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Logarithmic derivative of the Euler Γ-function in Clifford analysis

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Abstract

The logarithmic derivative of the Γ-function, namely the ψ-function, has numerous applications. We define analogous functions in a four dimensional space. We cut lattices and obtain Clifford-valued functions. These functions are holomorphic cliffordian and have similar properties as the ψ-function. These new functions show links between well-known constants: the Euler gamma constant and some generalisations, $\zeta^R(2)$, $\zeta^R(3)$. We get also the Riemann zeta function and the Epstein zeta functions.

1. Introduction and Notations

1.1. Introduction

In one real or complex dimension, the finite difference equation $f(x) - f(x+\omega) = g(x)$ is satisfied by many basic functions in analysis.

In the case $g(x) = 0$, we have periodic functions, in case $g(x) = -nx^{n-1}$ we get the Bernoulli polynomials $B_n(x)$

$$B_n(x) - B_n(x+1) = -nx^{n-1}.$$

Taking $g(x) = -1/x$, we get the logarithmic derivative of the Euler Γ-function, commonly written $\psi(z) = \Gamma'(z)/\Gamma(z)$:

$$\psi(z) - \psi(z+1) = -\frac{1}{z}.$$

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The ψ function may be written in the form

$$\psi(z) = -\gamma - \frac{1}{z} + \sum_{k=1}^{+\infty} \Big(\frac{1}{k} - \frac{1}{z+k}\Big);$$

this shows that it is "half" of the cotangent function.

The cotangent is a generating function of the Bernoulli numbers $b_{2n} = (-1)^{n+1} B_{2n}(0)$

$$\operatorname{cotan} x = x^{-1} - \sum_{n=1}^{+\infty} \frac{4^n b_{2n}}{(2n)!} x^{2n-1}.$$

It is well known that the Taylor series of these functions is related to the zeta Riemann function:

$$i\pi \operatorname{cotan}(i\pi z) = \frac{1}{z} + 2z \sum_{n\in\mathbb{N}^*} \frac{1}{z^2+n^2}$$

$$i\psi(iz) = -i\gamma - \frac{1}{z} + \sum_{n\in\mathbb{N}^*} \Big(\frac{1}{in-z} - \frac{1}{in}\Big).$$

Now take $y \in \mathbb{R}$ and compute the real and imaginary parts of $i\psi(iy)$:

$$\mathcal{R}(i\psi(iy)) = -\frac{1}{2y} - \frac{1}{2} i\pi \operatorname{cotan}(i\pi y)$$

$$\Im(i\psi(iy)) = -\gamma + \sum_{k\geq 1} (-1)^k \zeta^R(2k+1) y^{2k}.$$

Then, $y \to i\psi(iy)$ has its real part link with trigonometric functions (i.e. one-fold periodic functions) and his imaginary part is a generating function for $\zeta^R(2k+1)$. Functions defined by the finite difference equation are basic in analytic number theory.

The goal of this work is to define and study some special functions but in a higher dimensional context. First we remark that $\mathbb{C}$ is the Clifford algebra associated to a one-dimensional antieuclidean space. Then, it is natural to replace $\mathbb{C}$ by a Clifford algebra. Second, what is the right choice for the space of functions? Holomorphic cliffordian functions (defined in the next paragraph) were designed with special functions in mind.

In this work, we study finite difference equations for 1, 2 and 3 dimensional lattices and obtain generalizations of the functions which are holomorphic cliffordian (cf: [LR1])

We want to stress an important fact. For one complex variable real and pure imaginary are symmetric. From the point of view of Clifford algebras it is a degenerated case: the vector space is one-dimensional. This explains that we consider $i\psi(iz)$, not $\psi(z)$.

1.2. Notations

Let $\mathbb{R}_{0,3}$ be the Clifford algebra generated by an antieuclidean space of dimension 3. Wet take a basis $\{e_0 = 1, e_1, e_2, e_3, e_{ij} \quad (i < j), \ e_1e_2e_3\}$ with $e_ie_j + e_je_i = -2\delta_{ij}$.

$$S = \mathbb{R}e_0 \text{ is the subspace of scalars;}$$

$$V = \mathbb{R}e_1 \oplus \mathbb{R}e_2 \oplus \mathbb{R}e_3 \text{ is the subspace of vectors.}$$

All functions in this work are functions $f : \Omega \mapsto \mathbb{R}_{0,3}$ where Ω is a domain in $S \oplus V$, holomorphic cliffordian defined in [LR1]. Recall that f is said to be left holomorphic cliffordian (H.C.F.) if

$$\forall x \in \Omega \quad D\Delta f(x) = 0$$

where Δ is the Laplacian in $\mathbb{R}^4 = S \oplus V$ and D is the generalized Cauchy-Riemann operator

$$D = \sum_{i=0}^{3} e_i \frac{\partial}{\partial x_i}.$$

Let $\omega_1, \omega_2, \omega_3$ be three $\mathbb{R}$-linearly independent paravectors, i.e. elements of $S \oplus V$, half periods of lattices.

We introduce the following translation operators:

$$(E_if)(x) = f(x - 2\omega_i) \quad i = 1, 2, 3$$

and the symmetry operator: $(Sf)(x) = f(-x)$.

We get the following commutation relations:

$$E_iE_j = E_jE_i \quad \forall i, j \qquad E_i = E_i^{-1} \qquad \forall i$$

And, formally:

$$\sum_{n\in\mathbb{N}} E_i^n = (I - E_i)^{-1}.$$

Afterwards, g will be an odd H.C.F., that is $Sg = -g$.

2. Formal series based on 1, 2, 3 dimensional lattices

The aim of this section is to give some algebraic properties of these formal series, like periodicity or pseudo-periodicity and symmetry.

Starting with an odd H.C.F. we define Z_i and H_i $(i = 1, 2, 3)$ functions on lattices of 1, 2, 3 dimensions and we study some formal relations. We do not care about convergence, we work with formal series.

The half lattices we take for our definitions are the same as in the work of Soeren Krausshar [Kr].

The proofs are easy by using the operators E_i, and S.

2.1. One dimensional lattice

Let $R_1 = 2\mathbb{Z}\omega_1$ be the entire lattice and $\Omega_1 = \{0\} \cup 2\mathbb{N}^*\omega_1$ be the half lattice, by deleting zero, $\Omega_1^* = 2\mathbb{N}^*\omega_1$.

Let's define Z_1 and H_1, Z_1 is odd, periodic of period $\mathbb{Z}2\omega_1$. Z_1 and H_1 are H.C.F.

$$Z_1 = \sum_{n_1\in\mathbb{Z}} E_1^{n_1} g$$
$$H_1 = g + \sum_{n_1\in\mathbb{N}^*} E_1^{n_1} g = (I - E_1)^{-1} g.$$

Proposition 2.1 *We have:*

$$(I - E_1)H_1 = g \tag{1}$$
$$(I - SE_1)H_1 = Z_1 \tag{2}$$

Proof. (1) follows immediately from the definition, but we may compute:

$$E_1H_1 = E_1 g + \sum_{n_1\in\mathbb{N}^*} E_1^{n_1+1} g = \sum_{n_1\in\mathbb{N}^*} E_1^{n_1} g$$
$$(I - E_1)H_1 = g + \sum_{n_1\in\mathbb{N}^*} E_1^{n_1} g - \sum_{n_1\in\mathbb{N}^*} E_1^{n_1} g = g$$

(2) arises from:

$$SE_1H_1 = E_1^{-1}SH_1 = E_1^{-1}(Sg + \sum_{n_1\in\mathbb{N}^*} E_1^{-n_1} Sg)$$
$$= E_1^{-1}\Big(-g - \sum_{n_1\in\mathbb{N}^*} E_1^{-n_1} g\Big) = -\sum_{n_1\in\mathbb{N}^*} E_1^{-n_1} g$$
$$(I - SE_1)H_1 = g + \sum_{n_1\in\mathbb{N}^*} E_1^{n_1} g + \sum_{n_1\in\mathbb{N}^*} E_1^{-n_1} g = Z_1.$$

■

2.2. Two dimensional lattices

Let $R_2 = 2\mathbb{Z}\omega_1 \oplus 2\mathbb{Z}\omega_2$ be the entire lattice and consider the half lattices

$$\Omega_2 = \Omega_1 \cup (2\mathbb{Z}\omega_1 \oplus 2\mathbb{N}^*\omega_2)$$
$$\Omega_2^* = \Omega_1^* \cup (2\mathbb{Z}\omega_1 \oplus 2\mathbb{N}^*\omega_2)$$

Let's define Z_2 and H_2 based on this lattice:

$$Z_2 = \sum_{\substack{n_1\in\mathbb{Z}\\ n_2\in\mathbb{Z}}} E_1^{n_1}E_2^{n_2} g = \sum_{n_2\in\mathbb{Z}} E_2^{n_2} Z_1\,, \quad H_2 = g + \sum_{n_1\in\mathbb{N}^*} E_1^{n_1} g + \sum_{\substack{n_1\in\mathbb{Z}\\ n_2\in\mathbb{N}^*}} E_1^{n_1}E_2^{n_2} g$$

Z_2 is odd, H.C.F. two-fold periodic of periods $2\omega_1$ and $2\omega_2$, H_2 is H.C.F. and is built by translations of vectors $2\omega_2$ on the function Z_1:

$$H_2 = H_1 + \sum_{n_2 \in \mathbb{N}^*} E_2^{n_2} Z_1.$$

Proposition 2.2 *We have:*

$$H_2 = H_1 + (I - E_2)^{-1} E_2 Z_1 \tag{3}$$

$$(I - E_1) H_2 = g \tag{4}$$

$$(I - E_2) H_2 = (I - E_2 S E_1) H_1 \tag{5}$$

$$(I - S E_1) H_2 = Z_2 \tag{6}$$

Proof.

(3) One uses the commutation relations in the previous equality.

(4) Start from (3) and proceed similarly.

$$(I - E_1)H_2 = (I - E_1)H_1 + (I - E_2)^{-1} E_2 (I - E_1) Z_1 = (I - E_1) H_1 = g.$$

(5) Start from (3) and apply some simplifications. Then one uses (2):

$$(I - E_2)H_2 = (I - E_2)H_1 + E_2 Z_1 = H_1 + E_2(Z_1 - H_1) = H_1 - E_2 S E_1 H_1.$$

(6) First, compute a part making use of (2):

$$\begin{aligned} SE_1H_2 &= SE_1H_1 + \sum_{n_2 \in \mathbb{N}^*} SE_1 E_2^{n_2} Z_1 \\ SE_1H_2 &= SE_1H_1 + \sum_{n_2 \in \mathbb{N}^*} E_2^{-n_2} SE_1 Z_1 \\ &= SE_1H_1 - \sum_{n_2 \in \mathbb{N}^*} E_2^{-n_2} Z_1 \\ &= H_1 - Z_1 - \sum_{n_2 \in \mathbb{N}^*} E_2^{-n_2} Z_1. \end{aligned}$$

Second, from this result we may infer using (3):

$$\begin{aligned} (I - SE_1)H_2 &= H_1 + \sum_{n_2 \in \mathbb{N}^*} E_2^{n} Z_1 - H_1 + Z_1 + \sum_{n_2 \in \mathbb{N}^*} E_2^{-n_2} Z_1 \\ &= \sum_{n \in \mathbb{Z}} E_2^{n_2} Z_1 = Z_2. \end{aligned}$$

■

2.3. Three dimensional lattices

Let $R_3 = 2\mathbb{Z}\omega_1 \oplus 2\mathbb{Z}\omega_2 \oplus 2\mathbb{Z}\omega_3$ be the entire lattice and consider the half lattice:

$$\Omega_3 = \Omega_2 \cup 2\mathbb{Z}\omega_1 \oplus 2\mathbb{Z}\omega_2 \oplus 2\mathbb{N}^*\omega_3$$
$$\Omega_3^* = \Omega_2^* \cup 2\mathbb{Z}\omega_1 \oplus 2\mathbb{Z}\omega_2 \oplus 2\mathbb{N}^*\omega_3$$

Let's define Z_3 and H_3 based on this lattice:

$$Z_3 = \sum_{\substack{n_1\in\mathbb{Z}\\ n_2\in\mathbb{Z}\\ n_3\in\mathbb{Z}}} E_1^{n_1}E_2^{n_2}E_3^{n_3}g = \sum_{n_3\in\mathbb{Z}} E_3^{n_3}Z_2$$
$$H_3 = H_2 + \sum_{\substack{n_1\in\mathbb{Z}\\ n_2\in\mathbb{Z}\\ n_3\in\mathbb{N}^*}} E_1^{n_1}E_2^{n_2}E_3^{n_3}g$$

Z_3 is an odd H.C.F. three-fold periodic with periods $2\omega_1, 2\omega_2, 2\omega_3$; H_3 is an H.C.F. and is built by translations of vectors $2\omega_3$ on the function Z_2:

$$H_3 = H_2 + \sum_{n_3\in\mathbb{N}^*} E_3^{n_3}Z_2.$$

Proposition 2.3 *We have:*

$$H_3 = H_2 + (I - E_3)^{-1}E_3Z_2 \tag{7}$$
$$(I - E_1)H_3 = g \tag{8}$$
$$(I - E_2)H_3 = (I - E_2SE_1)H_1 \tag{9}$$
$$(I - E_3)H_3 = (I - E_3SE_1)H_2 \tag{10}$$
$$(I - SE_1)H_3 = Z_3. \tag{11}$$

Proof.

