

Approximation and symbolic calculus for Toeplitz algebras on the Bergman space

Daniel Suárez

Abstract

If $f \in L^\infty(\mathbb{D})$ let T_f be the Toeplitz operator on the Bergman space L_a^2 of the unit disk \mathbb{D} . For a C^* -algebra $A \subset L^\infty(\mathbb{D})$ let $\mathfrak{T}(A)$ denote the closed operator algebra generated by $\{T_f : f \in A\}$. We characterize its commutator ideal $\mathfrak{C}(A)$ and the quotient $\mathfrak{T}(A)/\mathfrak{C}(A)$ for a wide class of algebras A . Also, for $n \geq 0$ integer, we define the n -Berezin transform $B_n S$ of a bounded operator S , and prove that if $f \in L^\infty(\mathbb{D})$ and $f_n = B_n T_f$ then $T_{f_n} \rightarrow T_f$.

1. Introduction and preliminaries

Suppose that A is a C^* -algebra with unit. The commutator ideal \mathfrak{C} is the closed bilateral ideal generated by the elements $[x, y] = xy - yx$, with $x, y \in A$. The quotient A/\mathfrak{C} is a commutative C^* -algebra with unit, which by the Gelfand-Naimark Theorem is isometrically isomorphic to $C(M)$, the algebra of continuous functions on some compact Hausdorff space M . Following the arrows

$$A \rightarrow A/\mathfrak{C} \xrightarrow{\cong} C(M)$$

we can associate to every $x \in A$ a function $f_x \in C(M)$, which is the ‘symbol’ referred to in the title of the paper. Since the algebra A is determined by \mathfrak{C} and $C(M)$, the study of these two objects is an important tool for a better understanding of A . The possible advantages of this point of view are that $C(M)$ can be treated by topological methods, since it depends exclusively on the space M , and that \mathfrak{C} is usually much smaller than A . Of course, the first step of this journey is to determine \mathfrak{C} and $C(M)$. The whole process is known as abelianization, and it can be carried out for a much wider class

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of algebras than C^* -algebras. In particular, these ideas have been widely studied in the context of Toeplitz algebras acting on the Hardy space H^2 (see [18, pp. 339-392]). The literature shows some partial attempts to develop a similar scheme for Toeplitz algebras acting on the Bergman space $L_a^2 = L_a^2(dA)$, where dA is the normalized area measure on \mathbb{D} (see [14, Ch. 4] for a general discussion). We give below a brief summary of known results.

Let $\mathfrak{L}(L_a^2)$ be the algebra of bounded operators on L_a^2 . If $\mathcal{B} \subset L^\infty(\mathbb{D})$ is a closed subalgebra, let $\mathfrak{T}(\mathcal{B})$ be the closed subalgebra of $\mathfrak{L}(L_a^2)$ generated by the Toeplitz operators $\{T_a : a \in \mathcal{B}\}$ and $\mathfrak{C}(\mathcal{B})$ be the commutator ideal of $\mathfrak{T}(\mathcal{B})$.

In [11] Coburn proved that $\mathfrak{C}(C(\overline{\mathbb{D}}))$ is the ideal of compact operators and $\mathfrak{T}(C(\overline{\mathbb{D}}))/\mathfrak{C}(C(\overline{\mathbb{D}}))$ is isomorphic to $C(\partial\mathbb{D})$. In [17] McDonald and Sundberg characterized the quotient $\mathfrak{T}(\mathcal{U})/\mathfrak{C}(\mathcal{U})$, where \mathcal{U} is the C^* -algebra in $L^\infty(\mathbb{D})$ generated by H^∞ . Later, the two papers by Axler and Zheng ([4], [5]) provided additional information on Coburn's and McDonald-Sundberg's theorems by giving characterizations of the respective commutator ideals in terms of the Berezin transform. We give precise statements of these results in Sections 6 and 7. In [20] the author showed that $\mathfrak{C}(L^\infty(\mathbb{D})) = \mathfrak{T}(L^\infty(\mathbb{D}))$. Despite these results, no systematic theory of abelianization has been given so far for Toeplitz algebras on the Bergman space. One of the purposes of this paper is to develop a general theory of abelianization for Toeplitz algebras $\mathfrak{T}(\mathcal{B})$, where \mathcal{B} belongs to a special class of C^* -algebras in $L^\infty(\mathbb{D})$ that we call hyperbolic. Our main goal is to explain the underlying phenomenon that is apparently common to Coburn's and McDonald-Sundberg's theorems, and to apply it to other hyperbolic algebras.

Let $\mathcal{A} \subset L^\infty(\mathbb{D})$ be the algebra of functions on \mathbb{D} that are uniformly continuous with respect to the pseudohyperbolic metric. If n is a nonnegative integer, we define the n -Berezin transform $B_n : \mathfrak{L}(L_a^2) \rightarrow \mathcal{A}$. This is a linear operator, and we show that if $a \in L^\infty(\mathbb{D})$ and $a_n = B_n T_a$, then T_{a_n} tends to T_a in operator norm. In particular, the Toeplitz algebras associated to $L^\infty(\mathbb{D})$ and \mathcal{A} coincide. This will allow us to reduce the study of $\mathfrak{T}(\mathcal{B})$ and $\mathfrak{C}(\mathcal{B})$ for some C^* -algebras $\mathcal{B} \subset L^\infty(\mathbb{D})$ that are not hyperbolic, to the case of hyperbolic algebras. Once the reduction is made, we can use the maximal ideal space of \mathcal{A} as a powerful tool to describe $\mathfrak{C}(\mathcal{B})$ and $\mathfrak{T}(\mathcal{B})/\mathfrak{C}(\mathcal{B})$. We begin fixing some notation.

For $z \in \mathbb{D}$ denote

$$\varphi_z(\omega) = \frac{z - \omega}{1 - \bar{z}\omega}.$$

The pseudohyperbolic metric on \mathbb{D} is defined as $\rho(z, \omega) = |\varphi_z(\omega)|$. This metric is invariant under the action of $\text{Aut}(\mathbb{D})$. Sometimes, especially in

estimates involving the triangle inequality, it will be useful to use the hyperbolic metric

$$h(z, \omega) = \log \frac{1 + \rho(z, \omega)}{1 - \rho(z, \omega)}, \quad z, \omega \in \mathbb{D}$$

instead of ρ . The passage from one metric to the other is justified because $f(x) = \log \frac{1+x}{1-x}$ is a strictly increasing function of $x \in (0, 1)$. For $z \in \mathbb{D}$, $r \in (0, 1)$ and $s \in (0, \infty)$ write

$$K(z, r) = \{\omega \in \mathbb{D} : \rho(z, \omega) \leq r\} \quad \text{and} \quad K_h(z, r) = \{\omega \in \mathbb{D} : h(z, \omega) \leq s\}$$

for the closed pseudohyperbolic (resp. hyperbolic) disk of center z and radius r (resp. s).

Let $\mathcal{B} \subset L^\infty(\mathbb{D})$ be a closed subalgebra, where by algebra we always mean a unitary algebra. The maximal ideal space of \mathcal{B} is

$$M(\mathcal{B}) = \{\alpha : \mathcal{B} \rightarrow \mathbb{C} : \alpha \text{ is linear, multiplicative and } \alpha(1) = 1\},$$

provided with the weak $*$ topology induced by the dual space of \mathcal{B} . It is a compact Hausdorff space. We can look at a function $f \in \mathcal{B}$ as a continuous function on $M(\mathcal{B})$ via the Gelfand transform

$$\hat{f}(\alpha) = \alpha(f) \quad (\alpha \in M(\mathcal{B})).$$

If $\mathcal{B} \subset C(\mathbb{D}) \cap L^\infty(\mathbb{D})$ separates points of \mathbb{D} then evaluations at points of \mathbb{D} are members of $M(\mathcal{B})$. So, \mathbb{D} is naturally imbedded into $M(\mathcal{B})$, and \hat{f} is an extension to the whole maximal space of the function f . Unless the contrary is stated we avoid writing the hat for the Gelfand transform and look at f as a function on $M(\mathcal{B})$. The algebra

$$\mathcal{A} = \{f \in L^\infty(\mathbb{D}) : f \text{ is uniformly continuous with respect to } \rho\}$$

will be a major protagonist of this paper. It is C^* -algebra such that \mathbb{D} is dense in $M(\mathcal{A})$. Indeed, there cannot be $\alpha \in M(\mathcal{A}) \setminus \overline{\mathbb{D}}$, because otherwise there is $f \in \mathcal{A}$ with $f(\alpha) = 0$ while $|f| \geq \delta > 0$ on \mathbb{D} (since \mathcal{A} is a C^* -algebra). Since such f is invertible in \mathcal{A} , it is not in the maximal ideal $\text{Ker } \alpha$. Further information on $M(\mathcal{A})$ can be found in [8].

If $a \in L^\infty(\mathbb{D})$ let M_a be the multiplication operator on $L^2(\mathbb{D})$ and T_a be the Toeplitz operator on L^2_a . That is, $T_a = P_+ M_a$, where $P_+ : L^2(\mathbb{D}) \rightarrow L^2_a$ is the Bergman projection. It is clear that $\|M_a\| = \|a\|_\infty$ and $\|T_a\| \leq \|a\|_\infty$. A big difference with Toeplitz operators on the Hardy space H^2 is that the latter inequality is not always an equality, although we still have that $T_a = 0$

only when $a = 0$. For $z \in \mathbb{D}$, the ‘change of variable operator’ is given by $U_z f = (f \circ \varphi_z)\varphi'_z$. That is,

$$(U_z f)(\omega) = f(\varphi_z(\omega)) \frac{|z|^2 - 1}{(1 - \bar{z}\omega)^2}.$$

Is easy to prove that $U_z T_a U_z = T_{a \circ \varphi_z}$ for every $a \in L^\infty(\mathbb{D})$, and since U_z is unitary and self-adjoint, then

$$(1.1) \quad (T_{a_1} \dots T_{a_n})_z = (U_z T_{a_1} U_z) \dots (U_z T_{a_n} U_z) = T_{a_1 \circ \varphi_z} \dots T_{a_n \circ \varphi_z}$$

for $a_1, \dots, a_n \in L^\infty(\mathbb{D})$. We will write

$$S_z = U_z T_a U_z \quad \text{for } S \in \mathfrak{L}(L_a^2).$$

The paper is organized as follows. The main results are Theorems 5.7, 6.4 and 6.5. In Section 2 we introduce the n -Berezin transform of a bounded operator and study its basic properties. If $a \in L^\infty(\mathbb{D})$, $B_n T_a$ coincides with $B_n(a)$, the more familiar n -Berezin transform of a function. In Section 3 we study the maximal ideal space of \mathcal{A} and use some of its features to define the notion of hyperbolic algebra. A characterization of these algebras is obtained in terms of interpolating sequences.

If $S \in \mathfrak{T}(\mathcal{B})$, where \mathcal{B} is a hyperbolic algebra, we construct in Section 4 a continuous map $\Psi_S^{\mathcal{B}}$ from the maximal ideal space of \mathcal{B} into $\mathfrak{T}(\mathcal{B})$, when provided with the strong operator topology, and study its interaction with the n -Berezin transform. We prove that $\Psi_S^{\mathcal{B}}$ is multiplicative as a function of S , which translates into a kind of asymptotic multiplicative behavior of B_n . This will be a fundamental tool for much of what follows.

Theorem 5.7 shows that $T_{B_n(a)}$ tends to T_a for $a \in L^\infty(\mathbb{D})$. As a consequence we obtain that if $B_n(a)$ belongs to a hyperbolic algebra \mathcal{B} for infinitely many values of n then $T_a \in \mathfrak{T}(\mathcal{B})$. This argument will reduce the study of $\mathfrak{T}(C)$ for some non-hyperbolic algebras $C \subset L^\infty(\mathbb{D})$ to the hyperbolic case.

Theorem 6.4 gives a characterization of $\mathfrak{C}(\mathcal{B})$ and $\mathfrak{T}(\mathcal{B})/\mathfrak{C}(\mathcal{B})$ when \mathcal{B} is hyperbolic. If S is a finite sum of finite products of Toeplitz operators with symbols in $L^\infty(\mathbb{D})$ and \mathcal{B} is a hyperbolic algebra, Theorem 6.5 provides a necessary and sufficient condition for $S \in \mathfrak{T}(\mathcal{B})$ and $S \in \mathfrak{C}(\mathcal{B})$.

Section 7 is devoted to applications of the previous results. It is shown that the theorem of McDonald-Sundberg and part of Coburn’s theorem are particular cases of Theorem 6.4. An example will be given to illustrate how Theorems 5.7 and 6.4 can be used to characterize $\mathfrak{C}(C)$ and $\mathfrak{T}(C)/\mathfrak{C}(C)$ for some C^* -algebras $C \subset L^\infty(\mathbb{D})$ that are not hyperbolic.

Finally, we give a partial result towards a possible characterization of the center of $\mathfrak{T}(L^\infty(\mathbb{D}))/\mathcal{K}$, where \mathcal{K} denotes the ideal of compact operators. We finish the paper posing some open problems.

2. The n -Berezin transform.

If n is a nonnegative integer and $z \in \mathbb{D}$, the function

$$K_z^{(n)}(\omega) = \frac{1}{(1 - \bar{z}\omega)^{2+n}} \quad (\omega \in \mathbb{D})$$

is the reproducing kernel of z in the weighted Bergman space $L_a^2(dA_n)$, where $dA_n(\omega) = (n + 1)(1 - |\omega|^2)^n dA(\omega)$. The n -Berezin transform of an operator $S \in \mathfrak{L}(L_a^2)$ is defined as

$$(2.1) \quad (B_n S)(z) \stackrel{\text{def}}{=} (n + 1)(1 - |z|^2)^{2+n} \sum_{j=0}^n \binom{n}{j} (-1)^j \langle S(\omega^j K_z^{(n)}), \omega^j K_z^{(n)} \rangle.$$

It is clear that $B_n S \in C^\infty(\mathbb{D})$ for every $S \in \mathfrak{L}(L_a^2)$. Using that

$$\sum_{j=0}^n \binom{n}{j} (-1)^j |\omega|^{2j} = (1 - |\omega|^2)^n$$

we see that if $S = T_a$, with $a \in L^\infty(\mathbb{D})$, then

$$\begin{aligned} (B_n a)(z) &\stackrel{\text{def}}{=} (B_n T_a)(z) \\ &= (n + 1)(1 - |z|^2)^{2+n} \sum_{j=0}^n \binom{n}{j} (-1)^j \int_D \frac{a(\omega) |\omega|^{2j}}{|1 - \bar{z}\omega|^{2(2+n)}} dA(\omega) \\ &= \int_D a(\omega) \frac{(1 - |z|^2)^{2+n}}{|1 - \bar{z}\omega|^{2(2+n)}} (n + 1)(1 - |\omega|^2)^n dA(\omega) \\ (2.2) \quad &= \int_D a(\varphi_z(\zeta)) (n + 1)(1 - |\zeta|^2)^n dA(\zeta), \end{aligned}$$

where the last equality comes from the change of variables $\omega = \varphi_z(\zeta)$. Since $dA_n(\xi)$ is a probability measure that tends to concentrate its mass at 0 when $n \rightarrow \infty$, then $(B_n a)(z)$ is an average of a satisfying $\|B_n(a)\|_\infty \leq \|a\|_\infty$ for all $a \in L^\infty(\mathbb{D})$. A straightforward calculation shows that B_n maps $L^\infty(\mathbb{D})$ into \mathcal{A} for every $n \geq 0$, and we will prove in Corollary 4.6 that the same holds for $\mathfrak{L}(L_a^2)$. The last expression in (2.2) clearly shows that $\|B_n(a) - a\|_\infty \rightarrow 0$ when $n \rightarrow \infty$ for every $a \in \mathcal{A}$. That is, the sequence $\{B_n\}$ works as an approximate identity for \mathcal{A} . In particular, $\lim_n \|T_{B_n(a)} - T_a\| = 0$ for $a \in \mathcal{A}$.

The 0-Berezin transform of an operator is the usual Berezin transform, which has been extensively used in recent research (see for instance [2], [4], [5] and [19]). The n -Berezin transforms of functions (not necessarily bounded) were introduced by Berezin in [6]. Many of the results of this section were

proved by Ahern, Flores and Rudin [2] for n -Berezin transforms of functions of several variables. However, the results here do not follow immediately from theirs, because there are *a priori* several ways to define $B_n S$ for $n \geq 1$ and $S \in \mathfrak{L}(L_a^2)$ so that (2.2) holds when $S = T_a$. If for instance $S \in \mathfrak{L}(L_a^2) \cap \mathfrak{L}(L_a^2(dA_n))$, then the usual Berezin transform of S with respect to the weighted Bergman space $L_a^2(dA_n)$ is $(1 - |z|^2)^{2+n} \langle SK_z^{(n)}, K_z^{(n)} \rangle_{dA_n}$, which differs from our definition of $B_n S$. It is precisely because of the results of this section (especially Proposition 2.4) that I convinced myself (and hopefully convince the reader) about (2.1) as the right definition of $B_n S$ for $S \in \mathfrak{L}(L_a^2)$.

Lemma 2.1 *Let $S \in \mathfrak{L}(L_a^2)$ and $n \geq 0$. Then*

$$(2.3) \quad (n + 2)(1 - |z|^2)B_n(S - T_{\bar{\omega}}ST_{\omega})(z) = (n + 1)B_{n+1}(T_{1-\bar{\omega}z}ST_{1-\omega\bar{z}})(z)$$

for every $z \in \mathbb{D}$.

Proof. A simple rearrangement of terms gives

$$\begin{aligned} & \sum_{j=0}^n \binom{n}{j} (-1)^j [\langle S(\omega^j K_z^{(n)}), \omega^j K_z^{(n)} \rangle - \langle S(\omega^{j+1} K_z^{(n)}), \omega^{j+1} K_z^{(n)} \rangle] \\ &= \langle SK_z^{(n)}, K_z^{(n)} \rangle + (-1)^{n+1} \langle S(\omega^{n+1} K_z^{(n)}), \omega^{n+1} K_z^{(n)} \rangle \\ & \quad + \sum_{j=1}^n \left[\binom{n}{j} + \binom{n}{j-1} \right] (-1)^j \langle S(\omega^j K_z^{(n)}), \omega^j K_z^{(n)} \rangle \\ &= \sum_{j=0}^{n+1} \binom{n+1}{j} (-1)^j \langle S(\omega^j K_z^{(n)}), \omega^j K_z^{(n)} \rangle. \end{aligned}$$

Multiplying by $(n + 2)(n + 1)(1 - |z|^2)^{3+n}$ and using that

$$T_{1-\omega\bar{z}}(\omega^j K_z^{(n+1)}) = \omega^j K_z^{(n)},$$

the above equality becomes (2.3). ■

Lemma 2.2 $B_n S_{\alpha} = (B_n S) \circ \varphi_{\alpha}$ for every $n \geq 0$, $S \in \mathfrak{L}(L_a^2)$ and $\alpha \in \mathbb{D}$.

Proof. We shall prove the lemma by induction on n . The easy identity

$$(2.4) \quad (1 - \varphi_{\alpha}(\omega)\bar{z})^{-1} = (1 - \alpha\bar{z})^{-1}(1 - \bar{\alpha}\omega)(1 - \overline{\varphi_{\alpha}(z)\omega})^{-1}$$

implies that

$$(U_{\alpha}K_z^{(0)})(\omega) = \frac{(|\alpha|^2 - 1)}{(1 - \bar{\alpha}\omega)^2(1 - \varphi_{\alpha}(\omega)\bar{z})^2} = \frac{(|\alpha|^2 - 1)}{(1 - \alpha\bar{z})^2} K_{\varphi_{\alpha}(z)}^{(0)}(\omega).$$

Thus

$$(B_0 S_{\alpha})(z) = (1 - |\varphi_{\alpha}(z)|^2)^2 \langle SK_{\varphi_{\alpha}(z)}^{(0)}, K_{\varphi_{\alpha}(z)}^{(0)} \rangle = (B_0 S)(\varphi_{\alpha}(z)).$$

This takes care of $n = 0$.

