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Weighted Sobolev-Lieb-Thirring inequalities

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Abstract

We give a weighted version of the Sobolev-Lieb-Thirring inequality for suborthonormal functions. In the proof of our result we use φ-transform of Frazier-Jawerth.

1. Introduction

In 1994 Edmunds and Ilyin proved a generalization of the Sobolev-Lieb-Thirring inequality.

Theorem 1.1 ([2]). *Let $n \in \mathbb{N}, s > 0$ and p with*

$$\max\left(1, \frac{n}{2s}\right) < p \le 1 + \frac{n}{2s}.$$

Then there exists a positive constant $c = c(p, n, s)$ such that for every family $\{\phi_i\}_{i=1}^N$ in $H^s(\mathbb{R}^n)$ which is orthonormal in $L^2(\mathbb{R}^n)$, we have

$$\left\{\int_{\mathbb{R}^n} \rho(x)^{p/(p-1)}\, dx\right\}^{2s(p-1)/n} \le c \sum_{i=1}^N \|(-\Delta)^{s/2}\phi_i\|^2 \tag{1.1}$$

where

$$\rho(x) = \sum_{i=1}^N |\phi_i(x)|^2.$$

In this theorem $H^s(\mathbb{R}^n)$ denotes the Sobolev space of order s and $\|\cdot\|$ is the norm of $L^2(\mathbb{R}^n)$. In [8] Lieb and Thirring proved this theorem for $s = 1$ and applied it to the problem of the stability of matter.

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Ghidaglia, Marion and Temam proved (1.1) for $s \in \mathbb{N}$ under the suborthonormal condition on $\{\phi_i\}$, where $\{\phi_i\}_{i=1}^N$ in $L^2(\mathbb{R}^n)$ is called suborthonormal if the inequality

$$\sum_{i,j=1}^{N} \xi_i \overline{\xi_j} (\phi_i, \phi_j) \leq \sum_{i=1}^{N} |\xi_i|^2$$

holds for all $\xi_i \in \mathbb{C}, i = 1, \ldots, N$ ([4]). They applied the inequality (1.1) to the estimate of the dimension of attractors associated with partial differential equations(c.f. [13]). In this paper we shall give a weighted version of (1.1) under suborthonormal condition on $\{\phi_i\}$. In the proof of our theorem we shall use Frazier-Jawerth's φ-transform ([3]).

For the statement of our result we need to recall the definition of A_p-weights (c.f. [5], [10]). By a cube in $\mathbb{R}^n$ we mean a cube which sides are parallel to coordinate axes. Let w be a non-negative, locally integrable function on $\mathbb{R}^n$. We say that w is an A_p-weight for $1 < p < \infty$ if there exists a positive constant C such that

$$\frac{1}{|Q|} \int_Q w(x)\, dx \left(\frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)} dx \right)^{p-1} \leq C$$

for all cubes $Q \subset \mathbb{R}^n$. The infimum of the constant C is called the A_p-constant of w. For example, $w(x) = |x|^\alpha$ is an A_p-weight when $-n < \alpha < n(p-1)$.

We say that w is an A_1-weight if there exists a positive constant C such that

$$\frac{1}{|Q|} \int_Q w(y)\, dy \leq C w(x) \qquad a.e.\ x \in Q$$

for all cubes $Q \subset \mathbb{R}^n$. The infimum of the constant C is called the A_1-constant of w. Let A_p be the class of A_p-weights. The inclusion $A_p \subset A_q$ holds for $p < q$.

For a nonnegative, locally integrable function w on $\mathbb{R}^n$ we define

$$L^p(w) = \left\{ f :\ \text{measurable on } \mathbb{R}^n, \int_{\mathbb{R}^n} |f(x)|^p w(x)\, dx < \infty \right\}.$$

For $\nu \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$ the cube Q defined by

$$Q = Q_{\nu k} = \{(x_1, \ldots, x_n) \,:\, k_i \leq 2^\nu x_i < k_i + 1,\ i = 1, \ldots, n\}$$

is called a dyadic cube in $\mathbb{R}^n$. Let $\mathcal{Q}$ be the set of all dyadic cubes in $\mathbb{R}^n$. For any $Q \in \mathcal{Q}$ there exists a unique $Q' \in \mathcal{Q}$ such that $Q \subset Q'$ and the side-length of Q' is double of that of Q. We call Q' the parent of Q.

For $s > 0$ and $f \in C_0^\infty(\mathbb{R}^n)$ we define via inverse Fourier transform

$$(-\Delta)^{s/2} f(x) = \mathcal{F}^{-1}(|\xi|^s \hat{f})(x).$$

Let $w \in A_2$ and $\mathcal{H}^s(w)$ be the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|f\|_{\mathcal{H}^s(w)} = \left\{ \int_{\mathbb{R}^n} |(-\Delta)^{s/2} f(x)|^2 w(x)\, dx + \|f\|^2 \right\}^{1/2}.$$

We remark that for $f \in C_0^\infty(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} |(-\Delta)^{s/2} f(x)|^2 w(x)\, dx < \infty$$

because

$$|(-\Delta)^{s/2} f(x)| \le \frac{c}{(1+|x|)^n} \quad (x \in \mathbb{R}^n)$$

and

$$\int_{\mathbb{R}^n} \frac{w(x)}{(1+|x|)^{2n}}\, dx < \infty$$

(c.f. [10, p. 209]).

Let $f \in \mathcal{H}^s(w)$ and $\{f_i\}_{i=1}^\infty$ be a sequence in $C_0^\infty(\mathbb{R}^n)$ such that

$$\|f - f_i\|_{\mathcal{H}^s(w)} \to 0 \ (i \to \infty).$$

This means that there exist $g_1 \in L^2(\mathbb{R}^n)$ and $g_2 \in L^2(w)$ such that

$$\|g_1 - f_i\| \to 0 \qquad \text{and} \qquad \int_{\mathbb{R}^n} \left| g_2(x) - (-\Delta)^{s/2} f_i(x) \right|^2 w(x)\, dx \to 0$$

as $i \to \infty$. We denote $(-\Delta)^{s/2} f = g_2$. We remark that $g_1 \equiv 0$ means $g_2 \equiv 0$. In fact, for any $\varphi \in C_0^\infty(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n} g_2 \overline{\varphi}\, dx = \lim_{i\to\infty} \int_{\mathbb{R}^n} (-\Delta)^{s/2} f_i \overline{\varphi}\, dx = \lim_{i\to\infty} \int_{\mathbb{R}^n} f_i \overline{(-\Delta)^{s/2} \varphi}\, dx = 0.$$

Hence we have $g_2 \equiv 0$. This means that we can identify $\mathcal{H}^s(w)$ as a subspace of $L^2(\mathbb{R}^n)$.

