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Calderón-Zygmund theory for non-integral operators and the H^∞ functional calculus

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Abstract

We modify Hörmander's well-known weak type (1,1) condition for integral operators (in a weakened version due to Duong and McIntosh) and present a weak type (p,p) condition for arbitrary operators.

Given an operator A on L_2 with a bounded H^∞ calculus, we show as an application the L_r-boundedness of the H^∞ calculus for all $r \in (p,q)$, provided the semigroup (e^{-tA}) satisfies suitable weighted $L_p \to L_q$-norm estimates with $2 \in (p,q)$.

This generalizes results due to Duong, McIntosh and Robinson for the special case $(p,q)=(1,\infty)$ where these weighted norm estimates are equivalent to Poisson-type heat kernel bounds for the semigroup (e^{-tA}). Their results fail to apply in many situations where our improvement is still applicable, e.g. if A is a Schrödinger operator with a singular potential, an elliptic higher order operator with bounded measurable coefficients or an elliptic second order operator with singular lower order terms.

1. Introduction and main results

The subject of this paper is an extension of Calderón-Zygmund operator theory to non-integral operators. Assume that Ω is a measurable subset of a space (Ω_1, d, μ) of homogeneous type. It is classical (at least for the case $\Omega = \Omega_1$) that an integral operator $T \in \mathfrak{L}(L_2(\Omega))$ is of weak type $(1,1)$ if its kernel $k(x,y)$ satisfies the well-known Hörmander condition

$$\int_{B(y,cd(y,y'))^c} |k(x,y)-k(x,y')|\, d\mu(x) \le C \quad \text{for all } \ y,y' \in \Omega \text{ and some } c>0\,.$$

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In practice, many integral operators satisfy the Hörmander condition, on the other hand there are numerous important examples of operators who do not but are still of weak type $(1,1)$ [F] [CR]. This motivated the following improvement due to Duong and McIntosh [DM] (see e.g. [H], [S] for other results in this direction): T is of weak type $(1,1)$ if there exist integral operators $(S_t)_{t>0}$ satisfying suitable Poisson bounds (and playing the role of approximations to the identity) such that the kernels $k_t(x,y)$ of the composite operators $T(I-S_t)$ satisfy the weakened Hörmander condition

$$\int_{B(y,ct^{1/m})^c} |k_t(x,y)|\, d\mu(x) \le C_0 \quad \text{for all } \ t>0\,,\ y\in\Omega \tag{1.1}$$

and some $c, m > 0$. In this paper we generalize this approach to non-integral operators T. The hypothesis on the S_t to be integral operators satisfying Poisson bounds is replaced by suitable weighted norm estimates. Instead of the weakened Hörmander condition (1.1) we will suppose a suitable maximal estimate in terms of the Hardy-Littlewood p-maximal operator

$$M_p f(x) := \sup_{r>0} N_{p,r} f(x)\,, \quad \text{where} \quad N_{p,r} f(x) := |B_{\Omega_1}(x,r)|^{-1/p}\, \|\, f\,\|_{L_p(B(x,r))}$$

with obvious modification for the case $p=\infty$. Finally, we shall replace the first order approximation of the identity

$$I - DS_t \qquad \text{where} \qquad DS_t \ := \ I - S_t$$

[which underlies the result based on (1.1)] by approximations of higher order $n\in\mathbb{N}$:

$$I - D^n S_t \qquad \text{where} \qquad D^n S_t \ := \ \sum_{k=0}^{n} \binom{n}{k} (-1)^k S_{kt}$$

(here we let $S_0 := I$). Observe that we formally have $\frac{D^n f(t)}{t^n} \to (-1)^n f^{(n)}(0)$ for $t\to 0$ and an arbitrary function f. This leads to the first main result of this paper which is the following weak type (p,p) criterion for non-integral operators T.

Theorem 1.1 *Let (Ω_1, d, μ) be a space of homogeneous type and dimension $D \ge 1$:*

$$|B_{\Omega_1}(x,\lambda r)| \ \le \ C\,\lambda^D\, |B_{\Omega_1}(x,r)| \quad \textit{for all } \ x\in\Omega_1\,,\ \lambda\ge 1\,,\ r>0\,.$$