(7) can be proved in the same way as (3). This may be taken as definition for H_3 and explains clearly the structure of the function.

(8) same as (4):

$$(I - E_1)H_3 = (I - E_1)H_2 + (I - E_3)^{-1}E_3(I - E_1)Z_2 = g$$

(9) same as (5):

$$\begin{aligned}(I - E_2)H_3 &= (I - E_2)H_2 + (I - E_3)^{-1}E_3(I - E_2)Z_2\\ &= (I - E_2)H_2 = (I - E_2SE_1)H_1\end{aligned}$$

(10) make use of (6):

$$(I-E_3)H_3 = (I-E_3)H_2+E_3Z_2 = H_2+E_3(Z_2-H_2) = H_2-E_3SE_1H_2$$

(11) make an analogous computation as in (6):

$$SE_1H_3 = SE_1H_2 + \sum_{n_3\in\mathbb{N}^*} SE_1E_3^{n_3}Z_2 = SE_1H_2 + \sum_{n_3\in\mathbb{N}^*} E_3^{-n_3}Z_2$$

$$\begin{aligned}(I-SE_1)H_3 &= H_2 + \sum_{n_3\in\mathbb{N}^*} E_3^{n_3}Z_2 - H_2 + Z_2 + \sum_{n_3\in\mathbb{N}^*} E_3^{-n_3}Z_2 \\ &= \sum_{n_3\in\mathbb{Z}} E_3^{n_3}Z_2 = Z_3.\end{aligned}$$

■

3. H_1, H_2, H_3 with converging terms

We now generalize the ψ function, more exactly the $z \to i\psi(iz)$ function to the context of higher dimensional spaces: namely clifford-valued and holomorphic cliffordian. We take $g(x) = -x^{-1}$ which is an odd H.C.F. on $(S\oplus V)^*$. The series which was introduced previously did not converge, thus we add polynomials which may be called convergence preserving factors to get absolutely locally uniformly converging series.

Definition 3.1 *We introduce the following functions:*

$$H_1(x) = -x^{-1} + \sum_{n\in\Omega_1^*}\{(n-x)^{-1} - n^{-1}\}$$

$$H_2(x) = -x^{-1} + \sum_{n\in\Omega_2^*}\{(n-x)^{-1} - n^{-1} - n^{-1}xn^{-1}\}$$

$$H_3(x) = -x^{-1} + \sum_{n\in\Omega_3^*}\{(n-x)^{-1} - n^{-1} - n^{-1}xn^{-1} - n^{-1}xn^{-1}xn^{-1}\}$$

where $n = 2n_1\omega_1$ *or* $n = 2n_1\omega_1 + 2n_2\omega_2$ *or* $n = 2n_1\omega_1 + 2n_2\omega_2 + 2n_3\omega_3$ *is the paravector corresponding to the lattice.*

If we modify the Z_1, Z_2, Z_3 functions by adding the converging factors, then we get the $\zeta_1, \zeta_2, \zeta_3$ functions defined in [LR2] and apply in [LR3]. (In the classical complex case, it coincides with the Weierstrass ζ function of the elliptic functions theory).

For further computations, we need the $(h|\nabla_x)$ operator, which is merely the derivative in the h direction:

$$(h|\nabla_x)(x) = h$$
$$(h|\nabla_x)(x^{-1}) = -x^{-1}hx^{-1}$$
$$(h|\nabla_x)(g(x)) = x^{-1}hx^{-1}.$$

We have some commutation relations:

$$E_i \circ (h|\nabla_x) = (h|\nabla_x) \circ E_i$$
$$S \circ (h|\nabla_x) = -(h|\nabla_x) \circ S$$

where all operators act on the variable x.

Now we look at our previous study but involving converging terms.

3.1. H_1 and Z_1 functions

We need a technical trick. We introduce two variables in our functions:

$$Z_1(x,y) = g(x+y) + \sum_{n_1 \in \mathbb{Z}^*} E_{1y}^{n_1}(g(x+y) - g(y))$$
$$H_1(x,y) = g(x+y) + \sum_{n_1 \in \mathbb{N}^*} E_{1y}^{n_1}(g(x+y) - g(y))$$

Of course, we find again our previously defined functions by putting $y = 0$: $Z_1(x) = Z_1(x,0)$ and $H_1(x) = H_1(x,0)$.

We emphasize explicitly the variable, for example:

$$E_{1,x}\, H(x,y) = H(x - 2\omega_1, y)$$

Here are some intermediate formulas, useful for the proof of the next proposition

(12) $$H_1(-2\omega_1, y) = 0 \qquad \forall y$$
(13) $$E_{1y}H_1(x,y) = H_1(x,y) - g(x+y) + E_{1y}g(y)$$
(14) $$E_{1x}H_1(x,y) - E_{1y}H_1(x,y) = -E_{1y}g(y)$$
(15) $$(I - S_{xy})H_1(x,y) = g(x+y) + Z_1(x,y)$$

Proposition 3.1

(16) $$(I - E_1)H_1(x) = x^{-1}$$
(17) $$(I - SE_1)H_1(x) = \zeta_1(x).$$

Proof of (12). . . (17).

(12)

$$H_1(-2\omega_1, y) = E_{1y}g(y) + \sum_{n_1\in\mathbb{N}^*} \Big(E_{1y}^{n_1+1}g(y) - E_{1y}^{n_1}g(y)\Big) = 0$$

(it is a telescopic sum) and

$$\lim_{n_1\to+\infty} E_{1y}^{n_1}g(y) = \lim_{n_1\to+\infty} -(y-2n_1\omega_1)^{-1} = 0$$

(13)

$$E_{1y}H_1(x,y) = E_{1y}g(x+y) + \sum_{n_1\in\mathbb{N}^*} E_{1y}^{n_1+1}(g(x+y)-g(y))$$

$$\begin{aligned} E_{1y}H_1(x,y) &= \sum_{n_1\in\mathbb{N}^*} E_{1y}^{n_1}(g(x+y)-g(y)) + E_{1y}g(y) \\ &= H_1(x,y) - g(x+y) + E_{1y}g(y) \end{aligned}$$

(14) We compare the translation action on the x variable and that on the y variable. Then we apply (12). This then leads to

$$E_{1x}H_1(x,y) - E_{1y}H_1(x,y) = \sum_{n_1\in\mathbb{N}^*} \Big(E_{1y}^{n_1+1}g(y) - E_1^{n_1}g(y)\Big) = -E_{1y}g(y).$$

(15) Let S_{xy} be the following symmetry operator:

$$S_{xy}H_1(x,y) = H_1(-x,-y).$$

We obtain

$$\begin{aligned} S_{xy}H_1(x,y) &= H_1(-x,-y) \\ &= -g(x+y) - \sum_{n_1\in\mathbb{N}^*} E_{1y}^{-n_1}(g(x+y)-g(y)) \\ (I-S_{xy})H_1(x,y) &= g(x+y) + Z_1(x,y) \end{aligned}$$

We get a summation on the entire lattice.

(16) $E_{1x}H_1(x,y) = H_1(x,y) - g(x+y)$; then take $y=0$, we get $E_1H_1(x) = H_1(x) - g(x)$

(17) Insert $y=0$ into (15):

$$\begin{aligned} (I-S_x)H_1(x) &= g(x) + Z_1(x) \\ (I-SE_1)H_1 &= H_1 - S(H_1 - g) \quad \text{following (16)} \\ &= H_1 - SH_1 + Sg \\ &= g + Z_1 - g = Z_1. \end{aligned}$$

■

3.2. H_2 and Z_2 functions

In analogy to the previously introduced function, we again introduce the functions:

$$\begin{aligned}
Z_2(x,y) = g(x+y) \quad &+ \sum_{\substack{n_1,n_2\in\mathbb{Z}\\(n_1,n_2)\neq(0,0)}} E_{1y}^{n_1}E_{2y}^{n_2}(g(x+y)-g(y)-(x|\nabla_y)g(y))\\
H_2(x,y) = g(x+y) \quad &+ \sum_{n_1\in\mathbb{N}^*} E_{1y}^{n_1}(g(x+y)-g(y)-(x|\nabla_y)g(y))\\
&+ \sum_{\substack{n_1\in\mathbb{Z}\\n_2\in\mathbb{N}^*}} E_{1y}^{n_1}E_{2y}^{n_2}(g(x+y)-g(y)-(x|\nabla_y)g(y))
\end{aligned}$$

and coming back to one variable functions:

$$\begin{aligned}
Z_2(x) &= Z_2(x,0)\\
H_2(x) &= H_2(x,0).
\end{aligned}$$

Put $E^n = E_1^{n_1}\circ E_2^{n_2}$, then

$$\begin{aligned}
Z_2(x,y) &= g(x+y) + \sum_{n\in R_2^*} E_y^n(g(x+y)-g(y)-(x|\nabla_y)g(y))\\
H_2(x,y) &= g(x+y) + \sum_{n\in\Omega_2^*} E_y^n(g(x+y)-g(y)-(x|\nabla_y)g(y)).
\end{aligned}$$

Remark:

The links between Z_2 and Z_1 as well as H_2 and H_1 are more intricate because we have to take care of convergence terms. In particular, the term $(x \mid \nabla y)g(y)$ does not exist in H_1 and Z_1.

$$\begin{aligned}
Z_2(x,y) = Z_1(x,y) &- \sum_{n_1\in\mathbb{Z}^*} E_{1y}^{n_1}(x|\nabla_y)g(y)\\
&+ \sum_{\substack{n_1\in\mathbb{Z}\\n_2\in\mathbb{Z}^*}} E_{1y}^{n_1}E_{2y}^{n_2}(g(x+y)-g(y)-(x|\nabla_y)g(y))\\
H_2(x,y) = H_1(x,y) &- \sum_{n_1\in\mathbb{N}^*} E_{1y}^{n_1}(x|\nabla_y)g(y)\\
&+ \sum_{\substack{n_1\in\mathbb{Z}\\n_2\in\mathbb{N}^*}} E_{1y}^{n_1}E_{2y}^{n_2}(g(x+y)-g(y)-(x|\nabla_y)g(y))\,.
\end{aligned}$$

We shall need the following equality:

$$H_2(-2\omega_1, y) = H_1(-2\omega_1, y) + \sum_{n_1\in\mathbb{N}^*} E_{1y}^{n_1}(2\omega_1|\nabla_y)g(y) + \sum_{\substack{n_1\in\mathbb{Z}\\ n_2\in\mathbb{N}^*}} E_{1y}^{n_1}E_{2y}^{n_2}(g(-2\omega_1+y) - g(y) + (2\omega_1|\nabla_y)g(y)).$$

In particular, for $y = 0$ we get in view of (12)

$$H_2(-2\omega_1) = \sum_{n_1\in\mathbb{N}^*} (2n_1\omega_1)^{-1}2\omega_1(2n_1\omega_1)^{-1} + \sum_{\substack{n_1\in\mathbb{Z}\\ n_2\in\mathbb{N}^*}} \big\{(-2n_1\omega_1 - 2n_2\omega_2 - 2\omega_1)^{-1} + (2n_1\omega_1 + 2n_2\omega_2)^{-1} + (2n_1\omega_1 + 2n_2\omega_2)^{-1}2\omega_1(2n_1\omega_1 + 2n_2\omega_2)^{-1}\big\}.$$

Here are some intermediate formulas but useful for the proof of the next proposition

(18) $E_{1y}H_2(x,y) = H_2(x,y) - g(x+y) + E_{1y}g(y) + E_{1y}(x|\nabla_y)g(y)$

(19) $E_{1x}H_2(x,y) - E_{1y}H_2(x,y) = -E_{1y}g(y) - E_{1y}(x|\nabla_y)g(y) + H_2(-2\omega_1,y)$

(20) $E_{1x}H_2(x,y) = H_2(x,y) - g(x+y) + H_2(-2\omega_1,y)$

(21) $(I - S_{xy})H_2(x,y) = g(x+y) + Z_2(x,y.)$

Proposition 3.2

$$(I - E_1)H_2(x) = x^{-1} - H_2(-2\omega_1) \tag{22}$$

$$(I - SE_1)H_2(x) = \zeta_2(x) - H_2(-2\omega_1). \tag{23}$$

Proof of (18)...(23).