The following is the main result of this paper.

Theorem 1.2. *Let $n \in \mathbb{N}, s > 0,$ and*

$$\max\left(1, \frac{n}{2s}\right) < p \le 1 + \frac{n}{2s}.$$

Let $w \in A_2$. If $2s < n$, then we assume that $w^{-n/(2s)} \in A_{n/(2s)}$. If $2s \ge n$, then we assume that $w^{-n/(2s)} \in A_p$ and

$$\int_{Q'} w\, dx \le 2^{2s} \int_Q w\, dx \tag{1.2}$$

for all dyadic cubes $Q \in \mathcal{Q}$ and its parent Q'.

Then there exists a positive constant c such that for every family $\{\phi_i\}_{i=1}^N$ *in* $\mathcal{H}^s(w)$ *which is suborthonormal in* $L^2(\mathbb{R}^n)$*, we have*

$$\left\{\int_{\mathbb{R}^n} \rho(x)^{p/(p-1)} w(x)^{n/(2s(p-1))}\, dx\right\}^{2s(p-1)/n} \leq c \sum_{i=1}^N \int_{\mathbb{R}^n} \left|(-\Delta)^{s/2}\phi_i(x)\right|^2 w(x)\, dx,$$

where

$$\rho(x) = \sum_{i=1}^N |\phi_i(x)|^2$$

and c depends only on n, s, p, A_2*-constant of* w*, and* $A_{n/(2s)}$ *or* A_p*-constant of* $w^{-n/(2s)}$.

When $2s < n$, an example of weight function w is given by $w(x) = |x|^\alpha$ for $-n + 2s < \alpha < 2s$. When $2s > n$, an example of weight function w is given by $w(x) = |x|^\alpha$ for $0 \leq \alpha < \min\{2s - n, n\}$(c.f. [12, Section 4]). When $2s = n$, the condition (1.2) means w is equivalent to a constant almost everywhere(c.f.[12, Proposition 4.1]).

2. Preliminaries

Let ψ be a function which satisfies the following conditions.

(A1) $\psi \in \mathcal{S}(\mathbb{R}^n)$.

(A2) $\operatorname{supp}\hat{\psi} \subset \{\xi \in \mathbb{R}^n : \frac{1}{2} \leq |\xi| \leq 2\}$

(A3) $|\hat{\psi}(\xi)| \geq c > 0$ if $\frac{3}{5} \leq |\xi| \leq \frac{5}{3}$.

(A4) $\sum_{\nu \in \mathbb{Z}} |\hat{\psi}(2^\nu \xi)|^2 = 1$ for all $\xi \neq 0$.

For $\nu \in \mathbb{Z}, k \in \mathbb{Z}^n$ and $Q = Q_{\nu k}$, we set

$$\psi_Q(x) = 2^{\nu n/2}\psi(2^\nu x - k) \quad (x \in \mathbb{R}^n).$$

Let M be the Hardy-Littlewood maximal operator, that is,

$$M(f)(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)|\, dy,$$

where f is a locally integrable function on $\mathbb{R}^n$ and the supremum is taken over all cubes Q which contain x.

Proposition 2.1. *(i) Let $1 < p < \infty$ and w be a non-negative locally integrable function on $\mathbb{R}^n$. Then there exists a positive constant c such that*

$$\int_{\mathbb{R}^n} M(f)^p w\,dx \le c \int_{\mathbb{R}^n} |f|^p w\,dx$$

for all $f \in L^p(w)$ if and only if $w \in A_p$. The constant c depends only on n, p and A_p-constant of w.

(ii) Let $1 < p < \infty$ and $w \in A_p$. Then there exists a $q \in (1,p)$ such that $w \in A_q$.

(iii) Let $0 < \tau < 1$ and f be a locally integrable function on $\mathbb{R}^n$ such that $M(f)(x) < \infty$ a.e.. Then $(M(f))^\tau \in A_1$ and the A_1-constant of $(M(f))^\tau$ depends only on n and τ.

(iv) Let $1 \le p < \infty$ and $w \in A_p$. Then there exists a positive constant c such that

$$\int_{2Q} w\,dx \le c \int_Q w\,dx$$

for all cubes $Q \in \mathbb{R}^n$, where $2Q$ denotes the double of Q and c depend only on n and A_p-constant of w.

The proofs of these facts are in [5, Chapter IV] or [10, Chapter V].

3. Proof of Theorem 1.2

The suborthonormal condition on $\{\phi_i\}$ is equivalent to the inequality

$$\sum_{i=1}^N |(\phi_i, f)|^2 \le \|f\|^2$$

for all $f \in L^2(\mathbb{R}^n)$ (c.f.[1, p57]). We shall prove the inequality

$$\left\{ \int_{\mathbb{R}^n} \rho(x)^{p/(p-1)} w(x)^{n/(2s(p-1))}\,dx \right\}^{2s(p-1)/n} \le cK^{2sp/n-1} \sum_{i=1}^N \int_{\mathbb{R}^n} \left|(-\Delta)^{s/2}\phi_i(x)\right|^2 w(x)\,dx \tag{3.1}$$

under the assumption

$$\sum_{i=1}^N |(\phi_i, f)|^2 \le K\|f\|^2 \tag{3.2}$$

for all $f \in L^2(\mathbb{R}^n)$ where K is a positive constant.

This is equivalent to the statement of Theorem 1.2. We remark that K may depend on $\{\phi_i\}$. For example, the inequality (3.1) says that

$$\left\{\int_{\mathbb{R}^n} |\phi|^{2p/(p-1)} w^{n/(2s(p-1))}\, dx\right\}^{2s(p-1)/n} \leq c\|\phi\|^{4sp/n-2} \int_{\mathbb{R}^n} \left|(-\Delta)^{s/2}\phi\right|^2 w dx \tag{3.3}$$

holds for all $\phi \in \mathcal{H}^s(w)$ under suitable condition on s, p, n and w because

$$|(\phi, f)|^2 \leq \|\phi\|^2 \|f\|^2$$

for all $f \in L^2(\mathbb{R}^n)$.