The main tool for the inductive step will be formula (2.3), that we rewrite as

$$(2.5) \quad (B_{n+1}S)(z) = c_n(1 - |z|^2)B_n[T_{(1-\bar{\omega}z)^{-1}}(S - T_{\bar{\omega}}ST_{\omega})T_{(1-\omega\bar{z})^{-1}}](z),$$

where $c_n = (n + 2)/(n + 1)$. By (1.1) then

$$\begin{aligned} & T_{(1-\bar{\omega}z)^{-1}}(U_{\alpha}SU_{\alpha} - T_{\bar{\omega}}U_{\alpha}SU_{\alpha}T_{\omega})T_{(1-\omega\bar{z})^{-1}} \\ &= U_{\alpha}T_{(1-\overline{\varphi_{\alpha}(\omega)z})^{-1}}[S - T_{\overline{\varphi_{\alpha}(\omega)}}ST_{\varphi_{\alpha}(\omega)}]T_{(1-\varphi_{\alpha}(\omega)\bar{z})^{-1}}U_{\alpha} = J. \end{aligned}$$

Then (2.4) yields

$$(2.6) \quad \begin{aligned} J &= |1 - \alpha\bar{z}|^{-2} U_{\alpha}T_{(1-\varphi_{\alpha}(z)\bar{\omega})^{-1}}[T_{1-\alpha\bar{\omega}}ST_{1-\bar{\alpha}\omega} - T_{\bar{\alpha}-\bar{\omega}}ST_{\alpha-\omega}]T_{(1-\overline{\varphi_{\alpha}(z)\omega})^{-1}}U_{\alpha} \\ &= \frac{(1 - |\alpha|^2)}{|1 - \alpha\bar{z}|^2} U_{\alpha}T_{(1-\varphi_{\alpha}(z)\bar{\omega})^{-1}}[S - T_{\bar{\omega}}ST_{\omega}]T_{(1-\overline{\varphi_{\alpha}(z)\omega})^{-1}}U_{\alpha}. \end{aligned}$$

Hence,

$$\begin{aligned} (B_{n+1}S_{\alpha})(z) &= c_n(1 - |z|^2)B_n(J)(z) \\ &= c_n(1 - |\varphi_{\alpha}(z)|^2)B_n(U_{\alpha}T_{(1-\varphi_{\alpha}(z)\bar{\omega})^{-1}}[S - T_{\bar{\omega}}ST_{\omega}]T_{(1-\overline{\varphi_{\alpha}(z)\omega})^{-1}}U_{\alpha})(z) \\ &= c_n(1 - |\varphi_{\alpha}(z)|^2)B_n(T_{(1-\varphi_{\alpha}(z)\bar{\omega})^{-1}}[S - T_{\bar{\omega}}ST_{\omega}]T_{(1-\overline{\varphi_{\alpha}(z)\omega})^{-1}})(\varphi_{\alpha}(z)) \\ &= B_{n+1}(S)(\varphi_{\alpha}(z)), \end{aligned}$$

where the first equality comes from (2.5) with $U_{\alpha}SU_{\alpha}$ instead of S , the second from (2.6), the third by inductive hypothesis and the last one from (2.5) with $\varphi_{\alpha}(z)$ instead of z . ■

Corollary 2.3 *If $S \in \mathfrak{L}(L^2_a)$ and $n \geq 0$ then $\|B_nS\|_{\infty} \leq (n + 1)2^n\|S\|$.*

Proof. Since $\|K_z^{(0)}\| = (1 - |z|^2)^{-1}$ then

$$|(B_0S)(z)| = (1 - |z|^2)^2|\langle S(K_z^{(0)}), K_z^{(0)} \rangle| \leq \|S\|.$$

Suppose that the corollary holds for n , and we shall see that it holds for $n + 1$. By (2.3) $(B_{n+1}S)(0) = (n + 2/n + 1)B_n(S - T_{\bar{\omega}}ST_{\omega})(0)$. Thus

$$\begin{aligned} |(B_{n+1}S)(0)| &\leq \frac{n + 2}{n + 1} (\|B_nS\|_{\infty} + \|B_n(T_{\bar{\omega}}ST_{\omega})\|_{\infty}) \\ &\leq \frac{n + 2}{n + 1} ((n + 1)2^n\|S\| + (n + 1)2^n\|T_{\bar{\omega}}ST_{\omega}\|) \\ &\leq (n + 2)2^{n+1}\|S\|. \end{aligned}$$

Replacing S by U_zSU_z the result follows from Lemma 2.2. ■

The (conformally) invariant Laplacian is $\tilde{\Delta} = (1 - |z|^2)^2 4\partial\bar{\partial}$, where ∂ and $\bar{\partial}$ are the traditional Cauchy-Riemann operators. So, when f is analytic on \mathbb{D} , $\partial f = f'$, $\partial\bar{f} = 0$, $\bar{\partial}f = \bar{f}'$ and $\bar{\partial}\bar{f} = 0$. It is easy to check that $(\tilde{\Delta}f) \circ \psi = \tilde{\Delta}(f \circ \psi)$ for every $\psi \in \text{Aut}(\mathbb{D})$.

Proposition 2.4 *Let $S \in \mathfrak{L}(L_a^2)$ and $n \geq 0$. Then*

$$(2.7) \quad \tilde{\Delta}B_nS = 4(n + 1)(n + 2)(B_nS - B_{n+1}S).$$

Proof. By Lemma 2.2 and the conformal invariance of $\tilde{\Delta}$ it is enough to prove that the equality holds at $z = 0$. Using the mentioned properties of ∂ and $\bar{\partial}$, a tedious but straightforward calculation gives

$$(2.8) \quad \begin{aligned} &\tilde{\Delta}[(1 - |z|^2)^{n+2}\langle S(\omega^j K_z^{(n)}), \omega^j K_z^{(n)} \rangle](0) \\ &= 4(n + 2)(-\langle S\omega^j, \omega^j \rangle + (n + 2)\langle S\omega^{j+1}, \omega^{j+1} \rangle) \end{aligned}$$

for every $0 \leq j \leq n$. So, writing $X_j = (-1)^j \langle S\omega^j, \omega^j \rangle$, we have

$$\begin{aligned} \tilde{\Delta}(B_nS)(0) &= 4(n + 1)(n + 2) \sum_{j=0}^n \binom{n}{j} [-X_j - (n + 2)X_{j+1}] \\ &= 4(n + 1)(n + 2) \left\{ -X_0 - (n + 2)X_{n+1} - \sum_{j=1}^n \left[\binom{n}{j} + (n + 2)\binom{n}{j-1} \right] X_j \right\}. \end{aligned}$$

On the other hand,

$$(B_nS - B_{n+1}S)(0) = -(n + 2)X_{n+1} + \sum_{j=0}^n \left[(n + 1)\binom{n}{j} - (n + 2)\binom{n + 1}{j} \right] X_j.$$

A comparison of the coefficients for each X_j gives the result. ■

Corollary 2.5 *If $S \in \mathfrak{L}(L_a^2)$ and $n \geq 1$ then*

$$(2.9) \quad B_nS = \left(1 - \frac{\tilde{\Delta}}{4n(n + 1)} \right) B_{n-1}S$$

and

$$(2.10) \quad B_nS = G_n(\tilde{\Delta})B_0S,$$

where

$$G_n(\lambda) = \prod_{k=1}^n \left(1 - \frac{\lambda}{4k(k + 1)} \right).$$

Proof. Formula (2.9) is a rewriting of (2.7), while (2.10) follows immediately from (2.9). ■

Lemma 2.6 *If $S \in \mathfrak{L}(L_a^2)$ and $n \geq 0$ then $\tilde{\Delta}B_0(B_nS) = B_0\tilde{\Delta}(B_nS)$.*

Proof. If $f = B_nS$, Corollary 2.3 and (2.7) imply that f and $\tilde{\Delta}f$ are bounded. Hence, Lemma 1 of [1] says that $\tilde{\Delta}B_0f = B_0\tilde{\Delta}f$. ■

Corollary 2.7 *Let $S \in \mathfrak{L}(L_a^2)$ and $k, j \geq 0$. Then $(B_kB_j)(S) = (B_jB_k)(S)$.*

Proof. Combine (2.10) with the previous lemma. ■

3. Algebras related to the maximal ideal space of \mathcal{A}

For the next two subsections, if $E \subset M(\mathcal{A})$ then \overline{E} denotes the closure of E in the space $M(\mathcal{A})$.

Since the $M(\mathcal{A})$ -topology agrees with the Euclidean topology on \mathbb{D} , \overline{E} has the same meaning in both topologies when $E \subset r\mathbb{D}$ for some $0 < r < 1$. Later on, we will have to distinguish between closures in different spaces. A sequence $\{z_n\} \subset \mathbb{D}$ is separated if $\rho(z_n, z_k) \geq \delta > 0$ for $n \neq k$.

3.1. One-to-one maps from \mathbb{D} into $M(\mathcal{A})$

Lemma 3.1 *Let $E, F \subset \mathbb{D}$. Then $\overline{E} \cap \overline{F} = \emptyset$ if and only if $\rho(E, F) > 0$.*

Proof. If $\overline{E} \cap \overline{F} = \emptyset$ then there is $f \in \mathcal{A}$ such that $f \equiv 1$ on E and $f \equiv 0$ on F . The uniform ρ -continuity of f implies that

$$\rho(E, F) = \rho(\overline{E} \cap \mathbb{D}, \overline{F} \cap \mathbb{D}) > 0.$$

Now suppose that $\rho(E, F) \geq \alpha > 0$ and consider the function

$$f(z) = \begin{cases} 1 & \text{if } \rho(z, E) \leq \alpha/2 \\ 0 & \text{if } \rho(z, E) > \alpha/2 \end{cases}$$

Simple estimates show that $B_n(f) \rightarrow 1$ uniformly on $\{z : \rho(z, E) < \alpha/4\}$ and $B_n(f) \rightarrow 0$ uniformly on $\{z : \rho(z, F) < \alpha/4\}$. Since $B_n(f) \in \mathcal{A}$, it separates \overline{E} from \overline{F} for n big enough, showing that they are disjoint. ■

Let $x \in M(\mathcal{A})$ and suppose that (z_α) is a net in \mathbb{D} that tends to x . We can think of (φ_{z_α}) as a net in the product space $M(\mathcal{A})^\mathbb{D}$. By compactness there is a convergent subnet $(\varphi_{z_{\alpha_\beta}})$, meaning that there is some function $\varphi : \mathbb{D} \rightarrow M(\mathcal{A})$ such that $f \circ \varphi_{z_{\alpha_\beta}} \rightarrow f \circ \varphi$ pointwise on \mathbb{D} for every $f \in \mathcal{A}$.

We aim to show that the whole net (z_α) tends to φ and that φ does not depend on the net. So, suppose that (ω_γ) is another net in \mathbb{D} converging to x such that φ_{ω_γ} tends to some $\psi \in M(\mathcal{A})^\mathbb{D}$. If $\varphi \neq \psi$ there is

some $\xi \in \mathbb{D}$ such that $\varphi(\xi) \neq \psi(\xi)$. Then there are closed disjoint neighborhoods $U, V \subset M(\mathcal{A})$ of $\varphi(\xi)$ and $\psi(\xi)$, respectively. Since $\varphi_{z_{\alpha\beta}}(\xi) \rightarrow \varphi(\xi)$ and $\varphi_{\omega_\gamma}(\xi) \rightarrow \psi(\xi)$, there are tails of both nets satisfying

$$E = \{\varphi_{z_{\alpha\beta}}(\xi) : \beta \geq \beta_0\} \subset U \quad \text{and} \quad F = \{\varphi_{\omega_\gamma}(\xi) : \gamma \geq \gamma_0\} \subset V.$$

By Lemma 3.1 then $\rho(E, F) \geq \rho(U \cap \mathbb{D}, V \cap \mathbb{D}) > 0$. Since for every $z, \omega \in \mathbb{D}$ there is a constant $c_\xi > 0$ such that

$$\rho(\varphi_z(\xi), \varphi_\omega(\xi)) < c_\xi \rho(z, \omega),$$

then

$$\rho(E, F) \leq c_\xi \inf\{\rho(z_{\alpha\beta}, \omega_\gamma) : \beta \geq \beta_0, \gamma \geq \gamma_0\} = 0,$$

where the last equality holds because both nets $(z_{\alpha\beta})$ and (ω_γ) tend to x . We obtain a contradiction and consequently $\varphi = \psi$. The map φ will be denoted φ_x , and notice that $\varphi_x(0) = \lim \varphi_{z_\alpha}(0) = \lim z_\alpha = x$.

The following lemma is in [20, Lemma 2.1].

Lemma 3.2 *Let \mathcal{S} be a separated sequence and $0 < \sigma < 1$. Then there is a finite decomposition $\mathcal{S} = \mathcal{S}_1 \cup \dots \cup \mathcal{S}_N$ such that for every $1 \leq j \leq N$: $\rho(z, \omega) > \sigma$ for all $z \neq \omega$ in \mathcal{S}_j .*

Lemma 3.3 *Every $x \in M(\mathcal{A})$ is in the closure of some separated sequence.*

Proof. Suppose that $x \in M(\mathcal{A})$ and let (ω_α) be a net in \mathbb{D} such that $\omega_\alpha \rightarrow x$. Take a separated sequence \mathcal{S} such that $\rho(z, \mathcal{S}) < 1/8$ for every $z \in \mathbb{D}$, and for each ω_α pick some z_α in \mathcal{S} such that $\rho(z_\alpha, \omega_\alpha) < 1/8$ for every α . Therefore there is $\xi_\alpha \in 8^{-1}\mathbb{D}$ so that $\omega_\alpha = \varphi_{z_\alpha}(\xi_\alpha)$. Taking subnets we can assume that $\xi_\alpha \rightarrow \xi$ with $|\xi| \leq 1/8$. We claim that $\varphi_{z_\alpha}(\xi)$ tends to x . Indeed, if $f \in \mathcal{A}$ then

$$|f(\varphi_{z_\alpha}(\xi)) - f(x)| \leq |f(\varphi_{z_\alpha}(\xi)) - f(\varphi_{z_\alpha}(\xi_\alpha))| + |f(\omega_\alpha) - f(x)|,$$

where the first summand tends to 0 because $\rho(\varphi_{z_\alpha}(\xi), \varphi_{z_\alpha}(\xi_\alpha)) = \rho(\xi, \xi_\alpha) \rightarrow 0$, and the second summand tends to 0 because $\omega_\alpha \rightarrow x$. Thus, x is in the closure of the sequence $\mathcal{T} = \{\varphi_{z_n}(\xi) : z_n \in \mathcal{S}\}$. By Lemma 3.2 we can split $\mathcal{S} = \mathcal{S}_1 \cup \dots \cup \mathcal{S}_N$, where for each $1 \leq j \leq N$, $\rho(z_1, z_2) > 1/2$ when $z_1, z_2 \in \mathcal{S}_j$ are different. We also have the corresponding decomposition $\mathcal{T} = \mathcal{T}_1 \cup \dots \cup \mathcal{T}_N$, where $\mathcal{T}_j = \{\varphi_z(\xi) : z \in \mathcal{S}_j\}$. Hence, there is at least one j_0 such that x is in the closure of \mathcal{T}_{j_0} . The lemma will follow if we show that \mathcal{T}_{j_0} is a separated sequence. If $z_1, z_2 \in \mathcal{S}_{j_0}$ are different then

$$\begin{aligned} \rho(z_1, z_2) &\leq \rho(z_1, \varphi_{z_1}(\xi)) + \rho(\varphi_{z_1}(\xi), \varphi_{z_2}(\xi)) + \rho(\varphi_{z_2}(\xi), z_2) \\ &= 2|\xi| + \rho(\varphi_{z_1}(\xi), \varphi_{z_2}(\xi)). \end{aligned}$$

So, $\rho(\varphi_{z_1}(\xi), \varphi_{z_2}(\xi)) \geq (1/2) - 2|\xi| \geq 1/4$, proving our claim. ■

Lemma 3.4 *Let (z_α) be a net in \mathbb{D} converging to $x \in M(\mathcal{A})$. Then*

- (i) φ_x is a continuous one-to-one map from \mathbb{D} into $M(\mathcal{A})$,
- (ii) $f \circ \varphi_x \in \mathcal{A}$ for every $f \in \mathcal{A}$,
- (iii) $f \circ \varphi_{z_\alpha} \rightarrow f \circ \varphi_x$ uniformly on compact sets of \mathbb{D} for every $f \in \mathcal{A}$.

Proof. Suppose that $\omega \in \mathbb{D}$ and $f \in \mathcal{A}$. Given $\varepsilon > 0$ there is $\delta > 0$ such that $|f(u) - f(v)| < \varepsilon$ if $\rho(u, v) < \delta$. Take $\omega_1 \in K(\omega, \delta)$. Since $\rho(\varphi_{z_\alpha}(\omega_1), \varphi_{z_\alpha}(\omega)) = \rho(\omega_1, \omega) < \delta$ then $|f(\varphi_{z_\alpha}(\omega_1)) - f(\varphi_{z_\alpha}(\omega))| < \varepsilon$ for every α . Then

$$\begin{aligned} |f(\varphi_x(\omega_1)) - f(\varphi_x(\omega))| &\leq |f(\varphi_x(\omega_1)) - f(\varphi_{z_\alpha}(\omega_1))| + |f(\varphi_{z_\alpha}(\omega_1)) - f(\varphi_{z_\alpha}(\omega))| \\ &\quad + |f(\varphi_{z_\alpha}(\omega)) - f(\varphi_x(\omega))| \\ &\leq |f(\varphi_x(\omega_1)) - f(\varphi_{z_\alpha}(\omega_1))| + |f(\varphi_{z_\alpha}(\omega)) - f(\varphi_x(\omega))| + \varepsilon \end{aligned}$$

for every α . Taking limits in α we get $|f(\varphi_x(\omega_1)) - f(\varphi_x(\omega))| \leq \varepsilon$ when $\rho(\omega_1, \omega) < \delta$. This proves the continuity of φ_x and (ii).

To prove that φ_x is one-to-one, for an arbitrary $0 < r < 1$ we will construct a function $f \in \mathcal{A}$ (depending on r) such that $(f \circ \varphi_x)(\omega) = \omega$ when $|\omega| < r$. It is convenient to deal with the hyperbolic metric h instead of ρ . Write $s = \log \frac{1+r}{1-r}$. By Lemma 3.2 there is a sequence $\{z_n\}$ in \mathbb{D} whose closure contains x and such that $h(z_n, z_m) > 5s$ if $n \neq m$. Therefore

$$(3.1) \quad h(K_h(z_n, 2s), K_h(z_m, 2s)) \geq s \text{ if } n \neq m.$$

Take $g \in C(\mathbb{D})$ such that $g(\omega) = \omega$ if $h(\omega, 0) < s$ (i.e.: if $|\omega| < r$) and $g(\omega) = 0$ if $h(\omega, 0) > 2s$. Thus $g \circ \varphi_{z_n}$ is supported in $K_h(z_n, 2s)$ and

$$f = \sum_{n \geq 1} (g \circ \varphi_{z_n}) \in C(\mathbb{D}).$$

Since g is uniformly continuous with respect to the Euclidean metric then it is h -uniformly continuous. Hence, given $\varepsilon > 0$ there is δ , with $0 < \delta < s/2$, such that

$$(3.2) \quad |g(\xi_1) - g(\xi_2)| < \varepsilon \text{ if } h(\xi_1, \xi_2) < \delta.$$

Let $\omega_1, \omega_2 \in \mathbb{D}$ such that $h(\omega_1, \omega_2) < \delta$. By (3.1) $K_h(\omega_1, \delta)$ cuts at most one of the disks $K_h(z_n, 2s)$. If it doesn't cut any, then $f(\omega_1) = f(\omega_2) = 0$. If it cuts $K_h(z_{n_0}, 2s)$, then $f(\omega_1) - f(\omega_2) = g(\varphi_{z_{n_0}}(\omega_1)) - g(\varphi_{z_{n_0}}(\omega_2))$, and since $h(\varphi_{z_{n_0}}(\omega_1), \varphi_{z_{n_0}}(\omega_2)) = h(\omega_1, \omega_2) < \delta$ then (3.2) says that $|f(\omega_1) - f(\omega_2)| < \varepsilon$. Thus $f \in \mathcal{A}$.

If k is any positive integer and $|\omega| < r$ then $h(0, \omega) < s$ and $\varphi_{z_k}(\omega) \in K_h(z_k, s)$. So, (3.1) and the inclusion: $\text{supp}(g \circ \varphi_{z_n}) \subset K_h(z_n, 2s)$ imply that $(g \circ \varphi_{z_n})(\varphi_{z_k}(\omega)) = 0$ for $n \neq k$. Consequently

$$f(\varphi_{z_k}(\omega)) = (g \circ \varphi_{z_k})(\varphi_{z_k}(\omega)) = g(\omega) = \omega.$$

Thus, if (z_α) is a net of points in $\{z_n\}$ that tends to x then $(f \circ \varphi_{z_\alpha})(\omega) = \omega$ for every α and every $\omega \in r\mathbb{D}$. Therefore $(f \circ \varphi_x)(\omega) = \omega$ when $\omega \in r\mathbb{D}$.

Suppose that (iii) fails. This means that there are $f \in \mathcal{A}$, $0 < r < 1$ and $\varepsilon > 0$ such that $|(f \circ \varphi_{z_\alpha})(\xi_\alpha) - (f \circ \varphi_x)(\xi_\alpha)| > \varepsilon$ for some points $\xi_\alpha \in r\mathbb{D}$.

We can also assume that $\xi_\alpha \rightarrow \xi$. Since $(f \circ \varphi_{z_\alpha})(\xi) \rightarrow (f \circ \varphi_x)(\xi)$, this contradicts the uniform ρ -continuity of f . ■

3.2. The hyperbolic parts

Definition. If $x, y \in M(\mathcal{A})$ define $\rho(x, y) = \sup \rho(\mathcal{S}, \mathcal{T})$, where \mathcal{S} and \mathcal{T} run over all the separated sequences in \mathbb{D} so that $x \in \overline{\mathcal{S}}$ and $y \in \overline{\mathcal{T}}$. Defining $h(x, y)$ in analogous fashion, we have

$$h(x, y) = \log \frac{1 - \rho(x, y)}{1 + \rho(x, y)}.$$

Lemma 3.5 *Let $x, y \in M(\mathcal{A}) \setminus \mathbb{D}$. Then*

- (1) $\rho(x, y) = a < 1$ if and only if $y = \varphi_x(\omega)$ for some ω with $|\omega| = a$.
- (2) $y = \varphi_x(\xi)$ with $\xi \in \mathbb{D}$ if and only if every separated sequences \mathcal{S}, \mathcal{T} such that $x \in \overline{\mathcal{S}}$ and $y \in \overline{\mathcal{T}}$ satisfy $\rho(\mathcal{T}, \{\varphi_{z_n}(\xi) : z_n \in \mathcal{S}\}) = 0$.
- (3) $h(\varphi_x(\xi_1), \varphi_x(\xi_2)) = h(\xi_1, \xi_2)$ for every $\xi_1, \xi_2 \in \mathbb{D}$.
- (4) h is a $[0, +\infty]$ -valued metric on $M(\mathcal{A})$.

Proof. (1). Suppose that $\rho(x, y) = a < 1$ and take $b \in (a, 1)$. The continuity of φ_x implies that $\varphi_x(\overline{b\mathbb{D}})$ is compact. So, if $y \notin \varphi_x(\overline{b\mathbb{D}})$ there are closed disjoint neighborhoods U of $\varphi_x(\overline{b\mathbb{D}})$ and V of y . Let \mathcal{S} and \mathcal{T} be separated sequences in \mathbb{D} such that $x \in \overline{\mathcal{S}}$ and $y \in \overline{\mathcal{T}}$. If (z_α) is a net in \mathcal{S} that tends to x then $\varphi_{z_\alpha}(\xi) \rightarrow \varphi_x(\xi)$ for every $\xi \in \overline{b\mathbb{D}}$. By a compactness argument $\varphi_{z_\alpha}(\overline{b\mathbb{D}}) \subset U$ for a tail $(z_\alpha)_{\alpha \geq \alpha_0}$ of the original net. Let $\mathcal{S}_1 = \{z_n \in \mathcal{S} : z_n = z_\alpha \text{ for some } \alpha \geq \alpha_0\}$. Then $x \in \overline{\mathcal{S}_1}$ and $\varphi_{z_n}(\overline{b\mathbb{D}}) \subset U$ for every $z_n \in \mathcal{S}_1$. This means that

$$(3.3) \quad K(z_n, b) \subset U \text{ for every } z_n \in \mathcal{S}_1.$$

On the other hand, since V is a neighborhood of y then

$$(3.4) \quad y \in \overline{\mathcal{T}_1}, \quad \text{where } \mathcal{T}_1 = \{z \in \mathcal{T} : z \in V\}.$$

Since U and V are disjoint, (3.3) and (3.4) say that $\rho(\mathcal{S}_1, \mathcal{T}_1) \geq b > a$, contradicting the definition of $\rho(x, y) = a$. Since $b \in (a, 1)$ is arbitrary then $y \in \varphi_x(\overline{a\mathbb{D}})$, so $y = \varphi_x(\omega)$ with $|\omega| \leq a$.