(18) Apply the translation action and keep the remaining terms.

(19) Apply the translation action on x, then on y. After that a comparison of the terms leads to compare term by term:

$$E_{1x}\,H_2(x,y) - E_{1y}H_2(x,y) = E_{1x}H_1(x,y) - E_{1y}H_1(x,y) - \sum_{n_1\in\mathbb{N}^*} E_{1y}^{n_1}((x-2\omega_1)|\nabla_y)g(y) + \sum_{n_1\in\mathbb{N}^*} E_{1y}^{n_1+1}(x|\nabla_y)g(y) + \sum_{\substack{n_1\in\mathbb{Z}\\ n_2\in\mathbb{N}^*}} E_{1y}^{n_1}E_{2y}^{n_2}(g(x+y-2\omega_1) - g(y) - ((x-2\omega_1)|\nabla_y)g(y) - \sum_{\substack{n_1\in\mathbb{Z}\\ n_2\in\mathbb{N}^*}} E_{1y}^{n_1}E_{2y}^{n_2}(g(x+y) - g(y) - (x|\nabla_y)g(y)).$$

$$\begin{aligned}E_{1x}\,H_2(x,y) - E_{1y}H_2(x,y) &= -E_{1y}g(y) - E_{1y}(x|\nabla_g)(y)\\ &- \sum_{n_1\in\mathbb{N}^*} E_{1y}^{n_1}(-2\omega_1|\nabla_y)g(y)\\ &+ \sum_{\substack{n_1\in\mathbb{Z}\\ n_2\in\mathbb{N}^*}} E_{1y}^{n_1}E_{2y}^{n_2}\{g(x+y-2\omega_1) - g(x+y) - (-2\omega_1|\nabla_y)g(y)\}.\end{aligned}$$

But this last summation is independent of x because

$$\sum_{n_1\in\mathbb{Z}} E_{1y}^{n_1}(g(x+y-2\omega_1) - g(x+y)) = 0.$$

Thus, we get

$$\begin{aligned}&\sum_{\substack{n_1\in\mathbb{Z}\\ n_2\in\mathbb{N}^*}} E_{1y}^{n_1}\,E_{2y}^{n_2}\{g(x+y-2\omega_1) - g(x+y) - (-2\omega_1|\nabla_y)g(y)\}\\ &\qquad = \sum_{\substack{n_1\in\mathbb{Z}\\ n_2\in\mathbb{N}^*}} E_{1y}^{n_1}E_{2y}^{n_2}(g(y-2\omega_1) - g(y) - (-2\omega_1|\nabla_y)g(y)).\end{aligned}$$

At last, in the two series we recognize the term: $H_2(-2\omega_1, y)$.

For the next computation, first notice the equality:

$$S_{xy}\circ E_{1y}(x|\nabla_y)g(y) = E_{1y}^{-1}(x|\nabla_y)g(y)$$

which is a consequence of:

$$\begin{aligned}S_{xy}\circ E_{1y}(x|\nabla_y)g(y) &= -(x|\nabla_y)g(-y-2\omega_1)\\ &= (x|\nabla_y)g(y+2\omega_1) = E_{1y}^{-1}(x|\nabla_y)g(y)\end{aligned}$$

(21)
$$\begin{aligned}S_{xy}H_2(x,y) =& S_{xy}H_1(x,y) + \sum_{n_1\in\mathbb{N}^*} E_{1y}^{-n_1}(x|\nabla_y)g(y)\\ &- \sum_{\substack{n_1\in\mathbb{Z}\\ n_2\in\mathbb{N}^*}} E_{1y}^{n_1}E_{2y}^{-n_2}(g(x+y) + g(y) - (x|\nabla_y)g(y)).\end{aligned}$$

Then we obtain:

$$\begin{aligned}(I - S_{xy})H_2(x,y) &= g(x+y) + Z_1(x,y) - \sum_{n_1\in\mathbb{Z}^*} E_{1y}^{n_1}(x|\nabla_y)g(y)\\ &+ \sum_{\substack{n_1\in\mathbb{Z}\\ n_2\in\mathbb{Z}^*}} E_{1y}^{n_1}E_{2y}^{n_2}(g(x+y) - g(y) - (x|\nabla_y)g(y)).\end{aligned}$$

We get a summation on the entire lattice.

For (22) and (23), put $y = 0$ and we easily get: $SH_2 = H_2 - g - Z_2$ and $E_1H_2 = H_2 - g + H_2(-2\omega_1)$. ∎

3.3. H_3 and Z_3 functions

As previously we introduce:

$$Z_3(x,y) = g(x+y) + \sum_{\substack{(n_1,n_2,n_3)\in\mathbb{Z}^3\\(n_1,n_2,n_3)\neq(0,0,0)}} E_{1y}^{n_1}E_{2y}^{n_2}E_{3y}^{n_3}\big[g(x+y)-g(y) - (x|\nabla_y)g(y) - \frac{1}{2}(x|\nabla_y)^2 g(y)^2 g(y)\big].$$

For the sake of clarity, we put:

$$h(x,y) = g(x+y) - g(y) - (x|\nabla_y)g(y) - \frac{1}{2}(x|\nabla_y)^2 g(y)$$

$$H_3(x,y) = g(x,y) + \sum_{n_1\in\mathbb{N}^*} E_{1y}^{n_1} h(x,y) + \sum_{\substack{n_1\in\mathbb{Z}\\ n_2\in\mathbb{N}^*}} E_{1y}^{n_1}E_{2y}^{n_2} h(x,y) + \sum_{\substack{n_1,n_2\in\mathbb{Z}\\ n_3\in\mathbb{N}^*}} E_{1y}^{n_1}E_{2y}^{n_2}E_{3y}^{n_3} h(x,y)$$

and coming back to one variable functions:

$$Z_3(x) = Z_3(x,0)\,, \qquad H_3(x) = H_3(x,0).$$

Remark: For the next calculation, it is interesting to give the links between Z_3 and Z_2, as well as H_3 and H_2, because we will use again the formulas which have been obtained in section 3.2.

$$Z_3(x,y) = Z_2(x,y) - \sum_{\substack{n_1,n_2\in\mathbb{Z}\\(n_1,n_2)\neq(0,0)}} E_{1y}^{n_1}E_{2y}^{n_2}\,\frac{1}{2}(x|\nabla_y)^2 g(y) + \sum_{\substack{(n_1,n_2)\in\mathbb{Z}^2\\ n_3\in\mathbb{Z}^*}} E_{1y}^{n_1}E_{2y}^{n_2}E_{3y}^{n_3}\, h(x,y)$$

$$H_3(x,y) = H_2(x,y) - \sum_{n_1\in\mathbb{N}^*} E_{1y}^{n_1}\,\frac{1}{2}(x|\nabla_y)^2 g(y) - \sum_{\substack{n_1\in\mathbb{Z}\\ n_2\in\mathbb{N}^*}} E_{1y}^{n_1}E_{2y}^{n_2}\,\frac{1}{2}(x|\nabla_y)^2 g(y) - \sum_{\substack{(n_1,n_2)\in\mathbb{Z}^2\\ n_3\in\mathbb{N}^*}} E_{1y}^{n_1}E_{2y}^{n_2}E_{3y}^{n_3}\, h(x,y).$$

Let x be $-2\omega_1$, thus we get $H_3(-2\omega_1,y)$:

$$H_3(-2\omega_1,y) = H_2(-2\omega_1,y) - \sum_{n_1\in\mathbb{N}^*} E_{1y}^{n_1}\,\frac{1}{2}(2\omega_1|\nabla_y)^2 g(y) - \sum_{\substack{n_1\in\mathbb{Z}\\ n_2\in\mathbb{N}^*}} E_{1y}^{n_1}E_{2y}^{n_2}\,\frac{1}{2}(2\omega_1|\nabla_y)^2 g(y) + \sum_{\substack{(n_1,n_2)\in\mathbb{Z}^2\\ n_3\in\mathbb{N}^*}} h(-2\omega_1,y).$$

Here are some intermediate formulas but useful for the proof of the next proposition:

$$E_{1y}H_3(x,y) = H_3(x,y) - g(x+y) + E_{1y}\left[g(y) + (x|\nabla_y)g(y) + \frac{1}{2}(x|\nabla_y)^2 g(y)\right] \tag{24}$$

$$E_{1x}H_3(x,y) - E_{1y}H_3(x,y) = -E_{1y}\, g(y) - E_{1y}\,(x|\nabla_y)g(y) - E_{1y}\,\frac{1}{2}(x|\nabla_y)^2 g(y) + H_3(-\omega_1,y) + (x|\nabla_\omega)_{\omega=-2\omega_1}\, H_3(\omega,y) \tag{25}$$

$$E_{1x}H_3(x,y) = H_3(x,y) - g(x+y) + H_3(-2\omega_1,y) + (x|\nabla_\omega)_{\omega=-2\omega_1}\, H_3(\omega,y)(I-S_{xy})H_3(x,y) \tag{26}$$

$$= g(x+y) + Z_3(x,y). \tag{27}$$

Proposition 3.3

$$(I-E_1)H_3(x) \quad = x^{-1} - H_3(-2\omega_1) - (x|\nabla_\omega)_{\omega=-2\omega_1}\, H_3(\omega) \tag{28}$$

$$(I-SE_1)H_3(x) = \zeta_3(x) - H_3(-2\omega_1) - (x|\nabla_\omega)_{\omega=-2\omega_1}\, H_3(\omega) \tag{29}$$

Proof of (24) ... (29)

(24) We use the link between H_3 and H_2 and we apply a translation action on the variable y. In the right hand side of the equality, we use formula (18) and we group together the terms to get H_3 and then we keep the extra terms

$$E_{1y}\ H_3(x,y) = E_{1y}H_2(x,y) - \sum_{n_1\in\mathbb{N}^*} E_{1y}^{n_1+1}\ \frac{1}{2}(x|\nabla_y)^2 g(y) - \sum_{\substack{n_1\in\mathbb{Z}\\ n_2\in\mathbb{N}^*}} E_{1y}^{n_1}E_{2y}^{n_2}\ \frac{1}{2}(x|\nabla_y)^2 g(y) + \sum_{\substack{(n_1,n_2)\in\mathbb{Z}^2\\ n_3\in\mathbb{N}^*}} E_{1y}^{n_1}E_{2y}^{n_2}E_{3y}^{n_3}\, h(x,y).$$

Finally, we immediately obtain:

$$\begin{aligned} E_{1y}H_3(x,y) = {} & H_2(x,y) - g(x+y) + E_{1y}g(y) + E_{1y}(x|\nabla_y)g(y) \\ & - \sum_{n_1\in\mathbb{N}^*} E_{1y}^{n_1}\ \frac{1}{2}(x|\nabla_y)^2 g(y) + E_{1y}\ \frac{1}{2}(x|\nabla_y)^2 g(y) \\ & - \sum_{\substack{n_1\in\mathbb{Z}\\ n_2\in\mathbb{N}^*}} E_{1y}^{n_1}E_{2y}^{n_2}\frac{1}{2}(x|\nabla_y)^2 g(y) \\ & + \sum_{\substack{(n_1,n_2)\in\mathbb{Z}^2\\ n_3\in\mathbb{N}^*}} E_{1y}^{n_1}E_{2y}^{n_2}E_{3y}^{n_3}\ h(x,y). \end{aligned}$$