First we assume $\phi_i \in C_0^\infty(\mathbb{R}^n), i = 1, \ldots, N$. Let

$$V(x) = \delta_1 \rho(x)^{1/(p-1)} w(x)^{n/(2s(p-1))}$$

where the value of the constant $\delta_1 > 0$ will be given later. Since

$$\rho(x) = \sum_{i=1}^N |\phi_i(x)|^2$$

is a bounded function with compact support and $w^{n/(2s(p-1))}$ is locally integrable by the assumption $w^{-n/(2s)} \in A_p$, we have

$$\int_{\mathbb{R}^n} V^p w^{-n/(2s)}\, dx < \infty.$$

We may also assume that

$$0 < \int_{\mathbb{R}^n} V^p w^{-n/(2s)}\, dx.$$

By (ii) of Proposition 2.1 there exists a constant κ such that

$$1 < \kappa < p \qquad \text{and} \qquad w^{-n/(2s)} \in A_{p/\kappa}.$$

We set

$$v(x) = M(V^\kappa)(x)^{1/\kappa}.$$

Then (i) of Proposition 2.1 leads to

$$\int_{\mathbb{R}^n} v^p w^{-n/(2s)} dx = \int_{\mathbb{R}^n} M(V^\kappa)^{p/\kappa} w^{-n/(2s)}\, dx \leq c_1 \int_{\mathbb{R}^n} V^p w^{-n/(2s)}\, dx < \infty. \tag{3.4}$$

Furthermore v is an A_1-weight by (iii) of Proposition 2.1.

We have the following lemmas.

Lemma 3.1. *For $s > 0$ and $w \in A_2$ there exists a positive constant α such that*

$$\alpha \sum_{Q\in\mathcal{Q}} |Q|^{-2s/n}|(f,\psi_Q)|^2 \frac{1}{|Q|}\int_Q w\,dx \le \int_{\mathbb{R}^n} \left|(-\Delta)^{s/2} f\right|^2 w\,dx$$

for all $f \in C_0^\infty(\mathbb{R}^n)$, where α is given by

$$\alpha^{-1} = c \max_{|\sigma|\le n} \|\partial^\sigma \hat{\psi}\|_\infty^2$$

and c is a constant depending only on n, s and A_2-constant of w.

Lemma 3.2. *For $v \in A_2$ there exist positive constants β and β' such that*

$$\beta' \sum_{Q\in\mathcal{Q}} |(f,\psi_Q)|^2 \frac{1}{|Q|}\int_Q v\,dx \le \int_{\mathbb{R}^n} |f|^2 v\,dx \le \beta \sum_{Q\in\mathcal{Q}} |(f,\psi_Q)|^2 \frac{1}{|Q|}\int_Q v\,dx$$

for all $f \in C_0^\infty(\mathbb{R}^n)$, where β is given by

$$\beta = c \max_{|\sigma|\le n} \|\partial^\sigma \hat{\psi}\|_\infty^2$$

and c is a constant depending only on n and A_2-constant of v.

The proof of Lemmas 3.1 and 3.2 are in [11, Prop. 2.2 and Lemma 3.2]. We shall give the proof in Section 5 for the reader's convenience because the dependence of ψ in α and β is not explained in [11].

For $f \in C_0^\infty(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} |f|^2 V\,dx \le \int_{\mathbb{R}^n} |f|^2 v\,dx \le \beta \sum_{Q\in\mathcal{Q}} |(f,\psi_Q)|^2 \frac{1}{|Q|}\int_Q v\,dx,$$

where we used Lemma 3.2. Hence by Lemma 3.1

$$\begin{aligned} &\int_{\mathbb{R}^n} \left|(-\Delta)^{s/2} f\right|^2 w\,dx - \int_{\mathbb{R}^n} V|f|^2\,dx \\ &\ge \alpha \sum_{Q\in\mathcal{Q}} |Q|^{-2s/n}|(f,\psi_Q)|^2 \frac{1}{|Q|}\int_Q w\,dx - \beta \sum_{Q\in\mathcal{Q}} |(f,\psi_Q)|^2 \frac{1}{|Q|}\int_Q v\,dx. \end{aligned} \tag{3.5}$$

Now we set

$$\mathcal{I} = \left\{ Q \in \mathcal{Q} \,:\, \beta \int_Q v\,dx > \alpha |Q|^{-2s/n} \int_Q w\,dx \right\}. \tag{3.6}$$

Let $\{\mu_k\}_{1\le k}$ be the non-decreasing rearrangement of

$$\left\{ \alpha |Q|^{-2s/n-1} \int_Q w\,dx - \beta |Q|^{-1} \int_Q v\,dx \right\}_{Q\in\mathcal{I}}.$$

We will show that this rearrangement is possible in the proof of Lemma 3.3.

When
$$\mu_k = \alpha|Q|^{-2s/n-1}\int_Q w\,dx - \beta|Q|^{-1}\int_Q v\,dx,$$
we define $\Psi_k = \psi_Q$. Then we have by (3.5)

$$\begin{aligned}
(3.7)\quad & \sum_{i=1}^N \int_{\mathbb{R}^n} |(-\Delta)^{s/2}\phi_i|^2 w\,dx - \sum_{i=1}^N \int_{\mathbb{R}^n} V|\phi_i|^2\,dx \\
&\ge \sum_{i=1}^N \sum_{Q\in\mathcal{Q}} |(\phi_i,\psi_Q)|^2 \left\{\alpha|Q|^{-2s/n-1}\int_Q w\,dx - \beta|Q|^{-1}\int_Q v\,dx\right\} \\
&\ge \sum_{i=1}^N \sum_k \mu_k|(\phi_i,\Psi_k)|^2 = \sum_k \mu_k \sum_{i=1}^N |(\phi_i,\Psi_k)|^2 \\
(3.8)\quad &\ge -K\|\psi\|^2 \sum_k |\mu_k| \;\ge -K\|\psi\|^2 \left(\sum_k |\mu_k|^\gamma\right)^{1/\gamma},
\end{aligned}$$

where $\gamma = p - n/(2s) \in (0,1]$ and we used (3.2).

Now the following lemma holds.

Lemma 3.3.
$$\sum_k |\mu_k|^\gamma \le c\int_{\mathbb{R}^n} v^p w^{-n/(2s)}\,dx,$$
where c is given by
$$c = c' \max_{|\sigma|\le n} \|\partial^\sigma \hat{\psi}\|_\infty^{n/s+2p}$$
and c' depends only on n, s, p and w.