Reciprocally, suppose that $y = \varphi_x(\omega)$ with $|\omega| = a$, and let \mathcal{S}, \mathcal{T} be separated sequence in \mathbb{D} such that $x \in \overline{\mathcal{S}}$ and $y \in \overline{\mathcal{T}}$. If (z_α) is a net in \mathcal{S} that tends to x then $\varphi_{z_\alpha}(\omega) \rightarrow y$. Thus $y \in \overline{\mathcal{T}_1}$, where $\mathcal{T}_1 = \{\varphi_{z_n}(\omega) : z_n \in \mathcal{S}\}$. So, $y \in \overline{\mathcal{T}_1} \cap \overline{\mathcal{T}} \neq \emptyset$ and by Lemma 3.1, $\rho(\mathcal{T}_1, \mathcal{T}) = 0$. That is, given $\varepsilon > 0$ there are $z_n \in \mathcal{S}$ and $\omega_n \in \mathcal{T}$ such that $\rho(\varphi_{z_n}(\omega), \omega_n) < \varepsilon$, which yields

$$\rho(z_n, \omega_n) \leq \rho(z_n, \varphi_{z_n}(\omega)) + \rho(\varphi_{z_n}(\omega), \omega_n) < |\omega| + \varepsilon = a + \varepsilon.$$

So, $\rho(\mathcal{S}, \mathcal{T}) \leq a$ and by definition $\rho(x, y) \leq a$.

(2). The necessity follows from Lemma 3.1. If $y \neq \varphi_x(\xi)$ then $\rho(y, \varphi_x(\xi)) \neq 0$ and there are separated sequences $\mathcal{T}_1, \mathcal{T}_2$ such that $\varphi_x(\xi) \in \overline{\mathcal{T}_1}$, $y \in \overline{\mathcal{T}_2}$ and $\rho(\mathcal{T}_1, \mathcal{T}_2) \geq \delta > 0$. Let \mathcal{S} be a separated sequence such that $x \in \overline{\mathcal{S}}$. Therefore x is in the closure of $\mathcal{S}_1 = \{z_n : \rho(\varphi_{z_n}(\xi), \mathcal{T}_1) < \delta/2\}$, because if $x \in \overline{\mathcal{S} \setminus \mathcal{S}_1}$ then

$$\varphi_x(\xi) \in \overline{\{\varphi_{z_n}(\xi) : z_n \in \mathcal{S} \setminus \mathcal{S}_1\}} \cap \overline{\mathcal{T}_1}$$

while

$$\rho(\{\varphi_{z_n}(\xi) : z_n \in \mathcal{S} \setminus \mathcal{S}_1\}, \mathcal{T}_1) \geq \delta/2,$$

which contradicts Lemma 3.1. So, for $z_n \in \mathcal{S}_1$, $\rho(\varphi_{z_n}(\xi), \mathcal{T}_2) \geq \delta/2$.

(3). Fix $\xi_1, \xi_2 \in \mathbb{D}$. By Lemma 3.2 there is a separated sequence $\mathcal{S} = \{z_k\}$ such that $x \in \overline{\mathcal{S}}$ and $h(z_n, z_m) \geq h(\xi_1, \xi_2) + h(0, \xi_1) + h(0, \xi_2)$ if $n \neq m$. Since

$$\begin{aligned} h(z_n, z_m) &\leq h(z_n, \varphi_{z_n}(\xi_1)) + h(\varphi_{z_n}(\xi_1), \varphi_{z_m}(\xi_2)) + h(\varphi_{z_m}(\xi_2), z_m) \\ &= h(0, \xi_1) + h(0, \xi_2) + h(\varphi_{z_n}(\xi_1), \varphi_{z_m}(\xi_2)), \end{aligned}$$

then $h(\varphi_{z_n}(\xi_1), \varphi_{z_m}(\xi_2)) \geq h(\xi_1, \xi_2)$ if $n \neq m$. Therefore

$$h(\{\varphi_{z_n}(\xi_1)\}_{n \geq 1}, \{\varphi_{z_m}(\xi_2)\}_{m \geq 1}) = h(\varphi_{z_n}(\xi_1), \varphi_{z_n}(\xi_2)) = h(\xi_1, \xi_2),$$

implying that $h(\varphi_x(\xi_1), \varphi_x(\xi_2)) \geq h(\xi_1, \xi_2)$. For the other inequality let $\mathcal{T}_1, \mathcal{T}_2$ be separated sequences such that $\varphi_x(\xi_1) \in \overline{\mathcal{T}_1}$ and $\varphi_x(\xi_2) \in \overline{\mathcal{T}_2}$. For a separated sequence \mathcal{S} such that $x \in \overline{\mathcal{S}}$ and $\varepsilon > 0$ write

$$\mathcal{S}' = \{z_n \in \mathcal{S} : h(\varphi_{z_n}(\xi_1), \mathcal{T}_1) < \varepsilon, h(\varphi_{z_n}(\xi_2), \mathcal{T}_2) < \varepsilon\}$$

and $\mathcal{S}'' = \mathcal{S} \setminus \mathcal{S}'$.

By (2) $x \notin \overline{\mathcal{S}''}$. So, $x \in \overline{\mathcal{S}'}$ and

$$h(\mathcal{T}_1, \mathcal{T}_2) \leq h(\varphi_{z_n}(\xi_1), \varphi_{z_n}(\xi_2)) + 2\varepsilon = h(\xi_1, \xi_2) + 2\varepsilon.$$

That is, $h(\varphi_x(\xi_1), \varphi_x(\xi_2)) \leq h(\xi_1, \xi_2) + 2\varepsilon$. (4). We must prove only that given $x, y, z \in M(\mathcal{A})$,

$$(3.5) \quad h(x, y) \leq h(x, z) + h(z, y)$$

The inequality is obvious if its right member is infinite. Otherwise (1) says that $x = \varphi_z(\xi_1)$ and $y = \varphi_z(\xi_2)$ for some $\xi_1, \xi_2 \in \mathbb{D}$. Then (3.5) becomes

$$h(\varphi_z(\xi_1), \varphi_z(\xi_2)) \leq h(\varphi_z(\xi_1), \varphi_z(0)) + h(\varphi_z(0), \varphi_z(\xi_2)),$$

which holds by (3). ■

Definition. If $x \in M(\mathcal{A})$ define the hyperbolic part of x as

$$H(x) = \{\varphi_x(\omega) : \omega \in \mathbb{D}\}.$$

Observe that (1) of Lemma 3.5 implies that

$$H(x) = \{y \in M(\mathcal{A}) : \rho(x, y) < 1\} = \{y \in M(\mathcal{A}) : h(x, y) < \infty\}$$

and by (4) of the same lemma, $\{H(x) : x \in M(\mathcal{A})\}$ is a partition of $M(\mathcal{A})$. In fact if $z \in H(x) \cap H(y)$ then for any $u \in H(x)$,

$$h(u, y) \leq h(u, x) + h(x, z) + h(z, y) < \infty.$$

So, $H(x) \subset H(y)$ and by symmetry they coincide.

Lemma 3.6 *The map $x \mapsto \varphi_x$ from $M(\mathcal{A})$ into $M(\mathcal{A})^{\mathbb{D}}$ is continuous.*

Proof. Let (x_α) be a net in $M(\mathcal{A})$ that tends to x and $\xi \in \mathbb{D}$. We must show that if (x_β) is a subnet such that $\varphi_{x_\beta}(\xi) \rightarrow y$ then $y = \varphi_x(\xi)$. Let $\mathcal{S} = \{z_n\}$ and $\mathcal{T} = \{\omega_n\}$ be separated sequences such that $x \in \overline{\mathcal{S}}$ and $y \in \overline{\mathcal{T}}$. For $\delta > 0$ write

$$U = \bigcup_{n \geq 1} K(z_n, \delta) \quad \text{and} \quad V = \bigcup_{n \geq 1} K(\omega_n, \delta).$$

Since there is $f \in \mathcal{A}$ such that $f(z_n) = 0$ for all n and $f \equiv 1$ on $\mathbb{D} \setminus U$ then $\overline{U} \supset \{m \in M(\mathcal{A}) : |f(m)| < 1/2\}$, a neighborhood of x . So, \overline{U} is a neighborhood of x and by the same reason \overline{V} is a neighborhood of y . Since $x_\beta \rightarrow x$ and $\varphi_{x_\beta}(\xi) \rightarrow y$, there is β_0 such that for every $\beta \geq \beta_0$,

- (i) $\varphi_{x_\beta}(\xi) \in \overline{V}$, and
- (ii) $x_\beta \in \overline{\mathcal{S}_\beta}$, where $\mathcal{S}_\beta = \{z_n(\beta)\}_{n \geq 1}$ is a separated sequence in U .

Assume that $\beta \geq \beta_0$. Since

$$\varphi_{x_\beta}(\xi) \in \overline{\{\varphi_{z_n(\beta)}(\xi)\}_{n \geq 1}} \cap \overline{(\cup_n K(\omega_n, \delta))}$$

then Lemma 3.1 says that $\rho(\{\varphi_{z_n(\beta)}(\xi)\}, \mathcal{T}) \leq \delta$. So, there is n_0 such that $\rho(\varphi_{z_{n_0}(\beta)}(\xi), \mathcal{T}) < 2\delta$. On the other hand, by definition of U and (ii) there is some $z_{k_0} \in \mathcal{S}$ such that $\rho(z_{k_0}, z_{n_0}(\beta)) \leq \delta$. Since there is $c_\xi > 0$ such that

$$\rho(\varphi_{z_{k_0}}(\xi), \varphi_{z_{n_0}(\beta)}(\xi)) \leq c_\xi \rho(z_{k_0}, z_{n_0}(\beta)) \leq c_\xi \delta,$$

then $\rho(\varphi_{z_{k_0}}(\xi), \mathcal{T}) \leq (c_\xi + 2)\delta$. This shows that

$$\rho(\{\varphi_{z_n}(\xi) : z_n \in \mathcal{S}\}, \mathcal{T}) \leq (c_\xi + 2)\delta,$$

and since $\delta > 0$ is arbitrary, $\rho(\{\varphi_{z_n}(\xi) : z_n \in \mathcal{S}\}, \mathcal{T}) = 0$. Since \mathcal{S} and \mathcal{T} are arbitrary separated sequences such that $x \in \overline{\mathcal{S}}$ and $y \in \overline{\mathcal{T}}$ then (2) of Lemma 3.5 tells us that $y = \varphi_x(\xi)$. ■

3.3. Hyperbolic algebras

A closed self-adjoint subalgebra \mathcal{B} of \mathcal{A} that separates the points of \mathbb{D} and contains the constants will be called a *prehyperbolic* algebra. For such \mathcal{B} , Theorem 4.28 of [13] implies that whenever $b \in \mathcal{B}$ is invertible in \mathcal{A} then the inverse belongs to \mathcal{B} . Hence, the disk is dense in $M(\mathcal{B})$, because if there exists $y \in M(\mathcal{B})$ that is not in the closure of \mathbb{D} then there is $f \in \mathcal{B}$ such that $f(y) = 0$ and $|f| \geq \delta > 0$ on \mathbb{D} . Since clearly f is invertible in \mathcal{A} , then so is in \mathcal{B} and consequently f cannot vanish anywhere in $M(\mathcal{B})$, a contradiction.

The inclusion of \mathcal{B} in \mathcal{A} induces by transposition a projection $\pi : M(\mathcal{A}) \rightarrow M(\mathcal{B})$. Since $\pi(\mathbb{D}) = \mathbb{D}$ is dense in $M(\mathcal{B})$ then π is onto. For a set $E \subset \mathbb{D}$ we write \overline{E}^M , with $M = M(\mathcal{A})$ or $M(\mathcal{B})$, to distinguish between closures in the corresponding space. No distinction will be made for the closure of sets in \mathbb{C} .

A closed set $F \subset M(\mathcal{A})$ will be called saturated if $H(x) \subset F$ whenever $x \in F$. If $\pi : M(\mathcal{A}) \rightarrow M(\mathcal{B})$ is the natural projection, write

$$G_{\mathcal{B}} = \{y \in M(\mathcal{B}) : \pi^{-1}(y) \text{ is a singleton}\}$$

and

$$\Gamma_{\mathcal{B}} = \{y \in M(\mathcal{B}) : \pi^{-1}(y) \text{ is saturated}\}.$$

That is, if $y \in M(\mathcal{B})$ then $y \in G_{\mathcal{B}}$ if and only if \mathcal{B} separates every $x \in \pi^{-1}(y)$ from any other point of $M(\mathcal{A})$ (so $\pi^{-1}(y) = \{x\}$), and $y \in \Gamma_{\mathcal{B}}$ if and only if $b \circ \varphi_x$ is constant for all $x \in \pi^{-1}(y)$ and $b \in \mathcal{B}$. Since no single point is a saturated set then $G_{\mathcal{B}} \cap \Gamma_{\mathcal{B}} = \emptyset$. In addition, there could be points in $M(\mathcal{B})$ that are not in $G_{\mathcal{B}} \cup \Gamma_{\mathcal{B}}$. We will be interested in the cases that exclude the last possibility.

Definition. A prehyperbolic algebra \mathcal{B} will be called hyperbolic if $M(\mathcal{B}) = G_{\mathcal{B}} \cup \Gamma_{\mathcal{B}}$. That is, if $\pi^{-1}(\pi(x)) = \{x\}$ or contains $H(x)$ for every $x \in M(\mathcal{A})$.

Lemma 3.7 *Let $\mathcal{B} \subset \mathcal{A}$ be a prehyperbolic algebra. Then*

- (1) $\Gamma_{\mathcal{B}}$ is closed,
- (2) the restriction $\pi_0 : \pi^{-1}(G_{\mathcal{B}}) \rightarrow G_{\mathcal{B}}$ of π is an onto homeomorphism.

Proof. (1). If x is in the closure of $\pi^{-1}(\Gamma_{\mathcal{B}})$ take a net (x_{α}) in $\pi^{-1}(\Gamma_{\mathcal{B}})$ that tends to x . By definition of $\Gamma_{\mathcal{B}}$, $f \circ \varphi_{x_{\alpha}}$ is constant for every $f \in \mathcal{B}$. Hence, if $\omega \in \mathbb{D}$ and $f \in \mathcal{B}$, Lemma 3.6 gives

$$f(x) - f(\varphi_x(\omega)) = \lim f(x_{\alpha}) - f(\varphi_{x_{\alpha}}(\omega)) = 0,$$

implying that $f \circ \varphi_x \equiv f(x)$, so $x \in \pi^{-1}(\Gamma_{\mathcal{B}})$. That is, $\pi^{-1}(\Gamma_{\mathcal{B}})$ is closed in $M(\mathcal{A})$ and then $\pi(\pi^{-1}(\Gamma_{\mathcal{B}}))$ is closed in $M(\mathcal{B})$.

(2). By definition of $G_{\mathcal{B}}$, π_0 is one-to-one and onto, so we must show that $\pi_0^{-1} : G_{\mathcal{B}} \rightarrow \pi^{-1}(G_{\mathcal{B}})$ is continuous. Let (y_{α}) be a net in $G_{\mathcal{B}}$ such that $y_{\alpha} \rightarrow y \in G_{\mathcal{B}}$ and let $x_{\alpha} \in \pi^{-1}(G_{\mathcal{B}})$ such that $\pi(x_{\alpha}) = y_{\alpha}$. If $(x_{\alpha_{\beta}})$ is a convergent subnet of (x_{α}) , say to x , then $y_{\alpha_{\beta}} = \pi(x_{\alpha_{\beta}}) \rightarrow \pi(x) = y$. So, $x \in \pi^{-1}(y)$, but since $y \in G_{\mathcal{B}}$ then $\pi^{-1}(y) = \{x\}$. Hence every convergent subnet of (x_{α}) tends to x , and then $x_{\alpha} \rightarrow x$. ■

Proposition 3.8 *Let $\mathcal{B} \subset \mathcal{A}$ be a prehyperbolic algebra and $y \in M(\mathcal{B})$. The following conditions are equivalent*

- (a₁) $y \in \Gamma_{\mathcal{B}}$.
- (a₂) $f \circ \varphi_{z_{\alpha}} \rightarrow c \in \mathbb{C}$ uniformly on compact sets for every net (z_{α}) in \mathbb{D} tending to y and every $f \in \mathcal{B}$.
- (a₃) For every separated sequence \mathcal{S} such that $y \in \overline{\mathcal{S}}^{M(\mathcal{B})}$ and every $f \in \mathcal{B}$ there is a subsequence $\{z_n\}$ of \mathcal{S} (depending on f) such that $f \circ \varphi_{z_n} \rightarrow c \in \mathbb{C}$ pointwise on \mathbb{D} .

Proof. (a₁) \Rightarrow (a₂). If $y \in \Gamma_{\mathcal{B}}$ then $\pi^{-1}(y)$ is saturated. Let (z_{α}) be a net in \mathbb{D} such that $z_{\alpha} \rightarrow y$ in $M(\mathcal{B})$ and take a subnet $(z_{\alpha_{\beta}})$ that converges in $M(\mathcal{A})$, say to x . Thus $\pi(z_{\alpha_{\beta}}) \rightarrow \pi(x) = y$ in $M(\mathcal{B})$, saying that $x \in \pi^{-1}(y)$. Since $H(x) \subset \pi^{-1}(y)$ (because it is saturated) then

$$f(\varphi_x(\xi)) = \lim f(\varphi_{z_{\alpha_{\beta}}}(\xi)) = \text{const.} = \lim f(\varphi_{z_{\alpha_{\beta}}}(0)) = \lim f(z_{\alpha_{\beta}}) = f(y)$$

for every $f \in \mathcal{B}$ and $\xi \in \mathbb{D}$. This proves that whenever $(z_{\alpha_{\beta}})$ is a subnet of (z_{α}) that converges in $M(\mathcal{A})$ then $f \circ \varphi_{z_{\alpha_{\beta}}} \rightarrow f(y)$ pointwise. By Lemma 3.4 the convergence is also uniform on compact sets, and consequently $f \circ \varphi_{z_{\alpha}} \rightarrow f(y)$ in that way.

(a₂)⇒(a₃). If $y \in \overline{\mathcal{S}}^{M(\mathcal{B})}$ there is a net (z_α) in \mathcal{S} such that $z_\alpha \rightarrow y$ in $M(\mathcal{B})$. If $f \in \mathcal{B}$ then by (a₂), $f \circ \varphi_{z_\alpha} \rightarrow c \in \mathbb{C}$ uniformly on compact sets. Therefore for any positive integer n there is some z_α (that we rename as z_n) such that

$$\sup\{|(f \circ \varphi_{z_n})(\omega) - c| : |\omega| \leq 1 - n^{-1}\} \leq n^{-1}.$$

Therefore $\{z_n\}$ is a subsequence of \mathcal{S} that satisfies (a₃).

(a₃)⇒(a₁). We will show that (a₃) fails when (a₁) fails. If $y \notin \Gamma_{\mathcal{B}}$ there is $x \in \pi^{-1}(y)$ such that $H(x) \not\subset \pi^{-1}(y)$. Therefore there is $f \in \mathcal{B}$ such that $f \circ \varphi_x \neq \text{const.}$, or what is the same, $(f \circ \varphi_x)(\omega) \neq f(x)$ for some $\omega \in \mathbb{D}$. Put $\eta = |(f \circ \varphi_x)(\omega) - f(x)| > 0$. If \mathcal{S} is any separated sequence such that $x \in \overline{\mathcal{S}}^{M(\mathcal{A})}$ and we take

$$\mathcal{S}_1 = \{z \in \mathcal{S} : |(f \circ \varphi_{z_n})(\omega) - f(z_n)| \geq \eta/2\}$$

then $x \in \overline{\mathcal{S}}_1^{M(\mathcal{A})}$. Hence $y = \pi(x) \in \overline{\mathcal{S}}_1^{M(\mathcal{B})}$ and (a₃) fails for \mathcal{S}_1 and f . ■

Suppose that f is a continuous function from $M(\mathcal{A})$ into a topological space T . If \mathcal{B} is a hyperbolic algebra, the restriction $f|_{\mathbb{D}}$ admits a continuous extension from $M(\mathcal{B})$ into T if and only if $f(\pi^{-1}(y)) = \text{const.}$ for every $y \in \Gamma_{\mathcal{B}}$. In particular, for $T = \mathbb{C}$ we obtain that $f \in \mathcal{A}$ belongs to \mathcal{B} if and only if $f(\pi^{-1}(y)) = \text{const.}$ for every $y \in \Gamma_{\mathcal{B}}$.

Let $\mathcal{B} \subset L^\infty(\mathbb{D})$ be a closed algebra. A sequence $\{z_n\} \subset \mathbb{D}$ is called interpolating for \mathcal{B} if for every $\{\eta_n\} \in \ell^\infty$ there exists $f \in \mathcal{B}$ such that $f(z_n) = \eta_n$ for every n . It is clear that if \mathcal{B} is a subalgebra of \mathcal{A} then every interpolating sequence for \mathcal{B} must be separated and that every separated sequence is interpolating for \mathcal{A} . We say that $f \in \mathcal{A}$ separates two sets $E, F \subset M(\mathcal{A})$ when $\overline{f(E)} \cap \overline{f(F)} = \emptyset$.

Proposition 3.9 *Let $\mathcal{B} \subset \mathcal{A}$ be a prehyperbolic algebra. For $y \in M(\mathcal{B})$ consider the following conditions*

(b₁) $y \in G_{\mathcal{B}}$.

(b₂) *There is an interpolating sequence $\mathcal{S} = \{z_n\}$ for \mathcal{B} , whose closure in $M(\mathcal{B})$ contains y , such that for every $\delta > 0$ sufficiently small there exists $f \in \mathcal{B}$ that separates $\{z_n\}$ from $\mathbb{D} \setminus \bigcup_n K(z_n, \delta)$.*

Then (b₂) implies (b₁), and if \mathcal{B} is hyperbolic, (b₁) implies (b₂).

