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A note on the existence of H-bubbles via perturbation methods

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Abstract

We study the problem of existence of surfaces in $\mathbb{R}^3$ parametrized on the sphere $\mathbb{S}^2$ with prescribed mean curvature H in the perturbative case, i.e. for $H = H_0 + \varepsilon H_1$, where H_0 is a nonzero constant, H_1 is a C^2 function and ε is a small perturbation parameter.

1. Introduction

In this paper we are interested in the existence of H-bubbles, namely of $\mathbb{S}^2$-type parametric surfaces in $\mathbb{R}^3$ with prescribed mean curvature H. This geometrical problem is motivated by some models describing capillarity phenomena and has the following analytical formulation: given a function $H \in C^1(\mathbb{R}^3)$, find a smooth nonconstant function $\omega : \mathbb{R}^2 \to \mathbb{R}^3$ which is conformal as a map on $\mathbb{S}^2$ and solves the problem

$$(P_H) \qquad \begin{cases} \Delta\omega = 2H(\omega)\, \omega_x \wedge \omega_y, & \text{in } \mathbb{R}^2, \\ \displaystyle\int_{\mathbb{R}^2} |\nabla\omega|^2 < +\infty, \end{cases}$$

where

$$\begin{aligned} \omega_x &= \Big(\frac{\partial\omega_1}{\partial x}, \frac{\partial\omega_2}{\partial x}, \frac{\partial\omega_3}{\partial x}\Big), & \omega_y &= \Big(\frac{\partial\omega_1}{\partial y}, \frac{\partial\omega_2}{\partial y}, \frac{\partial\omega_3}{\partial y}\Big), \\ \Delta\omega &= \omega_{xx} + \omega_{yy}, & \nabla\omega &= (\omega_x, \omega_y), \end{aligned}$$

and $\wedge$ denotes the exterior product in $\mathbb{R}^3$.

Brezis and Coron [4] proved that for constant nonzero mean curvature $H(u) \equiv H_0$ the only nonconstant solutions are spheres of radius $|H_0|^{-1}$.

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While the Plateau and the Dirichlet problems have been largely studied both for H constant and for H nonconstant (see [3, 4, 10, 12, 13, 14, 15, 16]), problem (P_H) in the case of nonconstant H has been investigated only recently, see [5, 6, 7]. In [5] Caldiroli and Musina proved the existence of H-bubbles with minimal energy under the assumptions that $H \in C^1(\mathbb{R}^3)$ satisfies

$$(i) \quad \sup_{u\in\mathbb{R}^3} |\nabla H(u+\xi)\cdot u\, u| < 1 \quad \text{for some } \xi \in \mathbb{R}^3,$$
$$(ii) \quad H(u) \to H_\infty \quad \text{as } |u| \to \infty \quad \text{for some } H_\infty \in \mathbb{R},$$
$$(iii) \quad c_H = \inf_{\substack{u\in C^1_c(\mathbb{R}^2,\mathbb{R}^3)\\ u\neq 0}} \sup_{s>0} \mathcal{E}_H(su) < \frac{4\pi}{3H_\infty^2}$$

where

$$\mathcal{E}_H(u) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 + 2\int_{\mathbb{R}^2} Q(u)\cdot u_x \wedge u_y$$

and $Q : \mathbb{R}^3 \to \mathbb{R}^3$ is any vector field such that $\operatorname{div} Q = H$.

The perturbative method introduced by Ambrosetti and Badiale [1, 2] was used in [7] to treat the case in which H is a small perturbation of a constant, namely

$$H(u) = H_\varepsilon(u) = H_0 + \varepsilon H_1(u),$$

where $H_0 \in \mathbb{R}\setminus\{0\}$, $H_1 \in C^2(\mathbb{R}^3)$, and ε is a small real parameter. This method allows to find critical points of a functional f_ε of the type $f_\varepsilon(u) = f_0(u) - \varepsilon G(u)$ in a Banach space by studying a finite dimensional problem. More precisely, if the unperturbed functional f_0 has a finite dimensional manifold of critical points Z which satisfies a nondegeneracy condition, it is possible to prove, for $|\varepsilon|$ sufficiently small, the existence of a smooth function $\eta_\varepsilon(z) : Z \to (T_z Z)^\perp$ such that any critical point $\bar z \in Z$ of the function

$$\Phi_\varepsilon : Z \to \mathbb{R}, \quad \Phi_\varepsilon(z) = f_\varepsilon\big(z + \eta_\varepsilon(z)\big)$$

gives rise to a critical point $u_\varepsilon = \bar z + \eta_\varepsilon(\bar z)$ of f_ε, i.e. the perturbed manifold $Z_\varepsilon := \{z + \eta_\varepsilon(z) : \ z \in Z\}$ is a natural constraint for f_ε. Furthermore Φ_ε can be expanded as

$$\Phi_\varepsilon(z) = b - \varepsilon\Gamma(z) + o(\varepsilon) \quad \text{as } \varepsilon \to 0 \tag{1.1}$$

where $b = f_0(z)$ and Γ is the Melnikov function defined as the restriction of the perturbation G on Z, namely $\Gamma = G\big|_Z$. For problem (P_{H_ε}), Γ is given by

$$\Gamma : \mathbb{R}^3 \to \mathbb{R}, \quad \Gamma(p) = \int_{|p-q|<\frac{1}{|H_0|}} H_1(q)\, dq.$$

In [7] Caldiroli and Musina studied the functional Γ giving some existence results in the perturbative setting for problem (P_{H_ε}). They prove that for $|\varepsilon|$ small there exists a smooth H_ε-bubble if one of the following conditions holds

1) H_1 has a nondegenerate stationary point and $|H_0|$ is large;

2) $\max_{p\in\partial K} H_1(p) < \max_{p\in K} H_1(p)$ or $\min_{p\in\partial K} H_1(p) > \min_{p\in K} H_1(p)$ for some non-empty compact set $K \subset \mathbb{R}^3$ and $|H_0|$ is large;

3) $H_1 \in L^r(\mathbb{R}^3)$ for some $r \in [1, 2]$.