The proof of this lemma will be given in Section 4. By Lemma 3.3 and (3.4) the last quantity in (3.8) is estimated from below by

$$\begin{aligned}
-cK&\left(\int_{\mathbb{R}^n} V^p w^{-n/(2s)}\,dx\right)^{1/\gamma} \\
&= -cK\delta_1^{p/(p-n/(2s))}\left(\int_{\mathbb{R}^n} \rho^{p/(p-1)} w^{n/(2s(p-1))}\,dx\right)^{1/(p-n/(2s))},
\end{aligned}$$

where
$$c = c'\|\psi\|^2 \max_{|\sigma|\le n} \|\partial^\sigma \hat{\psi}\|_\infty^{(4ps+2n)/(2ps-n)}$$
and c' depends only on n, s, p and w. We may take the infimum of the above constant with respect to possible ψ and replace c by this infimum.

Let

$$\delta_1 = \delta_2 K^{1-2sp/n} \left(\int_{\mathbb{R}^n} \rho^{p/(p-1)} w^{n/(2s(p-1))}\, dx \right)^{2s(p-n/(2s)-1)/n},$$

where δ_2 is a positive constant. Then we have by (3.7)

$$\begin{aligned}
&\sum_{i=1}^{N} \int_{\mathbb{R}^n} |(-\Delta)^{s/2}\phi_i|^2 w\, dx \\
&\quad \geq \delta_2 K^{1-2sp/n} \left(\int_{\mathbb{R}^n} \rho^{p/(p-1)} w^{n/(2s(p-1))}\, dx \right)^{2s(p-1)/n} \\
&\quad\quad - cK\delta_2^{p/(p-n/(2s))} K^{-2sp/n} \left(\int_{\mathbb{R}^n} \rho^{p/(p-1)} w^{n/(2s(p-1))}\, dx \right)^{2s(p-1)/n} \\
&\quad = \{\delta_2 - c\delta_2^{p/(p-n/(2s))}\} K^{1-2sp/n} \left(\int_{\mathbb{R}^n} \rho^{p/(p-1)} w^{n/(2s(p-1))}\, dx \right)^{2s(p-1)/n}.
\end{aligned}$$

If we take δ_2 small enough, then we get the inequality (3.1) because $1 < p/(p-n/(2s))$.

Next we shall show (3.1) for $\phi_i \in \mathcal{H}^s(w)$, $i = 1, \dots, N$. First we show

$$\mathcal{H}^s(w) \subset L^{2p/(p-1)}(w^{n/(2s(p-1))}). \tag{3.9}$$

Let $h \in \mathcal{H}^s(w)$. Then there exists a sequence $\{h_m\}_{m=1}^{\infty} \subset C_0^{\infty}(\mathbb{R}^n)$ such that $\|h - h_m\|_{\mathcal{H}^s(w)} \to 0$ $(m \to \infty)$. Since we proved that (3.3) holds for $h_m \in C_0^{\infty}(\mathbb{R}^n)$, we get

$$\left\{ \int_{\mathbb{R}^n} |h_m|^{2p/(p-1)} w^{n/(2s(p-1))}\, dx \right\}^{2s(p-1)/n} \leq c\|h_m\|^{4sp/n-2} \int_{\mathbb{R}^n} \left|(-\Delta)^{s/2} h_m\right|^2 w\, dx,$$

where c does not depend on h_m. Since $4sp/n - 2 > 0$ and $\{h_m\}$ is a Cauchy sequence in $\mathcal{H}^s(w)$, the above inequality says that $\{h_m\}$ is a Cauchy sequence in $L^{2p/(p-1)}(w^{n/(2s(p-1))})$. Let g be the limit of $\{h_m\}$ in $L^{2p/(p-1)}(w^{n/(2s(p-1))})$. For any compact set E in $\mathbb{R}^n$ we have

$$\begin{aligned}
\int_E |g - h_m|\, dx \leq &\left(\int_E |g - h_m|^{2p/(p-1)} w^{n/(2s(p-1))}\, dx \right)^{(p-1)/(2p)} \\
&\times \left(\int_E w^{-n/(2s(p+1))}\, dx \right)^{(p+1)/(2p)}.
\end{aligned}$$

Since $w^{-n/(2s)}$ is locally integrable by the assumption $w^{-n/(2s)} \in A_{n/(2s)}$ or $w^{-n/(2s)} \in A_p$, we get $h_m \to g$ in $L^1_{loc}(\mathbb{R}^n)$ as $m \to \infty$. Hence we have $g = h$ and (3.9).

Furthermore we have

$$\left\{\int_{\mathbb{R}^n}|h|^{2p/(p-1)}w^{n/(2s(p-1))}\,dx\right\}^{2s(p-1)/n}\le c\|h\|^{4sp/n-2}\int_{\mathbb{R}^n}\left|(-\Delta)^{s/2}h\right|^2 w\,dx. \tag{3.10}$$

We fix a positive number ε. Let $\chi_1,\dots,\chi_N$ be functions in $C_0^\infty(\mathbb{R}^n)$ such that

$$\sum_{i=1}^N\|\phi_i-\chi_i\|^2_{\mathcal{H}^s(w)}<\varepsilon.$$

Now the inequalities

$$\begin{aligned}\sum_{i=1}^N|(\chi_i,f)|^2&\le 2\sum_{i=1}^N|(\chi_i-\phi_i,f)|^2+2\sum_{i=1}^N|(\phi_i,f)|^2\\&\le 2\sum_{i=1}^N\|\chi_i-\phi_i\|^2\|f\|^2+2K\|f\|^2\le 2(K+\varepsilon)\|f\|^2\end{aligned} \tag{3.11}$$

hold for all $f\in L^2(\mathbb{R}^n)$. On the other hand

$$\begin{aligned}&\left\{\int_{\mathbb{R}^n}\left(\sum_{i=1}^N|\phi_i-\chi_i|^2\right)^{p/(p-1)}w^{n/(2s(p-1))}\,dx\right\}^{2s(p-1)/n}\\&\quad\le\left\{\sum_{i=1}^N\left(\int_{\mathbb{R}^n}|\phi_i-\chi_i|^{2p/(p-1)}w^{n/(2s(p-1))}\,dx\right)^{(p-1)/p}\right\}^{2sp/n}\\&\quad\le N^{2sp/n-1}\sum_{i=1}^N\left(\int_{\mathbb{R}^n}|\phi_i-\chi_i|^{2p/(p-1)}w^{n/(2s(p-1))}\,dx\right)^{2s(p-1)/n}\\&\quad\le cN^{2sp/n-1}\sum_{i=1}^N\|\phi_i-\chi_i\|^{4sp/n-2}\int_{\mathbb{R}^n}\left|(-\Delta)^{s/2}\phi_i-(-\Delta)^{s/2}\chi_i\right|^2w\,dx\\&\quad\le cN^{2sp/n-1}\varepsilon^{2sp/n-1}\sum_{i=1}^N\int_{\mathbb{R}^n}\left|(-\Delta)^{s/2}\phi_i-(-\Delta)^{s/2}\chi_i\right|^2w\,dx\\&\quad\le cN^{2sp/n-1}\varepsilon^{2sp/n},\end{aligned}$$