Proof. (b₂)⇒(b₁). Let $y \in M(\mathcal{B})$ and \mathcal{S} as in (b₂). We claim that $\pi^{-1}(y) \subset \overline{\mathcal{S}}^{M(\mathcal{A})}$, because otherwise there is $x \in \pi^{-1}(y)$ and a separated sequence $\mathcal{T} \subset \mathbb{D}$, with $x \in \overline{\mathcal{T}}^{M(\mathcal{A})}$, such that $\rho(\mathcal{S}, \mathcal{T}) \geq \alpha > 0$. The continuity of π implies that $y = \pi(x) \in \overline{\mathcal{T}}^{M(\mathcal{B})}$, but this is not possible because by hypothesis there is $f \in \mathcal{B}$ such that $\overline{f(\mathcal{S})} \cap \overline{f(\mathcal{T})} = \emptyset$, which contradicts $y \in \overline{\mathcal{S}}^{M(\mathcal{B})} \cap \overline{\mathcal{T}}^{M(\mathcal{B})}$.

Now suppose that there are two different points $x_1, x_2 \in \pi^{-1}(y)$. Then there is a disjoint decomposition $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$, where

$$x_1 \in \overline{\mathcal{S}}_1^{M(\mathcal{A})} \quad \text{and} \quad x_2 \in \overline{\mathcal{S}}_2^{M(\mathcal{A})}.$$

Since \mathcal{S} is interpolating for \mathcal{B} there exists $f \in \mathcal{B}$ that separates \mathcal{S}_1 from \mathcal{S}_2 , leading to the same contradiction obtained before. Hence, $\pi^{-1}(y)$ is a single point.

(b₁) \Rightarrow (b₂) for \mathcal{B} hyperbolic. If $y \in G_{\mathcal{B}}$ then $\pi^{-1}(y) = \{x\}$ for some $x \in M(\mathcal{A})$. Since $\pi^{-1}(\Gamma_{\mathcal{B}})$ is closed in $M(\mathcal{A})$ (by Lemma 3.7) and $x \notin \pi^{-1}(\Gamma_{\mathcal{B}})$ then there is a closed neighborhood F of x in $M(\mathcal{A})$ such that $F \cap \pi^{-1}(\Gamma_{\mathcal{B}}) = \emptyset$. Hence there is $f \in \mathcal{A}$ such that $f \equiv 1$ on F and $f \equiv 0$ on $\pi^{-1}(\Gamma_{\mathcal{B}})$.

Let $\mathcal{T} \subset \mathbb{D}$ be a separated sequence such that $x \in \overline{\mathcal{T}}^{M(\mathcal{A})}$. Since $f \equiv 1$ on a neighborhood of x then $x \in \overline{\mathcal{S}}^{M(\mathcal{A})}$, where

$$\mathcal{S} = \{z \in \mathcal{T} : f(z) = 1\} = \{z_n\}.$$

Hence, $y = \pi(x) \in \overline{\mathcal{S}}^{M(\mathcal{B})}$. Observe also that $\overline{\mathcal{S}}^{M(\mathcal{A})} \subset F \subset \pi^{-1}(G_{\mathcal{B}})$.

Let $\{\eta_n\}$ be an arbitrary sequence in ℓ^∞ and take $g \in \mathcal{A}$ such that $g(z_n) = \eta_n$ for every n . Since $f \equiv 0$ on $\pi^{-1}(\Gamma_{\mathcal{B}})$ then so is $h = fg \in \mathcal{A}$, and consequently $h \in \mathcal{B}$. In addition, $h(z_n) = f(z_n)g(z_n) = \eta_n$ for every n , which shows that \mathcal{S} is interpolating for \mathcal{B} . Since f is ρ -uniformly continuous and $f(z_n) = 1$ for all n then

$$\bigcup K(z_n, \delta) \subset \{z : |f(z)| > 1/2\}$$

when $\delta > 0$ is small enough. Take $a \in \mathcal{A}$ such that

$$(3.6) \quad a(z_n) = 1 \text{ for all } n, \text{ and } a \equiv 0 \text{ on } \mathbb{D} \setminus \bigcup_n K(z_n, \delta).$$

Since $f \equiv 0$ on $\pi^{-1}(\Gamma_{\mathcal{B}})$ then

$$\pi^{-1}(\Gamma_{\mathcal{B}}) \subset \overline{\{z : |f(z)| < 1/4\}}^{M(\mathcal{A})} \subset \overline{\bigcup_n K(z_n, \delta)}^{M(\mathcal{A})},$$

implying that $a \equiv 0$ on $\pi^{-1}(\Gamma_{\mathcal{B}})$. Hence $a \in \mathcal{B}$ and (3.6) says that it separates \mathcal{S} from $\mathbb{D} \setminus \bigcup_n K(z_n, \delta)$. So (b₂) holds. ■

Propositions 3.8 and 3.9 provide criteria to decide whether a given pre-hyperbolic algebra is hyperbolic or not. Let us summarize these criteria in the next corollary.

Corollary 3.10 *A prehyperbolic algebra \mathcal{B} is hyperbolic if and only if every $y \in M(\mathcal{B})$ satisfies some of the conditions (a₁), (a₂), (a₃) or some of the conditions (b₁), (b₂).*

4. Operator-valued compact maps

We recall that if $S \in \mathfrak{L}(L_a^2)$ and $z \in \mathbb{D}$ then $S_z = U_z S U_z$, where $U_z f = (f \circ \varphi_z) \varphi'_z$. Consider the map $\Psi_S : \mathbb{D} \rightarrow \mathfrak{L}(L_a^2)$ given by $\Psi_S(z) = S_z$. We will study the possibility to extend Ψ_S continuously to $M(\mathcal{A})$ when $\mathfrak{L}(L_a^2)$ is provided with the weak or the strong operator topology (WOT and SOT, respectively). We will also look for a possible extension to $M(\mathcal{B})$, where \mathcal{B} is an arbitrary hyperbolic algebra.

Theorem 4.1 *Let (E, d) be a metric space and $f : \mathbb{D} \rightarrow E$ be a continuous map. Then f admits a continuous extension from $M(\mathcal{A})$ into E if and only if f is uniformly (ρ, d) continuous and $\overline{f(\mathbb{D})}$ is compact.*

Proof. Suppose that $f \in C(M(\mathcal{A}), E)$. Since \mathbb{D} is dense in the compact space $M(\mathcal{A})$ then $\overline{f(\mathbb{D})} = f(M(\mathcal{A}))$ is compact. If f is not uniformly (ρ, d) continuous there are two sequences $z_n, \omega_n \in \mathbb{D}$ such that $\rho(z_n, \omega_n) \rightarrow 0$ and $d(f(z_n), f(\omega_n)) \geq \delta > 0$ for every n . By the continuity of f on \mathbb{D} the sequence does not accumulate on \mathbb{D} . Let $x \in \overline{\{z_n\}^{M(\mathcal{A})}} \setminus \mathbb{D}$ and (z_α) be a subnet of $\{z_n\}$ that tends to x . Since every z_α is some $z_{n(\alpha)}$, writing $\omega_\alpha = \omega_{n(\alpha)}$ we have a subnet (ω_α) of the sequence $\{\omega_n\}$ such that

$$(4.1) \quad \rho(z_\alpha, \omega_\alpha) \rightarrow 0 \text{ and } d(f(z_\alpha), f(\omega_\alpha)) \geq \delta \text{ for all } \alpha.$$

The first condition in (4.1) implies that $g(\omega_\alpha) \rightarrow g(x)$ for every $g \in \mathcal{A}$, meaning that $\omega_\alpha \rightarrow x$ in $M(\mathcal{A})$. Since f is continuous on $M(\mathcal{A})$ then $\lim f(\omega_\alpha) = f(x) = \lim f(z_\alpha)$, contradicting (4.1).

Now assume that f is uniformly (ρ, d) continuous on \mathbb{D} and $\overline{f(\mathbb{D})}$ is compact. For $x \in M(\mathcal{A})$ write

$$F(x) \stackrel{\text{def}}{=} \{\lambda \in E : f(z_\alpha) \rightarrow \lambda, \text{ for some net } z_\alpha \rightarrow x, z_\alpha \in \mathbb{D}\}.$$

The compactness of $\overline{f(\mathbb{D})}$ assures that $F(x)$ is nonempty. Then F is a multivalued function defined on $M(\mathcal{A})$, and a standard diagonal argument shows that f can be extended continuously to $M(\mathcal{A})$ if and only if $F(x)$ is single-valued for every $x \in M(\mathcal{A})$. So, let $x \in M(\mathcal{A})$ and assume that there are $\lambda_1, \lambda_2 \in F(x)$ such that $d(\lambda_1, \lambda_2) = \alpha > 0$. Let $B(\lambda, r)$ denote the open ball in E of center $\lambda \in E$ and radius $r > 0$, and consider the sets

$$V_i = \{z \in \mathbb{D} : f(z) \in B(\lambda_i, \alpha/4)\}, \quad i = 1, 2.$$

Since $\lambda_i \in F(x)$ then $x \in \overline{V_i}^{M(\mathcal{A})}$ for $i = 1, 2$. Lemma 3.1 then tells us that $\rho(V_1, V_2) = 0$. On the other hand,

$$d(f(V_1), f(V_2)) \geq d(B(\lambda_1, \alpha/4), B(\lambda_2, \alpha/4)) \geq \frac{\alpha}{2}.$$

By the uniform (ρ, d) -continuity of f , the last inequality implies that $\rho(V_1, V_2) > 0$, a contradiction. ■

Lemma 4.2 For $z, \alpha \in \mathbb{D}$ put $\lambda = \lambda(z, \alpha) = (\alpha\bar{z} - 1)/(1 - z\bar{\alpha})$. Then $U_{\varphi_z(\alpha)}U_z = V_\lambda U_\alpha$, where $(V_\lambda f)(\omega) = \lambda f(\lambda\omega)$ for $f \in L_a^2$.

Proof. Since the function $\varphi_{\varphi_z(\alpha)} \circ \varphi_z \circ \varphi_\alpha$ is an automorphism that fixes the origin, it must be a rotation. A little bit of algebra shows that this function maps λ to 1. Since $\varphi_{\varphi_z(\alpha)}$ is its own inverse then $\varphi_z \circ \varphi_\alpha(\lambda\omega) = \varphi_{\varphi_z(\alpha)}(\omega)$. Therefore

$$\begin{aligned} (U_{\varphi_z(\alpha)}U_z f)(\omega) &= (f \circ \varphi_z \circ \varphi_{\varphi_z(\alpha)})(\omega) \varphi'_z(\varphi_{\varphi_z(\alpha)}(\omega)) \varphi'_{\varphi_z(\alpha)}(\omega) \\ &= (f \circ \varphi_z \circ \varphi_z \circ \varphi_\alpha)(\lambda\omega) \varphi'_z(\varphi_z \circ \varphi_\alpha(\lambda\omega)) \varphi'_z(\varphi_\alpha(\lambda\omega)) \varphi'_\alpha(\lambda\omega) \lambda \\ &= (f \circ \varphi_\alpha)(\lambda\omega) \varphi'_\alpha(\lambda\omega) \lambda = (V_\lambda U_\alpha f)(\omega), \end{aligned}$$

where the third equality holds because since $\varphi_z \circ \varphi_z = id$ then $(\varphi'_z \circ \varphi_z)\varphi'_z = 1$. ■

Lemma 4.3 Let $f \in L_a^2$ and $\varepsilon > 0$. Then there is $\delta = \delta(f, \varepsilon) > 0$ such that

$$\rho(z_1, z_2) < \delta \Rightarrow \|U_{z_1}f - U_{z_2}f\| < \varepsilon.$$

Proof. Since the polynomials are dense in L_a^2 and $\|U_z\| = 1$ for every $z \in \mathbb{D}$, it is enough to assume that f is a polynomial. If $\rho(z_1, z_2) < \delta$ then $z_2 = \varphi_{z_1}(\alpha)$ with $|\alpha| < \delta$. By the previous lemma,

$$(I - U_{\varphi_{z_1}(\alpha)}U_{z_1})f(\omega) = f(\omega) - f\left(\frac{\alpha - \lambda\omega}{1 - \bar{\alpha}\lambda\omega}\right) \left(\frac{|\alpha|^2 - 1}{1 - \bar{\alpha}\lambda\omega}\right) \lambda,$$

where λ comes from the lemma. When $\alpha \rightarrow 0$ we have $\lambda(z_1, \alpha) \rightarrow -1$ uniformly in z_1 , so the above expression tends to 0 uniformly in z_1 and ω . Hence,

$$\|U_{z_1}f - U_{\varphi_{z_1}(\alpha)}f\| = \|(U_{\varphi_{z_1}(\alpha)}U_{z_1} - I)f\| < \varepsilon$$

if $|\alpha|$ is small enough. That is, if δ is small enough. ■

Proposition 4.4 Let $S \in \mathfrak{L}(L_a^2)$. Then the map $\Psi_S : \mathbb{D} \rightarrow (\mathfrak{L}(L_a^2), WOT)$ extends continuously to $M(\mathcal{A})$.

Proof. The closed ball $B(0, \|S\|) \subset \mathfrak{L}(L_a^2)$ of center 0 and radius $\|S\|$ is compact and metrizable with the WOT-topology. Since $\Psi_S(\mathbb{D})$ is contained in $B(0, \|S\|)$, Theorem 4.1 reduces the problem to show that Ψ_S is uniformly continuous from the disk with the pseudo-hyperbolic metric into $B(0, \|S\|)$ with the weak operator topology. This amounts to see that for every $f, g \in L_a^2$, the function $z \mapsto \langle S_z f, g \rangle$ is uniformly continuous from (\mathbb{D}, ρ) into $(\mathbb{C}, |\cdot|)$. For $z_1, z_2 \in \mathbb{D}$ we have

$$U_{z_1}S U_{z_1} - U_{z_2}S U_{z_2} = U_{z_1}S(U_{z_1} - U_{z_2}) + (U_{z_1} - U_{z_2})S U_{z_2} = A + B.$$

If $f, g \in L_a^2$ then $|\langle Af, g \rangle| \leq \|U_{z_1}S\| \|(U_{z_1} - U_{z_2})f\|_2 \|g\|_2$ and $|\langle Bf, g \rangle| = |\langle f, B^*g \rangle| \leq \|f\|_2 \|U_{z_2}S^*\| \|(U_{z_1} - U_{z_2})g\|_2$. By Lemma 4.3 both expressions can be made small if we take $\rho(z_1, z_2)$ small enough. ■

Theorem 4.5 *Let $S \in \mathfrak{T}(\mathcal{A})$. Then the map*

$$\Psi_S : \mathbb{D} \rightarrow (\mathfrak{L}(L_a^2), SOT)$$

extends continuously to $M(\mathcal{A})$. In addition, $\Psi_S(M(\mathcal{A})) \subset \mathfrak{T}(\mathcal{A})$.

Proof. First suppose that $S = T_a$, with $a \in \mathcal{A}$. If $z \in \mathbb{D}$ tends to $x \in M(\mathcal{A})$, Lemma 3.4 says that $a \circ \varphi_z \rightarrow a \circ \varphi_x$ uniformly on compact sets. Thus, if $f \in L_a^2$ and $0 < r < 1$,

$$\|(T_{a \circ \varphi_z} - T_{a \circ \varphi_x})f\|^2 \leq \sup_{rD} |a \circ \varphi_z - a \circ \varphi_x|^2 \|f\|^2 + 2\|a\|_\infty^2 \int_{D \setminus rD} |f|^2 dA.$$

We can choose some $r = r(f, \|a\|_\infty)$ close enough to 1 so that the second term is smaller than a given $\varepsilon > 0$, and for such r the first term tends to 0 as $z \rightarrow x$. Since

$$\Psi_{S+T} = \Psi_S + \Psi_T,$$

the case of a polynomial in Toeplitz operators reduces to the case $S = T_{a_1} \dots T_{a_k}$, where $a_j \in \mathcal{A}$ and $\|a_j\|_\infty \leq 1$ for $j = 1, \dots, k$. Consider the operators

$$S_j = \begin{cases} T_{a_1 \circ \varphi_z} \dots T_{a_{j-1} \circ \varphi_z} T_{a_j \circ \varphi_x} \dots T_{a_k \circ \varphi_x} & \text{if } 1 \leq j \leq k \\ T_{a_1 \circ \varphi_z} \dots T_{a_k \circ \varphi_z} & \text{if } j = k + 1 \end{cases}$$

If $f \in L_a^2$ then

$$\|(S_{k+1} - S_1)f\| \leq \sum_{j=1}^k \|(S_{j+1} - S_j)f\|,$$

and since we have proved that $T_{a_j \circ \varphi_z} - T_{a_j \circ \varphi_x} \rightarrow 0$ in the strong operator topology as $z \rightarrow x$, then

$$\begin{aligned} \|(S_{j+1} - S_j)f\| &= \|T_{a_1 \circ \varphi_z} \dots T_{a_{j-1} \circ \varphi_z} (T_{a_j \circ \varphi_z} - T_{a_j \circ \varphi_x}) T_{a_{j+1} \circ \varphi_x} \dots T_{a_k \circ \varphi_x} f\| \\ &\leq \|(T_{a_j \circ \varphi_z} - T_{a_j \circ \varphi_x}) T_{a_{j+1} \circ \varphi_x} \dots T_{a_k \circ \varphi_x} f\| \rightarrow 0 \end{aligned}$$

when $z \rightarrow x$. Finally, if $S \in \mathfrak{T}(\mathcal{A})$ is arbitrary, given $\varepsilon > 0$ there is a polynomial in Toeplitz operators with symbols in \mathcal{A} , say T , such that $\|S - T\| < \varepsilon$. By Proposition 4.4 there is some $S_x \in \mathfrak{L}(L_a^2)$ (i.e.: $S_x = \Psi_S(x)$) such that

$$S_z - T_z \rightarrow S_x - T_x \quad \text{weakly when } z \rightarrow x.$$

Weak limits do not increase norms, so $\|S_x - T_x\| \leq \varepsilon$. The result follows because $\|S_z - T_z\| < \varepsilon$ for all $z \in \mathbb{D}$ and $T_z \rightarrow T_x$ strongly when $z \rightarrow x$. ■

Corollary 4.6 *If $S \in \mathfrak{L}(L_a^2)$ and $n \geq 0$ is an integer then $B_n S \in \mathcal{A}$. Besides, $B_n S_x = (B_n S) \circ \varphi_x$ for every $x \in M(\mathcal{A})$.*

Proof. By (2.1) and Lemma 2.2

$$(B_n S)(z) = ((B_n S) \circ \varphi_z)(0) = (B_n S_z)(0) = (n+1) \sum_{j=0}^n \binom{n}{j} (-1)^j \langle S_z \omega^j, \omega^j \rangle.$$

Since by Proposition 4.4 the map $z \mapsto \langle S_z \omega^j, \omega^j \rangle$ extends continuously to $M(\mathcal{A})$, it belongs to \mathcal{A} for every $0 \leq j \leq n$. For the second assertion take a net (z_α) in \mathbb{D} that tends to x and then take limits in the equality $(B_n S_{z_\alpha})(\xi) = (B_n S)(\varphi_{z_\alpha}(\xi))$ for each fixed $\xi \in \mathbb{D}$. The first term tends to $(B_n S_x)(\xi)$ because Proposition 4.4 says that

$$z \mapsto \langle S_z \omega^j K_\xi^{(n)}, \omega^j K_\xi^{(n)} \rangle$$

extends continuously to $M(\mathcal{A})$, and the second term tends to $(B_n S)(\varphi_x(\xi))$ because $B_n S \in \mathcal{A}$. ■

Corollary 4.7 *If $S \in \mathfrak{L}(L_a^2)$ and $x \in M(\mathcal{A})$ the following conditions are equivalent*

- (i) $S_u = \lambda I$ for every $u \in H(x)$
- (ii) $S_u = \lambda I$ for some $u \in H(x)$
- (iii) $B_0 S \equiv \lambda$ on $H(x)$.

Proof. Since $H(u) = H(x)$ when $u \in H(x)$ then every $v \in H(x)$ has the form $v = \varphi_u(\omega)$ for some $\omega \in \mathbb{D}$. By the previous corollary

$$(B_0 S)(v) = (B_0 S)(\varphi_u(\omega)) = (B_0 S_u)(\omega).$$

This identity and the fact that B_0 acts in a one-to-one fashion on $\mathfrak{L}(L_a^2)$ give all the equivalences. ■

Since for $a \in \mathcal{A}$ we have

$$(T_a)_z^* = T_{\bar{a} \circ \varphi_z} \rightarrow T_{\bar{a} \circ \varphi_x} = (T_a)_x^*$$

in the *SOT*-topology when $z \rightarrow x$, then also $(T_z)^* \rightarrow (T_x)^*$ in the *SOT*-topology for all $T \in \mathfrak{T}(\mathcal{A})$. Also, since the product of a *WOT*-convergent and a *SOT*-convergent net in $\mathfrak{L}(L_a^2)$ tends weakly to the product of the limits, Proposition 4.4 and Theorems 4.5 imply that

$$(4.2) \quad S_x T_x = (ST)_x \quad \text{and} \quad T_x S_x = (TS)_x$$

for every $S \in \mathfrak{L}(L_a^2)$, $T \in \mathfrak{T}(\mathcal{A})$ and $x \in M(\mathcal{A})$. This fails if we only assume $S, T \in \mathfrak{L}(L_a^2)$.

Indeed, consider the operator defined by $Sf(\omega) = f(-\omega)$. Since $S^2 = I$ then $(S^2)_x = I$ for every $x \in M(\mathcal{A})$. On the other hand, since $SK_z^{(0)} = K_{-z}^{(0)}$ then

$$(B_0S)(z) = (1 - |z|^2)^2 \langle K_{-z}^{(0)}, K_z^{(0)} \rangle = \frac{(1 - |z|^2)^2}{(1 + |z|^2)^2}.$$

So $(B_0S)(z) \rightarrow 0$ when $|z| \rightarrow 1$, and then $(B_0S)(x) = 0$ for all $x \in M(\mathcal{A}) \setminus \mathbb{D}$. Corollary 4.7 then tells us that $S_x = 0$ for $x \in M(\mathcal{A}) \setminus \mathbb{D}$, making (4.2) impossible for this choice of S and $T = S$.

