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Corrigenda: $(n, 2)$-sets have full Hausdorff dimension

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1. Introduction

In [1] the author claimed that an $(n, 2)$-set must have full Hausdorff dimension. However, as pointed out by Terence Tao and John Bueti, the proof contains an error. More precisely, on page 389, the argument doesn't really show that $P_k^\delta \subset \Pi_i^{\widetilde{C}\delta}$. In this note we outline how one can correct this, by constructing families of plates so that their intersections with a given one contain line segments of fixed length. The price we pay is a weaker result. Namely, we show that the Hausdorff dimension of an $(n, 2)$-set is at least $(2n+3)/3$, which is, nevertheless, an improvement on the previously known $(2n+2)/3$.

As in [1], the Hausdorff dimension bound is a consequence of the following which should replace Proposition 4.1 in [1]

Proposition 1.1 *Suppose E is a set in $\mathbb{R}^n$, $\lambda \leq 1$ and $\mathcal{B} = \{P_j\}_{j=1}^M$ is a δ-separated set in $\mathcal{G}_n$ with diam($\mathcal{B}$) $\leq 1/2$, such that for each j there is a plate P_j^δ satisfying*

$$|P_j^\delta \cap E| \geq \lambda |P_j^\delta|.$$

Then

$$|E| \geq C_\epsilon^{-1} \delta^\epsilon \lambda^\alpha M^{(2n-3)/(6(n-2))} \delta^{n-2},$$

where α is a positive constant depending on n.

2. Preliminaries

Our terminology and notation are the same as in [1]. The only difference is that $P^{l,\delta}$ denotes a plate of dimensions $l \times l \times \delta \times \cdots \times \delta$, $1 \leq l \leq 4$, $0 < \delta \ll 1$. Also, when we write $x \gtrapprox_\delta y$ we mean $x \gtrsim |\log \delta|^{-\alpha} y$, for some positive α. As is customary, C denotes positive constants not necessarily the same each time they occur.

We will make use of the following.

Lemma 2.1 *Suppose E is a set in $\mathbb{R}^n$, $\beta, \kappa \leq 1$ and $\mathcal{E} = \{P_j\}_{j=1}^M$ is an η-separated subset of $\mathcal{G}_n$ with diam$(\mathcal{E}) \leq 1/2$, such that for each j there is a plate $P_j^{l,\eta}$ satisfying*

$$|P_j^{l,\eta} \cap E \setminus T_e^\beta(z)| \geq \kappa |P_j^{l,\eta}|$$

for all $e \in S^{n-1}$, $z \in \mathbb{R}^n$. Then

$$|E| \gtrapprox_\beta \beta^{2(n-2)/3} \kappa |\mathcal{E}|^{1/3} \eta^{n-2}.$$

Proof. This is a 2-dimensional version of Bourgain's "bush" argument. The proof is almost identical to the proof of the result in [2], so we omit it. ■

3. Proof of the proposition

First, by an argument analogous to that of [1, page 386], one shows that there is a family $\mathcal{C}' \subset \{P_j^\delta\}_{j=1}^M$, with $|\mathcal{C}'| \geq M/2$ so that for each $P_j^\delta \in \mathcal{C}'$ there is a set $A_j' \subset P_j^\delta \cap E$ of measure $|A_j'| \gtrsim \lambda\delta^{n-2}$, such that for each $x \in A_j'$

$$|\{y \in P_j^\delta \cap E \setminus B(x, c_0) : |\{k : [x,y] \subset P_k^\delta\}| \geq \mu_0\}| \gtrsim \lambda\delta^{n-2},$$

where c_0 is a small fixed constant, $[x, y]$ is the line segment joining x and y, and

$$\mu_0 \sim M|E|^{-2}\lambda^2\delta^{2(n-2)}. \tag{3.1}$$

Then using the pigeonhole principle as in [1, page 387], we conclude that there is a number ρ with $\delta \leq \rho \leq 1$, a family $\mathcal{C} \subset \mathcal{C}'$ with $|\mathcal{C}| \gtrapprox_\delta M$, and a subset $A_j \subset A_j'$ with $|A_j| \gtrapprox_\delta \lambda\delta^{n-2}$ so that for each $P_j^\delta \in \mathcal{C}$ and each $x \in A_j$

$$\Big|\Big\{y \in P_j^\delta \cap E \setminus B(x, c_0) : \\ |\{k : [x,y] \subset P_k^\delta \text{ and } \rho \leq d(P_j, P_k) \leq 2\rho\}| \gtrapprox_\delta \mu_0\Big\}\Big| \gtrapprox_\delta \lambda\delta^{n-2}.$$

Next, for each $P_j^\delta \in \mathcal{C}$, let

$$\mathcal{D}_j = \{P_k^\delta : \rho \leq d(P_j, P_k) \leq 2\rho \\ \text{and } P_k^\delta \cap P_j^\delta \text{ contains a line segment of length at least } c_0\}.$$