where we used (3.10). Therefore

$$\begin{aligned}&\left\{\int_{\mathbb{R}^n}\left(\sum_{i=1}^N|\phi_i|^2\right)^{p/(p-1)}w^{n/(2s(p-1))}\,dx\right\}^{2s(p-1)/n}\\&\quad\le\left\{\int_{\mathbb{R}^n}\left(2\sum_{i=1}^N|\phi_i-\chi_i|^2+2\sum_{i=1}^N|\chi_i|^2\right)^{p/(p-1)}w^{n/(2s(p-1))}\,dx\right\}^{2s(p-1)/n}\end{aligned}$$

$$
\begin{aligned}
&\le 2^{2sp/n}\Bigg[\bigg\{\int_{\mathbb{R}^n}\Big(\sum_{i=1}^N|\phi_i-\chi_i|^2\Big)^{p/(p-1)} w^{n/(2s(p-1))}\,dx\bigg\}^{(p-1)/p} \\
&\qquad + \bigg\{\int_{\mathbb{R}^n}\Big(\sum_{i=1}^N|\chi_i|^2\Big)^{p/(p-1)} w^{n/(2s(p-1))}\,dx\bigg\}^{(p-1)/p}\Bigg]^{2sp/n} \\
&\le 2^{4sp/n-1}\bigg\{\int_{\mathbb{R}^n}\Big(\sum_{i=1}^N|\phi_i-\chi_i|^2\Big)^{p/(p-1)} w^{n/(2s(p-1))}\,dx\bigg\}^{2s(p-1)/n} \\
&\qquad + 2^{4sp/n-1}\bigg\{\int_{\mathbb{R}^n}\Big(\sum_{i=1}^N|\chi_i|^2\Big)^{p/(p-1)} w^{n/(2s(p-1))}\,dx\bigg\}^{2s(p-1)/n} \\
&\le c2^{4sp/n-1}N^{2sp/n-1}\varepsilon^{2sp/n} \\
&\qquad + c2^{6sp/n-2}(K+\varepsilon)^{2sp/n-1}\sum_{i=1}^N\int_{\mathbb{R}^n}|(-\Delta)^{s/2}\chi_i|^2 w\,dx,
\end{aligned} \tag{3.12}
$$

where we used (3.11) and (3.1) for χ_i. Since

$$
\begin{aligned}
&\sum_{i=1}^N\int_{\mathbb{R}^n}|(-\Delta)^{s/2}\chi_i|^2 w\,dx \\
&\quad\le 2\sum_{i=1}^N\int_{\mathbb{R}^n}\big|(-\Delta)^{s/2}\chi_i-(-\Delta)^{s/2}\phi_i\big|^2 w\,dx + 2\sum_{i=1}^N\int_{\mathbb{R}^n}\big|(-\Delta)^{s/2}\phi_i\big|^2 w\,dx \\
&\quad\le 2\varepsilon + 2\sum_{i=1}^N\int_{\mathbb{R}^n}\big|(-\Delta)^{s/2}\phi_i\big|^2 w\,dx,
\end{aligned}
$$

we have by (3.12)

$$
\begin{aligned}
&\bigg\{\int_{\mathbb{R}^n}\Big(\sum_{i=1}^N|\phi_i|^2\Big)^{p/(p-1)} w^{n/(2s(p-1))}\,dx\bigg\}^{2s(p-1)/n} \\
&\qquad\le c2^{4sp/n-1}N^{2sp/n-1}\varepsilon^{2sp/n} + c2^{6sp/n-1}(K+\varepsilon)^{2sp/n-1}\varepsilon \\
&\qquad\quad + c2^{6sp/n-1}(K+\varepsilon)^{2sp/n-1}\sum_{i=1}^N\int_{\mathbb{R}^n}|(-\Delta)^{s/2}\phi_i|^2 w\,dx.
\end{aligned}
$$

Since we can take ε arbitrary small, we conclude

$$
\begin{aligned}
&\bigg\{\int_{\mathbb{R}^n}\Big(\sum_{i=1}^N|\phi_i|^2\Big)^{p/(p-1)} w^{n/(2s(p-1))}\,dx\bigg\}^{2s(p-1)/n} \\
&\qquad\le c2^{6sp/n-1}K^{2sp/n-1}\sum_{i=1}^N\int_{\mathbb{R}^n}|(-\Delta)^{s/2}\phi_i|^2 w\,dx.
\end{aligned}
$$

Hence we get (3.1).

4. Proof of Lemma 3.3

The arguments of the proof are similar to those in [11] and [12]. First we consider the case $n > 2s$. For $\lambda > 0$ we set

$$\mathcal{I}_\lambda = \{Q \in \mathcal{Q} \,:\, \alpha|Q|^{-2s/n-1}\int_Q w\,dx - \beta|Q|^{-1}\int_Q v\,dx < -\lambda\}. \tag{4.1}$$

