in A} V_\xi = \emptyset$. Moreover, (b) automatically implies that $|\{\xi < \omega^+ : \lambda(U_\xi) > t\}| = \aleph_1$ for every $0 < t < 1$, whence $|\{\xi < \omega^+ : \tilde{\lambda}(V_\xi) > t\}| = \aleph_1$ for every $0 < t < 1$.

Consider $\mathcal{A} = \{A \subset \omega^+ : \cap_{\xi \in A} V_\xi \neq \emptyset\}$. Clearly, $\mathcal{A}$ is an adequate family (see [11, p. 1116]) such that every element of $\mathcal{A}$ is a countable subset of ω^+. Moreover, there are elements $A \in \mathcal{A}$ with $|A| = \aleph_0$. Indeed, apply a well-known result from measure theory (see Lemma 8) and the fact that $\{\xi < \omega^+ : \tilde{\lambda}(V_\xi) > \delta\}$ is infinite for some (in fact, every) $0 < \delta < 1$.

So, if $K = \{\mathbf{1}_A : A \in \mathcal{A}\} \subset \Sigma(\{0,1\}^{\omega^+}) \subset \ell_\infty^c(\omega^+)$, then K is a Corson compact space with respect to the w*-topology $\sigma(\ell_\infty(\omega^+), \ell_1(\omega^+))$. Define the continuous map $T : \Omega \to K$ so that, for every $x \in \Omega$, $T(x) = \mathbf{1}_{A_x}$, where $A_x = \{\xi \in \omega^+ : x \in V_\xi\}$. Observe that $A_x \in \mathcal{A}$ and, so, $T(x) \in K$, $\forall x \in \Omega$.

Let $L = T(\Omega) \subset K$. Then L is a Corson compact space without property (M), because L is nonseparable but L is the support of μ, where $\mu = T(\tilde{\lambda})$ is the probability on K image of $\tilde{\lambda}$ under T. So, as $L \subset K$, K is also a Corson compact space without property (M).

Let I be the space ω^+, with the discrete topology, and $X = \ell_\infty^c(I)$. Then, the dual space X^* is

$$X^* = \ell_1(I) \oplus_1 M_R(cI \setminus I),$$

where $M_R(cI \setminus I)$ is the space of Radon Borel measures ν on βI such that $\mathrm{supp}(\nu) \subset cI \setminus I$ and $\oplus_1$ means ℓ_1-sum (that is, if a Banach space Y has the decomposition $Y = Y_1 \oplus_1 Y_2$ and $y \in Y$, with $y = y_1 + y_2$ and $y_1 \in Y_1, y_2 \in Y_2$, then $\|y\| = \|y_1\| + \|y_2\|$). Observe that $\ell_1(I) \oplus_1 M_R(cI \setminus I)$ can be considered as a 1-complemented closed subspace of $(\ell_\infty(I))^* = \ell_1(I) \oplus_1 M_R(\beta I \setminus I)$.

The bidual of X is

$$X^{**} = \ell_\infty(I) \oplus_\infty M_R(cI \setminus I)^*,$$

where $\oplus_\infty$ means ℓ_∞-sum (that is, if a Banach space Y has the decomposition $Y = Y_1 \oplus_\infty Y_2$ and $y \in Y$, with $y = y_1 + y_2$ and $y_1 \in Y_1, y_2 \in Y_2$, then $\|y\| = \sup\{\|y_1\|, \|y_2\|\}$). Let $\pi_1, \pi_2 : X^{**} \to X^{**}$ be the canonical projections onto $\ell_\infty(I)$ and $M_R(cI \setminus I)^*$, respectively. The subspaces $\pi_1(X^{**}) = \ell_\infty(I)$ and $\pi_2(X^{**}) = M_R(cI\setminus I)^*$ are w*-closed subspaces of X^{**}. Moreover, the w*-topology $\sigma(X^{**}, X^*)$ coincides on $\pi_1(X^{**}) = \ell_\infty(I)$ with the $\sigma(\ell_\infty(I), \ell_1(I))$-topology. For $x \in X^{**}$ we write $x = (x_1, x_2)$, with $\pi_1(x) = x_1 \in \ell_\infty(I)$ and $\pi_2(x) = x_2 \in M_R(cI \setminus I)^*$. So, if $J : X \to X^{**}$ is the canonical embedding and $f \in X$, then $J(f) = (f_1, f_2)$, where $f_1 = \pi_1(f) = f$ and $\pi_2(f) = f_2$ is such that $f_2(\nu) = \nu(\check{f}) = \int_{cI\setminus I} \check{f} d\nu$, for every $\nu \in M_R(cI \setminus I)$.

The map $\phi : \ell_\infty(I) \to X^{**}$ such that $\phi(f) = (f, 0)$, $\forall f \in \ell_\infty(I)$, is an isomorphism between $\ell_\infty(I)$ and $\pi_1(X^{**})$, for the norm-topologies and also for the $\sigma(\ell_\infty(I), \ell_1(I))$-topology of $\ell_\infty(I)$ and the w*-topology of $\pi_1(X^{**})$. So, $H := \phi(K) = \{(k, 0) : k \in K\} \subset B(X^{**})$ is a Corson compact space without property (M), which is homeomorphic to K. Since the family $\mathcal{A}$ is adequate (in particular, $B \in \mathcal{A}$ if $B \subset A$ and $A \in \mathcal{A}$), the subset $\{\mathbf{1}_A : A \in \mathcal{A}, A \text{ finite}\}$ of K is dense in K. So, as $J(\mathbf{1}_A) = (\mathbf{1}_A, 0)$ when $A \subset \omega^+$ is finite, we get that $H \cap J(X)$ is w*-dense in H, because

$$\phi(\{\mathbf{1}_A : A \in \mathcal{A}, A \text{ finite}\}) = \{(\mathbf{1}_A, 0) : A \in \mathcal{A}, A \text{ finite}\} = \\ = J(\{\mathbf{1}_A : A \in \mathcal{A}, A \text{ finite}\}) \subset H \cap J(X).$$

Claim 1. $d(H, J(X)) = \frac{1}{2}$.