Lemma 4.8 *Let $S \in \mathfrak{L}(L_a^2)$ and $x \in M(\mathcal{A})$. Suppose that there is some $n_0 \geq 0$ such that $(B_{n_0}S) \circ \varphi_x = g$ harmonic. Then $(B_nS) \circ \varphi_x = g$ for every $n \geq 0$.*

Proof. By Corollary 4.6, $\tilde{\Delta}(B_{n_0}S_x) = \tilde{\Delta}g = 0$, which together with (2.7) yields $B_{n_0+1}S_x = B_{n_0}S_x = g$. Then $B_nS_x = g$ for every $n \geq n_0$. Thus $B_0(B_nS_x) = B_0g = g$ for $n \geq n_0$, implying that

$$g = \lim_{n \rightarrow \infty} B_0B_nS_x = \lim_{n \rightarrow \infty} B_nB_0S_x = B_0S_x,$$

where the second equality follows from Corollary 2.7 and the last one because since $B_0S_x \in \mathcal{A}$ by Corollary 4.6, then $B_n(B_0S_x) \rightarrow B_0S_x$ uniformly. Taking $n_0 = 0$, we have proved above that $B_nS_x = g$ for every $n \geq 0$. ■

By the lemma we can add two more equivalences to Corollary 4.7, saying that $B_nS \equiv \lambda$ on $H(x)$ for every (or for some) $n \geq 0$.

Theorem 4.9 *Let $S \in \mathfrak{T}(\mathcal{A})$ and \mathcal{B} be a hyperbolic algebra. Then the following conditions are equivalent,*

- (1) $S_x = \lambda I$ when $x \in \pi^{-1}(y)$ for every $y \in \Gamma_{\mathcal{B}}$, where $\lambda \in \mathbb{C}$ depends only on y ,
- (2) there is a continuous map $\Psi_{\mathcal{B}}^S : M(\mathcal{B}) \rightarrow (\mathfrak{T}(\mathcal{A}), SOT)$ such that $\Psi_{\mathcal{B}}^S \circ \pi = \Psi_S$,
- (3) $B_nS \in \mathcal{B}$ for some $n \geq 0$,
- (4) $B_nS \in \mathcal{B}$ for all $n \geq 0$.

If $S \in \mathfrak{L}(L_a^2)$ the theorem holds replacing $(\mathfrak{T}(\mathcal{A}), SOT)$ by $(\mathfrak{L}(L_a^2), WOT)$ in (2).

Proof. If (1) holds then for every $y \in M(\mathcal{B})$ and $x \in \pi^{-1}(y)$, S_x is an operator that only depends on y . Hence $\Psi_{\mathcal{B}}^S(y) = S_x$ is well defined and satisfies the equality in (2). The continuity of $\Psi_{\mathcal{B}}^S$ from $M(\mathcal{B})$ into any of the metric spaces $(\mathfrak{T}(\mathcal{A}), SOT)$ or $(\mathfrak{L}(L_a^2), WOT)$ (according to the hypothesis) follows from the respective continuity of Ψ_S , which is given by Theorem 4.5 and Proposition 4.4.

Now suppose that (2) holds. This means that if $y \in M(\mathcal{B})$ then S_x is the same operator T for every $x \in \pi^{-1}(y)$. Since $\varphi_x(\mathbb{D}) \subset \pi^{-1}(y)$ for $y \in \Gamma_{\mathcal{B}}$, then $S_{\varphi_x(\omega)} = T$ for every $\omega \in \mathbb{D}$. Corollary 4.6 then says that

$$(B_0S)(\varphi_x(\omega)) = (B_0S_{\varphi_x(\omega)})(0) = (B_0T)(0)$$

for every $x \in \pi^{-1}(y)$ and $\omega \in \mathbb{D}$. Writing $\lambda = (B_0T)(0)$, we obtain that $B_0S \equiv \lambda$ on $H(x)$ for every $x \in \pi^{-1}(y)$. Hence B_0S is constant on $\pi^{-1}(y)$ for every $y \in \Gamma_{\mathcal{B}}$, meaning that $(B_0S)|_D$ extends continuously to $M(\mathcal{B})$. Since the Gelfand-Naimark Theorem identifies \mathcal{B} with $C(M(\mathcal{B}))$ then $B_0S \in \mathcal{B}$. This proves (3) for $n = 0$. If (3) holds for some $n_0 \geq 0$ then $B_{n_0}S = \lambda_y \in \mathbb{C}$ on $\pi^{-1}(y)$ for every $y \in \Gamma_{\mathcal{B}}$. Lemma 4.8 then implies that the same happens with B_nS for all $n \geq 0$. This proves (4). Finally, if (4) holds then $(B_0S)|_{\pi^{-1}(y)} = \lambda_y \in \mathbb{C}$ for $y \in \Gamma_{\mathcal{B}}$. In particular, $B_0S \equiv \lambda_y$ on $H(x)$ for every $x \in \pi^{-1}(y)$. Then (1) follows from Corollary 4.7. ■

If $S \in \mathfrak{L}(L_a^2)$ satisfies the conditions of the theorem then the map $z \mapsto S_z$ admits a continuous extension to $M(\mathcal{B})$ given by $\Psi_S^{\mathcal{B}}$. Write

$$\Psi_S^{\mathcal{B}}(y) = \widehat{S}_y^{\mathcal{B}}$$

when $y \in M(\mathcal{B})$. If $\mathcal{B} = \mathcal{A}$ we keep the previous notation $\Psi_S(y) = S_y$ for $y \in M(\mathcal{A})$. Also, since it is clear that we can identify $\widehat{S}_z^{\mathcal{B}}$ with S_z when $z \in \mathbb{D}$, we do not make this notation distinction for $z \in \mathbb{D}$. Observe that if $y \in M(\mathcal{B})$ and (z_α) is a net in \mathbb{D} that tends to y in $M(\mathcal{B})$, then $\widehat{S}_y^{\mathcal{B}}$ admits the two alternative and equivalent expressions

$$\widehat{S}_y^{\mathcal{B}} = \lim_{\alpha} S_{z_\alpha},$$

a WOT-limit in general and a SOT-limit if $S \in \mathfrak{T}(\mathcal{A})$, or

$$\widehat{S}_y^{\mathcal{B}} = S_x \text{ for some (or all) } x \in \pi^{-1}(y),$$

where $\pi : M(\mathcal{A}) \rightarrow M(\mathcal{B})$ is the natural projection. Also, if $b \in \mathcal{B}$ we can look at b as a continuous function on $M(\mathcal{B})$ or on $M(\mathcal{A})$. If $\mathcal{B} \neq \mathcal{A}$ we write $\widehat{b}^{\mathcal{B}}$ when we need to distinguish the domain of the function, otherwise b will be looked as a function on $M(\mathcal{A})$. Of course, if $z \in \mathbb{D}$ then $b(z)$ has the same meaning either way.

If \mathcal{B} is a hyperbolic algebra, $b \in \mathcal{B}$ and $y \in \Gamma_{\mathcal{B}}$, then for every $x \in \pi^{-1}(y)$ we have

$$(T_b)_x = T_{b \circ \varphi_x} = \lambda I$$

with $\lambda \in \mathbb{C}$ depending only on y (actually $\lambda = \widehat{b}^{\mathcal{B}}(y)$). Since $\mathfrak{T}(\mathcal{B})$ is generated by these Toeplitz operators, the same holds for every $S \in \mathfrak{T}(\mathcal{B})$. Theorem 4.9 then says that $B_nS \in \mathcal{B}$ when $S \in \mathfrak{T}(\mathcal{B})$, for every nonnegative integer n .

5. Approximation and truncation by Toeplitz operators

If $A \subset L^\infty(\mathbb{D})$ is a subalgebra, we write $\mathfrak{T}_0(A)$ for the algebra generated by the Toeplitz operators T_a , with $a \in A$, without taking closure. In [4] Axler and Zheng found simple but very ingenious estimates for the norm of operators in $\mathfrak{T}_0(L^\infty(\mathbb{D}))$. The present section (especially Lemmas 5.2 and 5.5) makes heavy use of their method.

5.1. Norm estimates and truncation

The following lemma is a particular case of Lemma 4.2.2 in [21].

Lemma 5.1 *If $c < 0$ and $t > -1$ then*

$$J_{c,t}(z) = \int_D \frac{(1 - |\omega|^2)^t}{|1 - z\bar{\omega}|^{2+t+c}} dA(\omega), \quad z \in \mathbb{D},$$

is bounded.

The next result appeared in [4] for $p = 6$. The proof sketched here is a standard modification of that proof involving Lemma 5.1.

Lemma 5.2 *Let $p > 4$. Then there is a constant $C_p < \infty$ such that if $S \in \mathfrak{L}(L^2_a)$, then*

$$(5.1) \quad \int_D \frac{|(SK_z^{(0)})(w)|}{\sqrt{1 - |w|^2}} dA(w) \leq \frac{C_p \|S_z 1\|_p}{\sqrt{1 - |z|^2}}$$

for all $z \in \mathbb{D}$ and

$$(5.2) \quad \int_D \frac{|(SK_z^{(0)})(w)|}{\sqrt{1 - |z|^2}} dA(z) \leq \frac{C_p \|S_w^* 1\|_p}{\sqrt{1 - |w|^2}}$$

for all $w \in \mathbb{D}$.

Proof. To prove (5.1) let $S \in \mathfrak{L}(L^2_a)$ and fix $z \in \mathbb{D}$. Since

$$U_z 1 = (|z|^2 - 1)K_z^{(0)} \quad \text{and} \quad U_z U_z = I$$

then

$$U_z S_z 1 = (|z|^2 - 1)SK_z^{(0)}.$$

Thus

$$\int_D \frac{|(SK_z^{(0)})(w)|}{\sqrt{1 - |w|^2}} dA(w) = \frac{1}{1 - |z|^2} \int_D \frac{|(S_z 1)(\varphi_z(w))| |\varphi_z'(w)|}{\sqrt{1 - |w|^2}} dA(w).$$

Making the substitution $w = \varphi_z(\lambda)$ in the last integral and using Holder's inequality with $1/p + 1/q = 1$, we obtain

$$\begin{aligned} \int_D \frac{|(SK_z^{(0)})(w)|}{\sqrt{1-|w|^2}} dA(w) &= \frac{1}{\sqrt{1-|z|^2}} \int_D \frac{|(S_z 1)(\lambda)|}{|1-\bar{z}\lambda|\sqrt{1-|\lambda|^2}} dA(\lambda) \\ &\leq \frac{\|S_z 1\|_p}{\sqrt{1-|z|^2}} \left(\int_D \frac{dA(\lambda)}{|1-\bar{z}\lambda|^{q(1-|\lambda|^2)^{q/2}}} \right)^{1/q} \\ &= \frac{\|S_z 1\|_p}{\sqrt{1-|z|^2}} J(z)^{1/q}, \end{aligned}$$

where

$$J(z) = \int_D \frac{(1-|\lambda|^2)^{-q/2}}{|1-\bar{z}\lambda|^{2-(q/2)+(3/2)q-2}} dA(\lambda).$$

Since $p > 4$ then $q < 4/3$, which yields $q/2 < 2/3 < 1$ and $(3/2)q - 2 < 0$. By Lemma 5.1 there is $J_q > 0$ such that $J(z) \leq J_q$ for every $z \in \mathbb{D}$. This proves (5.1) with $C_p = J_q^{1/q}$. Replace S with S^* and interchange the roles of w and z in (5.1) to obtain

$$\int_D \frac{|(S^* K_w^{(0)})(z)|}{\sqrt{1-|z|^2}} dA(z) \leq \frac{C_p \|S_w^* 1\|_p}{\sqrt{1-|w|^2}}.$$

Then use the equality $(S^* K_w^{(0)})(z) = \overline{(SK_z^{(0)})(w)}$ to obtain (5.2). ■

Lemma 5.3 *Let $S \in \mathfrak{L}(L_a^2)$, $a, b \in L^\infty(\mathbb{D})$ and $p > 4$. Then*

$$\|T_b S T_a\|_{\mathfrak{L}(L_a^2)} \leq C_p (\|a\|_\infty \|b\|_\infty)^{\frac{1}{2}} \sup_{z \in D} \{\|S_z 1\|_p |a(z)|\}^{\frac{1}{2}} \sup_{\omega \in D} \{\|S_\omega^* 1\|_p |b(\omega)|\}^{\frac{1}{2}},$$

where C_p is the constant of Lemma 5.2.

Proof. For $f \in L_a^2$ and $w \in D$, we have

$$\begin{aligned} (S T_a f)(w) &= \langle S T_a f, K_w^{(0)} \rangle = \langle a f, S^* K_w^{(0)} \rangle \\ &= \int_D f(z) a(z) \overline{(S^* K_w^{(0)})(z)} dA(z) \\ &= \int_D f(z) a(z) (S K_z^{(0)})(w) dA(z). \end{aligned}$$

Thus, if M_b denotes the multiplication operator,

$$(M_b S T_a) f(w) = \int_D f(z) a(z) b(w) (S K_z^{(0)})(w) dA(z).$$

If $\Phi(z, w) = |a(z)b(w)(SK_z^{(0)})(w)|$ and $h(z) = (1 - |z|^2)^{-1/2}$ then (5.1) yields

$$\begin{aligned} \int_D \Phi(z, w)h(w) dA(w) &\leq C_p \|b\|_\infty \|S_z 1\|_p |a(z)| h(z) \\ &\leq C_p \|b\|_\infty \sup_{z \in D} \{ \|S_z 1\|_p |a(z)| \} h(z), \end{aligned}$$

and by (5.2)

$$\begin{aligned} \int_D \Phi(z, w)h(z) dA(w) &\leq C_p \|a\|_\infty \|S_w^* 1\|_p |b(w)| h(w) \\ &\leq C_p \|a\|_\infty \sup_{\omega \in D} \{ \|S_w^* 1\|_p |b(w)| \} h(w). \end{aligned}$$

By Schur’s theorem (see the proof in [21, p. 42]) the operator $M_b ST_a$ satisfies an inequality as in the lemma. The result follows because

$$\|(T_b ST_a)f\|_{L^2} \leq \|(M_b ST_a)f\|_{L^2}$$

for every $f \in L_a^2$. ■

Suppose that $1 < p < p' < \infty$, $f \in L^p(\mathbb{D})$ and $0 < r < 1$. Split the integral

$$\|f\|_p^p = \|f\chi_{D \setminus rD}\|_p^p + \|f\chi_{rD}\|_p^p,$$

where χ_E denotes the characteristic function of the set E . Taking $\alpha = p'/p$ and $\beta = p'/(p' - p)$ we have $\alpha^{-1} + \beta^{-1} = 1$. By Holder’s inequality

$$\|f\chi_{D \setminus rD}\|_p^p \leq \|f\|_{\alpha p}^p \|\chi_{D \setminus rD}\|_\beta = \|f\|_{p'}^p (1 - r^2)^{1 - \frac{p}{p'}},$$

and consequently

$$(5.3) \quad \|f\|_p^p \leq \|f\|_{p'}^p (1 - r^2)^{1 - \frac{p}{p'}} + \|f\chi_{rD}\|_p^p.$$

Proposition 5.4 *Suppose that $S \in \mathfrak{K}_0(L^\infty(\mathbb{D}))$ and $F \subset M(\mathcal{A})$ is a closed saturated set such that $B_0 S \equiv 0$ on F . Given $\varepsilon > 0$ there is an open neighborhood Ω of F in $M(\mathcal{A})$ such that if $U \subset \Omega \cap \mathbb{D}$ is measurable, then*

$$(5.4) \quad \|T_{a\chi_U} S\|_{\mathfrak{L}(L_a^2)} < \varepsilon \quad \text{and} \quad \|ST_{a\chi_U}\|_{\mathfrak{L}(L_a^2)} < \varepsilon$$

for every $a \in L^\infty(\mathbb{D})$ with $\|a\|_\infty \leq 1$.

Proof. Since F is saturated and $B_0 S \equiv 0$ on F , Proposition 4.4 and Corollary 4.7 say that $S_z \xrightarrow{\text{wOT}} S_x = 0$ when $z \rightarrow x \in F$, with $z \in \mathbb{D}$. Thus $S_z 1 \rightarrow 0$ weakly in L_a^2 and consequently

$$(5.5) \quad S_z 1 \rightarrow 0 \quad \text{uniformly on compact sets as } z \rightarrow x \quad (z \in \mathbb{D})$$

for every $x \in F$.

Write

$$S = \sum_{i=1}^m \prod_{j=1}^{n_i} T_{a_j^i},$$

with $a_j^i \in L^\infty(\mathbb{D})$, and fix p, p' with $4 < p < p'$. Then

$$(5.6) \quad \|S_z 1\|_{p'} = \left\| \sum_{i=1}^m \prod_{j=1}^{n_i} T_{a_j^i \circ \varphi_z} 1 \right\|_{p'} \leq \sum_{i=1}^m \prod_{j=1}^{n_i} c_{p'} \|a_j^i\|_\infty = c,$$

where $c_{p'}$ is the norm of the analytic projection P_+ acting on $L^{p'}(\mathbb{D})$. For $0 < r < 1$, (5.3) yields

$$\|S_z 1\|_p^p \leq \|S_z 1\|_{p'}^p (1-r)^{1-\frac{p}{p'}} + \|(S_z 1)\chi_{rD}\|_p^p.$$

By (5.6) there is r close enough to 1 so that the first member of the sum is smaller than $\varepsilon/2$, while (5.5) and the compactness of F imply that there is a neighborhood Ω of F so that the second member is smaller than $\varepsilon/2$ for every $z \in \Omega \cap \mathbb{D}$. In particular, if $U \subset \Omega \cap \mathbb{D}$ this holds for every $z \in U$. Since $\|a\|_\infty \leq 1$, Lemma 5.3 gives

$$\|ST_{a\chi_U}\|^2 \leq C_p^2 \sup\{\|S_z 1\|_p : z \in U\} \sup_D \|S_\omega^* 1\|_p \leq C_p^2 c \varepsilon^{1/p},$$

where c comes from (5.6) with S^* instead of S , and C_p is the constant of Lemma 5.3. To prove the first inequality of (5.4) observe that $B_0 S^* = \overline{B_0 S}$ also satisfies the hypothesis of the proposition and $\|T_{a\chi_U} S\| = \|S^* T_{\bar{a}\chi_U}\|$. ■

5.2. Approximation properties of the k -Berezin transforms

Lemma 5.5 *Suppose that $\{S_k\}$ is a bounded sequence in $\mathfrak{L}(L_a^2)$ such that $\|B_0 S_k\|_\infty \rightarrow 0$ when $k \rightarrow \infty$. Then*

$$\sup_{z \in D} |(S_k)_z 1| \rightarrow 0$$

uniformly on compact subsets of \mathbb{D} when $k \rightarrow \infty$.

Proof. Since there is a constant C such that $\|S_k\| \leq C$ for every k , then it is enough to prove that for every $S \in \mathfrak{L}(L_a^2)$, $\eta > 0$ and $r \in (0, 1)$, there is a function $c(r, \eta) > 0$, independent of S , such that

$$(5.7) \quad \sup_{z \in D} |(S_z 1)(u)| \leq c(r, \eta) \|B_0 S\|_\infty + \eta \|S\|$$

when $u \in r\mathbb{D}$.

Since

$$(5.8) \quad K_z^{(0)}(w) = \sum_{m=0}^{\infty} (m+1) \bar{z}^m \omega^m,$$

then for $z, \lambda \in \mathbb{D}$ we have

$$(B_0 S)(\varphi_z(\lambda)) = (B_0 S_z)(\lambda) = (1 - |\lambda|^2)^2 \sum_{j,m=0}^{\infty} (j+1)(m+1) \langle S_z \omega^j, \omega^m \rangle \bar{\lambda}^j \lambda^m,$$

where the first equality comes from Lemma 2.2. Then, for $0 < \delta < 1/2$ (to be chosen later) we obtain

$$\begin{aligned} \int_{\delta D} \frac{(B_0 S)(\varphi_z(\lambda)) \bar{\lambda}^n}{(1 - |\lambda|^2)^2} dA(\lambda) &= \sum_{j,m=0}^{\infty} (j+1)(m+1) \langle S_z \omega^j, \omega^m \rangle \int_{\delta D} \bar{\lambda}^{j+n} \lambda^m dA(\lambda) \\ &= \sum_{j=0}^{\infty} (j+1) \langle S_z \omega^j, \omega^{j+n} \rangle \delta^{2j+2n+2} \\ &= \delta^{2n+2} \left(\langle S_z 1, \omega^n \rangle + \sum_{j=1}^{\infty} (j+1) \langle S_z \omega^j, \omega^{j+n} \rangle \delta^{2j} \right). \end{aligned}$$

Since $0 < \delta < 1/2$ and $\|\omega^j\| = (j+1)^{-1/2}$ then

$$\begin{aligned} |\langle S_z 1, \omega^n \rangle| &\leq \frac{1}{\delta^{2n+2}} \|B_0 S\|_{\infty} \int_{\delta D} \frac{\delta^n dA(\lambda)}{(1 - |\lambda|^2)^2} + \|S\| \sum_{j=1}^{\infty} (j+1) \|\omega^j\| \|\omega^{j+n}\| \delta^{2j} \\ (5.9) \quad &\leq 2\delta^{-n} \|B_0 S\|_{\infty} + \delta \|S\|, \end{aligned}$$

where the last inequality holds because $\sum_{j=1}^{\infty} \delta^{2j} \leq \delta$ when $0 < \delta < 1/2$. By (5.8)

$$(S_z 1)(u) = \langle S_z 1, K_u^{(0)} \rangle = \sum_{n \geq 0} (n+1) \langle S_z 1, \omega^n \rangle u^n,$$

implying that

$$(5.10) \quad |(S_z 1)(u)| \leq \sum_{0 \leq n \leq N-1} (n+1) |\langle S_z 1, \omega^n \rangle| + \sum_{n \geq N} (n+1)^{1/2} \|S_z\| r^n$$

for $z \in \mathbb{D}$, $u \in r\mathbb{D}$ and $N \geq 1$. Since $r \in (0, 1)$ we can fix some integer $N = N(r, \eta)$ big enough so that the second sum is bounded by $(\eta/2) \|S\|$. Using (5.9) in (5.10) we get

$$\begin{aligned} |(S_z 1)(u)| &\leq N \sum_{0 \leq n \leq N-1} |\langle S_z 1, \omega^n \rangle| + (\eta/2) \|S\| \\ &\leq 2N^2 \delta^{-N} \|B_0 S\|_{\infty} + N^2 \delta \|S\| + (\eta/2) \|S\| \end{aligned}$$

for $z \in \mathbb{D}$ and $u \in r\mathbb{D}$. Choosing $\delta = \delta(r, \eta) < \min\{\eta/2N^2, 1/2\}$ we obtain (5.7) with $c(r, \eta) = 2N^2 \delta^{-N}$. ■