(25) Furthermore, we get:

$$E_{1x}\; H_3(x,y) - E_{1y}\, H_3(x,y) = E_{1x}\, H_2(x,y) - E_{1y}\, H_2(x,y)$$
$$- \sum_{n_1\in\mathbb{N}^*} E_{1y}^{n_1} E_{1x}\; \frac{1}{2}(x|\nabla_y)^2 g(y) + \sum_{n_1\in\mathbb{N}^*} E_{1y}^{n_1+1}\; \frac{1}{2}(x|\nabla_y)^2 g(y)$$
$$- \sum_{\substack{n_1\in\mathbb{Z}\\ n_2\in\mathbb{N}^*}} E_{1y}^{n_1} E_{2y}^{n_2} E_{1x}\; \frac{1}{2}(x|\nabla_y)^2 g(y) + \sum_{\substack{n_1\in\mathbb{Z}\\ n_2\in\mathbb{N}^*}} E_{1y}^{n_1} E_{2y}^{n_2}\; \frac{1}{2}(x|\nabla_y)^2 g(y)$$
$$+ \sum_{\substack{(n_1,n_2)\in\mathbb{Z}^2\\ n_3\in\mathbb{N}^*}} E_{1y}^{n_1} E_{2y}^{n_2} E_{3y}^{n_3} E_{1x}\, h(x,y) - \sum_{\substack{(n_1,n_2)\in\mathbb{Z}^2\\ n_3\in\mathbb{N}^*}} E_{1y}^{n_1} E_{2y}^{n_2} E_{3y}^{n_3}\, h(x,y).$$

A termwise comparison of the sums thus leads to:

$$E_{1x} H_3(x,y) - E_{1y} H_3(x,y) = E_{1x} H_2(x,y) - E_{1y} H_2(x,y)$$
$$- \sum_{n_1\in\mathbb{N}^*} E_{1y}^{n_1} \frac{1}{2}(x-2\omega_1|\nabla_y)^2 g(y) + \sum_{n_1\in\mathbb{N}^*} E_{1y}^{n_1}\; \frac{1}{2}(x|\nabla_y)^2 g(y)$$
$$- \sum_{\substack{n_1\in\mathbb{Z}\\ n_2\in\mathbb{N}^*}} E_{1y}^{n_1} E_{2y}^{n_2} \frac{1}{2}(x-2\omega_1|\nabla_y)^2 g(y) + \sum_{\substack{n_1\in\mathbb{Z}\\ n_2\in\mathbb{N}^*}} E_{1y}^{n_1} E_{2y}^{n_2} \frac{1}{2}(x|\nabla_y)^2 g(y)$$
$$+ \sum_{\substack{(n_1,n_2)\in\mathbb{Z}^2\\ n_3\in\mathbb{N}^*}} E_{1y}^{n_1} E_{2y}^{n_2} E_{3y}^{n_3} \big(h(x-2\omega_1,y) - h(x,y)\big) - E_{1y}\; \frac{1}{2}(x|\nabla_y)^2 g(y).$$

In view of

$$\frac{1}{2}(x-2\omega_1|\nabla_y)^2 g(y) = \frac{1}{2}(x|\nabla_y)^2 g(y) + $$
$$+ \frac{1}{2}(-2\omega_1|\nabla_y)^2 g(y) + (x|\nabla_y)(-2\omega_1|\nabla_y) g(y)$$

we may infer:

$$E_{1x} H_3(x,y) - E_{1y}\, H_3(x,y) = -E_{1y}\, g(y) - E_{1y}(x|\nabla_y) g(y) + H_2(-2\omega_1,y)$$
$$- \sum_{n_1\in\mathbb{N}^*} E_{1y}^{n_1} \frac{1}{2}(-2\omega_1|\nabla_y)^2 g(y) - \sum_{n_1\in\mathbb{N}^*} E_{1y}^{n_1} (x|\nabla_y)(-2\omega_1|\nabla_y)\, g(y)$$
$$- E_{1y} \frac{1}{2}(x|\nabla_y)^2\, g(y) - \sum_{\substack{n_1\in\mathbb{Z}\\ n_2\in\mathbb{N}^*}} E_{1y}^{n_1} E_{2y}^{n_2}\; \frac{1}{2}(-2\omega_1|\nabla_y)^2\, g(y)$$
$$- \sum_{\substack{n_1\in\mathbb{Z}\\ n_2\in\mathbb{N}^*}} E_{1y}^{n_1} E_{2y}^{n_2} (x|\nabla_y)(-2\omega_1|\nabla_y)\, g(y) + \sum_{\substack{(n_1,n_2)\in\mathbb{Z}^2\\ n_3\in\mathbb{N}^*}} E_{1y}^{n_1} E_{2y}^{n_2} E_{3y}^{n_3} \Big[\, g(x+y-2\omega_1)$$
$$- g(x+y) - (-2\omega_1|\nabla_y) g(y) - (x|\nabla_y)(-2\omega_1|\nabla_y) g(y) - \frac{1}{2}(-2\omega_1|\nabla_y)^2 g(y)\Big].$$

Notice that

$$\sum_{n_1\in\mathbb{Z}} E_{1y}^{n_1}\big(g(x+y-2\omega_1)-g(x+y)\big)=0.$$

Thus, this sum does not depend on x. Therefore it is possible to substitute $g(y-2\omega_1)-g(y)$ in the last sum, and we get:

$$\begin{aligned}
&E_{1x}\,H_3(x,y)-E_{1y}\,H_3(x,y)=-E_{1y}\,g(y)-E_{1y}(x|\nabla_y)\,g(y)\\
&-E_{1y}\,\frac{1}{2}(x|\nabla_y)^2g(y)+H_3(-2\omega_1,y)-\sum_{n_1\in\mathbb{N}^*}E_{1y}^{n_1}(x|\nabla_y)(-2\omega_1|\nabla_y)g(y)\\
&-\sum_{\substack{n_1\in\mathbb{Z}^2\\ n_2\in\mathbb{N}^*}}E_{1y}^{n_1}E_{2y}^{n_2}(x|\nabla_y)(-2\omega_1|\nabla_y)g(y)\\
&-\sum_{\substack{n_2\in\mathbb{Z}^2\\ n_3\in\mathbb{N}^*}}E_{2y}^{n_2}E_{3y}^{n_3}\sum_{n_1\in\mathbb{Z}}E_{1y}^{n_1}(x|\nabla_y)(-2\omega_1|\nabla_y)g(y).
\end{aligned}$$

■

Lemma 1

$$(x|\nabla_\omega)_{\omega=-2\omega_1}H_2(\omega,y)=0.$$

Proof.

$$\begin{aligned}
(x|\nabla_\omega)H_2(\omega,y)=&(\omega+y)^{-1}x(\omega+y)^{-1}\\
&+\sum_{n_1\in\mathbb{N}^*}E_{1y}^{n_1}\big[(\omega+y)^{-1}x(\omega+y)^{-1}-y^{-1}xy^{-1}\big]\\
&+\sum_{\substack{n_1\in\mathbb{Z}\\ n_2\in\mathbb{N}^*}}E_{1y}^{n_1}E_{2y}^{n_1}\big[(\omega+y)^{-1}x(\omega+y)^{-1}-y^{-1}xy^{-1}\big]
\end{aligned}$$

the sum is zero when $\omega=-2\omega_1$. ■

Lemma 2

$$\begin{aligned}
(x|\nabla_\omega)_{\omega=-2\omega_1}H_3(\omega,y)=&-\sum_{n_1\in\mathbb{N}^*}E_{1y}^{n_1}(x|\nabla_y)(-2\omega_1|\nabla_y)g(y)\\
&-\sum_{\substack{n_1\in\mathbb{Z}\\ n_2\in\mathbb{N}^*}}E_{1y}^{n_1}E_{2y}^{n_2}(x|\nabla_y)(-2\omega_1|\nabla_y)g(y)\\
&-\sum_{\substack{n_1\in\mathbb{Z}\\ n_3\in\mathbb{N}^*}}E_{2y}^{n_2}E_{3y}^{n_3}\sum_{n_1\in\mathbb{Z}}E_{1y}^{n_1}(x|\nabla_y)(-2\omega_1|\nabla_y)g(y)).
\end{aligned}$$

Proof.

$$(x|\nabla_\omega)\, H_3(\omega,y) = (x|\nabla_\omega)\, H_2(\omega,y) - \sum_{n_1\in\mathbb{N}^*} E_{1y}^{n_1}\frac{1}{2}(x|\nabla_\omega)(\omega|\nabla_y)^2 g(y)$$
$$- \sum_{\substack{n_1\in\mathbb{Z}\\ n_2\in\mathbb{N}^*}} E_{1y}^{n_1}E_{2y}^{n_2}\frac{1}{2}(x|\nabla_\omega)(\omega|\nabla_y)^2 g(y) + \sum_{\substack{(n_1,n_2)\in\mathbb{Z}^2\\ n_3\in\mathbb{N}^*}} E_{1y}^{n_1}E_{2y}^{n_2}E_{3y}^{n_3}\times$$
$$\times\left[(\omega+y)^{-1}x(\omega+y)^{-1} - y^{-1}xy^{-1} - \frac{1}{2}(x|\nabla_\omega)(\omega|\nabla_y)^2 g(y)\right].$$

We make the following computation:

$$\frac{1}{2}\,(x|\nabla_\omega)(\omega|\nabla_y)^2 g(y) = (x|\nabla_\omega)(-y^{-1}\omega y^{-1}\omega y^{-1})$$
$$= -y^{-1}xy^{-1}\omega y^{-1} - y^{-1}\omega y^{-1}xy^{-1}s = (x|\nabla_y)(\omega|\nabla_y)g(y).$$

Now we insert $\omega = -2\omega_1$, and we get:

$$(x|\nabla_\omega)_{\omega=-2\omega_1} H_3(\omega,y) = -\sum_{n_1\in\mathbb{N}^*} E_{1y}^{n_1}(x|\nabla_y)(\omega|\nabla_y)g(y)$$
$$- \sum_{\substack{n_1\in\mathbb{Z}\\ n_2\in\mathbb{N}^*}} E_{1y}^{n_1}E_{2y}^{n_2}(x|\nabla_y)(\omega|\nabla_y)g(y)$$
$$- \sum_{\substack{n_2\in\mathbb{Z}\\ n_3\in\mathbb{N}^*}} E_{2y}^{n_2}E_{3y}^{n_3}\sum_{n_1\in\mathbb{Z}} E_{1y}^{n_1}(x|\nabla_y)(\omega|\nabla_y)g(y)).$$

■

We have got the equality (25).

(26) is the result of (24) and (25).

(27)
$$S_{xy}\, H_3(x,y) = S_{xy}H_2(x,y) + \sum_{n_1\in\mathbb{N}^*} E_{1y}^{-n_1}\frac{1}{2}(x|\nabla_y)^2 g(y)$$
$$+ \sum_{\substack{n_1\in\mathbb{Z}\\ n_2\in\mathbb{N}^*}} E_{1y}^{n_1}E_{2y}^{-n_2}\frac{1}{2}(x|\nabla_y)^2 g(y) + \sum_{\substack{(n_1,n_2)\in\mathbb{Z}\\ n_3\in\mathbb{N}^*}} E_{1y}^{n_1}E_{2y}^{n_2}E_{3y}^{-n_3}h(x,y)$$

(28) Insert $y = 0$ into (26).

(29) comes from (27) and (28)

$$(I - SE_1)H_3(x) = H_3(x) - S\big[H_3(x) - g(x) + H_3(-2\omega_1) + (x|\nabla_\omega)_{\omega=-2\omega_1}H_3(\omega)\big]$$
$$(I - SE_1)H_3(x) = g(x) + Z_3(x) + Sg(x) - H_3(-2\omega_1) - (x|\nabla_\omega)_{\omega=-2\omega_1}H_3(\omega)$$
$$(I - SE_1)H_3(x) = Z_3(x) - H_3(-2\omega_1) - (x|\nabla_\omega)_{\omega=-2\omega_1}H_3(\omega).$$

■

4. Asymptotic study of converging factors

In the following paragraphs, we shall take a special lattice, built on an orthonormal basis. This choice allows us to compute things explicitly. It is a lattice with "complex multiplication" in the sense of the classical one complex variable theory. It is an open problem to get explicit results for other lattices.

Put:

$$\omega_1 = \frac{1}{2}\, e_1, \quad \omega_2 = \frac{1}{2}\, e_2, \quad \omega_3 = \frac{1}{2}\, e_3$$

From the following computations we shall get well-known or not so much well-known constants, in the same manner that the γ Euler constant ensues from the ψ function. If the series converges, then the constant is simply the summation of this series. But in case of non convergence, we can expand the partial sum by using the Euler-Mac-Laurin formula and this expansion defines a well defined constant. For example, we can expand the sum

$$\sum_{n=1}^{N} \frac{1}{n} = \ln N + \gamma - \frac{1}{2N} - \frac{1}{12N^2} + o\Big(\frac{1}{N^2}\Big)$$

thus this expansion defines the Euler constant γ. So, for convenience's sake, we define

$$\gamma^{\alpha}_{r,d}$$

r is the dimension of the lattice, d is the degree of the denominator which is always homogeneous α is nothing or a multiindice according to generic multivariable of the numerator: We draw attention to the fact that these numbers are well-known in number theory. The fifth series and the following ones are values of Epstein zeta functions attached to orthonormal lattices. But, the point of view here, is to look at them like Euler γ-constant.