Indeed, pick $f \in K$ and assume that $f = \mathbf{1}_A$, for some $A \in \mathcal{A}$. If $|A| < \aleph_0$, clearly $\phi(f) = (f, 0) = J(f)$, that is, $\phi(f) \in J(X)$. Suppose that $|A| = \aleph_0$. Then $d(\phi(f), J(X)) = \frac{1}{2}$ because:
(a) Clearly, $\|\phi(f) - \frac{1}{2}J(f)\| = \frac{1}{2}$, whence $d(\phi(f), J(X)) \le \frac{1}{2}$.
(b) On the other hand, $\|\phi(f) - J(g)\| \ge \frac{1}{2}$ for every $g \in X$. Indeed, let $g \in X$ and assume that $\|\phi(f) - J(g)\| \le \frac{1}{2}$. Then $\|f - g\| \le \frac{1}{2}$ in $\ell_\infty(I)$, which implies that $\frac{1}{2} \le g$ on A (because $f = \mathbf{1}_A$) and so $\check{g} \ge \frac{1}{2}$ on $\overline{A}^{\beta I}$. Since $|A| = \aleph_0$, we can pick $p \in \overline{A}^{\beta I} \setminus I \subset cI \setminus I$. Let $\delta_p \in M_R(cI \setminus I)$ be such that $\delta_p(h) = \check{h}(p)$ for every $h \in \ell_\infty(I)$. Notice that $\|\delta_p\| = 1$. Then, if $J(g) = (g, g_2)$, we have

$$|(\phi(f) - J(g))(\delta_p)| = |-g_2(\delta_p)| = |-\int_{cI\setminus I} \check{g} \cdot d(\delta_p)| = |-\check{g}(p)| \ge \frac{1}{2}.$$

Finally, recall that there are elements $A \in \mathcal{A}$ with $|A| = \aleph_0$.

Claim 2. $d(\overline{\text{co}}^{w^*}(H), J(X)) = 1$.

Indeed, first $d(\overline{\text{co}}^{w^*}(H), J(X)) \leq 1$ because $\overline{\text{co}}^{w^*}(H) \subset B(X^{**})$. On the other hand, let $\nu := \phi(\mu)$ be the probability on $\phi(L)$ image of μ under ϕ. Since $\phi(L) \subset B(\pi_1(X^{**}))$ and $\pi_1(X^{**})$ is a convex w*-closed subset of X^{**}, we conclude that $\overline{\text{co}}^{w^*}(\phi(L)) \subset B(\pi_1(X^{**}))$. So, as $r(\nu) \in \overline{\text{co}}^{w^*}(\phi(L))$, we get that $r(\nu) = (z_0, 0)$ for some $z_0 \in B(\ell_\infty(I))$. If $\xi \in I$, define $\pi_\xi : X^{**} \to \mathbb{R}$ by $\pi_\xi(f_1, f_2) = f_1(\xi)$, for all $(f_1, f_2) \in X^{**} = \ell_\infty(I) \oplus_\infty M_R(cI \setminus I)^*$. Observe that π_ξ is a w*-continuous linear map on X^{**}. So

$$z_0(\xi) = \pi_\xi(z_0, 0) = \pi_\xi(r(\nu)) = \int_{\phi(L)} \pi_\xi(k) d\nu = \int_L k(\xi) d\mu = \tilde{\lambda}(V_\xi).$$

Thus, for every $0 < t < 1$ we have, by construction, $|\{\xi \in I : z_0(\xi) > t\}| = |\{\xi \in I : \tilde{\lambda}(V_\xi) > t\}| = \aleph_1$, and this implies that $\|z_0 - g\| \geq 1$ in $\ell_\infty(I)$, for every $g \in X = \ell_\infty^c(I)$, whence $\|(z_0, 0) - J(g)\| \geq 1$ for every $g \in X$, that is, $d((z_0, 0), J(X)) \geq 1$. Finally, we obtain $d(\overline{\text{co}}^{w^*}(H), J(X)) \geq 1$ because $(z_0, 0) \in \overline{\text{co}}^{w^*}(\phi(L)) \subset \overline{\text{co}}^{w^*}(H)$.

And this completes the proof. ∎

Remark. Theorem 1 gives, under CH, a negative answer to the following question posed in Problem 3 of [7]: if X is a Banach space and $H \subset X$ a ϵ-WRK, is $\text{co}(H)$ a ϵ-WRK?

We need the following well known result from measure theory.

Lemma 8 *Let (Ω, Σ, μ) be a measure space with μ positive and finite and $\{A_n\}_{n<\omega} \subset \Sigma$ be a sequence of measurable sets with $\mu(A_n) > \delta > 0$ for all $n < \omega$ and some $\delta > 0$. Then there exists an infinite subset $I \subset \omega$ such that $\cap_{n \in I} A_n \neq \emptyset$.*

Proof. Consider the sequence $B_n = \cup_{k \geq n} A_k,\ n \geq 1$. The sequence $\{B_n\}_{n \geq 1}$ is decreasing and $\mu(B_n) > \delta$ for every $n \geq 1$. Hence $\mu(\cap_{n<\omega} B_n) \geq \delta$ and therefore $\cap_{n<\omega} B_n \neq \emptyset$. Choose $w \in \cap_{n<\omega} B_n$ and inductively a sequence $\{A_{n_k}\}_{k<\omega},\ n_k < n_{k+1}$, such that $w \in A_{n_k}$ for all $k < \omega$. Then $I = \{n_k : k < \omega\}$ is the desired infinite subset. ∎

Proposition 9 *Let I be an infinite set and $X = (c_0(I), \|\cdot\|_\infty)$. Then every w^*-compact subset $K \subset X^{**}$ satisfies $d(K, X) = d(\overline{co}^{w^*}(K), X)$.*

Proof. First, recall that if $f \in X^{**} = \ell_\infty(I)$, then

$$d(f, X) = \sup\{|\check{f}(p)| : p \in \beta I \setminus I\}.$$

Suppose that there exists a w*-compact subset $K \subset B(X^{**})$ such that $d(K,X) < d(\overline{\text{co}}^{w^*}(K),X)$. Then we can find two real numbers a,b such that

$$d(K,X) < a < b < d(\overline{\text{co}}^{w^*}(K),X) \leq 1.$$

Pick $z_0 \in \overline{\text{co}}^{w^*}(K)$ such that $d(z_0,X) > b$. So, there exist $\epsilon > 0$ and $p_0 \in \beta I \setminus I$ such that $|\check{z}_0(p_0)| > b+\epsilon$, for example, $\check{z}_0(p_0) > b+\epsilon$. Let $U \subset I$ be such that $p_0 \in \overline{U}^{\beta I}$ and $z_0(j) > b+\epsilon$, $\forall j \in U$. Let μ be a Radon Borel probability on K such that $z_0 = r(\mu)$ and denote $A_j := \{k \in K : k(j) \geq b\}$, $j \in U$, which is a closed subset of K.