Then we have for $Q \in \mathcal{I}_\lambda$

$$\alpha|Q|^{-2s/n-1}\int_Q w\,dx < |Q|^{-1}\int_Q (\beta v - \lambda)_+\,dx,$$

where

$$(\beta v - \lambda)_+(x) = \max\{0, \beta v(x) - \lambda\}.$$

Since $p = n/(2s) + \gamma, \gamma \in (0,1]$, and

$$\beta^{-p}\gamma\int_0^\infty \int_{\beta v > \lambda} (\beta v - \lambda)^{n/(2s)} w^{-n/(2s)}\,dx \lambda^{\gamma-1} d\lambda \le \int_{\mathbb{R}^n} v^p w^{-n/(2s)}\,dx < \infty,$$

we have

$$\int_{\mathbb{R}^n} (\beta v - \lambda)_+^{n/(2s)} w^{-n/(2s)}\,dx < \infty$$

for all $\lambda > 0$. By the assumption $w^{-n/(2s)} \in A_{n/(2s)}$ and (ii) of Proposition 2.1, there exists a $\kappa' \in (1, n/(2s))$ such that $w^{-n/(2s)} \in A_{n/(2s\kappa')}$. We set

$$v_\lambda^*(x) = M((\beta v - \lambda)_+^{\kappa'})(x)^{1/\kappa'}.$$

Then

$$\int_{\mathbb{R}^n} (v_\lambda^*)^{n/(2s)} w^{-n/(2s)}\,dx \le c_1 \int_{\mathbb{R}^n} (\beta v - \lambda)_+^{n/(2s)} w^{-n/(2s)}\,dx < \infty \tag{4.2}$$

and $v_\lambda^* \in A_1$ by (iii) of Proposition 2.1, where c_1 depends only on n, s and $A_{n/(2s)}$-constant of $w^{-n/(2s)}$.

We can show that $\mathcal{I}_\lambda$ is a finite set as follows. Let $Q \in \mathcal{I}_\lambda$. Then we have

$$\begin{aligned}\alpha|Q|^{-2s/n}\int_Q w\,dx &\le \int_Q v_\lambda^*\,dx \\ &\le \left\{\int_Q (v_\lambda^*)^{n/(2s)} w^{-n/(2s)}\,dx\right\}^{2s/n}\left\{\int_Q w^{n/(n-2s)}\,dx\right\}^{(n-2s)/n}.\end{aligned}$$

Since $w^{-n/(2s)} \in A_{n/(2s)}$, the last quantity is bounded by

$$c_2\left\{\int_Q (v_\lambda^*)^{n/(2s)} w^{-n/(2s)}\,dx\right\}^{2s/n} |Q| \left(\int_Q w^{-n/(2s)}\,dx\right)^{-2s/n} \\ \leq c_2\left\{\int_Q (v_\lambda^*)^{n/(2s)} w^{-n/(2s)}\,dx\right\}^{2s/n} |Q|^{-2s/n}\int_Q w\,dx,$$

where we used the inequality

$$1 \leq \frac{1}{|Q|}\int_Q w\,dx \left(\frac{1}{|Q|}\int_Q w^{-n/(2s)}\,dx\right)^{2s/n}.$$

The above calculation says

$$1 \leq c_3 \int_Q (v_\lambda^*)^{n/(2s)} w^{-n/(2s)}\,dx,$$

where $c_3 = c'\alpha^{-n/(2s)}$ and c' is the $A_{n/(2s)}$-constant of $w^{-n/(2s)}$.

First we assume that $\mathcal{I}_\lambda$ includes infinite disjoint cubes $\{Q_i\}_{i=1}^\infty$. Then we have

$$\infty = \sum_{i=1}^\infty 1 \leq \sum_{i=1}^\infty c_3 \int_{Q_i} (v_\lambda^*)^{n/(2s)} w^{-n/(2s)}\,dx \leq c_3 \int_{\mathbb{R}^n} (v_\lambda^*)^{n/(2s)} w^{-n/(2s)}\,dx < \infty.$$

This is a contradiction. Hence $\mathcal{I}_\lambda$ does not include infinite disjoint cubes.

Next we assume that there exist infinite cubes $\{Q_i\}_{i=1}^\infty \subset \mathcal{I}_\lambda$ such that $Q_i \neq Q_j$ $(i \neq j)$ and $Q_1 \subset Q_2 \subset Q_3 \subset \cdots$. Let $\tilde{Q}_i$ be a half size dyadic sub-cube of Q_{i+1} such that $Q_i \cap \tilde{Q}_i = \emptyset$. Since $Q_{i+1} \in \mathcal{I}_\lambda$, we have

$$\alpha |Q_{i+1}|^{-2s/n} \int_{Q_{i+1}} w\,dx \leq \int_{Q_{i+1}} v_\lambda^*\,dx.$$

Now we get

$$\int_{Q_{i+1}} v_\lambda^*\,dx \leq \int_{3\tilde{Q}_i} v_\lambda^*\,dx \leq c_4 \int_{\tilde{Q}_i} v_\lambda^*\,dx,$$

where we used the doubling property of v_λ^*. Since

$$|Q_{i+1}|^{-2s/n} \int_{Q_{i+1}} w\,dx \geq 2^{-2s} |\tilde{Q}_i|^{-2s/n} \int_{\tilde{Q}_i} w\,dx,$$

we get

$$c_5 |\tilde{Q}_i|^{-2s/n} \int_{\tilde{Q}_i} w\,dx \leq \int_{\tilde{Q}_i} v_\lambda^*\,dx.$$

The similar calculation as before leads to

$$1 \le c_6 \int_{\tilde{Q}_i} (v_\lambda^*)^{n/(2s)} w^{-n/(2s)}\, dx,$$

where $c_6 = c''\alpha^{-n/(2s)}$ and c'' depends only on n, s, and w. Since $\{\tilde{Q}_i\}_{i=1}^\infty$ is a set of infinite disjoint cubes, we have a contradiction as before. Hence any sequence in $\mathcal{I}_\lambda$ such that $Q_1 \subset Q_2 \subset Q_3 \subset \cdots$ has a maximal element. Similarly we can show that any sequence in $\mathcal{I}_\lambda$ such that $Q_1 \supset Q_2 \supset Q_3 \supset \cdots$ has a minimal element.

By these arguments the number of maximal cubes and minimal cubes in $\mathcal{I}_\lambda$ with respect to the inclusion relation is finite. Hence $\mathcal{I}_\lambda$ is a finite set. We remark that the non-decreasing rearrangement of $\mathcal{I}$ in (3.6) is possible because $\mathcal{I}_\lambda$ is a finite set for every $\lambda > 0$.

Let $N(\lambda) = \sharp\mathcal{I}_\lambda$, that is, the number of elements of $\mathcal{I}_\lambda$. Let $\tilde{\mathcal{I}}_\lambda$ be the set of all $Q \in \mathcal{I}_\lambda$ which satisfy the following condition: there exists a half size dyadic sub-cube $\tilde{Q} \subset Q$ such that $\tilde{Q} \notin \mathcal{I}_\lambda$ and $\tilde{Q}$ does not contain any dyadic cube in $\mathcal{I}_\lambda$. Then we have the following lemma.

Lemma 4.1. $\sharp\mathcal{I}_\lambda \le 2\sharp\tilde{\mathcal{I}}_\lambda$.