They prove that critical points of Γ give rise to solutions to (P_{H_ε}) for ε sufficiently small. Precisely the assumption that H_0 is large required in cases 1) and 2) ensures that if H_1 is not constant then Γ is not identically constant. If we let this assumption drop, it may happen that Γ is constant even if H_1 is not. This fact produces some loss of information because the first order expansion (1.1) is not sufficient to deduce the existence of critical points of Φ_ε from the existence of critical points of Γ. Instead of studying Γ we perform a direct study of Φ_ε which allows us to prove some new results. In the first one, we assume that H_1 vanishes at ∞ and has bounded gradient, and prove the existence of a solution without branch points. Let us recall that a branch point for a solution ω to (P_H) is a point where $\nabla\omega = 0$, i.e. a point where the surface parametrized by ω fails to be immersed.

Theorem 1.1 *Let $H_0 \in \mathbb{R} \setminus \{0\}$, $H_1 \in C^2(\mathbb{R}^3)$ such that*

$$(H1) \quad \lim_{|p|\to\infty} H_1(p) = 0;$$
$$(H2) \quad \nabla H_1 \in L^\infty(\mathbb{R}^3, \mathbb{R}^3).$$

Let $H_\varepsilon = H_0 + \varepsilon H_1$. Then for $|\varepsilon|$ sufficiently small there exists a smooth H_ε-bubble without branch points.

With respect to case 1) of [7] we require neither nondegeneracy of critical points of H_1 nor largeness of H_0. With respect to case 2) we do not assume that H_0 is large; on the other hand our assumption $(H1)$ implies 2). Moreover we do not assume any integrability condition of type 3). With respect to the result proved in [5], we have the same kind of behavior of H_1 at ∞ (see (ii) and assumption $(H1)$) but we do not need any assumption of type (iii); on the other hand in [5] it is not required that the prescribed curvature is a small perturbation of a constant.

The following results give some conditions on H_1 in order to have two or three solutions.

Theorem 1.2 *Let $H_0 \in \mathbb{R} \setminus \{0\}$, $H_1 \in C^2(\mathbb{R}^3)$ such that $(H1)$, $(H2)$,*

$(H3)$ $\operatorname{Hess} H_1(p)$ *is positive definite for any* $p \in B_{1/|H_0|}(0)$,

$(H4)$ $H_1(p) > 0$ *in* $B_{1/|H_0|}(0)$,

hold. Then for $|\varepsilon|$ sufficiently small there exist at least three smooth H_ε-bubbles without branch points.

Remark 1.3 *If we assume $(H1)$, $(H2)$, and, instead of $(H3)-(H4)$, that $H_1(0) > 0$ and* $\operatorname{Hess} H_1(0)$ *is positive definite, then we can prove that for $|H_0|$ sufficiently large and $|\varepsilon|$ sufficiently small there exist at least three smooth H_ε-bubbles without branch points.*

Theorem 1.4 *Let $H_0 \in \mathbb{R} \setminus \{0\}$, $H_1 \in C^2(\mathbb{R}^3)$ such that $(H1)$ and $(H2)$ hold. Assume that there exist $p_1, p_2 \in \mathbb{R}^3$ such that*

$$(H5) \qquad \int_{B(p_1,1/|H_0|)} H_1(\xi)\, d\xi > 0 \qquad \text{and} \qquad \int_{B(p_2,1/|H_0|)} H_1(\xi)\, d\xi < 0.$$

Then for $|\varepsilon|$ sufficiently small there exist at least two smooth H_ε-bubbles without branch points.

Remark 1.5 *If we assume $(H1)$, $(H2)$, and, instead of $(H5)$, that there exist $p_1, p_2 \in \mathbb{R}^3$ such that $H_1(p_1) > 0$ and $H_1(p_2) < 0$, then we can prove that for $|H_0|$ sufficiently large and $|\varepsilon|$ sufficiently small there exist at least two smooth H_ε-bubbles without branch points.*

The present paper is organized as follows. In Section 2 we introduce some notation and recall some known facts whereas Section 3 is devoted to the proof of Theorems 1.1, 1.2, and 1.4.

2. Notation and known facts

In the sequel we will take $H_0 = 1$; this is not restrictive since we can do the change $H_1(u) = H_0 \widetilde{H}_1(H_0 u)$. Hence we will always write

$$H_\varepsilon(u) = 1 + \varepsilon H(u),$$

where $H \in C^2(\mathbb{R}^3)$. Let us denote by ω the function $\omega : \mathbb{R}^2 \to \mathbb{S}^2$ defined as

$$\omega(x,y) = \big(\mu(x,y)x, \mu(x,y)y, 1-\mu(x,y)\big), \quad \text{where} \quad \mu(x,y) = \frac{2}{1+x^2+y^2}.$$

Note that ω is a conformal parametrization of the unit sphere and solves

$$(2.1) \qquad \begin{cases} \Delta\omega = 2\,\omega_x \wedge \omega_y & \text{on } \mathbb{R}^2 \\ \displaystyle\int_{\mathbb{R}^2} |\nabla\omega|^2 < +\infty. \end{cases}$$

Problem (2.1) has in fact a family of solutions of the form $\omega\circ\phi+p$ where $p\in\mathbb{R}^3$ and ϕ is any conformal diffeomorphism of $\mathbb{R}^2\cup\{\infty\}$. For $s\in(1,+\infty)$, we will set $L^s:=L^s(\mathbb{S}^2,\mathbb{R}^3)$, where any map $v\in L^s$ is identified with the map $v\circ\omega:\mathbb{R}^2\to\mathbb{R}^3$ which satisfies

$$\int_{\mathbb{R}^2}|v\circ\omega|^s\mu^2=\int_{\mathbb{S}^2}|v|^s.$$

We will use the same notation for v and $v\circ\omega$. By $W^{1,s}$ we denote the Sobolev space $W^{1,s}(\mathbb{S}^2,\mathbb{R}^3)$ endowed (according to the above identification) with the norm

$$\|v\|_{W^{1,s}}=\left[\int_{\mathbb{R}^2}|\nabla v|^s\mu^{2-s}\right]^{1/s}+\left[\int_{\mathbb{R}^2}|v|^s\mu^2\right]^{1/s}.$$

If s' is the conjugate exponent of s, i.e. $1/s+1/s'=1$, the duality product between $W^{1,s}$ and $W^{1,s'}$ is given by

$$\langle v,\varphi\rangle=\int_{\mathbb{R}^2}\nabla v\cdot\nabla\varphi+\int_{\mathbb{R}^2}v\cdot\varphi\,\mu^2\quad\text{for any } v\in W^{1,s}\text{ and }\varphi\in W^{1,s'}.$$