Claim. $\mu(A_j) > \frac{\epsilon}{1-b}$, $\forall j \in U$.

Indeed, let $\pi_j : \ell_\infty(I) \to \mathbb{R}$, $j \in I$, be such that $\pi_j(f) = f(j)$ for every $f \in \ell_\infty(I)$. Observe that π_j is a w*-continuous linear map on $\ell_\infty(I)$, for every $j \in I$. Thus, for every $j \in U$ we have

$$\begin{aligned} z_0(j) = \pi_j(z_0) = \pi_j(r(\mu)) = \int_K \pi_j(k)d\mu = \int_K k(j)d\mu = \\ = \int_{A_j} k(j)d\mu + \int_{K\setminus A_j} k(j)d\mu \leq \mu(A_j) + (1-\mu(A_j))b, \end{aligned}$$

and this implies

$$\mu(A_j) \geq \frac{z_0(j)-b}{1-b} > \frac{\epsilon}{1-b}.$$

Let $V_0 \subset U$ be an arbitrary infinite subset. By Lemma 8 there exists an infinite countable subset $N_0 \subset V_0$ such that $\emptyset \neq \cap_{j\in N_0} A_j \subset K$. Pick $x_0 \in \cap_{j\in N_0} A_j$. Then for every $q \in \overline{N_0}^{\beta I} \setminus I$ we have $\check{x}_0(q) \geq b$, which implies $d(x_0,X) \geq b$, a contradiction, because x_0 belongs to K. ■

If (X,τ) is a topological space, a subset $K \subset X$ is said to be *regular in* X if and only if the interior set $\text{int}(K)$ is dense in K.

Corollary 10 *Let I be an infinite set, $H \subset \beta I \setminus I$ a compact subset which is regular in $\beta I \setminus I$, and $Y_H = \{f \in \ell_\infty(I) : \check{f}_{|H} = 0\}$. Then for every w^*-compact subset $K \subset \ell_\infty(I)$ we have $d(K,Y_H) = d(\overline{\text{co}}^{w^*}(K),Y_H)$.*

Proof. First, observe that $d(z,Y_H) = \sup\{|\check{z}(x)| : x \in H\}$ for every $z \in \ell_\infty(I)$. Suppose that there exist a w*-compact subset $K \subset B(\ell_\infty(I))$ and real numbers a,b such that:

$$d(K,Y_H) < a < b < d(\overline{\text{co}}^{w^*}(K),Y_H) \leq 1.$$

Let $z_0 \in \overline{\text{co}}^{w^*}(K)$ be such that $d(z_0,Y_H) > b$. Since $\text{int}(H)$ is dense in H, there exists $p_0 \in \text{int}(H)$ such that, for example, $\check{z}_0(p_0) > b+\epsilon$, for some

$\epsilon > 0$. Let $U \subset I$ be an infinite subset such that $p_0 \in \overline{U}^{\beta I} \setminus I \subset H$ and $z_0(j) > b + \epsilon$, $\forall j \in U$. By an argument similar to that of Proposition 9, we find an infinite countable subset $N_0 \subset U$ and a vector $x_0 \in K$ such that $\check{x}_0(q) \geq b$, for every $q \in \overline{N_0}^{\beta I} \setminus I \subset H$, which implies $d(x_0, Y_H) \geq b$, a contradiction, because $x_0 \in K$ and $d(K, Y_H) \leq a < b$. ■

We now prove Theorem 3 and Corollary 4.

Proof of Theorem 3. Suppose that there exist a w*-compact subset $K \subset B(X^{**})$ and real numbers a, b such that:

$$d(K, X) < a < b < d(\overline{\text{co}}^{w^*}(K), X).$$

Pick $z_0 \in \overline{\text{co}}^{w^*}(K)$ with $d(z_0, X) > b$. Since $X \in J$ we can choose a sequence $\{x_n^*\}_{n\geq 1} \subset \mathfrak{S}(B(X^*), z_0, b)$ such that $x_n^* \xrightarrow{w^*} 0$. Let $T : X \to c_0 := c_0(\mathbb{N})$ be such that $T(x) = (x_n^*(x))_{n\geq 1}$, $\forall x \in X$. Clearly, T is a linear continuous map with $\|T\| \leq 1$. Let $L = T^{**}(K)$, which is a w*-compact subset of $B(\ell_\infty)$.

Claim 1. $d(L, c_0) \leq d(K, X)$.

Indeed, let $c_0^\perp = \{f \in c_0^{***} : \langle f, u\rangle = 0,\ \forall u \in c_0\}$ and pick $v \in B(c_0^\perp)$. Then $\|T^{***}(v)\| \leq 1$ and for every $x \in X$ we have:

$$\langle T^{***}(v), x\rangle = \langle v, T^{**}x\rangle = \langle v, Tx\rangle = 0.$$

So, $T^{***}(B(c_0^\perp)) \subset B(X^\perp)$. Hence, if $k \in K$ and $T^{**}(k) =: h \in L$ we have:

$$\begin{aligned} d(h, c_0) &= \sup\{\langle v, h\rangle : v \in B(c_0^\perp)\} = \\ &= \sup\{\langle v, T^{**}(k)\rangle : v \in B(c_0^\perp)\} = \sup\{\langle T^{***}(v), k\rangle : v \in B(c_0^\perp)\} \leq \\ &\leq \sup\{\langle w, k\rangle : w \in B(X^\perp)\} = d(k, X). \end{aligned}$$

Claim 2. If $w_0 := T^{**}(z_0) \in \overline{\text{co}}^{w^*}(L)$, then $d(w_0, c_0) \geq b$.