$\gamma_{1,1} = \gamma$	series	$\displaystyle\sum_{n_1=1}^{\infty} \frac{1}{n_1}$,	Euler constant
$\gamma_{1,p} = \zeta^R(p)$	series	$\displaystyle\sum_{n_1=1}^{\infty} \frac{1}{n_1^p}$,	values of ζ Riemann function.
$\gamma_{2,2}$	series	$\displaystyle\sum_{n_1,n_2=1}^{\infty} \frac{1}{n_1^2+n_2^2}$	
$\gamma^{1,0}_{2,2}$	series	$\displaystyle\sum_{n_1,n_2=1}^{\infty} \frac{n_1+n_2}{n_1^2+n_2^2}$	

$\gamma_{3,2}$	series	$\displaystyle\sum_{n_1,n_2,n_3=1}^{\infty} \frac{1}{n_1^2+n_2^2+n_3^2}$
$\gamma_{3,2}^{1,0,0}$	series	$\displaystyle\sum_{n_1,n_2,n_3=1}^{\infty} \frac{n_1+n_2+n_3}{n_1^2+n_2^2+n_3^2}$
$\gamma_{3,6}^{3,0,0}$	series	$\displaystyle\sum_{n_1,n_2,n_3=1}^{\infty} \frac{n_1^3+n_2^3+n_3^3}{(n_1^2+n_2^2+n_3^2)^3}$
$\gamma_{3,6}^{2,1,0}$	series	$\displaystyle\sum_{n_1,n_2,n_3=1}^{\infty} \frac{(n_1^2+n_2^2)n_3+(n_2^2+n_3^2)n_1+(n_3^2+n_1^2)n_2}{(n_1^2+n_2^2+n_3^2)^3}$
$\gamma_{2,6}^{3,0}$	series	$\displaystyle\sum_{n_1,n_2=1}^{\infty} \frac{n_1^3+n_2^3}{(n_1^2+n_2^2)^3}$
$\gamma_{2,6}^{2,1}$	series	$\displaystyle\sum_{n_1,n_2=1}^{\infty} \frac{n_1n_2(n_1+n_2)}{(n_1^2+n_2^2)^3}.$

4.1. ψ_1 function

The aim of this section is to replace the non convergent series corresponding to the convergence terms by an asymptotically equivalent function which is obtained, like in the Euler-Mac-Laurin formula, by an explicit although hard integral calculus.

The case of the ψ function is well known, and it will be our model for the ψ_2 and ψ_3 functions.

The comparison between the functions H_i and ψ_i gives us some interesting connection between the previously introduced constants.

Here we have $\omega_1 = \dfrac{1}{2}\, e_1$ with $e_1^2 = -1$

$$H_1(x) = -x^{-1} + \sum_{n_1\in\mathbb{N}^*} \big((n_1e_1 - x)^{-1} - (n_1e_1)^{-1}\big).$$

We expect to find the well-known ψ function. Following the classical technique, we use the Euler–Mac-Laurin summation formula, comparing the finite sum and the integral:

$$\sum_{k=1}^{N} \frac{1}{k} \qquad \text{with} \qquad \int_1^N \frac{1}{t}\, dt = \ln N.$$

Recall this summation formula:

$$\sum_{k=1}^{n} f(k) = \int_1^n f(t)dt + \frac{1}{2}\big(f(1)+f(n)\big) + \sum_{h=1}^{r}(-1)^{h-1}\frac{b_{2h}}{(2h)!}\Big(f^{(2h-1)}(n) - f^{(2h-1)}(1)\Big) + \frac{1}{(2r+1)!}\int_1^n \varphi_{2r+1}(t) f^{(2r+1)}(t)dt$$

where b_{2h} are the Bernoulli numbers and φ_{2r+1} is the Fourier series

$$\varphi_{2r+1}(x) = (-1)^{r+1}(2r+1)!\sum_{n=1}^{\infty}\frac{\sin 2n\pi x}{(2n\pi)^{2r+1}}$$

which is the periodic extension of the Bernoulli polynomial B_{2r+1} from $[0,1]$.

For $f(t) = \dfrac{1}{t}$, we have:

$$\sum_{k=1}^{n}\frac{1}{k} = \int_1^n \frac{1}{t}dt + \frac{1}{2}\Big(1+\frac{1}{n}\Big) + \text{constant} + o\Big(\frac{1}{n}\Big) = \ln(n) + \text{constant} + \frac{1}{2n} + o\Big(\frac{1}{n}\Big).$$

The substitution of the diverging series containing convergence terms of H_1 by his asymptotic development leads us to the definition:

Definition 4.1

$$\psi_1(x) = \lim_{N\to+\infty}\left\{\sum_{n_1=0}^{N}(n_1e_1 - x)^{-1} + (\ln N)e_1\right\}.$$

Of course the difference $H_1(x) - \psi_1(x)$ is interesting

$$H_1(x) - \psi_1(x) = \lim_{N\to+\infty}\Big(\sum_{k=1}^{N}\frac{1}{k} - \ln N\Big)e_1 = \gamma\, e_1.$$

It is not surprising to recover the Euler constant.

Be careful about the computation of the ψ_1 function: we are in a Clifford algebra which is non-commutative!

4.2. ψ_2 function

Here we have $\omega_1 = \frac{1}{2}\, e_1$, $\omega_2 = \frac{1}{2}\, e_2 \quad e_ie_j + e_je_i = -2\delta_{ij}$. It is easy to write down H_2:

$$H_2(x) = -x^{-1} + \sum_{n_1\in\mathbb{N}^*} ((n_1e_1 - x)^{-1} - (n_1e_1)^{-1}) - \sum_{n_1\in\mathbb{N}^*} (n_1e_1)^{-1}x(n_1e_1)^{-1} + \sum_{\substack{n_1\in\mathbb{Z}\\ n_2\in\mathbb{N}^*}} ((\overrightarrow{n} - x)^{-1} - \overrightarrow{n}^{-1} - \overrightarrow{n}^{-1}x\overrightarrow{n}^{-1})$$

where $\overrightarrow{n} = n_1e_1 + n_2e_2$. We may write also:

$$H_2(x) = H_1(x) - \gamma_{1,2}\, e_1xe_1 + \sum_{\substack{n_1\in\mathbb{Z}\\ n_2\in\mathbb{N}^*}} ((\overrightarrow{n} - x)^{-1} - \overrightarrow{n}^{-1} - \overrightarrow{n}^{-1}x\overrightarrow{n}^{-1}).$$

Recall that $\gamma_{1,2}$ is

$$\sum_{k\in\mathbb{N}^*} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

Now we use an iterated Euler–Mac-Laurin summation formula, and substitute the convergence terms by their asymptotic equivalent. First we study some double sums: For

$$\overrightarrow{n}^{-1} = \frac{-n_1e_1 - n_2e_2}{n_1^2 + n_2^2}$$

we have:

$$\sum_{n_1=-N}^{N}\sum_{n_2=1}^{N} \overrightarrow{n}^{-1} = \sum_{n_1=-N}^{N}\sum_{n_2=1}^{N} \frac{-n_2}{n_1^2+n_2^2}\, e_2 = -2\sum_{n_1,n_2=1}^{N} \frac{n_2}{n_1^2+n_2^2}\, e_2 - \sum_{n_2=1}^{N}\frac{1}{n_2}\, e_2.$$

Use a process of symmetrization:

$$\sum_{n_1=-N}^{N}\sum_{n_2=1}^{N} \overrightarrow{n}^{-1} = -\sum_{n_1,n_2=1}^{N} \frac{n_1+n_2}{n_1^2+n_2^2}\, e_2 - \sum_{n_2=1}^{N}\frac{1}{n_2}\, e_2$$

For

$$\overrightarrow{n}^{-1}x\overrightarrow{n}^{-1} = \frac{n_1^2}{(n_1^2+n_2^2)^2}\, e_1xe_1 + \frac{n_1n_2}{(n_1^2+n_2^2)^2}(e_1xe_2 + e_2xe_1) + \frac{n_2^2}{(n_1^2+n_2^2)^2}\, e_2xe_2\,,$$

we have:

$$\sum_{n_1=-N}^{N}\sum_{n_2=1}^{N}\overrightarrow{n}^{-1}x\overrightarrow{n}^{-1}=\sum_{n_1=-N}^{N}\sum_{n_2=1}^{N}\frac{n_1^2}{(n_1^2+n_2^2)^2}\ e_1xe_1+\frac{n_2^2}{(n_1^2+n_2^2)^2}e_2xe_2$$

$$=\sum_{n_1,n_2=1}^{N}\frac{2n_1^2}{(n_1^2+n_2^2)^2}\ e_1xe_1+\frac{2n_2^2}{(n_1^2+n_2^2)^2}\ e_2xe_2+\sum_{n_2=1}^{N}\frac{1}{n_2^2}e_2xe_2.$$

We then get

$$\sum_{n_1=-N}^{N}\sum_{n_2=1}^{N}\overrightarrow{n}^{-1}x\overrightarrow{n}^{-1}=\sum_{n_1,n_2=1}^{N}\frac{1}{n_1^2+n_2^2}\ (e_1xe_1+e_2xe_2)+\sum_{n_2=1}^{N}\frac{1}{n_2^2}e_2xe_2.$$

To estimate double sums, let's write Euler–Mac-Laurin two times for a symmetric function $f(x,y)=f(y,x)$:

$$\begin{aligned}\sum_{p=1}^{N}\sum_{q=1}^{N}f(p,q)&=\int_1^N\int_1^N f(x,y)dxdy+\int_1^N f(x,1)dx+\int_1^N f(x,N)dx\\&+\frac{1}{4}\big(f(1,1)+2f(1,N)+f(N,N)\big)+b_2\int_1^N\big[\partial_1 f(N,y)-\partial_1 f(1,y)\big]dy\\&+\frac{b_2}{2}\big(\partial_1 f(N,N)+\partial_1 f(N,1)-\partial_1 f(1,N)-\partial_1 f(1,1)\big)\\&+\frac{b_2^2}{4}\big(\partial_1\partial_2 f(N,N)-2\partial_1\partial_2 f(1,N)+\partial_1\partial_2(1,1)\big)\\&+\frac{1}{3}\int_1^N\int_1^N\varphi_3(t)\partial_1^3 f(t,u)\,dtdu\\&+\frac{1}{6}\int_1^N\varphi_3(t)\big(\partial_1^3 f(t,1)+\partial_1^3 f(t,N)\big)\,dt\\&+\frac{1}{36}\int_1^N\int_1^N\varphi_3(t)\varphi_3(u)\partial_1^3\partial_2^3 f(t,u)\,dtdu\end{aligned}$$

use the bound:

$$|\varphi_3(t)|\ \le\ \big(1+\frac{1}{2}\big)b_2.$$

Taking this formula with $f(x,y)=\dfrac{x+y}{x^2+y^2}$, we obtain:

$$\begin{aligned}\sum_{n_1=1}^{N}\sum_{n_2=1}^{N}\frac{n_1+n_2}{n_1^2+n_2^2}&=\int_1^N\int_1^N\frac{x+y}{x^2+y^2}dxdy\\&+\int_1^N\frac{1+y}{1+y^2}dy+\int_1^N\frac{N+y}{N^2+y^2}dy+\text{constant}+O\big(\frac{1}{N}\big).\end{aligned}$$

Compute the integral:

$$\iint \frac{x+y}{x^2+y^2}dxdy = \frac{1}{2}(x+y)\ln(x^2+y^2) + x\,\mathrm{Arctan}\Big(\frac{y}{x}\Big) + y\,\mathrm{Arctan}\Big(\frac{x}{y}\Big)$$

and

$$\int \frac{x+y}{x^2+y^2}dx = \frac{1}{2}\ln(x^2+y^2) + \mathrm{Arctan}\Big(\frac{x}{y}\Big).$$