Let Q be any smooth vector field on $\mathbb{R}^3$ such that $\operatorname{div}Q=H$. The energy functional associated to problem

$$(P_\varepsilon)\qquad\begin{cases}\Delta u=2\big(1+\varepsilon H(u)\big)\,u_x\wedge u_y, & \text{in } \mathbb{R}^2,\\ \displaystyle\int_{\mathbb{R}^2}|\nabla u|^2<+\infty, &\end{cases}$$

is given by

$$\mathcal{E}_\varepsilon(u)=\frac{1}{2}\int_{\mathbb{R}^2}|\nabla u|^2+2\mathcal{V}_1(u)+2\varepsilon\mathcal{V}_H(u),\qquad u\in W^{1,3},$$

where

$$\mathcal{V}_H(u)=\int_{\mathbb{R}^2}Q(u)\cdot u_x\wedge u_y$$

has the meaning of an algebraic volume enclosed by the surface parametrized by u with weight H (it is independent of the choice of Q); in particular

$$\mathcal{V}_1(u)=\frac{1}{3}\int_{\mathbb{R}^2}u\cdot u_x\wedge u_y.$$

In [7], Caldiroli and Musina studied some regularity properties of $\mathcal{V}_H$ on the space $W^{1,3}$. In particular they proved the following properties.

a) For $H \in C^1(\mathbb{R}^3)$, the functional $\mathcal{V}_H$ is of class C^1 on $W^{1,3}$ and the Fréchet differential of $\mathcal{V}_H$ at $u \in W^{1,3}$ is given by

$$d\mathcal{V}_H(u)\varphi = \int_{\mathbb{R}^2} H(u)\,\varphi \cdot u_x \wedge u_y \quad \text{for any } \varphi \in W^{1,3} \tag{2.2}$$

and admits a unique continuous and linear extension on $W^{1,3/2}$ defined by (2.2). Moreover for every $u \in W^{1,3}$ there exists $\mathcal{V}'_H(u) \in W^{1,3}$ such that

$$\langle \mathcal{V}'_H(u), \varphi\rangle = \int_{\mathbb{R}^2} H(u)\,\varphi \cdot u_x \wedge u_y \quad \text{for any } \varphi \in W^{1,3/2}. \tag{2.3}$$

b) For $H \in C^2(\mathbb{R}^3)$, the map $\mathcal{V}'_H : W^{1,3} \to W^{1,3}$ is of class C^1 and

$$\langle \mathcal{V}''_H(u)\cdot\eta, \varphi\rangle = \int_{\mathbb{R}^2} H(u)\varphi\cdot(\eta_x\wedge u_y + u_x\wedge\eta_y) + \int_{\mathbb{R}^2}(\nabla H(u)\cdot\eta)\varphi\cdot(u_x \wedge u_y)$$
$$\text{for any } u, \eta \in W^{1,3} \text{ and } \varphi \in W^{1,3/2}. \tag{2.4}$$

Hence for all $u \in W^{1,3}$, $\mathcal{E}'_\varepsilon(u) \in W^{1,3}$ and for any $\varphi \in W^{1,3/2}$

$$\langle \mathcal{E}'_\varepsilon(u), \varphi\rangle = \int_{\mathbb{R}^2} \nabla u\cdot\nabla\varphi + 2\int_{\mathbb{R}^2}\varphi\cdot u_x\wedge u_y + 2\varepsilon\int_{\mathbb{R}^2} H(u)\,\varphi\cdot u_x \wedge u_y.$$

As remarked in [7], critical points of $\mathcal{E}_\varepsilon$ in $W^{1,3}$ give rise to bounded weak solutions to (P_ε) and hence by the regularity theory for H-systems (see [9]) to classical conformal solutions which are $C^{3,\alpha}$ as maps on $\mathbb{S}^2$.

The unperturbed problem, i.e. (P_ε) for $\varepsilon = 0$, has a 9-dimensional manifold of solutions given by

$$Z = \{R\omega \circ L_{\lambda,\xi} + p : \ R \in SO(3),\ \lambda > 0,\ \xi \in \mathbb{R}^2,\ p \in \mathbb{R}^3\}$$

where $L_{\lambda,\xi} z = \lambda(z-\xi)$ (see [11]). In [11] the nondegeneracy condition $T_u Z = \ker \mathcal{E}''_0(u)$ for any $u \in Z$ (where $T_u Z$ denotes the tangent space of Z at u) is proved (see also [8]).

As observed in [7], in performing the finite dimensional reduction, the dependence on the 6-dimensional conformal group can be neglected since any H-system is conformally invariant. Hence we look for critical points of $\mathcal{E}_\varepsilon$ constrained on a three-dimensional manifold Z_ε just depending on the translation variable $p \in \mathbb{R}^3$.

3. Proof of Theorem 1.1

We start by constructing a perturbed manifold which is a natural constraint for $\mathcal{E}_\varepsilon$.

Lemma 3.1 *Assume $H \in C^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ and $\nabla H \in L^\infty(\mathbb{R}^3, \mathbb{R}^3)$. Then there exist $\varepsilon_0 > 0$, $C_1 > 0$, and a C^1 map $\eta : (-\varepsilon_0, \varepsilon_0) \times \mathbb{R}^3 \to W^{1,3}$ such that for any $p \in \mathbb{R}^3$ and $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$*

$$\mathcal{E}'_\varepsilon\big(\omega + p + \eta(\varepsilon, p)\big) \in T_\omega Z, \tag{3.1}$$

$$\eta(\varepsilon, p) \in (T_\omega Z)^\perp, \tag{3.2}$$

$$\int_{\mathbb{S}^2} \eta(\varepsilon, p) = 0, \tag{3.3}$$

$$\|\eta(\varepsilon, p)\|_{W^{1,3}} \leq C_1|\varepsilon|. \tag{3.4}$$

Moreover if we assume that the limit of H at ∞ exists and

$$\lim_{|p|\to\infty} H(p) = 0 \tag{3.5}$$

we have that $\eta(\varepsilon, p)$ converges to 0 in $W^{1,3}$ as $|p| \to \infty$ uniformly with respect to $|\varepsilon| < \varepsilon_0$.