Indeed, let $\{e_n\}_{n\geq 1}$ be the canonical basis of ℓ_1, which satisfies $T^*(e_n) = x_n^*$, $\forall n \geq 1$. Since $x_n^* \in \mathfrak{S}(B(X^*), z_0, b)$, then

$$\langle w_0, e_n\rangle = \langle T^{**}(z_0), e_n\rangle = \langle z_0, T^*(e_n)\rangle = \langle z_0, x_n^*\rangle \geq b. \tag{2.1}$$

Let ψ be a w*-limit point of $\{e_n\}_{n\geq 1}$ in (ℓ_∞^*, w^*). Clearly, $\psi \in B(c_0^\perp)$ and also $\psi(w_0) \geq b$ by (2.1). So, $d(w_0, c_0) \geq b$.

Therefore, $L \subset B(\ell_\infty)$ is a w*-compact subset such that

$$d(L, c_0) \leq d(K, X) < a < b \leq d(w_0, c_0) \leq d(\overline{\text{co}}^{w^*}(L), c_0),$$

a contradiction to Proposition 9. ■

Of course, not every Banach space has property J. Indeed, if X is a non-reflexive Grothendieck Banach space (for example, if $X = \ell_\infty(I)$ with I infinite), then clearly X does not have property J. Moreover, X cannot be isomorphically embedded into a Banach space with property J.

However, the family of Banach spaces fulfilling property J is very large. For example, this family includes the class of Banach spaces X whose dual unit ball $(B(X^*), \mathrm{w}^*)$ is angelic in the w*-topology. Recall that every WCG (even every WLD) Banach space belongs to this class (see [2]).

Proof of Corollary 4. The proof of this fact is standard and well known. Let us prove that if $z_0 \in B(X^{**}) \setminus X$ and $0 < b < d(z_0, X)$, then

$$0 \in \overline{\mathfrak{S}(B(X^*), z_0, b)}^{\sigma(X^*,X)}.$$

Find $\psi \in S(X^\perp) \subset X^{***}$ such that $\psi(z_0) > b$. As $B(X^*)$ is w*-dense in $B(X^{***})$ and $\psi(z_0) > b$, then

$$\psi \in \overline{\mathfrak{S}(B(X^*), z_0, b)}^{\sigma(X^{***},X^{**})},$$

whence we obtain

$$0 \in \overline{\mathfrak{S}(B(X^*), z_0, b)}^{\sigma(X^*,X)},$$

because $\psi \in X^\perp$. Finally, it is enough to apply the fact that $(B(X^*), \mathrm{w}^*)$ is angelic. ■

Now we prove some auxiliary facts. If X is a Banach space, let $I_X : X \to X$ denote the identity map of X, $J_X : X \to X^{**}$ the canonical embedding of X into X^{**} and $R_X : X^{***} \to X^*$ the canonical restriction map such that $\langle R_X(z), x\rangle = \langle z, J_X(x)\rangle$, for every $z \in X^{***}$ and every $x \in X$. Notice that $R_X = (J_X)^*$ and that $R_X \circ J_{X^*} = I_{X^*}$.

It is well-known that $J_{X^*}(X^*)$ is 1-complemented into X^{***}, by means of the projection $P_X : X^{***} \to X^{***}$ such that $P_X = J_{X^*} \circ R_X$. Since $\ker(P_X) = \{z \in X^{***} : \langle z, J_X(x)\rangle = 0,\ \forall x \in X\} = X^\perp$, we have the decomposition $X^{***} = X^\perp \oplus J_{X^*}(X^*)$. The subspace $X^\perp$ is complemented in X^{***} by means of the projection $Q_X : X^{***} \to X^{***}$ such that $Q_X = I_{X^{***}} - P_X$. Observe that $1 \le \|Q_X\| \le 2$ and that:

$$B(X^\perp) \subset Q_X(B(X^{***})) \subset \|Q_X\| \cdot B(X^\perp) \subset 2B(X^\perp).$$

Lemma 11 *Let X be a Banach space and $Q_X : X^{***} \to X^{***}$ be the canonical projection onto $X^\perp$. Assume that $Y \subset X$ is a closed subspace. Then, for every $u \in Y^{**}$ (considered Y^{**} as a subspace of X^{**}) we have:*

$$d(u, X) \le d(u, Y) \le \|Q_X\| \cdot d(u, X) \le 2d(u, X).$$

Proof. First, it is clear that $d(u,X) \leq d(u,Y)$, because $Y \subset X$.

In the following we distinguish X from $J_X(X)$, Y from $J_Y(Y)$, etc. Let $i : Y \to X$ denote the inclusion map. Then $i^* : X^* \to Y^*$ is a quotient map, $i^{**} : Y^{**} \to X^{**}$ is an inclusion map such that $(i^{**})_{\restriction Y} = i$, and $i^{***} : X^{***} \to Y^{***}$ is a quotient map such that $(i^{***})_{\restriction X^*} = i^*$. Observe that $i^{***}(B(X^{***})) = B(Y^{***})$. It is easy to see that $J_X \circ i = i^{**} \circ J_Y$ and that $J_{Y^*} \circ i^* = i^{***} \circ J_{X^*}$, whence we obtain

$$i^* \circ R_X = i^* \circ (J_X)^* = (J_X \circ i)^* = (i^{**} \circ J_Y)^* = (J_Y)^* \circ i^{***} = R_Y \circ i^{***}.$$

Claim. $Q_Y \circ i^{***} = i^{***} \circ Q_X$.

Indeed, we have

$$\begin{aligned} Q_Y \circ i^{***} &= (I_{Y^{***}} - J_{Y^*} \circ R_Y) \circ i^{***} = i^{***} - J_{Y^*} \circ R_Y \circ i^{***} = \\ &= i^{***} - J_{Y^*} \circ i^* \circ R_X = i^{***} - i^{***} \circ J_{X^*} \circ R_X = \\ &= i^{***} \circ (I_{X^{***}} - J_{X^*} \circ R_X) = i^{***} \circ Q_X. \end{aligned}$$

From the Claim we obtain $\|Q_Y\| \leq \|Q_X\|$ and

$$\begin{aligned} B(Y^\perp) \subset Q_Y(B(Y^{***})) &= Q_Y(i^{***}(B(X^{***}))) = \\ &= i^{***}(Q_X(B(X^{***}))) \subset i^{***}(\|Q_X\| \cdot B(X^\perp)). \end{aligned}$$

Thus, if $u \in Y^{**}$, we finally get

$$\begin{aligned} d(u, J_Y(Y)) &= \sup\{\langle z, u\rangle : z \in B(Y^\perp)\} \\ &\leq \sup\{\langle i^{***}(w), u\rangle : w \in \|Q_X\| \cdot B(X^\perp)\} \\ &= \|Q_X\| \cdot \sup\{\langle w, i^{**}(u)\rangle : w \in B(X^\perp)\} \\ &= \|Q_X\| \cdot d(i^{**}(u), J_X(X)) \\ &\leq 2d(i^{**}(u), J_X(X)). \end{aligned}$$

■

Let us prove our extension of the Krein-Šmulian Theorem.