Let Ω be a measurable subset of Ω_1, let $1\le p<p_o<q\le\infty$, $q_o\in(p,\infty]$ and $(S_t)_{t\ge 0}$ a family in $\mathfrak{L}(L_{p_o}(\Omega))$ satisfying $S_0 = I$ and the weighted norm estimate

$$\|\, P_{A(x,t^{1/m},k)}\, S_t\, P_{B(x,t^{1/m})}\,\|_{p\to q} \ \le \ |B_{\Omega_1}(x,t^{1/m})|^{\frac{1}{q}-\frac{1}{p}}\, g_t(k) \tag{1.2}$$

for all $x \in \Omega_1$, $t > 0$, $k \in \mathbb{N}_0$, *for some* $m > 0$ *and functions* $g_t : \mathbb{R}_{\geq 0} \to \mathbb{R}_+$ *such that* $\sum_k (k+1)^{D-1} g_t(k) \leq C$ *for all* $t > 0$. *Let* $T \in \mathfrak{L}(L_{p_o}(\Omega), L^w_{p_o}(\Omega))$ *satisfy*

$$N_{p',t^{1/m}/2}\Big((T\, D^n S_t)^* \, P_{B(z,4t^{1/m})^c} f \Big)(z) \;\leq\; C_1 \, (M_{q_o'} f)(x) \tag{1.3}$$

for all $t > 0$, $f \in L_{p'}$, $z \in \Omega_1$, $x \in B_{\Omega_1}(z, t^{1/m}/2)$ *and some* $n \in \mathbb{N}$. *Then we have* $T \in \mathfrak{L}(L_p(\Omega), L^w_p(\Omega))$.

Here and for the rest of the paper we denote by $\| \cdot \|_{p\to q}$ the $L_p(\Omega) \to L_q(\Omega)-$norm and by $A(x,r,k)$ the annular region $A(x,r,k) := B(x,(k+1)r) \setminus B(x,kr)$ in Ω . Moreover, we denote by L^w_p the weak L_p-spaces.

Some remarks on the comparison of the weakened Hörmander condition (1.1) and its substitute (1.3) are given in Section 2 where we also present a variant of Theorem 1.1 which uses a different substitute (Theorem 2.1).

As our second main result and as an application of Theorem 1.1 we will show that every sectorial operator A with a bounded H^∞ calculus on $L_2(\Omega)$ also has a bounded H^∞ functional calculus on $L_r(\Omega)$ for all $r \in (p,q)$, provided $2 \in (p,q)$ and the semigroup (e^{-tA}) satisfies weighted $L_p \to L_q$-norm estimates of the type (1.2) .

For a sectorial operator A in a Banach space X we say that A *has a bounded* $H^\infty(\Sigma_\nu)$ *calculus* if

$$\|F(A)\|_{\mathfrak{L}(X)} \;\leq\; C\, \|F\|_{H^\infty(\Sigma_\nu)} \quad \text{for all } \; F \in H^\infty(\Sigma_\nu)\,,$$

where $H^\infty(\Sigma_\nu)$ denotes the space of all bounded holomorphic functions on the sector $\Sigma_\nu := \{\, z\,;\, |\arg(z)| < \nu \,\}$. This notion was introduced by McIntosh [M], details on the construction of $F(A)$ may be found in Section 4 below. We will say that A has an H^∞ calculus if A has a bounded $H^\infty(\Sigma_\nu)$ calculus for some $\nu > 0$.

Since there are sectorial operators A which do not have an H^∞ calculus, there is an interest in manageable criteria to check for this property. Duong and McIntosh show in [DM] (see also [DR]) as an application of the weakened Hörmander condition (1.1) that every operator A with an H^∞ functional calculus on $L_2(\Omega)$, which generates a semigroup of integral operators (e^{-tA}) satisfying Poisson bounds, also has an H^∞ functional calculus on $L_p(\Omega)$ for all $p \in (1,\infty)$.

Since the validity of Poisson bounds implies that (e^{-tA}) acts on $L_p(\Omega)$ for all $p \in (1,\infty)$, this result cannot be applied in situations where the semigroup (e^{-tA}) does not act on $L_p(\Omega)$ for all $p \in (1,\infty)$. This happens, e.g., for

Schrödinger operators with bad potentials [ScV], second order elliptic operators with bad lower order terms [LSV], or higher order elliptic operators with bounded measurable coefficients [D2] [AT]. For other problems, e.g. p-independence of the spectrum of A or maximal regularity of $-A$, Poisson bounds have been successfully replaced by the assumption that the semigroup (e^{-tA}) satisfies weighted $L_p \to L_q$-norm estimates of the type (1.2); see, e.g., [ScV], [KV], [LSV], [BK1].

In this case, the second main result of this paper now allows to extend an H^∞ functional calculus of A from $L_2(\Omega)$ to $L_r(\Omega)$ for all $r \in (p,q)$, provided $2 \in (p,q)$.