Proof. Let us define the map

$$F = (F_1, F_2) : \mathbb{R} \times \mathbb{R}^3 \times W^{1,3} \times \mathbb{R}^6 \times \mathbb{R}^3 \to W^{1,3} \times \mathbb{R}^6 \times \mathbb{R}^3$$

given by

$$\langle F_1(\varepsilon, p, \eta, \lambda, \alpha), \varphi\rangle = \langle \mathcal{E}'_\varepsilon(\omega + p + \eta), \varphi\rangle - \sum_{i=1}^{6} \lambda_i \int_{\mathbb{R}^2} \nabla\varphi \cdot \nabla\tau_i + \alpha \cdot \int_{\mathbb{S}^2} \varphi,$$

for all $\varphi \in W^{1,3/2}$ and

$$F_2(\varepsilon, p, \eta, \lambda, \alpha) = \left(\int_{\mathbb{R}^2} \nabla\eta \cdot \nabla\tau_1, \ldots, \int_{\mathbb{R}^2} \nabla\eta \cdot \nabla\tau_6, \int_{\mathbb{S}^2} \eta \right)$$

where $\tau_1, \ldots, \tau_6$ are chosen in $T_\omega Z$ such that

$$\int_{\mathbb{R}^2} \nabla\tau_i \cdot \nabla\tau_j = \delta_{ij} \quad \text{and} \quad \int_{\mathbb{S}^2} \tau_i = 0 \quad i, j = 1, \ldots, 6$$

so that $T_\omega Z$ is spanned by $\tau_1, \ldots, \tau_6, e_1, e_2, e_3$. It has been proved by Caldiroli and Musina [7] that F is of class C^1 and that the linear continuous operator

$$\mathcal{L} : W^{1,3} \times \mathbb{R}^6 \times \mathbb{R}^3 \to W^{1,3} \times \mathbb{R}^6 \times \mathbb{R}^3$$
$$\mathcal{L} = \frac{\partial F}{\partial(\eta, \lambda, \alpha)}(0, p, 0, 0, 0)$$

i.e.

$$\langle \mathcal{L}_1(v,\mu,\beta),\varphi\rangle = \langle \mathcal{E}_0''(\omega)\cdot v,\varphi\rangle - \sum_{i=1}^{6}\mu_i\int_{\mathbb{R}^2}\nabla\varphi\cdot\tau_i - \beta\int_{\mathbb{S}^2}\varphi$$

for all $\varphi\in W^{1,3/2}$ and

$$\mathcal{L}_2(v,\mu,\beta) = \biggl(\int_{\mathbb{R}^2}\nabla v\cdot\nabla\tau_1,\dots,\int_{\mathbb{R}^2}\nabla v\cdot\nabla\tau_6,\int_{\mathbb{S}^2} v\biggr)$$

is invertible. Caldiroli and Musina applied the Implicit Function Theorem to solve the equation $F(\varepsilon,p,\eta,\lambda,\alpha)=0$ locally with respect to the variables ε,p, thus finding a C^1-function η on a neighborhood $(-\varepsilon_0,\varepsilon_0)\times B_R\subset\mathbb{R}\times\mathbb{R}^3$ satisfying (3.1), (3.2), and (3.3). We will use instead the Contraction Mapping Theorem, which allows to prove the existence of such a function η globally on $\mathbb{R}^3$, thanks to the fact that the operator $\mathcal{L}$ does not depend on p and hence it is invertible uniformly with respect to $p\in\mathbb{R}^3$.

We have that $F(\varepsilon,p,\eta,\lambda,\alpha)=0$ if and only if (η,λ,α) is a fixed point of the map $T_{\varepsilon,p}$ defined as

$$T_{\varepsilon,p}(\eta,\lambda,\alpha) = -\mathcal{L}^{-1}F(\varepsilon,p,\eta,\lambda,\alpha)+(\eta,\lambda,\alpha).$$

To prove the existence of η satisfying (3.1), (3.2), and (3.3), it is enough to prove that $T_{\varepsilon,p}$ is a contraction in some ball $B_\rho(0)$ with $\rho=\rho(\varepsilon)>0$ independent of p, whereas the regularity of $\eta(\varepsilon,p)$ follows from the Implicit Function Theorem.

We have that if $\|\eta\|_{W^{1,3}}\le\rho$

$$\begin{aligned}
\|T_{\varepsilon,p}(\eta,\lambda,\alpha)\,&\|_{W^{1,3}\times\mathbb{R}^6\times\mathbb{R}^3}\\
&\le C_2\|F(\varepsilon,p,\eta,\lambda,\alpha)-\mathcal{L}(\eta,\lambda,\alpha)\|_{W^{1,3}\times\mathbb{R}^6\times\mathbb{R}^3}\\
&\le C_2\|\mathcal{E}_\varepsilon'(\omega+p+\eta)-\mathcal{E}_0''(\omega)\eta\|_{W^{1,3}}\\
&\le C_2\bigl(\|\mathcal{E}_0'(\omega+\eta)-\mathcal{E}_0''(\omega)\eta\|_{W^{1,3}}+2|\varepsilon|\|\mathcal{V}_H'(\omega+p+\eta)\|_{W^{1,3}}\bigr)\\
&\le C_2\biggl(\int_0^1\|\mathcal{E}_0''(\omega+t\eta)-\mathcal{E}_0''(\omega)\|_{W^{1,3/2}}\,dt\|\eta\|_{W^{1,3}}\\
&\qquad\qquad+2|\varepsilon|\|\mathcal{V}_H'(\omega+p+\eta)\|_{W^{1,3}}\biggr)\\
&\le C_2\rho\sup_{\|\eta\|_{W^{1,3}}\le\rho}\|\mathcal{E}_0''(\omega+\eta)-\mathcal{E}_0''(\omega)\|_{W^{1,3/2}}\\
&\qquad\qquad+2C_2|\varepsilon|\sup_{\|\eta\|_{W^{1,3}}\le\rho}\|\mathcal{V}_H'(\omega+p+\eta)\|_{W^{1,3}}
\end{aligned}\tag{3.6}$$

where $C_2=\|\mathcal{L}^{-1}\|_{\mathcal{L}(W^{1,3}\times\mathbb{R}^6\times\mathbb{R}^3)}$.