Proof of Theorem 5. Suppose that there exist a closed subspace $Z \subset X$ and a w*-compact subset $K \subset B(X^{**})$ such that

$$d(\overline{\text{co}}^{w^*}(K), Z) > 5d(K, Z).$$

Then we can find $z_0 \in \overline{\text{co}}^{w^*}(K)$ and $a, b > 0$ such that

$$d(z_0, Z) > b > 5a > 5d(K, Z).$$

Pick $\psi \in S(Z^\perp(X^{***}))$ with $\psi(z_0) > b$.

Step 1. Since $\psi(z_0) > b$, there exists $x_1^* \in S(X^*)$ such that $x_1^*(z_0) > b$. So, as $z_0 \in \overline{\text{co}}^{w^*}(K)$ we can find $\eta_1 \in \text{co}(K)$ with

$$\eta_1 = \sum_{i=1}^{n_1} \lambda_{1i}\eta_{1i},\ \eta_{1i} \in K,\ \lambda_{1i} \geq 0,\ \sum_{i=1}^{n_1} \lambda_{1i} = 1,$$

such that $x_1^*(\eta_1) > b$. Since $d(\eta_{1i}, Z) < a$ we have the decomposition $\eta_{1i} = \eta_{1i}^1 + \eta_{1i}^2$ with $\eta_{1i}^1 \in Z$ and $\eta_{1i}^2 \in aB(X^{**})$.

Step 2. Let $Y_1 = [\{\eta_{1i}^1 : 1 \leq i \leq n_1\}] \subset Z$. Since $\dim(Y_1) \leq n_1 < \infty$, $\psi(z_0) > b$ and $\psi \in Y_1^{\perp}(X^{***})$, there exists $x_2^* \in S(X^*)$ such that $x_2^*(z_0) > b$ and $x^*_{2\restriction Y_1} = 0$. So, as $x_i^*(z_0) > b,\ i = 1, 2$, and $z_0 \in \overline{\text{co}}^{w^*}(K)$, we can find $\eta_2 \in \text{co}(K)$ with

$$\eta_2 = \sum_{i=1}^{n_2} \lambda_{2i}\eta_{2i},\ \eta_{2i} \in K,\ \lambda_{2i} \geq 0,\ \sum_{i=1}^{n_2} \lambda_{2i} = 1,$$

such that $x_i^*(\eta_2) > b, i = 1, 2$. Since $d(\eta_{2i}, Z) < a$ we have the decomposition $\eta_{2i} = \eta_{2i}^1 + \eta_{2i}^2$ with $\eta_{2i}^1 \in Z$ and $\eta_{2i}^2 \in aB(X^{**})$.

By reiteration, we obtain the sequences $\{x_n^*\}_{n\geq 1} \subset S(X^*)$, $\eta_k \in \text{co}(K)$ with

$$\eta_k = \sum_{i=1}^{n_k} \lambda_{ki}\eta_{ki},\ \eta_{ki} \in K,\ \lambda_{ki} \geq 0,\ \sum_{i=1}^{n_k} \lambda_{ki} = 1,$$
$$\eta_{ki} = \eta_{ki}^1 + \eta_{ki}^2 \text{ with } \eta_{ki}^1 \in Z \text{ and } \eta_{ki}^2 \in aB(X^{**}),\ k \geq 1,$$

such that $x_i^*(\eta_k) > b, i = 1, ..., k$, and $x^*_{k+1\restriction Y_k} = 0$, where

$$Y_k = [\{\eta_{ij_i}^1 : i = 1, ..., k; 1 \leq j_i \leq n_i\}] \subset Y_{k+1} \subset Z.$$

Let $Y = \overline{\cup_{k\geq 1} Y_k} \subset Z$ and $K_1 = (K + aB(X^{**})) \cap Y^{**}$. Then Y is a closed separable subspace of Z and K_1 is a w*-compact subset of Y^{**} (considered Y^{**} canonically embedded into $Z^{**} \subset X^{**}$). Observe that $\{\eta_{ij_i}^1 : i \geq 1, 1 \leq j_i \leq n_i\} \subset K_1$. By Lemma 11, since $K_1 \subset Y^{**}$ and $d(K_1, Z) \leq 2a$, we have $d(K_1, Y) \leq 4a$ (in fact, $d(K_1, Y) \leq 2\|Q_Z\|a \leq 2\|Q_X\|a \leq 4a$). As Y has property J (because Y is separable and, so, WCG, see Corollary 4), we get $d(\overline{\text{co}}^{w^*}(K_1), Y) = d(K_1, Y)$, whence $d(\overline{\text{co}}^{w^*}(K_1), Y) \leq 4a$.

Let η_0 be a w*-limit point of $\{\eta_k\}_{k\geq 1}$ in X^{**}.

Claim 1. $d(\eta_0, Y) \leq 5a$.

Indeed, first

$$\eta_0 \in \overline{\text{co}}^{w^*}(\{\eta_{ij_i} : i \geq 1, 1 \leq j_i \leq n_i\}) \subset \overline{\text{co}}^{w^*}(K_1) + aB(X^{**}).$$

On the other hand, $d(\overline{\text{co}}^{w^*}(K_1), Y) \leq 4a$. Hence, $d(\eta_0, Y) \leq 5a$.

Claim 2. $d(\eta_0, Y) \geq b$.

Indeed, let $\phi \in B(X^{***})$ be a w*-limit point of $\{x_n^*\}_{n\geq 1}$. Since $x_n^*(\eta_k) > b$ if $k \geq n$, then $x_n^*(\eta_0) \geq b$, $\forall n \geq 1$, whence $\phi(\eta_0) \geq b$. Moreover, $\phi \in Y^\perp(X^{***})$ because $x^*_{n+1|Y_n} = 0$ and $Y_n \subset Y_{n+1}$. Hence, $d(\eta_0, Y) \geq \phi(\eta_0) \geq b$.