Finally we get:

$$\sum_{n_1=1}^{N}\sum_{n_2=1}^{N}\frac{n_1+n_2}{n_1^2+n_2^2} = \big(\ln 2+\frac{\pi}{2}\big)N - \ln N + \gamma_{2,2}^{1,0} + O\big(\frac{1}{N}\big).$$

With the same summation formula with $f(x,y) = \dfrac{1}{x^2+y^2}$, we get:

$$\sum_{n_1=1}^{N}\sum_{n_2=1}^{N}\frac{1}{n_1^2+n_2^2} = \int_1^N\int_1^N \frac{1}{x^2+y^2}dxdy + \text{constant} + O\big(\frac{1}{N}\big).$$

The classical function Ti is defined by:

$$\mathrm{Ti}(x) = \int_0^x \frac{1}{t}\,\mathrm{Arctan}\,t\;dt.$$

Recalling that

$$\mathrm{Ti}(x) = \sum_{n=0}^{+\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)^2},$$

also the Catalan constant:

$$\mathrm{Ti}(1) = G = 0,91596559$$

The Ti function satisfies:

$$\forall x>0 \quad \mathrm{Ti}(x) - \mathrm{Ti}\,\Big(\frac{1}{x}\Big) = \frac{\pi}{2}\ln(x).$$

Then

$$\iint \frac{1}{x^2+y^2}dxdy = -\frac{1}{2}\Big(\mathrm{Ti}\big(\frac{x}{y}\big) + \mathrm{Ti}\big(\frac{y}{x}\big)\Big).$$

At last we get the asymptotic formula

$$\sum_{n_1=1}^{N}\sum_{n_2=1}^{N}\frac{1}{n_1^2+n_2^2} = \frac{\pi}{2}\ln N + \gamma_{2,2} + O\big(\frac{1}{N}\big)\,.$$

Definition 4.2

$$\psi_2(x) = \psi_1(x) + \lim_{N\to+\infty}\Big\{\sum_{n_1=-N}^{N}\sum_{n_2=1}^{N}(\overrightarrow{n} - x)^{-1} + \Big(\big(\frac{\pi}{2} + \ln 2\big)N - \ln N\Big)e_2 + (\ln N)\, e_2 - \frac{\pi}{2}\ln N(e_1xe_1 + e_2xe_2)\Big\}.$$

We also may write:

$$\psi_2(x) = \psi_1(x) + \lim_{N\to+\infty}\Big\{\sum_{n_1=-N}^{N}\sum_{n_2=1}^{N}(\overrightarrow{n} - x)^{-1} + \big(\frac{\pi}{2} + \ln 2\big)N\, e_2 - \frac{\pi}{2}\ln N(e_1xe_1 + e_2xe_2)\Big\}.$$

Now we do the same substraction as we did in the case of the one dimensional lattice. Here, we obtain the Euler constant and some other constants.

Proposition 4.1

$$H_2(x) - \psi_2(x) = \gamma e_1 + (\gamma + \gamma_{2,2}^{1,0})e_2 - (\gamma_{1,2} + \gamma_{2,2})\,(e_1xe_1 + e_2xe_2)$$

with:

$$\gamma_{2,2}^{1,0} = \lim_{N\to+\infty}\Big\{\sum_{n_1=1}^{N}\sum_{n_2=1}^{N}\frac{n_1+n_2}{n_1^2+n_2^2} - \big(\ln 2 + \frac{\pi}{2}\big)N + \ln N\Big\} \approx -1,004\ldots$$

and

$$\gamma_{2,2} = \lim_{N\to+\infty}\Big\{\sum_{n_1=1}^{N}\sum_{n_2=1}^{N}\frac{1}{n_1^2+n_2^2} - \frac{\pi}{2}\ln N\Big\} \approx -0,825\ldots$$

4.3. ψ_3 function

Here we have $\omega_1 = \frac{1}{2}e_1$, $\omega_2 = \frac{1}{2}e_2$, $\omega_3 = \frac{1}{2}e_3$, $e_ie_j + e_je_i = -2\delta_{ij}$.

In this paragraph, we shall use some polynomials $P_\alpha(x)$ introduced in [LR1]. We only need

$$P_{0200}(x) = e_1xe_1\ ; \quad P_{0020}(x) = e_2xe_2\ ; \quad P_{0002}(x) = e_3xe_3$$
$$P_{0300}(x) = e_1xe_1xe_1\ ; \quad P_{0030}(x) = e_2xe_2xe_2; \quad P_{0003}(x) = e_3xe_3xe_3$$
$$P_{0210}(x) = e_1xe_1xe_2 + e_1xe_2xe_1 + e_2xe_1xe_1$$
$$P_{0021}(x) = e_2xe_2xe_3 + e_2xe_3xe_2 + e_3xe_2xe_2.$$

Making an explicit expansion we get:

$$
\begin{aligned}
H_3(x) &= -x^{-1} + \sum_{n_1\in\mathbb{N}^*} [(n_1e_1 - x)^{-1} - (n_1e_1)^{-1}] \\
&- \sum_{n_1\in\mathbb{N}^*} [(n_1e_1)^{-1}x(n_1e_1)^{-1} + (n_1e_1)^{-1}x(n_1e_1)^{-1}x(n_1e_1)^{-1}] \\
&+ \sum_{\substack{n_1\in\mathbb{Z}\\ n_2\in\mathbb{N}^*}} [(n_1e_1+n_2e_2-x)^{-1} - (n_1e_1+n_2e_2)^{-1} - (n_1e_1+n_2e_2)^{-1}x(n_1e_1+n_2e_2)^{-1}] \\
&- \sum_{\substack{n_1\in\mathbb{Z}\\ n_2\in\mathbb{N}^*}} (n_1e_1+n_2e_2)^{-1}x(n_1e_1+n_2e_2)^{-1}x(n_1e_1+n_2e_2)^{-1} \\
&+ \sum_{\substack{(n_1,n_2)\in\mathbb{Z}^2\\ n_3\in\mathbb{N}^*}} [(\overrightarrow{n} - x)^{-1} - \overrightarrow{n}^{-1} - \overrightarrow{n}^{-1}x\overrightarrow{n}^{-1} - \overrightarrow{n}^{-1}x\overrightarrow{n}^{-1}x\overrightarrow{n}^{-1}],
\end{aligned}
$$

where $\overrightarrow{n} = n_1e_1 + n_2e_2 + n_3e_3$

$$
\begin{aligned}
H_3(x) =& H_2(x) + \gamma_{1,3}\, P_{0300}(x) + \sum_{\substack{n_1\in\mathbb{Z}\\ n_2\in\mathbb{N}^*}} \frac{n_2^3}{(n_1^2+n_2^2)^3}\, P_{0030}(x) \\
&+ \sum_{\substack{n_1\in\mathbb{Z}\\ n_2\in\mathbb{N}^*}} \frac{n_1^2 n_2}{(n_1^2+n_2^2)^3}\, P_{0210}(x) \\
&+ \sum_{\substack{(n_1,n_2)\in\mathbb{Z}^2\\ n_3\in\mathbb{N}^*}} [(\overrightarrow{n} - x)^{-1} - \overrightarrow{n}^{-1} - \overrightarrow{n}^{-1}x\overrightarrow{n}^{-1} - \overrightarrow{n}^{-1}x\overrightarrow{n}^{-1}x\overrightarrow{n}^{-1}].
\end{aligned}
$$

Recall that $\gamma_{1,3}$ is

$$\sum_{k\in\mathbb{N}^*} \frac{1}{k^3}.$$

We get:

$$
\begin{aligned}
H_3(x) =& H_2(x) + \gamma_{1,3}\big(P_{0300}(x) + P_{0030}(x)\big) + \sum_{\substack{n_1\in\mathbb{N}^*\\ n_2\in\mathbb{N}^*}} \frac{(n_1^3+n_2^3)}{(n_1^2+n_2^2)^3}\, P_{0030}(x) \\
&+ \sum_{\substack{n_1\in\mathbb{N}^*\\ n_2\in\mathbb{N}^*}} \frac{n_1n_2(n_1+n_2)}{(n_1^2+n_2^2)^3}\, P_{0210}(x) \\
&+ \sum_{\substack{(n_1,n_2)\in\mathbb{Z}^2\\ n_3\in\mathbb{N}^*}} \big[(\overrightarrow{n} - x)^{-1} - \overrightarrow{n}^{-1} - \overrightarrow{n}^{-1}x\overrightarrow{n}^{-1} - \overrightarrow{n}^{-1}x\overrightarrow{n}^{-1}x\overrightarrow{n}^{-1}\big].
\end{aligned}
$$

We have to study triple sums: For $\overrightarrow{n}^{-1} = \dfrac{-n_1e_1 - n_2e_2 - n_3e_3}{n_1^2+n_2^2+n_3^2}$,

$$\sum_{n_1,n_2=-N}^{N}\sum_{n_3=1}^{N}\overrightarrow{n}^{-1} = \sum_{n_1,n_2=-N}^{N}\sum_{n_3=1}^{N}\frac{-n_3}{n_1^2+n_2^2+n_3^2}\,e_3$$
$$= -\sum_{n_3=1}^{N}\frac{1}{n_3}\,e_3 - 4\sum_{n_2,n_3=1}^{N}\frac{n_3}{n_2^2+n_3^2}e_3 - 4\sum_{n_1,n_2,n_3=1}^{N}\frac{n_3}{n_1^2+n_2^2+n_3^2}\,e_3.$$

By a process of symmetrization:

$$\sum_{n_1,n_2=-N}^{N}\sum_{n_3=1}^{N}\overrightarrow{n}^{-1} = -\sum_{n_3=1}^{N}\frac{1}{n_3}e_3 - 2\sum_{n_2,n_3=1}^{N}\frac{n_2+n_3}{n_2^2+n_3^2}\,e_3$$
$$-\frac{4}{3}\sum_{n_1,n_2,n_3=1}^{N}\frac{n_1+n_2+n_3}{n_1^2+n_2^2+n_3^2}\,e_3.$$

For $\overrightarrow{n}^{-1}x\overrightarrow{n}^{-1} = \displaystyle\sum_{i,j=1}^{3}\frac{n_in_j}{(n_1^2+n_2^2+n_3^2)^2}\,e_ixe_j$,

$$\sum_{n_1,n_2=-N}^{N}\sum_{n_3=1}^{N}\overrightarrow{n}^{-1}x\overrightarrow{n}^{-1} = \sum_{n_1,n_2=-N}^{N}\sum_{n_3=1}^{N}\Big(\sum_{i=1}^{3}\frac{n_i^2}{(n_1^2+n_2^2+n_3^2)^2}e_ixe_i\Big)$$
$$= \sum_{n_3=1}^{N}\frac{1}{n_3^2}\,e_3xe_3 + \sum_{n_1,n_2=1}^{N}\frac{1}{n_1^2+n_2^2}\big(P_{0200}(x)+P_{0020}(x)+2P_{0002}(x)\big)$$
$$+\frac{4}{3}\sum_{n_1,n_2,n_3=1}^{N}\frac{1}{n_1^2+n_2^2+n_3^2}(P_{0200}(x)+P_{0020}(x)+P_{0002}(x)).$$

For $\overrightarrow{n}^{-1}x\overrightarrow{n}^{-1}x\overrightarrow{n}^{-1} = -\displaystyle\sum_{i,j,k=1}^{3}\frac{n_in_jn_k}{(n_1^2+n_2^2+n_3^2)^3}\,e_ixe_jxe_k$,

$$\sum_{n_1,n_2=-N}^{N}\sum_{n_3=1}^{N}\overrightarrow{n}^{-1}x\overrightarrow{n}^{-1}x\overrightarrow{n}^{-1} = \sum_{n_1,n_2=-N}^{N}\sum_{n_3=1}^{N}\Big\{-\frac{n_3^3}{(n_1^2+n_2^2+n_3^2)^3}P_{0003}(x)$$
$$-\frac{n_1^2n_3}{(n_1^2+n_2^2+n_3^2)^3}P_{0201}(x) - \frac{n_2^2n_3}{(n_1^2+n_2^2+n_3^2)^3}P_{0021}(x)\Big\}$$

$$= -\sum_{n_3=1}^{N} \frac{1}{n_3^3} P_{0003}(x) - 2 \sum_{n_1,n_3=1}^{N} \frac{n_1^3+n_3^3}{(n_1^2+n_3^2)^3} P_{0003}(x)$$
$$- \sum_{n_1,n_3=1}^{N} \frac{n_1 n_3 (n_1+n_3)}{(n_1^2+n_3^2)^3} \big(P_{0201}(x) + P_{0021}(x)\big)$$
$$- \frac{4}{2} \sum_{n_1,n_2,n_3=1}^{N} \frac{(n_1^2+n_2^2)n_3}{(n_1^2+n_2^2+n_3^2)^3} \big(P_{0201}(x) + P_{0021}(x)\big)$$
$$- \frac{4}{3} \sum_{n_1,n_2,n_3=1}^{N} \frac{n_1^3+n_2^3+n_3^3}{(n_1^2+n_2^2+n_3^2)^3} \quad P_{0003}(x).$$