For $(\eta_1,\lambda_1,\alpha_1)$, $(\eta_2,\lambda_2,\alpha_2)\in B_\rho(0)\subset W^{1,3}\times\mathbb{R}^6\times\mathbb{R}^3$ we have

$$\begin{aligned}
&\frac{\|T_{\varepsilon,p}(\eta_1,\lambda_1,\alpha_1)-T_{\varepsilon,p}(\eta_2,\lambda_2,\alpha_2)\|_{W^{1,3}\times\mathbb{R}^6\times\mathbb{R}^3}}{C_2\|\eta_1-\eta_2\|_{W^{1,3}}}\\
&\le \frac{\|\mathcal{E}'_\varepsilon(\omega+p+\eta_1)-\mathcal{E}'_\varepsilon(\omega+p+\eta_2)-\mathcal{E}''_0(\omega)(\eta_1-\eta_2)\|_{W^{1,3}}}{C_2\|\eta_1-\eta_2\|_{W^{1,3}}}\\
&\le \int_0^1 \|\mathcal{E}''_\varepsilon(\omega+p+\eta_2+t(\eta_1-\eta_2))-\mathcal{E}''_0(\omega)\|_{W^{1,3/2}}\,dt\\
&\le \int_0^1 \|\mathcal{E}''_0(\omega+p+\eta_2+t(\eta_1-\eta_2))-\mathcal{E}''_0(\omega)\|_{W^{1,3/2}}\,dt\\
&\qquad + 2|\varepsilon|\int_0^1 \|\mathcal{V}''_H(\omega+p+\eta_2+t(\eta_1-\eta_2))\|_{W^{1,3/2}}\,dt\\
&\le \sup_{\|\eta\|_{W^{1,3}}\le 3\rho}\|\mathcal{E}''_0(\omega+\eta)-\mathcal{E}''_0(\omega)\|_{W^{1,3/2}}+2|\varepsilon|\sup_{\|\eta\|_{W^{1,3}}\le 3\rho}\|\mathcal{V}''_H(\omega+p+\eta)\|_{W^{1,3/2}}.
\end{aligned}$$

From (2.3), (2.4), and the Hölder inequality it follows that there exists a positive constant C_3 such that for any $\eta\in W^{1,3}$, $p\in\mathbb{R}^3$

$$\|\mathcal{V}'_H(\omega+p+\eta)\|_{W^{1,3}}\le C_3\left[\left(\int_{\mathbb{R}^2}|H(\omega+p+\eta)|^{3/2}|\nabla\omega|^3\mu^{-1}\right)^{2/3}+\|\eta\|^2_{W^{1,3}}\right] \tag{3.7}$$

and

$$\begin{aligned}
\|\mathcal{V}''_H(\omega+p+\eta)\|_{W^{1,3/2}}\le C_3\Bigg[&\left(\int_{\mathbb{R}^2}|H(\omega+p+\eta)|^2|\nabla(\omega+\eta)|^2\right)^{1/2}\\
&+\left(\int_{\mathbb{R}^2}|\nabla H(\omega+p+\eta)|^{3/2}|\nabla(\omega+\eta)|^3\mu^{-1}\right)^{2/3}\Bigg].
\end{aligned}\tag{3.8}$$

Choosing $\rho_0>0$ such that

$$C_2\sup_{\|\eta\|_{W^{1,3}}\le 3\rho_0}\left\|\mathcal{E}''_0(\omega+\eta)-\mathcal{E}''_0(\omega)\right\|_{W^{1,3/2}}<\frac{1}{2}$$

and $\varepsilon_0>0$ such that

$$8C_2C_3\varepsilon_0\|H\|_{L^\infty(\mathbb{R}^3)}\|\omega\|^2_{W^{1,3}}<\min\left\{1,\rho_0,\frac{1}{8C_2C_3\varepsilon_0}\right\}, \tag{3.9}$$

$$\sup_{\substack{\|\eta\|_{W^{1,3}}\le\rho_0\\ p\in\mathbb{R}^3}}\left\|\mathcal{V}'_H(\omega+p+\eta)\right\|_{W^{1,3}}<\frac{\rho_0}{6\varepsilon_0C_2}, \tag{3.10}$$

$$\sup_{\substack{\|\eta\|_{W^{1,3}}\le 3\rho_0\\ p\in\mathbb{R}^3}}\left\|\mathcal{V}''_H(\omega+p+\eta)\right\|_{W^{1,3/2}}<\frac{1}{8\varepsilon_0C_2}, \tag{3.11}$$

we obtain that $T_{\varepsilon,p}$ maps the ball $\overline{B_{\rho_0}(0)}$ into itself for any $|\varepsilon|<\varepsilon_0$, $p\in\mathbb{R}^3$, and is a contraction there.

Hence it has a unique fixed point $\big(\eta(\varepsilon,p),\lambda(\varepsilon,p),\alpha(\varepsilon,p)\big) \in \overline{B_{\rho_0}(0)}$. From (3.6) we have that the following property holds

$(*)$ $T_{\varepsilon,p}$ maps a ball $\overline{B_\rho(0)} \subset W^{1,3}\times\mathbb{R}^6\times\mathbb{R}^3$ into itself whenever $\rho \le \rho_0$

$$\text{and } \rho > 4|\varepsilon| C_2 \sup_{\|\eta\|_{W^{1,3}}\le\rho} \big\|\mathcal{V}_H'(\omega+p+\eta)\|_{W^{1,3}}.$$

In particular let us set

$$\rho_\varepsilon = 5|\varepsilon| C_2 \sup_{\substack{\|\eta\|_{W^{1,3}}\le\rho_0\\ p\in\mathbb{R}^3}} \big\|\mathcal{V}_H'(\omega+p+\eta)\|_{W^{1,3}}. \tag{3.12}$$