Since $b > 5a$ we get a contradiction and this completes the proof. ■

Proof of Theorem 6. Suppose that there exist a closed subspace $Z \subset X$ and a w*-compact subset $K \subset B(X^{**})$, with $Z \cap K$ w*-dense in K, such that $d(\overline{\text{co}}^{w^*}(K), Z) > 2d(K, Z)$. Then we can find $z_0 \in \overline{\text{co}}^{w^*}(K)$ and $a, b > 0$ such that $d(z_0, Z) > b > 2a > 2d(K, Z)$. Pick $\psi \in S(Z^\perp(X^{***}))$ such that $\psi(z_0) > b$. We follow the argument of Theorem 5 with the following changes:

(i) As $Z \cap K$ is w*-dense in K we choose $\eta_k \in \text{co}(Z \cap K)$ with $\eta_k = \sum_{i=1}^{n_k} \lambda_{ki}\eta_{ki}$, $\eta_{ki} \in Z \cap K$ and $\lambda_{ki} \geq 0, \sum_{i=1}^{n_k} \lambda_{ki} = 1$;

(ii) Define

$$Y_k = [\{\eta_{ij_i} : i = 1, ..., k; 1 \leq j_i \leq n_i\}]\ , Y = \overline{\cup_{k\geq 1} Y_k} \subset Z \text{ and}$$
$$K_1 = \text{w}^*\text{-cl}(\{\eta_{ij_i} : i \geq 1, 1 \leq j_i \leq n_i\}) \subset Y^{**} \cap K.$$

Clearly, $d(K_1, Z) \leq d(K, Z) < a$, whence $d(K_1, Y) \leq 2d(K_1, Z) \leq 2a$ (in fact, $d(K_1, Y) \leq \|Q_Z\|a \leq \|Q_X\|a \leq 2a$). Since Y is separable, we have $d(\overline{\text{co}}^{w^*}(K_1), Y) = d(K_1, Y) \leq 2a$. Finally, every w*-limit point η_0 of $\{\eta_k\}_{k\geq 1}$ in X^{**} satisfies $\eta_0 \in \overline{\text{co}}^{w^*}(K_1)$, $d(\eta_0, Y) \leq 2a$ and $d(\eta_0, Y) \geq b$, a contradiction. ■

Remarks. (1) The argument of Theorem 5 in fact yields the following

$$d(\overline{\text{co}}^{w^*}(K), Z) \leq (2\|Q_Z\| + 1)d(K, Z) \leq (2\|Q_X\| + 1)d(K, Z) \leq 5d(K, Z).$$

In Theorem 6 we also obtain

$$d(\overline{\text{co}}^{w^*}(K), Z) \leq \|Q_Z\|d(K, Z) \leq \|Q_X\|d(K, Z) \leq 2d(K, Z).$$

(2) Let $Y \subset X$ be a subspace of the Banach space X and assume that $d(\overline{\text{co}}^{w^*}(K), X) \leq Md(K, X)$ for some $1 \leq M < \infty$ and every w*-compact subset $K \subset X^{**}$. Then using the fact that $d(z, X) \leq d(z, Y) \leq \|Q_X\|d(z, X) \leq 2d(z, X)$, for every $z \in Y^{**}$, it can be proved easily that $d(\overline{\text{co}}^{w^*}(K), Y) \leq M\|Q_X\|d(K, Y) \leq 2Md(K, X)$, for every w*-compact subset $K \subset Y^{**}$.

A subset $A \subset X^*$ is said to be *fragmented by the norm* of X^* (see [6, p. 81], [10]) if for every subset $B \subset A$ and every $\epsilon > 0$ there exists a w*-open subset $V \subset X^*$ such that $V \cap B \neq \emptyset$ and $\text{diam}(V \cap B) \leq \epsilon$, where $\text{diam}(V \cap B)$ means the diameter of $V \cap B$. In order to prove Corollary 13 and Theorem 7 we need the following lemma.

Lemma 12 *Let X be a Banach space, $Z \subset X^*$ a subspace and $K \subset B(X^*)$ a w^*-compact subset such that there exist $a, b > 0$ with:*

$$d(K, Z) < a < b < d(\overline{co}^{w^*}(K), Z).$$

Then there exist $z_0 \in \overline{co}^{w^*}(K)$ and $\psi \in S(Z^\perp(X^{**}))$ with $\psi(z_0) > b$ such that, if μ is a Radon Borel probability measure on K with barycentre $r(\mu) = z_0$, then: (a) μ is atomless; (b) if $H = supp(\mu)$, for every w^*-open subset V of X^* with $V \cap H \neq \emptyset$ there exists $\xi \in \overline{co}^{w^*}(V \cap H)$ such that $\psi(\xi) > b$; and (c) H is not fragmented by the norm of X^*.

Proof. Pick $z \in \overline{\mathrm{co}}^{w^*}(K)$ and $\psi \in S(Z^\perp(X^{**}))$ such that $\psi(z) > b + \epsilon$ for some $\epsilon > 0$. By the Bishop-Phelps theorem, there exists $\phi \in S(X^{**})$ with $\|\psi - \phi\| \leq \epsilon/4$ such that ϕ attains its maximum value on $\overline{\mathrm{co}}^{w^*}(K)$ in some $z_0 \in \overline{\mathrm{co}}^{w^*}(K)$. So:

$$\phi(z_0) \geq \phi(z) = \psi(z) + (\phi - \psi)(z) > b + \epsilon - \frac{1}{4}\epsilon = b + \frac{3}{4}\epsilon\ , \tag{2.2}$$

$$\psi(z_0) = \phi(z_0) + (\psi - \phi)(z_0) > b + \frac{3}{4}\epsilon - \frac{1}{4}\epsilon = b + \frac{1}{2}\epsilon \quad \text{and} \tag{2.3}$$

$$\forall k \in K, \phi(k) = \psi(k) + (\phi - \psi)(k) < a + \frac{1}{4}\epsilon < b + \frac{3}{4}\epsilon < \phi(z_0). \tag{2.4}$$

In particular, observe that $z_0 \notin K$ by (2.4).