Arguing as in [1, page 387] we show that

$$|\mathcal{D}_j| \gtrapprox_\delta (\lambda\rho\delta^{-1})^2\mu_0. \tag{3.2}$$

Now we are in a position to carry out a version of Wolff's "hairbush" argument. Namely, for each $P_j^\delta \in \mathcal{C}$ take a maximal δ/ρ-separated set of points $\{e_{ji}\}_i$ on the $(n-3)$-dimensional unit sphere $S^{n-1} \cap P_j^\perp$, and let

$$\Pi_{ji} = c_j + \Pi'_{ji},$$

where c_j is the center of P_j^δ and Π'_{ji} is the 3-plane spanned by e_{ji} and P_j. Using the fact that the intersection of each $P_k^\delta \in \mathcal{D}_j$ with P_j^δ contains a line segment of length at least c_0, one can indeed show that for every $P_k^\delta \in \mathcal{D}_j$ there exists an i such that $P_k^\delta \subset \Pi_{ji}^{C\delta}$. Therefore, letting

$$\mathcal{D}_{ji} = \{P_k^\delta \in \mathcal{D}_j : P_k^\delta \subset \Pi_{ji}^{C\delta}\},$$

we have

$$\mathcal{D}_j = \bigcup_i \mathcal{D}_{ji}.$$

Now for each $P_j^\delta \in \mathcal{C}$, let $P_j^{4,C\rho}$ be a plate with direction plane P_j, the same center as P_j^δ and the indicated dimensions. Proceeding as in [1, pages 390-391] one shows that for all $e \in S^{n-1}$, $z \in \mathbb{R}^n$

$$|P_j^{4,C\rho} \cap E \setminus T_e^\gamma(z)| \gtrsim \gamma^{n-3}\lambda^3\delta^{n-2} \sum_i |\mathcal{D}_{ji}|^{1/2},$$

where $\gamma = \lambda|\log\delta|^{-1}$. Using this, (3.1), (3.2) and the inequality

$$|\mathcal{D}_j| \le \sum_i |\mathcal{D}_{ji}| \lesssim \rho\delta^{-1} \sum_i |\mathcal{D}_{ji}|^{1/2}$$

we get

$$|P_j^{4,C\rho} \cap E \setminus T_e^\gamma(z)| \gtrapprox_\delta \lambda^{n+4} M |E|^{-2} \rho \delta^{3n-7}. \tag{3.3}$$

Now let $\mathcal{E}$ be a maximal $C\rho$-separated subset of $\{P_j : P_j^\delta \in \mathcal{C}\}$. Then

$$|\mathcal{E}| \gtrapprox_\delta (\delta\rho^{-1})^{2(n-2)} M.$$

So, rewriting (3.3) as

$$|P_j^{4,C\rho} \cap E \setminus T_e^\gamma(z)| \ge C_\epsilon^{-1}\delta^\epsilon \lambda^{n+4} M |E|^{-2} \rho^{3-n} \delta^{3n-7} |P_j^{4,C\rho}|,$$

we see that the family $\{P_j^{4,C\rho} : P_j \in \mathcal{E}\}$ satisfies the conditions of Lemma 2.1 with $l = 4$, $\eta = C\rho$, $\beta = \gamma = \lambda|\log\delta|^{-1}$ and

$$\kappa = C_\epsilon^{-1}\delta^\epsilon \lambda^{n+4} M |E|^{-2} \rho^{3-n} \delta^{3n-7}.$$

Hence, after some algebra,

$$|E| \geq C_\epsilon^{-1}\delta^\epsilon\lambda^{\alpha_1}(\delta\rho^{-1})^{(2n-7)/9}M^{4/9}\delta^{n-2}, \tag{3.4}$$

for some $\alpha_1 > 0$. Note that (3.3) trivially implies

$$|E| \geq C_\epsilon^{-1}\delta^\epsilon\lambda^{\alpha_2}(\rho\delta^{-1})^{1/3}M^{1/3}\delta^{n-2}$$

for some $\alpha_2 > 0$. So, if $\rho \geq \delta M^{1/(2(n-2))}$ then

$$|E| \geq C_\epsilon^{-1}\delta^\epsilon\lambda^{\alpha_2}M^{(2n-3)/(6(n-2))}\delta^{n-2}. \tag{3.5}$$

On the other hand, if $\rho \leq \delta M^{1/(2(n-2))}$ then (3.4) gives

$$|E| \geq C_\epsilon^{-1}\delta^\epsilon\lambda^{\alpha_1}M^{(2n-3)/(6(n-2))}\delta^{n-2}. \tag{3.6}$$

Combining (3.5) and (3.6) we complete the proof.

References

[1] Mitsis, T.: $(n, 2)$-sets have full Hausdorff dimension. *Rev. Mat. Iberoamericana* **20** (2004), 381–393.

[2] Mitsis, T.: Norm estimates for a Kakeya-type maximal operator. *Math. Nachr.*, to appear.

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