In view of (3.10) and (3.12), we have that for any $|\varepsilon|<\varepsilon_0$ and for any $p\in\mathbb{R}^3$

$$\rho_\varepsilon \le \rho_0 \quad\text{and}\quad \rho_\varepsilon > 4|\varepsilon| C_2 \sup_{\|\eta\|_{W^{1,3}}\le\rho_\varepsilon} \big\|\mathcal{V}_H'(\omega+p+\eta)\|_{W^{1,3}}$$

so that, due to $(*)$, $T_{\varepsilon,p}$ maps $\overline{B_{\rho_\varepsilon}(0)}$ into itself. From the uniqueness of the fixed point we have that for any $|\varepsilon|<\varepsilon_0$ and $p\in\mathbb{R}^3$

$$\|(\eta(\varepsilon,p),\lambda(\varepsilon,p),\alpha(\varepsilon,p))\|_{W^{1,3}\times\mathbb{R}^6\times\mathbb{R}^3} \le \rho_\varepsilon \le C_1|\varepsilon| \tag{3.13}$$

for some positive constant C_1 independent of p and hence $\|\eta(\varepsilon,p)\|_{W^{1,3}} \le \rho_\varepsilon \le C_1|\varepsilon|$ thus proving (3.4). Assume now (3.5) and set for any $p\in\mathbb{R}^3$

$$\rho_p = 8C_2C_3\varepsilon_0\bigg(\int_{\mathbb{R}^2}\sup_{|q-p|\le 1+C_0}|H(q)|^{3/2}|\nabla\omega|^3\mu^{-1}\bigg)^{2/3}$$

where C_0 is a positive constant such that $\|u\|_{L^\infty} \le C_0\|u\|_{W^{1,3}}$ for any $u\in W^{1,3}$. From (3.9) we have that

$$\rho_p < \min\bigg\{1,\rho_0,\frac{1}{8C_2C_3\varepsilon_0}\bigg\}.$$

Hence, due to (3.7), we have that for $|\varepsilon|<\varepsilon_0$ and $\|\eta\|_{W^{1,3}}\le\rho_p$

$$\begin{aligned}&4|\varepsilon|C_2\big\|\mathcal{V}_H'(\omega+p+\eta)\|_{W^{1,3}}\\ &\quad\le 4\varepsilon_0C_2C_3\bigg(\int_{\mathbb{R}^2}\sup_{|q-p|\le 1+C_0}|H(q)|^{3/2}|\nabla\omega|^3\mu^{-1}\bigg)^{2/3} + 4\varepsilon_0C_2C_3\rho_p^2 < \rho_p.\end{aligned}$$

From $(*)$ and the uniqueness of the fixed point, we deduce that

$$\|\eta(\varepsilon,p)\|_{W^{1,3}} \le \rho_p$$

for any $|\varepsilon|<\varepsilon_0$ and $p\in\mathbb{R}^3$. On the other hand, since H vanishes at ∞, by the definition of ρ_p we have that $\rho_p\to 0$ as $|p|\to\infty$, hence

$$\lim_{|p|\to\infty}\eta(\varepsilon,p) = 0 \quad\text{in } W^{1,3} \text{ uniformly for } |\varepsilon|<\varepsilon_0.$$

The proof of Lemma 3.1 is now complete. ∎

Remark 3.2 *The map η given in Lemma 3.1 satisfies*

$$\langle \mathcal{E}'_\varepsilon(\omega+p+\eta(\varepsilon,p)),\varphi\rangle - \sum_{i=1}^{6}\lambda_i(\varepsilon,p)\int_{\mathbb{R}^2}\nabla\varphi\cdot\nabla\tau_i + \alpha(\varepsilon,p)\cdot\int_{\mathbb{S}^2}\varphi, \quad \forall\varphi\in W^{1,3/2}$$

where $\big(\eta(\varepsilon,p),\lambda(\varepsilon,p),\alpha(\varepsilon,p)\big)\in\overline{B_{\rho_\varepsilon}(0)}\subset W^{1,3}\times\mathbb{R}^6\times\mathbb{R}^3$ being ρ_ε given in (3.12), hence

$$\begin{aligned}\int_{\mathbb{R}^2}\nabla(\omega+\eta(\varepsilon,p))\cdot\nabla\varphi + 2\int_{\mathbb{R}^2}\varphi\cdot(\omega+\eta(\varepsilon,p))_x\wedge(\omega+\eta(\varepsilon,p))_y\\ + 2\varepsilon\int_{\mathbb{R}^2}H(\omega+p+\eta(\varepsilon,p))\varphi\cdot(\omega+\eta(\varepsilon,p))_x\wedge(\omega+\eta(\varepsilon,p))_y\\ = \sum_{i=1}^{6}\lambda_i(\varepsilon,p)\int_{\mathbb{R}^2}\nabla\varphi\cdot\nabla\tau_i - \alpha(\varepsilon,p)\cdot\int_{\mathbb{S}^2}\varphi, \quad \forall\varphi\in W^{1,3/2},\end{aligned}$$

i.e. $\eta(\varepsilon,p)$ satisfies the equation

$$\Delta\eta(\varepsilon,p) = F(\varepsilon,p)$$

where

$$\begin{aligned}F(\varepsilon,p) = {}& 2(\omega+\eta(\varepsilon,p))_x\wedge(\omega+\eta(\varepsilon,p))_y - 2\omega_x\wedge\omega_y + \lambda(\varepsilon,p)\cdot\Delta\tau\\ & - \alpha(\varepsilon,p)\mu^2 + 2\varepsilon H(\omega+p+\eta(\varepsilon,p))(\omega+\eta(\varepsilon,p))_x\wedge(\omega+\eta(\varepsilon,p))_y\end{aligned}$$

in $\mathbb{R}^2$. Since $F(\varepsilon,p)\in L^{3/2}$ and, in view of (3.4) and (3.13), $F(\varepsilon,p)\to 0$ in $L^{3/2}$ as $\varepsilon\to 0$ uniformly with respect to p, by regularity we have that

$$\eta(\varepsilon,p)\in W^{2,3/2} \quad \text{and} \quad \eta(\varepsilon,p)\to 0 \text{ in } W^{2,3/2}$$

hence, by Sobolev embeddings, $F(\varepsilon,p)\in L^3$ and $F(\varepsilon,p)\to 0$ in L^3 as $\varepsilon\to 0$ uniformly with respect to p. Again by regularity

$$\eta(\varepsilon,p)\in W^{2,3} \quad \text{and} \quad \eta(\varepsilon,p)\to 0 \text{ in } W^{2,3}$$

hence $\eta(\varepsilon,p)\in C^{1,1/3}$ and

$$\eta(\varepsilon,p)\to 0 \quad \text{in } C^{1,1/3} \text{ as } \varepsilon\to 0 \text{ uniformly with respect to } p. \tag{3.14}$$

For any $\varepsilon\in(-\varepsilon_0,\varepsilon_0)$, let us define the perturbed manifold

$$Z_\varepsilon := \big\{\omega+p+\eta(\varepsilon,p):\ p\in\mathbb{R}^3\big\}.$$

From [7], we have that Z_ε is a natural constraint for $\mathcal{E}_\varepsilon$, namely any critical point $p\in\mathbb{R}^3$ of the functional

$$\Phi_\varepsilon:\mathbb{R}^3\to\mathbb{R}, \qquad \Phi_\varepsilon(p) = \mathcal{E}_\varepsilon\big(\omega+p+\eta(\varepsilon,p)\big)$$

gives rise to a critical point $u_\varepsilon = \omega+p+\eta(\varepsilon,p)$ of $\mathcal{E}_\varepsilon$.