Lemma 5.6 *Let $\{S_k\}$ be a sequence in $\mathfrak{L}(L_a^2)$ such that for some $p' > 4$,*

$$(5.11) \quad \|B_0 S_k\|_\infty \rightarrow 0, \quad \text{when } k \rightarrow \infty,$$

$$(5.12) \quad \sup_{z \in D} \|(S_k)_z 1\|_{p'} \leq C \quad \text{and} \quad \sup_{\omega \in D} \|(S_k^*)_\omega 1\|_{p'} \leq C,$$

where $C > 0$ does not depend on k . Then

$$\|S_k\|_{\mathfrak{L}(L_a^2)} \rightarrow 0 \quad \text{when } k \rightarrow \infty.$$

Proof. By (5.12) and Lemma 5.3 with $a = b = 1$,

$$\|S_k\|_{\mathfrak{L}(L_a^2)} \leq C_{p'} \sup_{z \in D} \{ \|(S_k)_z 1\|_{p'} \}^{1/2} \sup_{\omega \in D} \{ \|(S_k^*)_\omega 1\|_{p'} \}^{1/2} \leq C_{p'} C.$$

Hence, $\{S_k\}$ is a bounded sequence in $\mathfrak{L}(L_a^2)$ that satisfies (5.11). Under these conditions Lemma 5.5 says that

$$(5.13) \quad \sup_{z \in D} |(S_k)_z 1| \rightarrow 0 \quad \text{uniformly on compact sets of } \mathbb{D}.$$

Let p with $4 < p < p'$. By (5.3)

$$\sup_{z \in D} \|(S_k)_z 1\|_p^p \leq \sup_{z \in D} \|(S_k)_z 1\|_{p'}^p (1-r)^{1-\frac{p}{p'}} + \sup_{z \in D} \|[(S_k)_z 1] \chi_{rD}\|_p^p$$

for every $0 < r < 1$. By (5.12) the first member of the sum is bounded by

$$C^p (1-r)^{1-\frac{p}{p'}},$$

which can be made small by taking r close to 1, and by (5.13) the second member of the sum tends to 0 as $k \rightarrow \infty$. Therefore,

$$\sup_{z \in D} \|(S_k)_z 1\|_p \rightarrow 0 \quad \text{when } k \rightarrow \infty$$

for every $p \in (4, p')$. Using again Lemma 5.3, this time with p instead of p' , we obtain

$$\begin{aligned} \|S_k\|_{\mathfrak{L}(L_a^2)} &\leq C_p \sup_{z \in D} \{ \|(S_k)_z 1\|_p \}^{1/2} \sup_{\omega \in D} \{ \|(S_k^*)_\omega 1\|_p \}^{1/2} \\ &\leq C_p \sup_{z \in D} \{ \|(S_k)_z 1\|_p \}^{1/2} C^{1/2} \rightarrow 0 \end{aligned}$$

when $k \rightarrow \infty$, where the last inequality holds by (5.12), since $\| \cdot \|_p \leq \| \cdot \|_{p'}$. ■

Theorem 5.7 *If $a \in L^\infty(\mathbb{D})$ then $T_{B_k(a)} \rightarrow T_a$ in operator norm when $k \rightarrow \infty$. In particular, $\mathfrak{T}(\mathcal{A}) = \mathfrak{T}(L^\infty(\mathbb{D}))$.*

Proof. Write $S_k = T_{B_k(a)} - T_a$. Since Corollary 2.7 says that $B_0B_k = B_kB_0$ on $\mathfrak{L}(L^2_a)$ then

$$B_0S_k = B_0T_{B_k(a)} - B_0T_a = B_0B_k(a) - B_0(a) = B_kB_0(a) - B_0(a),$$

which tends uniformly to 0 when $k \rightarrow \infty$ because $B_0(a) \in \mathcal{A}$. That is, $\{S_k\}$ satisfies (5.11). On the other hand, if $p' > 4$ then

$$\|(S_k)_z 1\|_{p'} = \|P_+ M_{(B_k(a)-a) \circ \varphi_z} 1\|_{p'} \leq c_{p'} (\|B_k(a)\|_\infty + \|a\|_\infty) \leq 2c_{p'} \|a\|_\infty,$$

where $c_{p'}$ is the norm of the analytic projection P_+ acting on $L^{p'}(\mathbb{D})$ (see [21, p. 54]). Since

$$(S_k^*)_z = P_+ M_{\overline{(B_k(a)-a)} \circ \varphi_z}$$

then also

$$\|(S_k^*)_z 1\|_{p'} \leq 2c_{p'} \|a\|_\infty.$$

So, $\{S_k\}$ satisfies (5.12) and Lemma 5.6 then says that $\|S_k\|_{\mathfrak{L}(L^2_a)} \rightarrow 0$ as $k \rightarrow \infty$. ■

Remark 5.8 An obvious consequence of the theorem is that Theorems 4.5 and 4.9 hold for $S \in \mathfrak{T}(L^\infty(\mathbb{D}))$. The argument of Theorem 5.7 works word by word for any $S \in \mathfrak{L}(L^2_a)$ such that $T_{B_k}S - S$ satisfies (5.12) for some $p' > 4$. So, $T_{B_k}S \rightarrow S$ for such S . Maybe this holds for every $S \in \mathfrak{T}_0(L^\infty(\mathbb{D}))$, which would imply that $\mathfrak{T}(L^\infty(\mathbb{D}))$ is the closure of $\{T_a : a \in \mathcal{A}\}$.

6. Abelianization

Lemma 6.1 *Let $F \subset M(\mathcal{A}) \setminus \mathbb{D}$ be a closed saturated set, $\Omega \subset M(\mathcal{A})$ an open neighborhood of F and $k \geq 0$ an integer. Write $U = \Omega \cap \mathbb{D}$ and $\mathfrak{F} = \{a \in L^\infty(\mathbb{D}) : a \equiv 0 \text{ on } U\}$. Then*

$$B_k a \equiv 0 \text{ on } F \text{ for every } a \in \mathfrak{F}.$$

In particular, if \mathcal{B} is a hyperbolic algebra and $F = \pi^{-1}(\Gamma_{\mathcal{B}})$ then $B_k a \in \mathcal{B}$ and $T_a \in \mathfrak{T}(\mathcal{B})$.

Proof. By Lemma 4.8 it is enough to prove the lemma for $k = 0$. Let $x \in F$ and take a net (z_α) in \mathbb{D} such that $z_\alpha \rightarrow x$. We claim that for every $r \in (0, 1)$ there is $\alpha_0 = \alpha_0(r)$ such that $\varphi_{z_\alpha}(r\mathbb{D}) \subset \Omega$ for $\alpha \geq \alpha_0$. Otherwise there is a subnet (z_{α_β}) and points $\xi_\beta \in r\mathbb{D}$ such that $\varphi_{z_{\alpha_\beta}}(\xi_\beta) \notin \Omega$ for all β . We can assume that $\xi_\beta \rightarrow \xi_0$, with $|\xi_0| \leq r$. If $f \in \mathcal{A}$, the inequality

$$|f(\varphi_{z_{\alpha_\beta}}(\xi_\beta)) - f(\varphi_x(\xi_\beta))| \leq |f(\varphi_{z_{\alpha_\beta}}(\xi_\beta)) - f(\varphi_{z_{\alpha_\beta}}(\xi_0))| + |f(\varphi_{z_{\alpha_\beta}}(\xi_0)) - f(\varphi_x(\xi_0))|$$

and the uniform ρ -continuity of f imply that $f(\varphi_{z_{\alpha_\beta}}(\xi_\beta)) \rightarrow f(\varphi_x(\xi_0))$.

Therefore

$$\varphi_{z_{\alpha\beta}}(\xi_\beta) \rightarrow \varphi_x(\xi_0) \in H(x) \subset F,$$

and since Ω is a neighborhood of F then $\varphi_{z_{\alpha\beta}}(\xi_\beta) \in \Omega$ for $\beta \geq \beta_0$, a contradiction. So, if $a \in \mathfrak{F}$ and $0 < r < 1$, there is α_0 such that $(a \circ \varphi_{z_\alpha})(\omega) = 0$ for $|\omega| < r$ and $\alpha \geq \alpha_0$. Hence for $\alpha \geq \alpha_0$,

$$|(B_0a)(z_\alpha)| \leq \int_D |(a \circ \varphi_{z_\alpha})(\omega)| dA(\omega) = \int_{D \setminus rD} |(a \circ \varphi_{z_\alpha})(\omega)| dA(\omega) \leq \|a\|_\infty(1-r^2),$$

which can be made arbitrarily small by taking r close enough to 1. Therefore $(B_0a)(z_\alpha) \rightarrow 0$, but since also $(B_0a)(z_\alpha) \rightarrow (B_0a)(x)$ then $(B_0a)(x) = 0$, and this happens for all $x \in F$.

Now suppose that $F = \pi^{-1}(\Gamma_{\mathcal{B}})$, with \mathcal{B} a hyperbolic algebra. Since $B_k a \in \mathcal{A}$ identically vanishes on $\pi^{-1}(\Gamma_{\mathcal{B}})$ then $B_k a \in \mathcal{B}$. Consequently $T_{B_k a} \in \mathfrak{T}(\mathcal{B})$, and since by Theorem 5.7, $T_{B_k a} \rightarrow T_a$ as $k \rightarrow \infty$, then so is T_a . ■

Let $F \subset M(\mathcal{A})$ be a closed set. A set $U \subset \mathbb{D}$ will be called a relative neighborhood of F if there is some open neighborhood $\Omega \subset M(\mathcal{A})$ of F such that $U = \Omega \cap \mathbb{D}$. Since the disk is dense in $M(\mathcal{A})$ and Ω is open, it is clear that $\overline{U}^{M(\mathcal{A})}$ contains Ω , and consequently it is a neighborhood of F . Also, for $V \subset \mathbb{D}$ we will denote $V^c = \mathbb{D} \setminus V$.

Lemma 6.2 *Let $S = \sum_{i=1}^m \prod_{j=1}^{n_i} T_{a_j^i}$, with $a_j^i \in L^\infty(\mathbb{D})$ for $1 \leq i \leq m$ and $1 \leq j \leq n_i$, and $F \subset M(\mathcal{A})$ be a closed saturated set such that $B_0 S \equiv 0$ on F . Then given $\varepsilon > 0$ there exist relative neighborhoods U, V of F such that*

$$\left\| S - \left(\sum_{i=1}^m \prod_{j=1}^{n_i} T_{a_j^i \chi_{V^c}} \right) T_{\chi_{U^c}} \right\| < \varepsilon.$$

Proof. Without loss of generality we can assume that $\|a_j^i\|_\infty \leq 1$ for every i, j . By Proposition 5.4 there is a relative neighborhood U of F such that

$$(6.1) \quad \|S - ST_{\chi_{U^c}}\| = \|ST_{\chi_U}\| < \varepsilon.$$

By Lemma 6.1 and (4.2), for $1 \leq i \leq m$ each of the operators

$$S_k^i \stackrel{\text{def}}{=} \left(\prod_{j=k}^{n_i} T_{a_j^i} \right) T_{\chi_{U^c}}, \quad 1 \leq k \leq n_i, \quad S_{n_i+1}^i = T_{\chi_{U^c}}$$

satisfies $B_0 S_k^i = 0$ on F . Hence, a new use of Proposition 5.4 provides a relative neighborhood V of F such that

$$\|T_{a_k^i \chi_V} S_{k+1}^i\| \leq \varepsilon$$

for every $1 \leq i \leq m$ and $1 \leq k \leq n_i$.

Indeed, the proposition says that there are relative neighborhoods V_k^i of F that satisfy the inequality for each i and k , but it also says that their intersection satisfies the inequality. Therefore

$$\begin{aligned} & \|T_{a_1^i \chi_{V^c}} \cdots T_{a_{k-1}^i \chi_{V^c}} S_k^i - T_{a_1^i \chi_{V^c}} \cdots T_{a_k^i \chi_{V^c}} S_{k+1}^i\| \\ &= \|T_{a_1^i \chi_{V^c}} \cdots T_{a_{k-1}^i \chi_{V^c}} T_{a_k^i} S_{k+1}^i - T_{a_1^i \chi_{V^c}} \cdots T_{a_k^i \chi_{V^c}} S_{k+1}^i\| \\ &\leq \|T_{a_1^i \chi_{V^c}} \cdots T_{a_{k-1}^i \chi_{V^c}}\| \|(T_{a_k^i} - T_{a_k^i \chi_{V^c}}) S_{k+1}^i\| \\ &\leq \|T_{a_k^i \chi_V} S_{k+1}^i\| < \varepsilon, \end{aligned}$$

which leads to

$$\begin{aligned} & \|T_{a_1^i} \cdots T_{a_{n_i}^i} T_{\chi_{U^c}} - T_{a_1^i \chi_{V^c}} \cdots T_{a_{n_i}^i \chi_{V^c}} T_{\chi_{U^c}}\| \\ &\leq \sum_{k=1}^{n_i} \|T_{a_1^i \chi_{V^c}} \cdots T_{a_{k-1}^i \chi_{V^c}} S_k^i - T_{a_1^i \chi_{V^c}} \cdots T_{a_k^i \chi_{V^c}} S_{k+1}^i\| < n_i \varepsilon. \end{aligned}$$

Therefore

$$\left\| \left(\sum_{i=1}^m \prod_{j=1}^{n_i} T_{a_j^i} \right) T_{\chi_{U^c}} - \left(\sum_{i=1}^m \prod_{j=1}^{n_i} T_{a_j^i \chi_{V^c}} \right) T_{\chi_{U^c}} \right\| \leq \sum_{i=1}^m n_i \varepsilon.$$

Since $ST_{\chi_{U^c}} = (\sum_{i=1}^m \prod_{j=1}^{n_i} T_{a_j^i}) T_{\chi_{U^c}}$ and $\varepsilon > 0$ is arbitrary, the lemma follows from (6.1) and the above inequality. ■

If $\mathcal{B} \subset L^\infty(\mathbb{D})$ is a subalgebra, we write $\mathfrak{C}_0(\mathcal{B})$ for the bilateral ideal of $\mathfrak{T}_0(\mathcal{B})$ generated by commutators $[T_a, T_b] = T_a T_b - T_b T_a$, with $a, b \in \mathcal{B}$. Therefore, $\mathfrak{C}(\mathcal{B})$ is the closure of $\mathfrak{C}_0(\mathcal{B})$ in $\mathfrak{L}(L_a^2)$.

Lemma 6.3 *Let \mathcal{B} be a hyperbolic algebra. If $S \in \mathfrak{C}_0(L^\infty(\mathbb{D}))$ is such that $B_0 S \in \mathcal{B}$ and $\widehat{B_0 S^{\mathcal{B}}} \equiv 0$ on $\Gamma_{\mathcal{B}}$ then $S \in \mathfrak{C}(\mathcal{B})$.*

Proof. By hypothesis

$$S = \sum_{i=1}^m T_{b_1^i} \cdots T_{b_{n_i}^i} [T_{a_1^i}, T_{a_2^i}] T_{c_1^i} \cdots T_{c_{k_i}^i},$$

where n_i, k_i and m are some positive integers and all the symbols are in $L^\infty(\mathbb{D})$. If $\widehat{B_0 S^{\mathcal{B}}} \equiv 0$ on $\Gamma_{\mathcal{B}}$, Lemma 6.2 says that given $\varepsilon > 0$ there are relative neighborhoods U, V of $\Gamma_{\mathcal{B}}$ such that if

$$R = \sum_{i=1}^m T_{b_1^i \chi_{V^c}} \cdots T_{b_{n_i}^i \chi_{V^c}} [T_{a_1^i \chi_{V^c}}, T_{a_2^i \chi_{V^c}}] T_{c_1^i \chi_{V^c}} \cdots T_{c_{k_i}^i \chi_{V^c}} T_{\chi_{U^c}}$$

then $\|S - R\| < \varepsilon$. By Lemma 6.1 every Toeplitz operator involved in the last expression is in $\mathfrak{T}(\mathcal{B})$. So, $R \in \mathfrak{C}(\mathcal{B})$ and then so is S . ■

It is well known that if \mathcal{B}, \mathcal{D} are C^* -algebras and ϕ is a $*$ -homomorphism from \mathcal{B} to \mathcal{D} , then $\|\phi\| \leq 1$ and ϕ is an isometry if and only if ϕ is one-to-one [13, p. 100].

Theorem 6.4 *If \mathcal{B} is a hyperbolic algebra then*

- (1) $\mathfrak{C}(\mathcal{B}) = \{S \in \mathfrak{T}(\mathcal{B}) : \widehat{B_0 S^{\mathcal{B}}} \equiv 0 \text{ on } \Gamma_{\mathcal{B}}\} = \{S \in \mathfrak{T}(\mathcal{B}) : \widehat{S^{\mathcal{B}}}_y = 0 \text{ for all } y \in \Gamma_{\mathcal{B}}\}$.
- (2) $S - T_{B_0 S} \in \mathfrak{C}(\mathcal{B})$ for every $S \in \mathfrak{T}(\mathcal{B})$.
- (3) The C^* -algebras $\mathfrak{T}(\mathcal{B})/\mathfrak{C}(\mathcal{B})$ and $C(\Gamma_{\mathcal{B}})$ are isomorphic via $\phi : S + \mathfrak{C}(\mathcal{B}) \mapsto \widehat{B_0 S^{\mathcal{B}}}|_{\Gamma_{\mathcal{B}}}$.

Proof. (1). The equality of the last two sets follows from Corollary 4.7. Suppose first that $S \in \mathfrak{C}_0(\mathcal{B})$, so

$$S = \sum_{1 \leq i \leq n} A_i [T_{a_i}, T_{b_i}] B_i,$$

where $a_i, b_i \in \mathcal{B}$ and $A_i, B_i \in \mathfrak{T}_0(\mathcal{B})$. If $x \in \pi^{-1}(\Gamma_{\mathcal{B}})$ then $a_i \circ \varphi_x$ and $b_i \circ \varphi_x$ are constant functions for all $1 \leq i \leq n$. By (4.2) then

$$S_x = \sum_{1 \leq i \leq n} (A_i)_x [T_{a_i \circ \varphi_x}, T_{b_i \circ \varphi_x}] (B_i)_x = 0.$$

Since every $S \in \mathfrak{C}(\mathcal{B})$ can be approximated by operators of this form, then $S_x = 0$ for every $x \in \pi^{-1}(\Gamma_{\mathcal{B}})$. By Corollary 4.7 then $B_0 S \equiv 0$ on $\pi^{-1}(\Gamma_{\mathcal{B}})$, which is another way to say that $\widehat{B_0 S^{\mathcal{B}}} \equiv 0$ on $\Gamma_{\mathcal{B}}$. This proves the inclusion of the first set into the second one.

Suppose now that $S \in \mathfrak{T}(\mathcal{B})$ and $\widehat{B_0 S^{\mathcal{B}}} \equiv 0$ on $\Gamma_{\mathcal{B}}$. We can assume that $\|S\| = 1$. Let $0 < \varepsilon < 1$ and take $Q \in \mathfrak{T}_0(\mathcal{B})$ such that $\|Q - S\| < \varepsilon$. Since $Q \in \mathfrak{T}(\mathcal{B})$ then $\widehat{Q^{\mathcal{B}}}_y = \lambda I$ and $(\widehat{B_0 Q})^{\mathcal{B}}(y) = \lambda$ for every $y \in \Gamma_{\mathcal{B}}$, where $\lambda \in \mathbb{C}$ depends on y . Thus

$$(\widehat{T_{B_0 Q}}^{\mathcal{B}})_y = \lim_{z \rightarrow y} T_{(B_0 Q) \circ \varphi_z} = T_{(\widehat{B_0 Q})^{\mathcal{B}}(y)} = \lambda I.$$

Then

$$B_0(Q - T_{B_0 Q})^{\mathcal{B}} \equiv 0 \quad \text{on } \Gamma_{\mathcal{B}}$$

by Corollary 4.7, and since $\widehat{B_0 S^{\mathcal{B}}} \equiv 0$ on $\Gamma_{\mathcal{B}}$ then

$$B_0(\widehat{T_{B_0 S}})^{\mathcal{B}} \equiv 0 \quad \text{on } \Gamma_{\mathcal{B}}$$

by the same corollary.

So, if

$$S_1 = Q - T_{B_0Q} + T_{B_0S}$$

then $\widehat{B_0S_1}^{\mathcal{B}} \equiv 0$ on $\Gamma_{\mathcal{B}}$ and

$$(6.2) \quad \|S_1 - S\| \leq \|Q - S\| + \|T_{B_0S} - T_{B_0Q}\| \leq 2\|Q - S\| < 2\varepsilon.$$

In [20, Thm. 1.1] it is proved that

$$\mathfrak{C}(L^\infty(\mathbb{D})) = \mathfrak{T}(L^\infty(\mathbb{D})),$$

so it contains the identity I .

Since Theorem 5.7 implies that $\mathfrak{C}(L^\infty(\mathbb{D})) = \mathfrak{C}(\mathcal{A})$ then $I \in \mathfrak{C}(\mathcal{A})$. Consequently there is $R \in \mathfrak{C}_0(\mathcal{A})$ such that $\|R - I\| < \varepsilon$. Thus

$$(6.3) \quad \|RS_1 - S_1\| \leq \|R - I\| \|S_1\| < \varepsilon(\|S\| + 2\varepsilon) < 3\varepsilon.$$

Since $B_0S_1 \equiv 0$ on $\pi^{-1}(\Gamma_{\mathcal{B}})$, Corollary 4.7 says that $(S_1)_x = 0$ for all $x \in \pi^{-1}(\Gamma_{\mathcal{B}})$. By (4.2) then $(RS_1)_x = R_x(S_1)_x = 0$ for all $x \in \pi^{-1}(\Gamma_{\mathcal{B}})$, which means that

$$B_0(RS_1) \in \mathcal{B} \quad \text{and} \quad \widehat{B_0(RS_1)}^{\mathcal{B}} \equiv 0 \quad \text{on } \Gamma_{\mathcal{B}}.$$

But since $R \in \mathfrak{C}_0(\mathcal{A})$ and $S_1 \in \mathfrak{T}_0(\mathcal{A})$ then $RS_1 \in \mathfrak{C}_0(\mathcal{A})$, which together with Lemma 6.3 gives $RS_1 \in \mathfrak{C}(\mathcal{B})$. By (6.2) and (6.3), $\|RS_1 - S\| < 5\varepsilon$ and (1) follows.