(a) Let μ be a Radon Borel probability on K with barycentre $r(\mu) = z_0$ and suppose that μ has some atom, that is, there exist $0 < \lambda \leq 1$ and $k_0 \in K$ such that $\mu = \lambda \cdot \delta_{k_0} + \mu_1,\ \mu_1 \geq 0$. If $\lambda = 1$ then $\mu = \delta_{k_0}$, whence $r(\mu) = k_0 \in K$, which is impossible because $r(\mu) = z_0 \notin K$ by (2.4). So, $0 < \lambda < 1$, *i.e.,* $\mu_1 \neq 0$ and $\|\mu_1\| = 1 - \lambda > 0$. Then $\mu = \lambda \cdot \delta_{k_0} + (1 - \lambda)\frac{\mu_1}{\|\mu_1\|}$ and

$$z_0 = r(\mu) = \lambda k_0 + (1 - \lambda) r(\frac{\mu_1}{\|\mu_1\|}),$$

whence, since $\phi(k_0) < \phi(z_0)$ (by (2.4)) and $\phi(r(\frac{\mu_1}{\|\mu_1\|})) \leq \phi(z_0)$ (because $r(\frac{\mu_1}{\|\mu_1\|}) \in \overline{\mathrm{co}}^{w^*}(K)$), we get

$$\phi(z_0) = \lambda\phi(k_0) + (1 - \lambda)\phi(r(\frac{\mu_1}{\|\mu_1\|})) < \lambda\phi(z_0) + (1 - \lambda)\phi(z_0) = \phi(z_0),$$

a contradiction.

(b) Let $H = \mathrm{supp}(\mu)$ and suppose that there exists a w*-open subset V of X^* with $V \cap H \neq \emptyset$ such that $\psi(\xi) \leq b$, for every $\xi \in \overline{\mathrm{co}}^{w^*}(V \cap H)$. Let $\mu_1 = \mu_{\restriction V \cap H}$ denote the restriction of μ to $V \cap H$ (that is, $\mu_1(B) = \mu(B \cap V \cap H)$, for every Borel subset $B \subset K$) and $\mu_2 := \mu - \mu_1$. Observe that μ_1, μ_2 are positive measures such that $\mu_1 \neq 0$ (because $\emptyset \neq V \cap H = V \cap \text{supp}(\mu)$) and $\mu_2 \neq 0$ (if $\mu_2 = 0$, *i.e.*, $\mu = \mu_1 = \mu_{\restriction V\cap H}$, then $z_0 = r(\mu) \in \overline{\text{co}}^{w^*}(V \cap H)$) and $\psi(z_0) \leq b$, a contradiction to (2.3)). Thus, we have the decomposition $\mu = \mu_1 + \mu_2$ and so:

$$z_0 = r(\mu) = \|\mu_1\| \cdot r(\frac{\mu_1}{\|\mu_1\|}) + \|\mu_2\| \cdot r(\frac{\mu_2}{\|\mu_2\|}).$$

Since $r(\frac{\mu_1}{\|\mu_1\|}) \in \overline{\text{co}}^{w^*}(V \cap H)$, then $\psi(r(\frac{\mu_1}{\|\mu_1\|})) \leq b$, whence $\phi(r(\frac{\mu_1}{\|\mu_1\|})) \leq b + \frac{1}{4}\epsilon$ (because $\|\psi - \phi\| \leq \epsilon/4$). Therefore, taking into account that $r(\frac{\mu_2}{\|\mu_2\|}) \in \overline{\text{co}}^{w^*}(K)$ and (2.2) we get

$$\begin{aligned}\phi(z_0) &= \|\mu_1\|\phi(r(\frac{\mu_1}{\|\mu_1\|})) + \|\mu_2\|\phi(r(\frac{\mu_2}{\|\mu_2\|})) \leq \\ &\leq \|\mu_1\|(b + \frac{1}{4}\epsilon) + \|\mu_2\|\phi(z_0) < \|\mu_1\|\phi(z_0) + \|\mu_2\|\phi(z_0) = \phi(z_0),\end{aligned}$$

a contradiction.

(c) Let $\eta = b - a$ and suppose that H is fragmented by the norm of X^*. Then there exists a w*-open subset V such that $V \cap H \neq \emptyset$ and $\text{diam}(V \cap H) < \frac{\eta}{2}$. Therefore, if $h_0 \in V \cap H$, then $\overline{\text{co}}^{w^*}(V \cap H) \subset B(h_0; \eta/2)$ (=closed ball with center h_0 and radius $\eta/2$). Hence, for every $\xi \in \overline{\text{co}}^{w^*}(V \cap H)$ we have

$$\psi(\xi) \leq \psi(h_0) + \frac{\eta}{2} \leq d(h_0, Z) + \frac{\eta}{2} < a + \frac{\eta}{2} < b,$$

a contradiction to (b). ■

Corollary 13 *Let X be a Banach space, $Z \subset X^*$ a subspace and $K \subset X^*$ a w*-compact subset which is fragmented by the norm of X^*. Then $d(\overline{co}^{w^*}(K), Z) = d(K, Z)$.*

Proof. This follows immediately from Lemma 12. It also follows from [10, Theorem 2.3] where it is proved that $\overline{\text{co}}(K) = \overline{\text{co}}^{w^*}(K)$ whenever $K \subset X^*$ is w*-compact subset such that (K, w^*) is fragmented by the norm of X^*. ■

Now we prove Theorem 7. Observe that we cannot apply Theorem 3 because we do not know whether $\ell_1(I)$ has property J when I is uncountable (if I is countable it has because $\ell_1(I)$ is separable in this case). In fact, if we assume that there exists an uncountable measurable cardinal α (see [4, p. 186, 196] for definitions) and I is a set with $|I| = \alpha$, then it is easy to prove that $\ell_1(I)$ fails to have property J.

Proof of Theorem 7. First, observe that $X^* = \ell_\infty(I)$ and X^{**} is the space $M_R(\beta I)$ of Radon Borel measures on βI. Thus, X^{**} has the decomposition

$$X^{**} = \ell_1(I) \oplus_1 M_R(\beta I \setminus I).$$

Notice that the subspace $\ell_1(I)$ of this decomposition coincides with the space $J(X)$, $J : X \to X^{**}$ being the canonical inclusion. If $\mu \in M_R(\beta I)$, we write $\mu = \mu_1 + \mu_2$, where $\mu_1 \in \ell_1(I)$ and $\mu_2 = \mu_{\restriction \beta I \setminus I} \in M_R(\beta I \setminus I)$. So, $d(\mu, X) = \|\mu_2\|$.