Proposition 3.3 *Assume* $H \in C^2(\mathbb{R}^3)$, $\nabla H \in L^\infty(\mathbb{R}^3, \mathbb{R}^3)$, *and*

$$\lim_{|p|\to\infty} H(p) = 0.$$

Then for any $|\varepsilon| < \varepsilon_0$

$$\lim_{|p|\to\infty} \Phi_\varepsilon(p) = \text{const} = \mathcal{E}_0(\omega).$$

Proof. We have that

$$\begin{aligned}
\Phi_\varepsilon(p) &= \mathcal{E}_\varepsilon\big(\omega + p + \eta(\varepsilon,p)\big) \\
&= \mathcal{E}_0\big(\omega + p + \eta(\varepsilon,p)\big) + 2\varepsilon\mathcal{V}_H\big(\omega + p + \eta(\varepsilon,p)\big) \\
&= \frac{1}{2}\int_{\mathbb{R}^2} |\nabla\omega|^2 + \frac{1}{2}\int_{\mathbb{R}^2} |\nabla\eta(\varepsilon,p)|^2 + \int_{\mathbb{R}^2} \nabla\omega\cdot\nabla\eta(\varepsilon,p) \\
&\quad + \frac{2}{3}\int_{\mathbb{R}^2} \big(\omega + p + \eta(\varepsilon,p)\big)\cdot\big(\omega+\eta(\varepsilon,p)\big)_x \wedge \big(\omega+\eta(\varepsilon,p)\big)_y \\
&\quad + 2\varepsilon\big[\mathcal{V}_H(\omega+p) + \langle\mathcal{V}_H'(\omega+p), \eta(\varepsilon,p)\rangle + o\big(\|\eta(\varepsilon,p)\|_{W^{1,3}}\big)\big] \\
&= \frac{1}{2}\int_{\mathbb{R}^2} |\nabla\omega|^2 + \frac{2}{3}\int_{\mathbb{R}^2} \omega\cdot\omega_x\wedge\omega_y + \frac{1}{2}\int_{\mathbb{R}^2} |\nabla\eta(\varepsilon,p)|^2 \\
&\quad + \int_{\mathbb{R}^2} \nabla\omega\cdot\nabla\eta(\varepsilon,p) + \frac{2}{3}\int_{\mathbb{R}^2} \omega\cdot(\omega_x\wedge\eta(\varepsilon,p)_y + \eta(\varepsilon,p)_x\wedge\omega_y) \\
&\quad + \frac{2}{3}\int_{\mathbb{R}^2} \omega\cdot\eta(\varepsilon,p)_x\wedge\eta(\varepsilon,p)_y \\
&\quad + \frac{2}{3}\int_{\mathbb{R}^2} \eta(\varepsilon,p)\cdot(\omega+\eta(\varepsilon,p))_x\wedge(\omega+\eta(\varepsilon,p))_y \\
&\quad + 2\varepsilon\mathcal{V}_H(\omega+p) + 2\varepsilon\langle\mathcal{V}_H'(\omega+p),\eta(\varepsilon,p)\rangle + 2\varepsilon\, o\big(\|\eta(\varepsilon,p)\|_{W^{1,3}}\big)
\end{aligned} \tag{3.15}$$

where we have used the fact that

$$\int_{\mathbb{R}^2} p\cdot u_x\wedge u_y = 0 \quad \forall\, p\in\mathbb{R}^3,\ u\in W^{1,3},$$

(see [7], Lemma A.3). Notice that from Lemma 3.1 we have that

$$\int_{\mathbb{R}^2} |\nabla\eta(\varepsilon,p)|^2 \le \sqrt[3]{4\pi}\,\|\eta(\varepsilon,p)\|^2_{W^{1,3}} \underset{|p|\to\infty}{\longrightarrow} 0, \tag{3.16}$$

$$\left|\int_{\mathbb{R}^2} \nabla\omega\cdot\nabla\eta(\varepsilon,p)\right| \le \sqrt[6]{4\pi}\left(\int_{\mathbb{R}^2}|\nabla\omega|^2\right)^{1/2} \|\eta(\varepsilon,p)\|_{W^{1,3}} \underset{|p|\to\infty}{\longrightarrow} 0, \tag{3.17}$$

and, by the Hölder inequality and Lemma 3.1,

$$\left| \int_{\mathbb{R}^2} \omega \cdot (\omega_x \wedge \eta(\varepsilon,p)_y + \eta(\varepsilon,p)_x \wedge \omega_y) \right| \le 2\|\omega\|^2_{W^{1,3}} \|\eta(\varepsilon,p)\|_{W^{1,3}} \underset{|p|\to\infty}{\longrightarrow} 0, \tag{3.18}$$

$$\left| \int_{\mathbb{R}^2} \omega \cdot (\eta(\varepsilon,p)_x \wedge \eta(\varepsilon,p)_y) \right| \le \|\omega\|_{W^{1,3}} \|\eta(\varepsilon,p)\|^2_{W^{1,3}} \underset{|p|\to\infty}{\longrightarrow} 0, \tag{3.19}$$

and

$$\begin{aligned} &\left| \int_{\mathbb{R}^2} \eta(\varepsilon,p) \cdot (\omega + \eta(\varepsilon,p))_x \wedge (\omega + \eta(\varepsilon,p))_y \right| \\ &\qquad \le \|\omega + \eta(\varepsilon,p)\|^2_{W^{1,3}} \|\eta(\varepsilon,p)\|_{W^{1,3}} \underset{|p|\to\infty}{\longrightarrow} 0. \end{aligned} \tag{3.20}$$