(2). Let $y \in \Gamma_{\mathcal{B}}$. Since $S \in \mathfrak{T}(\mathcal{B})$ then $\widehat{S}_y^{\mathcal{B}} = \lambda I$. Thus

$$\widehat{(B_0S)}^{\mathcal{B}}(y) = \lambda \quad \text{and} \quad \widehat{(T_{B_0S})}_y^{\mathcal{B}} = T_{\widehat{(B_0S)}^{\mathcal{B}}(y)} = \lambda I.$$

The result then follows from (1).

(3). By (1) the map ϕ is well-defined and one-to-one. It is clear that ϕ is $*$ -linear. Suppose that $S, T \in \mathfrak{T}(\mathcal{B})$ and $y \in \Gamma_{\mathcal{B}}$. Then

$$\widehat{S}_y^{\mathcal{B}} = \lambda_S I \quad \text{and} \quad \widehat{T}_y^{\mathcal{B}} = \lambda_T I$$

for some $\lambda_S, \lambda_T \in \mathbb{C}$ that depend on y . Hence

$$\begin{aligned} \widehat{B_0(ST)}^{\mathcal{B}}(y) &= \lim_{z \rightarrow y} \langle S_z T_z 1, 1 \rangle = \langle \widehat{S}_y^{\mathcal{B}} \widehat{T}_y^{\mathcal{B}} 1, 1 \rangle \\ &= \langle \lambda_S \lambda_T 1, 1 \rangle = \lambda_S \lambda_T = \widehat{(B_0S)}^{\mathcal{B}}(y) \widehat{(B_0T)}^{\mathcal{B}}(y), \end{aligned}$$

and ϕ is multiplicative. If $f \in C(\Gamma_{\mathcal{B}})$ we can extend f to a continuous function F on $M(\mathcal{B})$. Therefore $F \in \mathcal{B}$ and

$$\phi(T_F + \mathfrak{C}(\mathcal{B})) = \widehat{B_0F}^{\mathcal{B}}|_{\Gamma_{\mathcal{B}}} = f.$$

So, ϕ is onto. ■

Theorem 6.5 *Let \mathcal{B} be a hyperbolic algebra and $S \in \mathfrak{T}_0(L^\infty(\mathbb{D}))$. Then*

- (1) $S \in \mathfrak{T}(\mathcal{B})$ if and only if $B_0S \in \mathcal{B}$.
- (2) $S \in \mathfrak{C}(\mathcal{B})$ if and only if $\widehat{B_0S}^\mathcal{B} \equiv 0$ on $\Gamma_\mathcal{B}$.

Proof. (1). We know the necessity from Theorem 4.9. Suppose that

$$S = \sum_{i=1}^m \prod_{j=1}^{n_i} T_{a_j^i},$$

where all $a_j^i \in L^\infty(\mathbb{D})$, and $B_0S \in \mathcal{B}$. Then $T_{B_0S} \in \mathfrak{T}(\mathcal{B})$ and

$$B_0(S - T_{B_0S})^\wedge^\mathcal{B} \equiv 0 \quad \text{on } \Gamma_\mathcal{B}.$$

Consequently Lemma 6.2 tells us that given $\varepsilon > 0$ there are relative neighborhoods U, V of $\Gamma_\mathcal{B}$ such that

$$\|S - T_{B_0S} - \sum_{i=1}^m \prod_{j=1}^{n_i} T_{a_j^i \chi_{V^c}} T_{\chi_{U^c}} + T_{(B_0S)\chi_{V^c}} T_{\chi_{U^c}}\| < \varepsilon.$$

By Lemma 6.1,

$$T_{a_j^i \chi_{V^c}}, T_{\chi_{U^c}}, T_{(B_0S)\chi_{V^c}} \in \mathfrak{T}(\mathcal{B})$$

for all $1 \leq i \leq m$ and $1 \leq j \leq n_i$. Therefore $S \in \mathfrak{T}(\mathcal{B})$.

(2). The necessity follows from (1) of Theorem 6.4. For the sufficiency, observe that it is implicit in the condition $\widehat{B_0S}^\mathcal{B} \equiv 0$ on $\Gamma_\mathcal{B}$ that $B_0S \in \mathcal{B}$. By the previous assertion then $S \in \mathfrak{T}(\mathcal{B})$. So, (1) of Theorem 6.4 says that $S \in \mathfrak{C}(\mathcal{B})$. ■

If \mathcal{B} is a hyperbolic algebra and $a \in \mathcal{A}$, then $a \in \mathcal{B}$ if and only if $B_0a \in \mathcal{B}$. Therefore the theorem says that $T_a \in \mathfrak{T}(\mathcal{B})$ if and only if $a \in \mathcal{B}$ and that $T_a \in \mathfrak{C}(\mathcal{B})$ if and only if $a \equiv 0$ on $\pi^{-1}(\Gamma_\mathcal{B})$.

The algebra $C(\overline{\mathbb{D}})$, of continuous functions on the closed disk is hyperbolic, its maximal ideal space identifies with $\overline{\mathbb{D}}$, and it is immediate that $\Gamma_{C(\overline{\mathbb{D}})} = \partial\mathbb{D}$ via this identification. Since by Coburn’s theorem $\mathfrak{C}(C(\overline{\mathbb{D}}))$ is the ideal of compact operators, then part (2) of the theorem says that $S \in \mathfrak{T}_0(L^\infty(\mathbb{D}))$ is compact if and only if

$$(B_0S)(z) \rightarrow 0 \quad \text{as } |z| \rightarrow 1^-.$$

That is, we recover the theorem of Axler and Zheng [4, Thm. 2.2]. It is clear that the above condition is equivalent to $S_x = 0$ for all $x \in M(\mathcal{A}) \setminus \mathbb{D}$, or what is the same, $S_z \rightarrow 0$ in the SOT-topology when $|z| \rightarrow 1$.

7. Applications

7.1. Continuous functions up to a boundary set

Suppose that $E \subset \partial\mathbb{D}$ is a closed set and consider the algebra C_E formed by the functions of \mathcal{A} that extend continuously to E . Then C_E is a hyperbolic algebra. If $id \in \mathcal{A}$ denotes the function $id(z) = z$ and for $\lambda \in \partial\mathbb{D}$ we write

$$M_\lambda = \{x \in M(\mathcal{A}) : id(x) = \lambda\}$$

for the fiber of λ over $M(\mathcal{A})$, then $M(C_E)$ consists of $M(\mathcal{A})/\sim$, where \sim is the equivalence relation that collapses M_λ to a single point (depending on λ) for each $\lambda \in E$. Thus, Γ_{C_E} can be identified with E . Theorem 6.4 then says that

$$\mathfrak{C}(C_E) = \{S \in \mathfrak{T}(C_E) : \lim_{z \rightarrow E} (B_0 S)(z) = 0\} \text{ and } \mathfrak{T}(C_E)/\mathfrak{C}(C_E) \simeq C(E).$$

As mentioned before, when $E = \partial\mathbb{D}$, the above isomorphism is part of Coburn’s theorem. Now consider the algebra CL_E^∞ formed by the functions in $L^\infty(\mathbb{D})$ that extend continuously to E . Since $CL_E^\infty \not\subset \mathcal{A}$, it is not a hyperbolic algebra. So, at a first sight it is not possible to apply our results to this algebra. Fortunately, Theorem 5.7 gives us a way to overcome this apparent difficulty. In fact, it is easy to prove that if $f \in CL_E^\infty$ then $B_k f \in C_E$ for every $k \geq 0$ and $(B_k f)(\lambda) = f(\lambda)$ for $\lambda \in E$. By Theorem 5.7 then $\mathfrak{T}(C_E) = \mathfrak{T}(CL_E^\infty)$ and $\mathfrak{C}(C_E) = \mathfrak{C}(CL_E^\infty)$.

7.2. The McDonald-Sundberg Theorem

Let \mathcal{U} be the C^* -subalgebra of $L^\infty(\mathbb{D})$ generated by $H^\infty = \{f \in L^\infty(\mathbb{D}) : f \text{ is analytic}\}$. The celebrated corona theorem of Carleson [10] states that \mathbb{D} is dense in $M(H^\infty)$, the maximal ideal space of H^∞ . This translates into the alternative description of \mathcal{U} as $C(M(H^\infty))$. Since Schwarz Lemma implies that $H^\infty \subset \mathcal{A}$ then $\mathcal{U} \subset \mathcal{A}$. Therefore \mathcal{U} is a prehyperbolic algebra and we aim to prove that it is hyperbolic.

Clearly, every interpolating sequence for H^∞ is interpolating for \mathcal{U} . The interpolating sequences for H^∞ were characterized by Carleson in [9]. Suppose that $x \in M(H^\infty) \setminus \mathbb{D}$ is in the closure of some interpolating sequence $\{z_n\}$ for H^∞ , where we can assume that $z_n \neq 0$ for all $n \geq 1$. It is known that the infinite product

$$b(\omega) = \prod_{n \geq 1} \frac{|z_n|}{z_n} \varphi_{z_n}(\omega)$$

represents a function $b \in H^\infty$ such that $b(z_n) = 0$ for all $n \geq 1$. This b is called an interpolating Blaschke product.

We also know (see [15, p. 404]) that if $\delta \in (0, 1)$ then there is $\varepsilon(\delta) > 0$ such that

$$|b(\omega)| \geq \varepsilon(\delta) \quad \text{for every } \omega \in \mathbb{D} \setminus \bigcup_{n \geq 1} K(z_n, \delta).$$

Thus x satisfies condition (b₂) of Proposition 3.9. On the other hand, if $x \in M(H^\infty) \setminus \mathbb{D}$ is not in the closure of any interpolating sequence for H^∞ , it is known that for every net (z_α) in \mathbb{D} that tends to x ,

$$f \circ \varphi_{z_\alpha} \rightarrow \lambda \in \mathbb{C}$$

uniformly on compact sets for every $f \in H^\infty$ (see [15, Ch. X]). Since \mathcal{U} is the C^* -algebra generated by H^∞ the same holds for every $f \in \mathcal{U}$. Thus x satisfies (a₂) of Proposition 3.8. Consequently Corollary 3.10 tells us that \mathcal{U} is hyperbolic and that $\Gamma_{\mathcal{U}}$ is formed by the points $x \in M(H^\infty)$ that are not in the closure of any interpolating sequence for H^∞ . Such points are usually called ‘trivial points’ because they can be characterized as the $x \in M(H^\infty)$ whose Gleason part (with respect to H^∞) is just $\{x\}$. For the definition and further information on Gleason parts the reader may consult the original paper of Hoffman [16] or Garnett’s book [15, Ch. X].

Theorem 6.4 now tells us that $\mathfrak{T}(\mathcal{U})/\mathfrak{C}(\mathcal{U}) \simeq C(\Gamma_{\mathcal{U}})$, a result obtained by McDonald and Sundberg in [17]. Theorem 6.4 also says that $\mathfrak{C}(\mathcal{U}) = \{S \in \mathfrak{T}(\mathcal{U}) : \widehat{B_0 S^{\mathcal{U}}} \equiv 0 \text{ on } \Gamma_{\mathcal{U}}\}$ and $S - T_{B_0 S} \in \mathfrak{C}(\mathcal{U})$, which are recent additions to the McDonald-Sundberg Theorem discovered by Axler and Zheng [5].

7.3. The algebra of nontangential limits

Consider the algebra $\mathcal{N} = \{f \in \mathcal{A} : f \text{ has nontangential limits a.e. on } \partial\mathbb{D}\}$. It is clear that \mathcal{N} is prehyperbolic, and we are going to use Corollary 3.10 to show that it is hyperbolic. To do so we need to characterize the interpolating sequences for \mathcal{N} . For $u \in \partial\mathbb{D}$ and $0 < \alpha < \pi/2$ let $\Lambda_\alpha(u) = \{u - \omega : |\arg \omega - \arg u| < \alpha, \text{ and } 0 < |u - \omega| < 1\}$ be an angular region with vertex u of total opening 2α . If $V \subset \mathbb{D}$ set

$$\text{NT}_\alpha(V) = \{u \in \partial\mathbb{D} : u \in \overline{V \cap \Lambda_\alpha(u)}\} \quad \text{and} \quad \text{NT}(V) = \bigcup_{0 < \alpha < \pi/2} \text{NT}_\alpha(V).$$

Geometrically, $\text{NT}(V)$ is the subset of $\partial\mathbb{D}$ that can be approached nontangentially by points of V . If $u \in \partial\mathbb{D}$, $0 < r < 1$ and $0 < \alpha < \pi/2$, there is some $0 < \beta < \pi/2$ depending on α and r such that the r -pseudohyperbolic neighborhood of $\Lambda_\alpha(u)$ is contained in $\Lambda_\beta(u)$. Thus

$$(7.1) \quad \text{NT}(V) = \text{NT}(\{z \in \mathbb{D} : \rho(z, V) \leq r\}).$$

We write $|E|$ for the one-dimensional Lebesgue measure of $E \subset \partial\mathbb{D}$.

Lemma 7.1 *A separated sequence $\mathcal{S} = \{z_n\}$ is interpolating for \mathcal{N} if and only if $|NT(\mathcal{S})| = 0$. If that is the case, for any $r > 0$ sufficiently small there exists $f \in \mathcal{N}$ that separates \mathcal{S} from $\mathbb{D} \setminus \bigcup_{n \geq 1} K(z_n, r)$.*

Proof. Suppose that $|NT(\mathcal{S})| = 0$ and $\rho(z_n, z_m) \geq \delta > 0$ for $n \neq m$. By (7.1) then $|NT(\bigcup_{n \geq 1} K(z_n, \delta/4))| = 0$. Take $f \in \mathcal{A}$ such that

$$f(z_n) = 1 \text{ for all } n \text{ and } f \equiv 0 \text{ on } \mathbb{D} \setminus \bigcup_{n \geq 1} K(z_n, \delta/4).$$

So, f has null nontangential limit a.e. on $\partial\mathbb{D}$. Thus $f \in \mathcal{N}$ and separates \mathcal{S} from $\mathbb{D} \setminus \bigcup_{n \geq 1} K(z_n, \delta/4)$. If $\{\eta_n\}$ is an arbitrary sequence and we take $g \in \mathcal{A}$ such that $g(z_n) = \eta_n$ for every n then $fg \in \mathcal{N}$ and $f(z_n)g(z_n) = \eta_n$ for every n . So, \mathcal{S} is interpolating for \mathcal{N} .

Now suppose that $|NT(\mathcal{S})| > 0$. If $0 < \alpha_k < \alpha_{k+1} \rightarrow \pi/2$ is a strictly increasing sequence, then $NT(\mathcal{S}) = \bigcup_k NT_{\alpha_k}(\mathcal{S})$. So, there is some $\alpha_k = \alpha$ such that $|NT_{\alpha}(\mathcal{S})| > 0$, and consequently there exists a compact set $E \subset NT_{\alpha}(\mathcal{S})$ of positive measure. That is, $u \in \overline{\Lambda_{\alpha}(u)} \cap \mathcal{S}$ for every $u \in E$. So, if $u \in E$ there is some $z_n \in \Lambda_{\alpha}(u) \cap \mathcal{S}$. Since $\Lambda_{\alpha}(u)$ is open, it is geometrically clear that there is an open neighborhood I_u of u in $\partial\mathbb{D}$ such that $z_n \in \Lambda_{\alpha}(v) \cap \mathcal{S}$ for every $v \in I_u$. By the compactness of E there is a finite set \mathcal{R}_1 in \mathcal{S} such that $\Lambda_{\alpha}(u) \cap \mathcal{R}_1 \neq \emptyset$ for every $u \in E$. If $r_1 = \max\{|z| : z \in \mathcal{R}_1\}$ and $\mathcal{S}_1 = \{z \in \mathcal{S} : |z| \leq r_1\}$ then we also have $\Lambda_{\alpha}(u) \cap \mathcal{S}_1 \neq \emptyset$ for every $u \in E$. We can repeat this process with $\mathcal{S} \setminus \mathcal{S}_1$ instead of \mathcal{S} to obtain $r_2 \in (r_1, 1)$ such that if $\mathcal{S}_2 = \{z \in \mathcal{S} : r_1 < |z| \leq r_2\}$ then $\Lambda_{\alpha}(u) \cap \mathcal{S}_2 \neq \emptyset$ for every $u \in E$. We keep going to construct a sequence $0 < r_1 < \dots < r_n < \dots < 1$ such that if $\mathcal{S}_n = \{z \in \mathcal{S} : r_{n-1} < |z| \leq r_n\}$ then

$$(7.2) \quad \Lambda_{\alpha}(u) \cap \mathcal{S}_n \neq \emptyset \text{ for every } u \in E.$$

The sequence $\{r_n\}$ must tend to 1 because if $r_n \leq r < 1$ for every n then $\{z : |z| \leq r\} \cap \mathcal{S}$ is infinite, which is not possible because \mathcal{S} is separated. Now take

$$\mathcal{T}_1 = \bigcup_{j \text{ odd}} \mathcal{S}_j \quad \text{and} \quad \mathcal{T}_2 = \bigcup_{j \text{ even}} \mathcal{S}_j.$$

Since (7.2) holds for all $n \geq 1$ then $E \subset NT_{\alpha}(\mathcal{T}_1) \cap NT_{\alpha}(\mathcal{T}_2)$, and since $|E| > 0$, the interpolation problem

$$f(z_n) = \begin{cases} 1 & \text{for } z_n \in \mathcal{T}_1 \\ 0 & \text{for } z_n \in \mathcal{T}_2 \end{cases}$$

cannot be solved by a function with nontangential limits almost everywhere on E . ■

Theorem 7.2 *The algebra \mathcal{N} is hyperbolic. In addition, $y \in M(\mathcal{N})$ is in $G_{\mathcal{N}}$ if and only if y is in the closure of some interpolating sequence for \mathcal{N} .*

Proof. Let $y \in M(\mathcal{N})$. If y is in the closure of an interpolating sequence for \mathcal{N} the previous lemma says that y satisfies condition (b₂) of Proposition 3.9, so $y \in G_{\mathcal{N}}$.

If y is not in the closure of an interpolating sequence for \mathcal{N} and \mathcal{S} is a separated sequence with $y \in \overline{\mathcal{S}^{M(\mathcal{N})}}$ then Lemma 7.1 says that $|\text{NT}(\mathcal{S})| > 0$. So, if $f \in \mathcal{N}$ there must be some point $u \in \text{NT}(\mathcal{S})$ such that f has nontangential limit λ at u , and for some $\alpha \in (0, \pi/2)$, $u \in \overline{\Lambda_{\alpha}(u) \cap \mathcal{S}}$. Let $\{z_n\}$ be a subsequence in $\Lambda_{\alpha}(u) \cap \mathcal{S}$ that tends to u . If $0 < r < 1$ then the argument preceding (7.1) says that there is some $\beta = \beta(\alpha, r) \in (0, \pi/2)$ such that

$$\bigcup_n K(z_n, r) \subset \Lambda_{\beta}(u).$$

So, $f(\varphi_{z_n}(\omega)) \rightarrow \lambda$ for $|\omega| \leq r$ when $n \rightarrow \infty$. Thus y satisfies (a₃) of Proposition 3.8, and consequently $y \in \Gamma_{\mathcal{N}}$. By Corollary 3.10 then \mathcal{N} is hyperbolic. ■

The nontangential limit function of $f \in \mathcal{N}$ will be denoted \tilde{f} . So, $\tilde{f} \in L^{\infty}(\partial\mathbb{D})$. Also, we write $z \xrightarrow{\text{nt}} u$ to indicate that z tends nontangentially to $u \in \partial\mathbb{D}$.

Lemma 7.3 *Let $f \in \mathcal{N}$. Then $\widehat{f^{\mathcal{N}}} \equiv 0$ on $\Gamma_{\mathcal{N}}$ if and only if $\tilde{f} = 0$.*

Proof. If there is $y \in \Gamma_{\mathcal{N}}$ such that $|\widehat{f^{\mathcal{N}}}(y)| = \delta > 0$ and \mathcal{S} is a separated sequence such that $y \in \overline{\mathcal{S}^{M(\mathcal{N})}}$, then y is in the $M(\mathcal{N})$ -closure of

$$\mathcal{S}_1 = \{z \in \mathcal{S} : |f(z)| > \delta/2\}.$$

Since $y \in \Gamma_{\mathcal{N}}$ then Theorem 7.2 and Lemma 7.1 imply that $|\text{NT}(\mathcal{S}_1)| > 0$, and since $|\tilde{f}| \geq \delta/2$ for almost every point of $\text{NT}(\mathcal{S}_1)$, the sufficiency holds.

Now suppose that $\tilde{f} \neq 0$, so there is some $\delta > 0$ such that $|\tilde{f}| > \delta$ on a set of positive measure. It is easy then to construct a separated sequence \mathcal{S} such that $|\text{NT}(\mathcal{S})| > 0$ and $|f(z)| > \delta/2$ for every $z \in \mathcal{S}$. The necessity will follow if we show that $\overline{\mathcal{S}^{M(\mathcal{N})}} \cap \Gamma_{\mathcal{N}} \neq \emptyset$, because for any y in the intersection we would have $|\widehat{f^{\mathcal{N}}}(y)| \geq \delta/2$.