Suppose that there exist a w*-compact subset $K \subset B(X^{**})$ and two numbers $a, b > 0$ such that:

$$d(K, X) < a < b < d(\overline{\mathrm{co}}^{w^*}(K), X).$$

By Lemma 12 we have the following Fact:

Fact. There exist $\psi \in S(X^\perp)$ and a w*-compact subset $\emptyset \neq H \subset K$ such that for every w*-open subset V with $V \cap H \neq \emptyset$ there exists $\xi \in \overline{\mathrm{co}}^{w^*}(V \cap H)$ with $\psi(\xi) > b$.

Step 1. By the Fact we can pick $\xi_1 \in \overline{\mathrm{co}}^{w^*}(H)$ with $\psi(\xi_1) > b$ and $x_1^* \in S(X^*)$ with $x_1^*(\xi_1) > b$. Now we choose

$$\eta_1 = \sum_{i=1}^{n_1} \lambda_{1i}\eta_{1i} \in \mathrm{co}(H), \quad \eta_{1i} \in H, \quad \lambda_{1i} \geq 0, \quad \sum_{i=1}^{n_1} \lambda_{1i} = 1,$$

such that $x_1^*(\eta_1) > b$. If $\eta_1 = \eta_1^1 + \eta_1^2$, with $\eta_1^1 \in \ell_1(I)$ and $\eta_1^2 \in M_R(\beta I \setminus I)$, then

$$\|\eta_1^2\| = d(\eta_1, X) \leq d(K, X) < a,$$

whence $\|\eta_1^1\| = \|\eta_1\| - \|\eta_1^2\| > b - a$, because $\|\eta_1\| \geq x_1^*(\eta_1) > b$. So, we can find $y_1 \in B(X^*) = B(\ell_\infty)$ with finite support $\mathrm{supp}(y_1) = \{\gamma_{11}, ..., \gamma_{1p_1}\} \subset I$ such that $y_1(\eta_1^1) > b - a$. Since $y_1(\eta_1^2) = 0$, we have

$$y_1(\eta_1) = y_1(\eta_1^1) > b - a,$$

whence it follows that $y_1(\eta_{1i}) > b - a$ for some $1 \leq i \leq n_1$.

Step 2. Let $V_1 = \{u \in X^{**} : y_1(u) > b - a\}$, which is a w*-open subset of X^{**} with $V_1 \cap H \neq \emptyset$, because $\eta_{1i} \in V_1 \cap H$ for some $1 \leq i \leq n_1$. By the Fact there exists $\xi_2 \in \overline{\mathrm{co}}^{w^*}(V_1 \cap H)$ with $\psi(\xi_2) > b$. Since $\psi(\xi_2) > b$ and $\psi(e_{\gamma_{1i}}) = 0,\ 1 \leq i \leq p_1$ (where $e_{\gamma_{1i}} \in \ell_1(I)$ is the unit vector such that $e_{\gamma_{1i}}(\gamma) = 1$, if $\gamma = \gamma_{1i}$, and $e_{\gamma_{1i}}(\gamma) = 0$, if $\gamma \neq \gamma_{1i}$), there exists $x_2^* \in B(X^*)$ with $x_2^*(\xi_2) > b$ and $x_2^*(e_{\gamma_{1i}}) = 0,\ 1 \leq i \leq p_1$. Clearly, we can choose

$$\eta_2 = \sum_{i=1}^{n_2} \lambda_{2i}\eta_{2i} \in \mathrm{co}(V_1 \cap H), \quad \eta_{2i} \in V_1 \cap H, \quad \lambda_{2i} \geq 0, \quad \sum_{i=1}^{n_2} \lambda_{2i} = 1,$$

such that $x_2^*(\eta_2) > b$.

As $y_1(\eta_{2i}) > b - a$, $1 \leq i \leq n_2$, we get $y_1(\eta_2) > b - a$. Let $\eta_2 = \eta_2^1 + \eta_2^2$, with $\eta_2^1 \in \ell_1(I)$, $\eta_2^2 \in M_R(\beta I \setminus I)$ and $\|\eta_2^2\| = d(\eta_2, X) \leq d(K, X) < a$. Since

$$\|\eta_2^1\| \geq |x_2^*(\eta_2^1)| = |x_2^*(\eta_2) - x_2^*(\eta_2^2)| \geq |x_2^*(\eta_2)| - |x_2^*(\eta_2^2)| > b - a,$$

and $x_2^* = 0$ on $\text{supp}(y_1)$, we can find $y_2 \in B(X^*)$ with finite support $\text{supp}(y_2) = \{\gamma_{21}, ..., \gamma_{2p_2}\} \subset I \setminus \text{supp}(y_1)$ such that $y_2(\eta_2^1) > b - a$. Hence, $y_2(\eta_2) = y_2(\eta_2^1) > b - a$ and this implies $y_2(\eta_{2i}) > b - a$ for some $1 \leq i \leq n_2$.

By reiteration, we obtain the sequence $\{y_k\}_{k\geq 1} \subset B(X^*)$ with pairwise disjoint supports and the sequence $\{\eta_k\}_{k\geq 1} \subset \text{co}(H) \subset B(X^{**})$ such that $y_n(\eta_k) > b - a$ for $k \geq n$.

Since $\|\sum_{i=1}^n y_i\| \leq 1$ (because the vectors $\{y_k\}_{k\geq 1} \subset B(\ell_\infty)$ have pairwise disjoint supports) and $(\sum_{i=1}^n y_i)(\eta_n) > n(b - a)$, $\forall n \geq 1$, we get $\|\eta_n\| > n(b - a)$, $\forall n \geq 1$, a contradiction, because $\|\eta_n\| \leq 1$. ■

Acknowledgements. The author would like to thank the referee for many suggestions which helped to improve this paper.

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Recibido: 6 de febrero de 2003
Revisado: 30 de abril de 2004

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Supported in part by DGICYT grant BFM2001-1284.