Moreover the Gauss-Green Theorem yields

$$\mathcal{V}_H(\omega + p) = -\int_{B_1} H(\xi + p)\, d\xi$$

so that by the Dominated Convergence Theorem we have that

$$\lim_{|p|\to\infty} \mathcal{V}_H(\omega + p) = 0. \tag{3.21}$$

From (2.3), Hölder's inequality, and Lemma 3.1, we have that

$$\begin{aligned} |\langle \mathcal{V}_H'(\omega + p), \eta(\varepsilon,p)\rangle| &= \left| \int_{\mathbb{R}^2} H(\omega + p)\, \eta(\varepsilon,p) \cdot \omega_x \wedge \omega_y \right| \\ &\le \|H\|_{L^\infty(\mathbb{R}^3)} \|\omega\|^2_{W^{1,3}} \|\eta(\varepsilon,p)\|_{W^{1,3}} \underset{|p|\to\infty}{\longrightarrow} 0. \end{aligned} \tag{3.22}$$

From (3.15)–(3.22), it follows that

$$\lim_{|p|\to\infty} \Phi_\varepsilon(p) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla \omega|^2 + \frac{2}{3} \int_{\mathbb{R}^2} \omega \cdot \omega_x \wedge \omega_y = \mathcal{E}_0(\omega).$$

The proposition is thereby proved. ■

Proof of Theorem 1.1. As already observed at the beginning of Section 2, it is not restrictive to take $H_0 = 1$. From Proposition 3.3 it follows that for $|\varepsilon| < \varepsilon_0$ either Φ_ε is constant (and hence we have infinitely many critical points) or it has a global maximum or minimum point. In any case Φ_ε has a critical point. Since Z_ε is a natural constraint for $\mathcal{E}_\varepsilon$, we deduce the existence of a critical point of $\mathcal{E}_\varepsilon$ for $|\varepsilon| < \varepsilon_0$ and hence of a solution to (P_ε). The H_ε-bubble ω_ε found in this way is of the form $\omega + p^\varepsilon + \eta(\varepsilon, p^\varepsilon)$ for some $p^\varepsilon \in \mathbb{R}^3$ where η is as in Lemma 3.1. Remark 3.2 yields that ω_ε is closed in $C^{1,1/3}(\mathbb{S}^2, \mathbb{R}^3)$-norm to the manifold $\{\omega + p : p \in \mathbb{R}^3\}$ for ε small. Since ω has no branch points, we deduce that ω_ε has no branch points. ■

To prove Theorems 1.2 and 1.4, we need the following expansion for Φ_ε:

$$\Phi_\varepsilon(p) = \mathcal{E}_0(\omega) - 2\varepsilon\Gamma(p) + O(\varepsilon^2) \tag{3.23}$$

as $\varepsilon \to 0$ uniformly in $p \in \mathbb{R}^3$ (see [7]).

Proof of Theorem 1.2. Let $\varepsilon > 0$ small. Assumption $(H4)$ implies that $\Gamma(0) > 0$ and hence from (3.23) we have that for ε small

$$\Phi_\varepsilon(0) < \mathcal{E}_0(\omega),$$

whereas from assumption $(H3)$ we have that $\operatorname{Hess}\Gamma(0)$ is positive definite so that Γ has a strict local minimum in 0 and hence from (3.23) Φ_ε has a strict local maximum in $B_r(0)$ for some $r > 0$ such that

$$\Phi_\varepsilon(p) < \Phi_\varepsilon(0) - c_\varepsilon < \mathcal{E}_0(\omega)$$

for $|p| = r$, where c_ε is some positive constant depending on ε. In particular Φ_ε has a mountain pass geometry. Moreover by Theorem 1.1,

$$\Phi_\varepsilon(p) \to \mathcal{E}_0(\omega) \quad \text{as} \quad |p| \to \infty,$$

and so Φ_ε must have a global minimum point. If the minimum point and the mountain pass point coincide then Φ_ε has infinitely many critical points. Otherwise Φ_ε has at least three critical points: a local maximum point, a global minimum point, and a mountain pass. If $\varepsilon < 0$ we find the inverse inequalities and hence we find that Φ_ε has a local minimum point, a global maximum point, and a mountain pass. As a consequence (P_ε) has at least three solutions provided $|\varepsilon|$ is sufficiently small. ∎

As observed in Remark 1.3, if $H_1(0) > 0$ and $\operatorname{Hess} H_1(0)$ is positive definite, by continuity we have that for H_0 sufficiently large $\Gamma(0) > 0$ and $\operatorname{Hess}\Gamma(0)$ is positive definite, so that we can still prove the existence of three solutions arguing as above.

Proof of Theorem 1.4. Assumption $(H5)$ implies that $\Gamma(p_1) > 0$ and $\Gamma(p_2) < 0$. Since

$$\Phi_\varepsilon(p) = \mathcal{E}_0(\omega) + 2\varepsilon\big(- \Gamma(p) + o(1)\big) \quad \text{as } \varepsilon \to 0,$$

we have for ε sufficiently small

$$\Phi_\varepsilon(p_1) < \mathcal{E}_0(\omega) \quad \text{and} \quad \Phi_\varepsilon(p_2) > \mathcal{E}_0(\omega)$$

if $\varepsilon > 0$ and the inverse inequalities if $\varepsilon < 0$. Since by Theorem 1.1

$$\Phi_\varepsilon(p) \to \mathcal{E}_0(\omega) \quad \text{as} \quad |p| \to \infty,$$

we conclude that Φ_ε must have a global maximum point and a global minimum point in $\mathbb{R}^3$. Since Z_ε is a natural constraint for $\mathcal{E}_\varepsilon$, we deduce the existence of two critical points of $\mathcal{E}_\varepsilon$ for $|\varepsilon|$ sufficiently small and hence of two solutions to (P_ε). ■

As observed in Remark 1.5, if $H_1(p_1) > 0$ and $H_1(p_2) < 0$, by continuity we have that for H_0 sufficiently large $\Gamma(p_1) > 0$ and $\Gamma(p_2) < 0$, so that we can still prove the existence of two solutions arguing as above.

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