Now we write down the three time iterated Euler–Mac-Laurin formula for a symmetric function $f(x, y, z)$:

$$\sum_{p,q,r=1}^{N} f(p,q,r) =$$
$$= \int_1^N \int_1^N \int_1^N f(x,y,z)\, dxdydz + \frac{3}{2} \int_1^N \int_1^N \big(f(x,y,1) + f(x,y,N)\big)\, dxdy$$
$$+ \frac{3}{4} \int_1^N \big(f(x,1,1) + 2f(x,1,N) + f(x,N,N)\big)\, dx$$
$$+ \frac{3}{2} b_2 \int_1^N \int_1^N \big(\partial_3 f(x,y,N) - \partial_3 f(x,y,1)\big)\, dxdy + \text{constant} + O(\frac{1}{N}).$$

We deduce:

$$\sum_{n_1,n_2,n_3=1}^{N} \frac{n_1^3+n_2^3+n_3^3}{(n_1^2+n_2^2+n_3^2)^3} = \int_1^N \int_1^N \int_1^N \frac{x^3+y^3+z^3}{(x^2+y^2+z^2)^3}\, dxdydz + \text{constant} + O(\frac{1}{N}).$$

We have:

$$\iiint \frac{x^3+y^3+z^3}{(x^2+y^2+z^2)^3}\, dxdydz$$
$$= \sum_{p.c.\{x,y,z\}} \frac{1}{4} \operatorname{Ti}\left(\frac{y+\sqrt{y^2+z^2}}{x+\sqrt{x^2+z^2}}\right) - \frac{1}{4} \operatorname{Ti}\left(\frac{y+\sqrt{y^2+z^2}}{-x+\sqrt{x^2+z^2}}\right)$$
$$- \frac{1}{8} \frac{x}{\sqrt{x^2+z^2}} \operatorname{Arctan}\left(\frac{y}{\sqrt{x^2+z^2}}\right) - \frac{1}{8} \frac{y}{\sqrt{y^2+z^2}} \operatorname{Arctan}\left(\frac{x}{\sqrt{y^2+z^2}}\right).$$

At last we get the asymptotic formula:

$$\sum_{n_1,n_2,n_3=1}^{N} \frac{n_1^3+n_2^3+n_3^3}{(n_1^2+n_2^2+n_3^2)^3} = \frac{3}{8}\pi \ln(N) + \gamma_{3,6}^{3,0,0} + O(\frac{1}{N}).$$

In the same way

$$\sum_{n_1,n_2,n_3=1}^{N} \frac{(n_1^2+n_2^2)n_3}{(n_1^2+n_2^2+n_3^2)^3} = \int_1^N\int_1^N\int_1^N \frac{(x^2+y^2)\,z}{(x^2+y^2+z^2)^3}\,dxdydz + \text{constant} + O(\frac{1}{N}).$$

Now we must integrate:

$$\iiint \frac{(x^2+y^2)\,z}{(x^2+y^2+z^2)^3}\,dxdydz$$
$$= \frac{1}{4}\,\mathrm{Ti}\left(\frac{y+\sqrt{y^2+z^2}}{x+\sqrt{x^2+z^2}}\right) - \frac{1}{4}\,\mathrm{Ti}\left(\frac{y+\sqrt{y^2+z^2}}{-x+\sqrt{x^2+z^2}}\right)$$
$$+ \frac{1}{8}\,\frac{x}{\sqrt{x^2+z^2}}\,\mathrm{Arctan}\left(\frac{y}{\sqrt{x^2+z^2}}\right) + \frac{1}{8}\,\frac{y}{\sqrt{y^2+z^2}}\,\mathrm{Arctan}\left(\frac{x}{\sqrt{y^2+z^2}}\right).$$

We have the asymptotic formula:

$$\sum_{n_1,n_2,n_3=1}^{N} \frac{(n_1^2+n_2^2)\,n_3}{(n_1^2+n_2^2+n_3^2)^3} = \frac{\pi}{8}\ln(N) + \frac{1}{3}\gamma_{3,6}^{2,1,0} + O(\frac{1}{N}).$$

Now for the homogeneous terms of degree -2:

$$\sum_{n_1,n_2,n_3=1}^{N} \frac{1}{n_1^2+n_2^2+n_3^2} = \int_1^N\int_1^N\int_1^N \frac{1}{x^2+y^2+z^2}\,dxdydz$$
$$+ \frac{3}{2}\int_1^N\int_1^N \left[\frac{1}{x^2+y^2+1} + \frac{1}{x^2+y^2+N^2}\right] dxdy + \text{constant} + O(\frac{1}{N}).$$

Computation of the integral:

$$\iiint \frac{1}{x^2+y^2+z^2}\,dxdydz =$$
$$= \sum_{p.c.\{x,y,z\}} -x\,\mathrm{Ti}\left(\frac{z+\sqrt{z^2+x^2}}{y+\sqrt{y^2+x^2}}\right) + x\,\mathrm{Ti}\left(\frac{z+\sqrt{z^2+x^2}}{-y+\sqrt{y^2+x^2}}\right)$$

$$\iint \frac{1}{x^2+y^2+1}\,dxdy = -\,\mathrm{Ti}\left(\frac{y+\sqrt{y^2+1}}{x+\sqrt{x^2+1}}\right) + \mathrm{Ti}\left(\frac{y+\sqrt{y^2+1}}{-x+\sqrt{x^2+1}}\right)$$

$$\iint \frac{1}{x^2+y^2+N^2}\,dxdy = -\,\mathrm{Ti}\left(\frac{y+\sqrt{y^2+N^2}}{x+\sqrt{x^2+N^2}}\right) + \mathrm{Ti}\left(\frac{y+\sqrt{y^2+N^2}}{-x+\sqrt{x^2+N^2}}\right).$$

The asymptotic formula is here:

$$\sum_{n_1,n_2,n_3=1}^{N} \frac{1}{n_1^2+n_2^2+n_3^2} = \left[-3G + \frac{3}{2}\left(\mathrm{Ti}\left((1+\sqrt{2})^2\right) + \mathrm{Ti}\left((-1+\sqrt{2})^2\right)\right)\right] N - \frac{9}{4}\pi \ \ln(N) + \gamma_{3,2} + O(\frac{1}{N}).$$

The last one is for homogeneous terms of degree -1:

$$\sum_{n_1,n_2,n_3=1}^{N} \frac{n_1+n_2+n_3}{n_1^2+n_2^2+n_3^2} = \int_1^N \int_1^N \int_1^N \frac{x+y+z}{x^2+y^2+z^2}\, dxdydz$$
$$+ \frac{3}{2}\int_1^N \int_1^N \left[\frac{x+y+1}{x^2+y^2+1} + \frac{x+y+N}{x^2+y^2+N^2}\right] dxdy$$
$$+ \frac{3}{4}\int_1^N \left[\frac{x+1+1}{x^2+1+1} + 2\frac{x+1+N}{x^2+1+N^2} + \frac{x+N+N}{x^2+N^2+N^2}\right] dx$$
$$+ \frac{3}{2}\, b_2 \int_1^N \int_1^N \left(\frac{\partial}{\partial z}\frac{x+y+z}{x^2+y^2+z^2}\right)_{(z=N)-(z=1)} dxdy$$
$$+ \text{constant} + O(\frac{1}{N}).$$

Computation of the integral:

$$\iiint \frac{x+y+z}{x^2+y^2+z^2}\, dxdydz = \sum_{p.c.\{x,y,z\}} \left\{ \frac{1}{2}xy \ \ln(x^2+y^2+z^2) \right.$$
$$+ \frac{1}{2}x\sqrt{x^2+z^2}\,\mathrm{Arctan}\left(\frac{y}{\sqrt{x^2+z^2}}\right) + \frac{1}{2}y\sqrt{y^2+z^2}\,\mathrm{Arctan}\left(\frac{x}{\sqrt{y^2+z^2}}\right)$$
$$\left. - \frac{1}{2}\, z^2 \ \mathrm{Ti}\left(\frac{y+\sqrt{y^2+z^2}}{x+\sqrt{x^2+z^2}}\right) + \frac{1}{2}\, z^2 \ \mathrm{Ti}\left(\frac{y+\sqrt{y^2+z^2}}{-x+\sqrt{x^2+z^2}}\right) \right\}.$$

$$\iint \frac{x+y+z}{x^2+y^2+z^2}\, dxdy = \frac{1}{2}(x+y) \ \ln(x^2+y^2+z^2)$$
$$+ \sqrt{x^2+z^2}\,\mathrm{Arctan}\left(\frac{y}{\sqrt{x^2+z^2}}\right) + \sqrt{y^2+z^2}\,\mathrm{Arctan}\left(\frac{x}{\sqrt{y^2+z^2}}\right)$$
$$- z \ \mathrm{Ti}\left(\frac{y+\sqrt{y^2+z^2}}{x+\sqrt{x^2+z^2}}\right) + z \ \mathrm{Ti}\left(\frac{y+\sqrt{y^2+z^2}}{-x+\sqrt{x^2+z^2}}\right)$$

$$\int \frac{x+y+z}{x^2+y^2+z^2}\, dx = \frac{1}{2} \ \ln(x^2+y^2+z^2) + \frac{y+z}{\sqrt{y^2+z^2}}\,\mathrm{Arctan}\left(\frac{x}{\sqrt{y^2+z^2}}\right)$$

$$\iint \frac{\partial}{\partial z}\left(\frac{x+y+z}{x^2+y^2+z^2}\right) dxdy =$$
$$= \frac{z}{\sqrt{x^2+z^2}} \operatorname{Arctan}\left(\frac{y}{\sqrt{x^2+z^2}}\right) + \frac{z}{\sqrt{y^2+z^2}} \operatorname{Arctan}\left(\frac{x}{\sqrt{y^2+z^2}}\right)$$
$$- \frac{x}{\sqrt{x^2+z^2}} \operatorname{Arctan}\left(\frac{y}{\sqrt{x^2+z^2}}\right) - \frac{y}{\sqrt{y^2+z^2}} \operatorname{Arctan}\left(\frac{x}{\sqrt{y^2+z^2}}\right)$$
$$- \operatorname{Ti}\left(\frac{y+\sqrt{y^2+z^2}}{x+\sqrt{x^2+z^2}}\right) + \operatorname{Ti}\left(\frac{y+\sqrt{y^2+z^2}}{-x+\sqrt{x^2+z^2}}\right).$$

Now we have our asymptotic formula:

$$\sum_{n_1,n_2,n_3=1}^{N} \frac{n_1+n_2+n_3}{n_1^2+n_2^2+n_3^2} = \Big[\frac{3}{2}\ln\frac{3}{2} + 3\sqrt{2}\operatorname{Arctan}\frac{\sqrt{2}}{2} + \frac{3}{4}\big(\operatorname{Ti}((\sqrt{2}+1)^2)+$$
$$+ \operatorname{Ti}((\sqrt{2}-1)^2)\big) - \frac{3}{2}G - \frac{3\pi}{4}\Big] N^2 + \Big[\frac{3}{2}\ln\frac{3}{4} + 3\sqrt{2}\operatorname{Arctan}\frac{\sqrt{2}}{2} +$$
$$+ \frac{3}{4}\big(\operatorname{Ti}((\sqrt{2}+1)^2) + \operatorname{Ti}((\sqrt{2}-1)^2)\big) - \frac{3}{2}G - \frac{33\pi}{16} - 3\Big] N$$
$$+ \big(\frac{3}{4} - \frac{\pi}{8}\big)\ln N + \gamma_{3,2}^{1,0,0} + O(\frac{1}{N}).$$

We can define the ψ_3 function:

Definition 4.3

$$\psi_3(x) = \psi_2(x) + \lim_{N\to+\infty}\Big\{ \sum_{n_1,n_2=-N}^{N} \sum_{n_3=1}^{N} (\overrightarrow{n} - x)^{-1} + \ln N\, e_3$$
$$+ 2\Big(\big(\frac{\pi}{2} + \ln 2\big)N - \ln N\Big)e_3 + \Big[2\ln\frac{3}{2} + 4\sqrt{2}\operatorname{Arctan}\frac{\sqrt{2}}{2} + \operatorname{Ti}((\sqrt{2}+1)^2)$$
$$+ \operatorname{Ti}((\sqrt{2}-1)^2) - 2G - \pi\Big] N^2 e_3 + \Big[2\ln\frac{3}{4} + 4\sqrt{2}\operatorname{Arctan}\frac{\sqrt{2}}{2} + \operatorname{Ti}((\sqrt{2}+1)^2)$$
$$+ \operatorname{Ti}((\sqrt{2}-1)^2) - 2G - \frac{11\pi}{4} - 4\Big] N e_3 + \big(1 - \frac{\pi}{6}\big)\ln N\, e_3$$
$$- \frac{\pi}{2}\ln N\,\big(P_{0200}(x) + P_{0020}(x) + 2P_{0002}(x)\big)$$
$$- 2\Big[\operatorname{Ti}\big((1+\sqrt{2})^2\big) + \operatorname{Ti}\big((-1+\sqrt{2})^2\big) - 2G\Big] N\big(P_{0200}(x) + P_{0020}(x) + P_{0002}(x)\big)$$
$$+ 3\pi\,\ln(N)\,\big(P_{0200}(x) + P_{0020}(x) + P_{0002}(x)\big)$$
$$+ \frac{\pi}{4}\ln N\,\big(P_{0201}(x) + P_{0021}(x)\big) + \frac{\pi}{4}\ln N P_{0003}(x)\Big\}$$

Another expression for the ψ_3 function:

$$\psi_3(x) = \psi_2(x) + \lim_{N\to+\infty}\Big\{ \sum_{n_1,n_2=-N}^{N}\sum_{n_3=1}^{N}(\overrightarrow{n} - x)^{-1} \; - \; \frac{\pi}{6}\ln N\; e_3 + \Big[2\ln\frac{3}{2}+$$
$$+ 4\sqrt{2}\,\mathrm{Arctan}\,\frac{\sqrt{2}}{2} + \mathrm{Ti}((\sqrt{2}+1)^2) + \mathrm{Ti}((\sqrt{2}-1)^2) - 2G - \frac{7\pi}{4} - 4\Big] N e_3$$
$$+ \Big[2\ln\frac{3}{2} + 4\sqrt{2}\,\mathrm{Arctan}\,\frac{\sqrt{2}}{2} + \mathrm{Ti}((\sqrt{2}+1)^2) + \mathrm{Ti}((\sqrt{2}-1)^2) - 2G - \pi\Big] N^2 e_3$$
$$+ \frac{\pi}{2}\ln N\,\big(P_{0200}(u) + P_{0020}(x)\big)$$
$$- 2\Big[\,\mathrm{Ti}\,\big((1+\sqrt{2})^2\big) + \mathrm{Ti}\,\big((-1+\sqrt{2})^2\big) - 2G\Big]\times$$
$$\times N\,\big(P_{0200}(x) + P_{0020}(x) + P_{0002}(x)\big)$$
$$+ 2\,\pi\,\ln(N)\,\big(P_{0200}(x) + P_{0020}(x) + P_{0002}(x)\big)$$
$$+ \frac{\pi}{4}\ln N\,\big(P_{0201}(x) + P_{0021}(x)\big) + \frac{\pi}{4}\ln N\,P_{0003}(x)\Big\}.$$

Proposition 4.2

$$\begin{aligned} H_3(x) - \psi_3(x) = {} & \gamma\, e_1 + (\gamma + \gamma_{2,2}^{1,0})\, e_2 + (\gamma + 2\gamma_{2,2}^{1,0} + \frac{4}{3}\gamma_{3,2}^{1,0,0})\, e_3 \\ & - \Big(\gamma_{1,2} + 2\,\gamma_{2,2} + \frac{4}{3}\,\gamma_{3,2}\Big)\;\big(P_{0200}(x) + P_{0020}(x) + P_{0002}(x)\big) \\ & + \gamma_{2,6}^{3,0}\, P_{0030}(x) + \Big(2\,\gamma_{2,6}^{3,0} + \frac{4}{3}\gamma_{3,6}^{3,0,0}\Big) P_{0003}(x) \\ & + \gamma_{2,6}^{2,1}\,\big(P_{0210}(x) + P_{0201}(x) + P_{0021}(x)\big) \\ & + \gamma_{1,3}\,\big(P_{0300}(x) + P_{0030}(x) + P_{0003}(x)\big) \\ & + \frac{2}{3}\,\gamma_{3,6}^{2,1,0}\,\big(P_{0201}(x) + P_{0021}(x)\big) \end{aligned}$$

with

$$\gamma_{2,6}^{3,0} = \sum_{\substack{n_1\in\mathbb{N}^*\\ n_2\in\mathbb{N}^*}} \frac{(n_1^3 + n_2^3)}{(n_1^2 + n_2^2)^3}$$

and

$$\gamma_{2,6}^{2,1} = \sum_{\substack{n_1\in\mathbb{N}^*\\ n_2\in\mathbb{N}^*}} \frac{n_1 n_2 (n_1 + n_2)}{(n_1^2 + n_2^2)^3}.$$

5. Special values

It is well known that the ψ function verifies

$$\psi(1) = -\gamma.$$

Using our definitions, this relation corresponds exactly to

$$\psi_1(-e_1) = -\gamma e_1.$$

So, it seems natural to write some analogous relations with ψ_2 and ψ_3.

Proposition 5.1 *We have the following special values:*

1) $\psi_1(-e_1) = -\gamma\, e_1$.

2) $\big(u\,|\nabla_x\big)_{x=-e_1}\psi_2(x) = \big(\gamma_{1,2}+\gamma_{2,2}\big)\,\big(P_{0200}(u)+P_{0020}(u)\big)$.

3)
$$\begin{aligned}\frac{1}{2}\big(u\,|\nabla_x\big)^2_{x=-e_1}\,\psi_3(x) = &-\gamma^{3,0}_{2,6}\,P_{0030}(u) - 2\,\gamma^{3,0}_{2,6}\,P_{0003}(u) - \gamma^{2,1}_{2,6}\,P_{0210}(u)\\ &-\Big(\gamma^{2,1}_{2,6}+\frac{2}{3}\gamma^{2,1,0}_{3,6}\Big)\,\big(P_{0201}(u)+P_{0021}(u)\big)\\ &-\Big(\gamma_{1,3}+\frac{4}{3}\gamma^{3,0,0}_{3,6}\Big)\,\big(P_{0300}(u)+P_{0030}(u)+P_{0003}(u)\big)\,.\end{aligned}$$

Proof. The two first relations follow respectively from $H_1(-e_1) = 0$ and $\big(u\,|\nabla_x\big)_{x=-e_1}H_2(x) = 0$ written previously.

For the third: $\frac{1}{2}\big(u\,|\nabla_x\big)^2_{x=-e_1}H_3(x) = 0$.

$$\begin{aligned}&\frac{1}{2}\big(u\,|\nabla_x\big)^2\,H_3(x) =\\ &-x^{-1}ux^{-1}ux^{-1} + \sum_{n_1\in\mathbb{N}^*}(n_1e_1-x)^{-1}u(n_1e_1-x)^{-1}u(n_1e_1-x)^{-1}\\ &-\sum_{n_1\in\mathbb{N}^*}(n_1e_1)^{-1}u(n_1e_1)^{-1}u(n_1e_1)^{-1}\\ &+\sum_{\substack{n_1\in\mathbb{Z}\\ n_2\in\mathbb{N}^*}}(n_1e_1+n_2e_2-x)^{-1}u(n_1e_1+n_2e_2-x)^{-1}u(n_1e_1+n_2e_2-x)^{-1}\\ &-\sum_{\substack{n_1\in\mathbb{Z}\\ n_2\in\mathbb{N}^*}}(n_1e_1+n_2e_2)^{-1}u(n_1e_1+n_2e_2)^{-1}u(n_1e_1+n_2e_2)^{-1}\\ &+\sum_{\substack{(n_1,n_2)\in\mathbb{Z}^2\\ n_3\in\mathbb{N}^*}}\big[(\vec{n}-x)^{-1}u(\vec{n}-x)^{-1}u(\vec{n}-x)^{-1}-\vec{n}^{-1}u\,\vec{n}^{-1}u\,\vec{n}^{-1}\big].\end{aligned}$$

This is zero when we put $x = -e_1$. ∎

6. Conclusion

The functions ψ_k are holomorphic cliffordian functions. They may be defined by

$$\psi_k(x) = \lim_{N\to\infty} \Big\{ \sum_{n\in\Omega_{k,N}} (n-x)^{-1} + \varphi_{k-1}(x,N) \Big\}.$$

Here $\Omega_{k,N}$ is the subset of the Ω_k lattice such that $0 \le n_j \le N$. The holomorphic cliffordian polynomial $\varphi_{k-1}(x,N)$ is of degree $k-1$. With it, the limit exists. The choice of this polynomial is a consequence of Euler–Mac-Laurin formula.

These functions may also be defined by

$$\psi_k(x) = -x^{-1} + \Big[\Big[\sum_{p=0}^{k-1} \sum_{n\in\Omega_k^*} (n^{-1}x)^p \, n^{-1} \Big]\Big] + \sum_{p=k}^{\infty} \sum_{n\in\Omega_k^*} (n^{-1}x)^p \, n^{-1}$$

where the symbol $[[\ldots]]$ means that non-converging terms must be substituted by finite terms in their asymptotic developments.

The ideas and algorithms seems to work for all dimensions. But we think that this paper would be less understandable if we would write it in general.

Notice that holomorphic cliffordian functions in $\mathbb{R}_{0,3}$ are defined by a third order operator unlike holomorphic functions in $\mathbb{C}$ which are defined by a first order operator. The substitution of constants by polynomials of degree two is clear in formulas giving $H_3(x) - \psi_3(x)$.

Proposition 5.1 shows that it is possible to compute special values of ψ_k functions, giving links between well-known or not so well-known constants. It will be interesting to understand more deeply these links.

Connections between ψ_k and ζ_k studied in [LR2], generating elliptic cliffordian functions are described above.

Quote W. Sprössig [Sp]:

> "The effective application of methods of Clifford analysis to partial differential equations needs systems of elementary functions which are Clifford regular and still satisfy most of the properties which we know from the complex plane".

Special function for higher dimensional spaces is a program reached in part in [Sp], [DSS], [So], [LR2], [LR3]. We think that "Clifford regular" should be understood in the sense of holomorphic cliffordian, it is the case in chapter 2 of [GS].

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References

[DSS] Delanghe, R., Sommen, F. and Soucek, V.: *Clifford algebra and spinor-valued functions. A function theory for the Dirac operator.* Mathematics and its Applications **53**. Kluwer, Dordrecht, 1992.

[Er] Erdélyi, A.: *Higher transcendental functions*. Bateman manuscript project. McGraw-Hill, New York-Toronto-London, 1953.

[GS] Gürlebeck, K. and Sprössig, W.: *Quaternionic and Clifford calculus for physicists and engineers*. John Wiley and Sons, 1997.

[Kr1] Krausshar, R. S.: *Eisenstein series in Clifford analysis*. Thesis, University of Aachen, 2000.

[Kr2] Krausshar, R. S.: On a new type of Eisentein series in Clifford analysis. *Z. Anal. Anwendungen* **20** (2001), no. 4, 1007–1029.

[LR1] Laville, G. and Ramadanoff, I.: Holomorphic cliffordian functions. *Adv. Appl. Clifford Algebras* **8** (1998), no. 2, 323–340.

[LR2] Laville, G. and Ramadanoff, I.: Elliptic cliffordian functions. *Complex Variables Theory Appl.* **45** (2001), no. 4, 297–318.

[LR3] Laville, G. and Ramadanoff, I.: Jacobi elliptic cliffordian functions. *Complex Var. Theory Appl.* **47** (2002), no. 9, 787–802.

[Le] Lewin, L.: *Polylogarithms and associated functions*. North Holland, New York-Amsterdam, 1981.

[So] Sommen, F.: Special functions in Clifford analysis and axial symmetry. *J. Math. Anal. Appl.* **130** (1988), no. 1, 100–133.

[Sp] Sprössig, W.: On operators and elementary functions in Clifford analysis. *Z. Anal. Anwendungen* **18** (1999), no. 2, 349–360.

[Wa] Walker, P. L.: *Elliptic functions*. John Wiley and Sons, Chichester, 1996.

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