Lemma 4.1 is proved in Rochberg and Taibleson's paper ([9, Lemma 1]).

Let $Q \in \tilde{\mathcal{I}}_\lambda$ and $\tilde{Q}$ be a dyadic cube which satisfies the condition in the definition of $\tilde{\mathcal{I}}_\lambda$. Then by similar calculations as before we get

$$1 \le c_6 \int_{\tilde{Q}} (v_\lambda^*)^{n/(2s)} w^{-n/(2s)}\, dx.$$

For every $Q \in \tilde{\mathcal{I}}_\lambda$ we choose a $\tilde{Q}$ as above. Let $\{\tilde{Q}_j\}_{j\in J}$ be the set of all such cubes $\tilde{Q}$. Then the cubes in $\{\tilde{Q}_j\}_{j\in J}$ are mutually disjoint. Therefore we get

$$\begin{aligned}\sharp\tilde{\mathcal{I}}_\lambda = \sharp J &\le \sum_{j\in J} c_6 \int_{\tilde{Q}_j} (v_\lambda^*)^{n/(2s)} w^{-n/(2s)}\, dx \\ &\le c_6 \int_{\mathbb{R}^n} (v_\lambda^*)^{n/(2s)} w^{-n/(2s)}\, dx \le c_7 \int_{\mathbb{R}^n} (\beta v - \lambda)_+^{n/(2s)} w^{-n/(2s)}\, dx,\end{aligned}$$

where we used (4.2). Hence we have

$$N(\lambda) \le 2c_7 \int_{\mathbb{R}^n} (\beta v - \lambda)_+^{n/(2s)} w^{-n/(2s)}\, dx.$$

Therefore we conclude

$$\begin{aligned}\sum_k |\mu_k|^\gamma &= \int_0^\infty \gamma\lambda^{\gamma-1} N(\lambda)\,d\lambda \\ &\le 2c_7 \int_0^\infty \int_{\beta v>\lambda} (\beta v - \lambda)_+^{n/(2s)} w^{-n/(2s)}\,dx \gamma\lambda^{\gamma-1} d\lambda \\ &\le c_8 \int_{\mathbb{R}^n} v^p w^{-n/(2s)}\,dx,\end{aligned}$$

where $c_8 = c'''\alpha^{-n/(2s)}\beta^p$ and c''' depends only on n, s, p and w.

Next we consider the case $n \le 2s$. We remark that $v(x) > 0$ for all $x \in \mathbb{R}^n$. In fact if $v(x_0) = 0$ at some point x_0, then by the definition of the maximal operator we have $V \equiv 0$, that is, $\phi_i \equiv 0, i = 1, \ldots, N$.

We also remark that $\mathcal{I}$ in (3.6) is not empty. In fact if $\mathcal{I}$ is empty, then we have

$$\beta \int_Q v\,dx \le \alpha|Q|^{-2s/n} \int_Q w\,dx$$

for all $Q \in \mathcal{Q}$. Let $Q_0 \in \mathcal{Q}$ and $Q_0 \subset Q_1 \subset Q_2 \subset \cdots$ be the infinite sequence of dyadic cubes such that Q_{i+1} is the parent of Q_i for all $i = 0, 1, 2, \ldots$. By (1.2) we have

$$|Q_{i+1}|^{-2s/n} \int_{Q_{i+1}} w\,dx \le |Q_i|^{-2s/n} \int_{Q_i} w\,dx \qquad \text{for all } i.$$

Hence we have

$$\beta \int_{Q_i} v\,dx \le \alpha|Q_0|^{-2s/n} \int_{Q_0} w\,dx \tag{4.3}$$

for all i. On the other hand, since $v \in A_1$, there exists a constant $d > 1$ such that

$$d \int_{Q_i} v\,dx \le \int_{Q_{i+1}} v\,dx$$

for all i (c.f. [5, p. 141]). Hence we have

$$d^i \int_{Q_0} v\,dx \le \int_{Q_i} v\,dx$$

and

$$\lim_{i\to\infty} \int_{Q_i} v\,dx = \infty,$$

which contradicts to (4.3). Therefore $\mathcal{I}$ is not empty.

Let $Q \in \mathcal{I}$ and Q' be the parent of Q. Then we have

$$\alpha|Q'|^{-2s/n} \int_{Q'} w\,dx \le \alpha|Q|^{-2s/n} \int_Q w\,dx < \beta \int_Q v\,dx \le \beta \int_{Q'} v\,dx,$$

where we used the assumption (1.2). Hence we have $Q' \in \mathcal{I}$, which means that $\mathcal{I}$ is an infinite set.

Lemma 4.2. *There exists a $c > 0$ such that*

$$\sum_{Q\in\mathcal{I}} \left(\frac{1}{|Q|}\int_Q v\,dx\right)^{\gamma} \le c\int_{\mathbb{R}^n} v^p w^{-n/(2s)}\,dx,$$

where $c = c'\alpha^{-n/(2s)}\beta^{n/(2s)}$ and c' depends only on n, p, s and w.

This lemma is proved in [12, Lemma 3.3]. Let $\mathcal{I}_\lambda$ be the set defined by (4.1).

Lemma 4.3. *For each $\lambda > 0$, $\mathcal{I}_\lambda$ is a finite set.*

Lemma 4.3 is easily proved by Lemma 4.2 (cf. [12, Lemma 3.4]). By Lemma 4.3 we can show that the non-decreasing rearrangement of $\mathcal{I}$ is possible.

By Lemma 4.2 we conclude

$$\begin{aligned}\sum_{k=1}^{\infty} |\mu_k|^{\gamma} &= c\sum_{Q\in\mathcal{I}} \left(\beta|Q|^{-1}\int_Q v\,dx - \alpha|Q|^{-2s/n-1}\int_Q w\,dx\right)^{\gamma}\\ &\le c\sum_{Q\in\mathcal{I}} \left(\beta|Q|^{-1}\int_Q v\,dx\right)^{\gamma} \le c\int_{\mathbb{R}^n} v^p w^{-n/(2s)}\,dx,\end{aligned}$$

where $c = c''\alpha^{-n/(2s)}\beta^p$ and c'' depends only on n, p, s and w. This ends the proof of Lemma 3.3.

5. Proof of Lemmas 3.1 and 3.2

In this section we give a proof of Lemmas 3.1 and 3.2. The following argument is in [11]. We use the following lemma.