Since \mathcal{N} is hyperbolic, if $\overline{\mathcal{S}^{M(\mathcal{N})}} \cap \Gamma_{\mathcal{N}} = \emptyset$ then $\overline{\mathcal{S}^{M(\mathcal{N})}} \subset G_{\mathcal{N}}$. So, Proposition 3.9 says that for every $y \in \overline{\mathcal{S}^{M(\mathcal{N})}} \setminus \mathcal{S}$ there is an interpolating sequence \mathcal{T}_y for \mathcal{N} , such that $y \in \overline{\mathcal{T}_y^{M(\mathcal{N})}}$. Hence, for every $0 < r < 1$ the $M(\mathcal{N})$ -closure of $\bigcup_{z \in \mathcal{T}_y} K(z, r)$ is a neighborhood of y (by Lemma 7.1). By the

compactness of $\overline{\mathfrak{S}}^{M(\mathcal{N})} \setminus \mathcal{S}$ there are finitely many interpolating sequences $\mathcal{T}_1, \dots, \mathcal{T}_N$ for \mathcal{N} such that the closure of

$$U \stackrel{\text{def}}{=} \bigcup_{1 \leq j \leq N} \bigcup_{z \in \mathcal{T}_j} K(z, r)$$

is a neighborhood of $\overline{\mathfrak{S}}^{M(\mathcal{N})} \setminus \mathcal{S}$. Thus there is $0 < \varrho < 1$ so that $\mathcal{S} \cap \{z \in \mathbb{D} : |z| \geq \varrho\}$ is contained in U . Together with (7.1) this yields

$$\text{NT}(\mathcal{S}) \subset \bigcup_{1 \leq j \leq N} \text{NT}\left(\bigcup_{z \in \mathcal{T}_j} K(z, r)\right) = \bigcup_{1 \leq j \leq N} \text{NT}(\mathcal{T}_j),$$

which is impossible because $|\text{NT}(\mathcal{S})| > 0$ while $|\text{NT}(\mathcal{T}_j)| = 0$ for $j = 1, \dots, N$. ■

Lemma 7.4 *If $S \in \mathfrak{T}(\mathcal{N})$ then for almost every $u \in \partial\mathbb{D}$ there is $\lambda(u) \in \mathbb{C}$ such that $S_z \xrightarrow{SOT} \lambda(u)I$ when $z \xrightarrow{\text{nt}} u$.*

Proof. Let $a \in \mathcal{N}$ and suppose that $u \in \partial\mathbb{D}$ is such that $a(z) \rightarrow \lambda \in \mathbb{C}$ when $z \xrightarrow{\text{nt}} u$. If $0 < \alpha < \pi/2$ and $0 < r < 1$ there is $\beta = \beta(\alpha, r)$ in $(\alpha, \pi/2)$ such that $\varphi_z(\omega) \in \Lambda_\beta(u)$ when $z \in \Lambda_\alpha(u)$ and $|\omega| \leq r$. Therefore $a \circ \varphi_z \rightarrow \lambda$ uniformly on $r\mathbb{D}$ when $z \rightarrow u$ inside $\Lambda_\alpha(u)$. Since r is arbitrary the convergence is uniform on compact sets, implying that $(T_a)_z = T_{a \circ \varphi_z} \rightarrow \lambda I$ in the *SOT*-topology when $z \rightarrow u$ inside $\Lambda_\alpha(u)$. Since α is arbitrary and the product of operators is continuous with respect to the *SOT*-topology, the lemma holds for every $S \in \mathfrak{T}_0(\mathcal{N})$. If $S \in \mathfrak{T}(\mathcal{N})$ take a sequence $\{S_n\}$ in $\mathfrak{T}_0(\mathcal{N})$ that converges to S . So, for every $n \geq 1$ there is a set $E_n \subset \partial\mathbb{D}$ of full measure such that

$$(S_n)_z \xrightarrow{SOT} \lambda_n(u)I \quad \text{when} \quad z \xrightarrow{\text{nt}} u \in E_n.$$

Therefore the set $E = \bigcap E_n$ has full measure, and given $\varepsilon > 0$ there is $n_0 = n_0(\varepsilon)$ such that if $u \in E$,

$$(7.3) \quad |\lambda_n(u) - \lambda_m(u)| \leq \lim_{z \xrightarrow{\text{nt}} u} \|(S_n)_z - (S_m)_z\| = \|S_n - S_m\| < \varepsilon$$

for all $n, m \geq n_0$. This implies that there is some $\lambda(u) \in \mathbb{C}$ such that $\lambda_n(u) \rightarrow \lambda(u)$ for every $u \in E$. If $f \in L^2_a$ has norm 1, $u \in E$ and $n \geq n_0$, (7.3) yields

$$\begin{aligned} \|S_z f - \lambda(u)f\| &\leq \|S_z f - (S_n)_z f\| + \|(S_n)_z f - \lambda_n(u)f\| + |\lambda_n(u) - \lambda(u)| \|f\| \\ &\leq \|S - S_n\| + |\lambda_n(u) - \lambda(u)| + \|(S_n)_z f - \lambda_n(u)f\| \\ &\leq 2\varepsilon + \|(S_n)_z f - \lambda_n(u)f\| \rightarrow 2\varepsilon \end{aligned}$$

when $z \xrightarrow{\text{nt}} u$. Thus $S_z f \rightarrow \lambda(u)f$ in L^2_a when $z \xrightarrow{\text{nt}} u \in E$ and the lemma holds for S . ■

Theorem 7.5 $\mathfrak{T}(\mathcal{N})/\mathfrak{C}(\mathcal{N}) \simeq L^\infty(\partial\mathbb{D})$ and

$$(7.4) \quad \mathfrak{C}(\mathcal{N}) = \{S \in \mathfrak{T}(\mathcal{N}) : \widetilde{B_0S} = 0\}$$

$$(7.5) \quad = \{S \in \mathfrak{T}(\mathcal{N}) : S_z \xrightarrow{SOT} 0, \text{ when } z \xrightarrow{nt} u \text{ for a.e. } u \in \partial\mathbb{D}\}.$$

Proof. Equality (7.4) follows immediately from Theorem 6.4 and Lemma 7.3.

By Lemma 7.4, for every $S \in \mathfrak{T}(\mathcal{N})$ there is a set $E_S \subset \partial\mathbb{D}$ of full measure and $\lambda_S : E_S \rightarrow \mathbb{C}$ such that

$$(7.6) \quad S_z \xrightarrow{SOT} \lambda_S(u)I \text{ when } z \xrightarrow{nt} u \in E_S.$$

Then $(B_0S)(z) = (B_0S_z)(0) = \langle S_z1, 1 \rangle \rightarrow \lambda_S(u)$ when $z \xrightarrow{nt} u \in E_S$, which means that $(\widetilde{B_0S})(u) = \lambda_S(u)$ for every $u \in E_S$. This proves (7.5).

Let $\Phi : \mathfrak{T}(\mathcal{N})/\mathfrak{C}(\mathcal{N}) \rightarrow L^\infty(\partial\mathbb{D})$ given by $\Phi(S + \mathfrak{C}(\mathcal{N})) = \widetilde{B_0S}$. By (7.4) Φ is well-defined and one-to-one. It is also clear that Φ is $*$ -linear. To prove that Φ is multiplicative let $S, T \in \mathfrak{T}(\mathcal{N})$ and use (7.6) to obtain

$$\widetilde{B_0(ST)}(u) = \lim_{z \xrightarrow{nt} u} \langle S_zT_z1, 1 \rangle = \lambda_S(u)\lambda_T(u) = (\widetilde{B_0S})(u)(\widetilde{B_0T})(u)$$

for every $u \in E_S \cap E_T$. Hence ϕ is a $*$ -homomorphism and we only need to show that it is onto. Let $a \in L^\infty(\partial\mathbb{D})$ and consider the Poisson integral

$$A(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|1 - ze^{-it}|^2} a(e^{it}) dt.$$

So, A is a bounded harmonic function such that $\tilde{A} = a$. Since A is uniformly continuous with respect to ρ then $A \in \mathcal{N}$. So, $T_A \in \mathfrak{T}(\mathcal{N})$ and

$$\Phi(T_A + \mathfrak{C}(\mathcal{N})) = \widetilde{B_0T_A} = \widetilde{B_0A} = \tilde{A} = a. \quad \blacksquare$$

Let \mathcal{U} be the algebra of the McDonald-Sundberg Theorem. Since every $f \in H^\infty$ has nontangential limits a.e. then $\mathcal{U} \subset \mathcal{N} \subset \mathcal{A}$. Therefore

$$\mathfrak{C}(\mathcal{U}) \subset \mathfrak{C}(\mathcal{N}) \subset \mathfrak{C}(\mathcal{A}).$$

We shall show that both inclusions are proper. The function

$$a = \sin \left(\log \frac{1 + |z|}{1 - |z|} \right)$$

is in \mathcal{A} but has no nontangential limit at any point of $\partial\mathbb{D}$ [8]. Hence,

$$T_a \in \mathfrak{C}(\mathcal{A}) \setminus \mathfrak{T}(\mathcal{N}).$$

The Shilov boundary of H^∞ , denoted ∂H^∞ , is the smallest closed set $F \subset M(H^\infty)$ such that

$$\|f\|_\infty = \sup_{x \in F} |\widehat{f}^{\mathcal{U}}(x)| \quad \text{for every } f \in H^\infty.$$

It is known that ∂H^∞ is properly contained in $\Gamma_{\mathcal{U}}$ [15, p. 438], and that a function $f \in \mathcal{U}$ satisfies $\widehat{f}^{\mathcal{U}} \equiv 0$ on ∂H^∞ if and only if its nontangential function vanishes a.e. on $\partial \mathbb{D}$ (see [3, Thm. 7] and [7, Coro. 1.3]). So, take $y \in \Gamma_{\mathcal{U}} \setminus \partial H^\infty$ and $f \in \mathcal{U}$ such that $\widehat{f}^{\mathcal{U}} \equiv 0$ on ∂H^∞ and $\widehat{f}^{\mathcal{U}}(y) = 1$. Since $f(z)$ has trivial nontangential limits almost everywhere then $T_f \in \mathfrak{C}(\mathcal{N})$ but since $\widehat{f}^{\mathcal{U}} \not\equiv 0$ on $\Gamma_{\mathcal{U}}$ then $T_f \notin \mathfrak{C}(\mathcal{U})$.

Let \mathcal{NL}^∞ be the algebra of functions in $L^\infty(\mathbb{D})$ that have nontangential limits a.e. on $\partial \mathbb{D}$. From the paragraph preceding (7.1) it easily follows that if $f \in \mathcal{NL}^\infty$ then $B_k f$ has the same nontangential limits as f a.e. on $\partial \mathbb{D}$ for every $k \geq 0$. Thus Theorem 5.7 tells us that

$$\mathfrak{T}(\mathcal{N}) = \mathfrak{T}(\mathcal{NL}^\infty) \quad \text{and} \quad \mathfrak{C}(\mathcal{N}) = \mathfrak{C}(\mathcal{NL}^\infty).$$

Moreover, let $E \subset \mathbb{D}$ be a set of positive measure. Then all of the above can be generalized (with similar proofs) for the algebras

$$\mathcal{NL}_E^\infty = \{f \in L^\infty(\mathbb{D}) : f \text{ has nontangential limits a.e. on } E\}$$

and

$$\mathcal{N}_E = \mathcal{NL}_E^\infty \cap \mathcal{A}.$$

Hence, we obtain a version of Theorem 7.5, where \mathcal{N} is replaced by \mathcal{N}_E or \mathcal{NL}_E^∞ and $\partial \mathbb{D}$ is replaced by E .

7.4. Constant on hyperbolic parts

Definition. If $F \subset M(\mathcal{A}) \setminus \mathbb{D}$ is a closed saturated set, define

$$\text{CO}(F) = \{f \in \mathcal{A} : f|_F = \text{const.}\}.$$

and

$$\text{COH}(F) = \{f \in \mathcal{A} : f|_{H(x)} = \text{const. for every } x \in F\}.$$

These notations stand for ‘constant on F ’ and ‘constant on hyperbolic parts of F ’, respectively. It is clear that $\text{CO}(F)$ and $\text{COH}(F)$ are hyperbolic algebras and that

$$F = \pi_1^{-1}(\Gamma_{\text{CO}(F)}) = \pi_2^{-1}(\Gamma_{\text{COH}(F)}),$$

where π_1 and π_2 are the projections from $M(\mathcal{A})$ onto the respective maximal ideal spaces.

If \mathcal{B} is a hyperbolic algebra and $\pi : M(\mathcal{A}) \rightarrow M(\mathcal{B})$ is the usual projection then

$$(7.7) \quad \{S \in \mathfrak{T}_0(\mathcal{A}) : B_0S|_{\pi^{-1}(\Gamma_{\mathcal{B}})} = 0\} \subset \mathfrak{C}(\mathcal{B}) \subset \{S \in \mathfrak{T}(\mathcal{A}) : B_0S|_{\pi^{-1}(\Gamma_{\mathcal{B}})} = 0\},$$

where the first inclusion follows from Theorem 6.5 and the second from Theorem 6.4. Observe that since the first set contains $\mathfrak{C}_0(\mathcal{B})$, it is dense in $\mathfrak{C}(\mathcal{B})$. The significance of $\text{CO}(F)$ and $\text{COH}(F)$ is given by the following

Proposition 7.6 *Let \mathcal{B} be a hyperbolic algebra and $F \subset M(\mathcal{A})$ be a closed saturated set. Then the following conditions are equivalent*

- (1) $F = \pi^{-1}(\Gamma_{\mathcal{B}})$,
- (2) $\mathfrak{C}(\mathcal{B}) = \mathfrak{C}(\text{COH}(F))$,
- (3) $\text{CO}(F) \subset \mathcal{B} \subset \text{COH}(F)$.

Proof. We prove first the equivalence between (1) and (2). If (1) holds then the comment following (7.7) says that $\{S \in \mathfrak{T}_0(\mathcal{A}) : B_0S|_F = 0\}$ is dense in both $\mathfrak{C}(\mathcal{B})$ and $\mathfrak{C}(\text{COH}(F))$, so they must coincide. If (2) holds, (7.7) implies that

$$\{S \in \mathfrak{T}_0(\mathcal{A}) : B_0S|_{\pi^{-1}(\Gamma_{\mathcal{B}})} = 0\} \subset \{S \in \mathfrak{T}(\mathcal{A}) : B_0S|_F = 0\}.$$

Therefore $F \subset \pi^{-1}(\Gamma_{\mathcal{B}})$, and a symmetrical argument gives the other inclusion, so (1) holds.

If (1) holds the functions of $\text{CO}(F)$ are continuous on $M(\mathcal{B})$ and the functions of \mathcal{B} are continuous on $M(\text{COH}(F))$. Since these are all C^* -algebras, (3) holds. If (3) holds then

$$\mathfrak{C}(\text{CO}(F)) \subset \mathfrak{C}(\mathcal{B}) \subset \mathfrak{C}(\text{COH}(F)),$$

so the proof of (2) reduces to show that $\mathfrak{C}(\text{CO}(F)) = \mathfrak{C}(\text{COH}(F))$. But this equality is a special case of the equivalence between (1) and (2). ■

Let us write COH for $\text{COH}(M(\mathcal{A}) \setminus \mathbb{D})$. In this case the last proposition says that $\mathfrak{C}(\text{COH}) = \mathfrak{C}(C(\overline{\mathbb{D}}))$, and this is the ideal of compact operators \mathcal{K} . Then Theorem 6.4 tells us that $S - T_{B_0S} \in \mathcal{K}$ for every $S \in \mathfrak{T}(\text{COH})$. In particular,

$$\mathfrak{T}(\text{COH})/\mathcal{K} = \{T_b + \mathcal{K} : b \in \text{COH}\}.$$

The center of an algebra \mathcal{B} is formed by the elements that commute with all the members of \mathcal{B} . Our next result relates $\mathfrak{T}(\text{COH})/\mathcal{K}$ with the center of $\mathfrak{T}(L^\infty(\mathbb{D}))/\mathcal{K}$.

Suppose that $S \in \mathcal{K}$ and for $z \in \mathbb{D}$ let $k_z^0 = (1 - |z|^2)K_z^{(0)}$. Since $\|k_z^0\| = 1$ and $k_z^0 \rightarrow 0$ weakly as $|z| \rightarrow 1$, then

$$|(B_0S)(z)| \leq \|Sk_z^0\| \rightarrow 0 \quad \text{when } |z| \rightarrow 1.$$

Therefore $S_x = 0$ for every $x \in M(\mathcal{A}) \setminus \mathbb{D}$.

Theorem 7.7 *Let $\mathfrak{J} = \{S \in \mathfrak{T}(L^\infty(\mathbb{D})) : S_x = 0 \text{ for } x \in M(\mathcal{A}) \setminus \mathbb{D}\}$. Then*

$$\{T_b + \mathcal{K} : b \in \text{COH}\} \subset \text{Center}(\mathfrak{T}(L^\infty(\mathbb{D}))/\mathcal{K}) \subset \{T_b + \mathfrak{J} : b \in \text{COH}\}$$

Proof. We prove first that if $S \in \mathfrak{T}(L^\infty(\mathbb{D}))$ and $b \in \text{COH}$ then $[S, T_b] \in \mathcal{K}$. Let $S_n \in \mathfrak{T}_0(\mathcal{A})$ such that $S_n \rightarrow S$. Since $(S_n T_b - T_b S_n) \rightarrow (S T_b - T_b S)$ we can assume that $S \in \mathfrak{T}_0(\mathcal{A})$. By (4.2),

$$(S T_b - T_b S)_x = S_x (T_b)_x - (T_b)_x S_x \text{ for every } x \in M(\mathcal{A}),$$

and since $(T_b)_x$ is a constant operator for every $x \in M(\mathcal{A}) \setminus \mathbb{D}$, then

$$[S, T_b]_x = 0 \quad \text{for } x \in M(\mathcal{A}) \setminus \mathbb{D}.$$

The comment after Theorem 6.5 then says that $[S, T_b]$ is compact. This proves that $\{T_b + \mathcal{K} : b \in \text{COH}\}$ is contained in the center of $\mathfrak{T}(L^\infty(\mathbb{D}))/\mathcal{K}$.

Now suppose that $S \in \mathfrak{T}(L^\infty(\mathbb{D}))$ is such that

$$S + \mathcal{K} \subset \text{Center}(\mathfrak{T}(L^\infty(\mathbb{D}))/\mathcal{K}).$$

This means that $S T_a - T_a S \in \mathcal{K}$ for every $a \in L^\infty(\mathbb{D})$. So,

$$S_x (T_a)_x - (T_a)_x S_x = 0 \quad \text{for every } x \in M(\mathcal{A}) \setminus \mathbb{D},$$

or equivalently,

$$(7.8) \quad S_z (T_a)_z - (T_a)_z S_z \xrightarrow{\text{SOT}} 0 \text{ as } |z| \rightarrow 1.$$

Let $x \in M(\mathcal{A}) \setminus \mathbb{D}$ and take a net (z_α) in \mathbb{D} converging to x . The closed ball of center 0 and radius $\|S\|$ in $\mathfrak{L}(L_a^2)$ admits a metric d with the SOT-topology. Since $S_{z_\alpha} \xrightarrow{\text{SOT}} S_x$ then for every integer $n \geq 1$ there is some point of the net, that we rename as z_n , such that $d(S_{z_n}, S_x) < 1/n$. So,

$$(7.9) \quad S_{z_n} \xrightarrow{\text{SOT}} S_x.$$

If $\{r_n\}$ is a sequence in $(0, 1)$ that tends to 1, we can assume (taking a subsequence of $\{z_n\}$ if needed) that $K(z_n, r_n) \cap K(z_j, r_j) = \emptyset$ if $n \neq j$. For an arbitrary $a \in L^\infty(\mathbb{D})$ consider the function

$$b(\omega) = \sum_{j \geq 1} (a \circ \varphi_{z_j})(\omega) \chi_{K(z_j, r_j)}(\omega).$$

Hence $(T_b)_{z_n} = T_{b \circ \varphi_{z_n}}$, where

$$\begin{aligned} (b \circ \varphi_{z_n})(\omega) &= a(\omega)\chi_{K(0,r_n)}(\omega) + \sum_{j:j \neq n} (a \circ \varphi_{z_j})(\varphi_{z_n}(\omega)) \chi_{K(\varphi_{z_n}(z_j),r_j)}(\omega) \\ &= g_n(\omega) + h_n(\omega). \end{aligned}$$

Since the support of h_n is disjoint from $K(0, r_n) = r_n\mathbb{D}$ then

$$|h_n(\omega)| \leq \|a\|_\infty \chi_{D \setminus r_n D}(\omega) \quad \text{for all } \omega \in \mathbb{D}.$$

Since $r_n \rightarrow 1$, it is clear that $T_{h_n} \xrightarrow{\text{SOT}} 0$ and $T_{g_n} \xrightarrow{\text{SOT}} T_a$. Thus

$$(7.10) \quad (T_b)_{z_n} = T_{g_n} + T_{h_n} \xrightarrow{\text{SOT}} T_a.$$

By (7.8)

$$S_{z_n}(T_a)_{z_n} - (T_a)_{z_n} S_{z_n} \xrightarrow{\text{SOT}} 0,$$

which together with (7.9) and (7.10) gives $S_x T_a - T_a S_x = 0$. This means that S_x commutes with every Toeplitz operator with symbol in $L^\infty(\mathbb{D})$. By [12, Thm. 10.28] then $S_x = \lambda I$ for some $\lambda \in \mathbb{C}$, and consequently $B_0 S \equiv \lambda$ on $H(x)$ by Corollary 4.7. Since $x \in M(\mathcal{A}) \setminus \mathbb{D}$ is arbitrary then $B_0 S \in \text{COH}$ and

$$(S - T_{B_0 S})_x = S_x - T_{(B_0 S) \circ \varphi_x} = \lambda I - \lambda I = 0$$

for every $x \in M(\mathcal{A}) \setminus \mathbb{D}$. That is, $S - T_{B_0 S} \in \mathfrak{I}$. ■

The concept of center plays an important role when studying localizations of C^* -algebras (see [13, Th. 7.47]). I believe that the ideal \mathfrak{I} in Theorem 7.7 is \mathcal{K} , so the inclusions of the theorem should be equalities. If $S \in \mathfrak{L}(L_a^2)$, the essential spectrum $\sigma_e(S)$ is the spectrum of $S + \mathcal{K}$ in the Calkin algebra $\mathfrak{L}(L_a^2)/\mathcal{K}$. Let $\sigma(S)$ denote the usual spectrum of S . Is it true that

$$\sigma_e(S) = \bigcup_{x \in M(\mathcal{A}) \setminus \mathbb{D}} \sigma(S_x) \quad \text{for every } S \in \mathfrak{L}(L^\infty(\mathbb{D}))?$$

There is strong evidence to support an affirmative answer. This holds for $S \in \mathfrak{L}(\text{COH})$, while the example preceding Lemma 4.8 shows that this fails for a general $S \in \mathfrak{L}(L_a^2)$. This example appeared in [4], where it is also shown that there is an infinite dimensional orthogonal projection P such that $B_0 P(z) \rightarrow 0$ when $|z| \rightarrow 1$. We do not know the answer even for a general Toeplitz operator with bounded symbol.

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Daniel Suárez
Departament de Matemàtiques
Universitat Autònoma de Barcelona
08193, Bellaterra, Barcelona (Spain)
dsuarez@mat.uab.es