Lemma 5.1. *Let $w \in A_2$ and $m \in C^n(\mathbb{R}^n \setminus \{0\})$. Suppose that*

$$B = \max_{|\sigma|\le n} \sup_{0<r} r^{2|\sigma|-n} \int_{r\le|\xi|\le 2r} \left|\left(\frac{\partial}{\partial\xi}\right)^{\sigma} m(\xi)\right|^2 d\xi < \infty.$$

Then the operator T defined by $\widehat{Tf}(\xi) = m(\xi)\hat{f}(\xi)$ is bounded from $L^2(w)$ to $L^2(w)$ and the operator norm $\|T\|$ is bounded by $CB^{1/2}$ where C is a constant which depends only on n and w.

The proof of Lemma 5.1 is in [6] or [7].

For $\nu \in \mathbb{Z}$ we define $\psi_\nu(x) = 2^{n\nu}\psi(2^\nu x)$. Let $w \in A_2$ and $s \geq 0$. Frazier and Jawerth proved that there exist positive constants c_1 and c_2 such that

$$c_1 \sum_{Q\in\mathcal{Q}} |Q|^{-2s/n}|(f,\psi_Q)|^2 \frac{1}{|Q|}\int_Q w\,dx \leq \int_{\mathbb{R}^n}\left\{\sum_{\nu\in\mathbb{Z}} 2^{2s\nu}|f*\psi_\nu(x)|^2\right\} w(x)\,dx$$

$$\leq c_2 \sum_{Q\in\mathcal{Q}} |Q|^{-2s/n}|(f,\psi_Q)|^2 \frac{1}{|Q|}\int_Q w\,dx$$

for all $f \in C_0^\infty(\mathbb{R}^n)$ where c_1 and c_2 depend only on n, s and w ([3, Proposition 10.14]).

We shall use the argument in Kurtz [6, p.242, p.243]. Let $\{r_\nu(t)\}$ be the Rademacher functions on $[0,1]$ indexed by $\nu \in \mathbb{Z}$ and

$$T_t f(x) = \sum_{\nu\in\mathbb{Z}} r_\nu(t) f * \psi_\nu(x).$$

Then T_t satisfies the condition of Lemma 5.1. Hence

$$\int_{\mathbb{R}^n} \left|T_t f(x)\right|^2 w(x)\,dx \leq CM \int_{\mathbb{R}^n} \left|f(x)\right|^2 w(x)\,dx,$$

for all $f \in C_0^\infty(\mathbb{R}^n)$ where

$$M = \max_{|\sigma|\leq n} \|\partial^\sigma \hat{\psi}\|_\infty^2$$

and C is a positive constant depending only on n and w. By integrating from 0 to 1 with respect to t, we get

$$\int_{\mathbb{R}^n}\left\{\sum_{\nu\in\mathbb{Z}} \left|f*\psi_\nu(x)\right|^2\right\} w(x)\,dx \leq CM \int_{\mathbb{R}^n} \left|f(x)\right|^2 w(x)\,dx.$$

By the duality argument and the fact $w^{-1} \in A_2$ we obtain

$$\int_{\mathbb{R}^n} \left|f(x)\right|^2 w(x)\,dx \leq CM \int_{\mathbb{R}^n}\left\{\sum_{\nu\in\mathbb{Z}} \left|f*\psi_\nu(x)\right|^2\right\} w(x)\,dx$$

for all $f \in C_0^\infty(\mathbb{R}^n)$. Hence we have

$$c_3 M^{-1} \int_{\mathbb{R}^n} |f|^2 w\,dx \leq \int_{\mathbb{R}^n}\left\{\sum_{\nu\in\mathbb{Z}} \left|f*\psi_\nu\right|^2\right\} w\,dx \leq c_4 M \int_{\mathbb{R}^n} |f|^2 w\,dx,$$

where c_3 and c_4 are constants depending only on n and w.

Therefore we get

$$c_3 M^{-1}\int_{\mathbb{R}^n}|(-\Delta)^{s/2}f|^2 w\,dx \le \int_{\mathbb{R}^n}\left\{\sum_{\nu\in\mathbb{Z}}\left|(-\Delta)^{s/2}f*\psi_\nu\right|^2\right\}w\,dx$$
$$\le c_4 M\int_{\mathbb{R}^n}\left|(-\Delta)^{s/2}f\right|^2 w\,dx$$

for all $f\in C_0^\infty(\mathbb{R}^n)$(c.f.[11]).

Let $\Phi\in\mathcal{S}(\mathbb{R}^n)$ satisfy $\operatorname{supp}\Phi\subset\{\xi : 1/4\le|\xi|\le 4\}$ and $\Phi(\xi)=1$ for $1/2\le|\xi|\le 2$. For $\nu\in\mathbb{Z}$ the multiplier $m_\nu(\xi)=2^{-s\nu}|\xi|^s\Phi(\xi/2^\nu)$ satisfies the condition of Lemma 5.1. Hence we have

$$\int_{\mathbb{R}^n}\left|(-\Delta)^{s/2}f*\psi_\nu(x)\right|^2 w(x)\,dx \le c_5\int_{\mathbb{R}^n}2^{2s\nu}\left|f*\psi_\nu(x)\right|^2 w(x)\,dx,$$

where

$$c_5 = c_6\inf_{\Phi}\max_{|\sigma|\le n}\|\partial^\sigma\Phi\|_\infty^2$$

and c_6 is a positive constant depending only on n, s and w and the infimum is taken over all possible Φ.

Similarly there exists a positive constant c_7 depending only on n, s and w such that

$$\int_{\mathbb{R}^n}2^{2s\nu}|f*\psi_\nu(x)|^2 w(x)\,dx \le c_7\int_{\mathbb{R}^n}|(-\Delta)^{s/2}f*\psi_\nu(x)|^2 w(x)\,dx.$$

Hence we get

$$c_8 M^{-1}\int_{\mathbb{R}^n}|(-\Delta)^{s/2}f|^2 w\,dx \le \sum_{Q\in\mathcal{Q}}|Q|^{-2s/n}|(f,\psi_Q)|^2\frac{1}{|Q|}\int_Q w\,dx$$
$$\le c_9 M\int_{\mathbb{R}^n}|(-\Delta)^{s/2}f|^2 w\,dx$$

for all $f\in C_0^\infty(\mathbb{R}^n)$, where c_8 and c_9 are positive constant depending only on n, s and w. This ends the proof of Lemmas 3.1 and 3.2.

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