Theorem 1.2 *Let (Ω_1, d, μ) be a space of homogeneous type and dimension $D \geq 1$:*

$$|B_{\Omega_1}(x, \lambda r)| \;\leq\; C\,\lambda^D\,|B_{\Omega_1}(x,r)| \quad \text{for all } \; x \in \Omega_1\,,\; \lambda \geq 1\,,\; r > 0\,.$$

Let Ω be a measurable subset of Ω_1, let $1 \leq p < 2 < q \leq \infty$ and $w \in [0, \frac{\pi}{2})$. Furthermore, let (e^{-tA}) be a bounded analytic semigroup of angle $\frac{\pi}{2} - w$ on $L_2(\Omega)$ satisfying the weighted norm estimates

$$\text{(1.4)} \quad ||\, P_{A(x,|z|^{1/m},k)}\, e^{-zA} P_{B(x,|z|^{1/m})}\, ||_{p\to q} \leq C_\theta\, |B_{\Omega_1}(x, |z|^{1/m})|^{\frac{1}{q}-\frac{1}{p}}\; (1+k)^{-\kappa_\theta}$$

$$\text{(1.5)} \quad ||\, P_{B(x,|z|^{1/m})}\, e^{-zA} P_{A(x,|z|^{1/m},k)}\, ||_{p\to q} \leq C_\theta\, |B_{\Omega_1}(x, |z|^{1/m})|^{\frac{1}{q}-\frac{1}{p}}\; (1+k)^{-\kappa_\theta}$$

for all $x \in \Omega_1$, $k \in \mathbb{N}_0$, $\theta > w$, $z \in \Sigma_{\frac{\pi}{2}-\theta}$ and some $m > 0$, $\kappa_\theta > D$. Furthermore, for all $\nu > w$, let A have a bounded $H^\infty(\Sigma_\nu)$ calculus on $L_2(\Omega)$. Then, for all $\nu > w$ and $r \in (p,q)$, A has a bounded $H^\infty(\Sigma_\nu)$ calculus on $L_r(\Omega)$.

For the verification of the L_2-hypotheses of Theorem 1.2 we recall that if (e^{-tA}) is a contractive analytic semigroup of angle $\frac{\pi}{2} - w$ [with $w \in [0, \frac{\pi}{2})$] on a Hilbert space such that A is one-one then A has a bounded $H^\infty(\Sigma_\nu)$ calculus for all $\nu > w$ [M].

Since, for the special case $(p,q) = (1,\infty)$, weighted norm estimates of the type in (1.4) and (1.5) are equivalent to Poisson heat kernel bounds for the semigroup (e^{-tA}) [see Prop. 3.5 below], our Theorem 1.2 generalizes the result of Duong and McIntosh [DM, Thm. 6]. In Section 2 we apply Theorem 1.2 to the above classes of elliptic operators and derive new results.

Besides Theorem 1.2, further applications of Theorem 1.1 to weak (p,p) estimates for Riesz-type transforms are given in [BK2].

The outline of this paper is as follows. In Section 2 we comment on our weak type (p,p) criterion Theorem 1.1 and discuss some modifications. Moreover, we apply Theorem 1.2 to some types of elliptic operators and derive new results. Section 3 contains our main tools which are the Calderón-Zygmund decomposition of L_p-functions and the interplay of weighted norm estimates and maximal functions as already employed in [BK1]. In Section 4 we prove the results in Section 1 and Section 2.

2. Modifications and applications

2.1. Substitute of the weakened Hörmander condition

Here we compare our weak type (p,p) criterion Theorem 1.1 in the case $(p,q) = (1,\infty)$ of integral operators with the following weak type $(1,1)$ criterion of Duong and McIntosh [DM, Thm. 2] which was already mentioned in the introduction.

Theorem A. *Let (Ω_1, d, μ) be a space of homogeneous type and dimension $D \geq 1$. Let Ω be a measurable subset of Ω_1, $p_o \in (1,\infty)$ and $(S_t)_{t>0}$ be a family in $\mathfrak{L}(L_{p_o}(\Omega))$ satisfying the weighted norm estimate*

$$(2.1)\qquad ||\, P_{A(x,t^{1/m},k)}\, S_t\, P_{B(x,t^{1/m})}\,||_{1\to\infty} \;\leq\; |B_{\Omega_1}(x,t^{1/m})|^{-1}\, g(k)$$

for all $x \in \Omega_1$, $t > 0$, $k \in \mathbb{N}_0$, for some $m > 0$ and a decreasing function $g : \mathbb{R}_{\geq 0} \to \mathbb{R}_+$ with $\sum_k (k+1)^{D-1} g(k) < \infty$. Let $T \in \mathfrak{L}(L_{p_o}(\Omega))$ such that the operators $T(I - S_t)$ have integral kernels $k_t(x,y)$ satisfying the weakened Hörmander condition

$$(2.2)\qquad \int_{B(y,ct^{1/m})^c} |k_t(x,y)|\, d\mu(x) \;\leq\; C_0 \quad \textit{for all } t > 0\,,\, y \in \Omega\,.$$

Then T is of weak type $(1,1)$.

For the equivalence of the weighted norm estimate (2.1) and the Poisson type heat kernel bound used in [DM] we refer to Proposition 3.5 below. Observe that the weakened Hörmander condition (2.2) implies the following estimate which is crucial in the proof of Theorem A :

$$(2.3)\qquad ||\, P_{B(z,t^{1/m})}\, (T\, DS_t)^*\, P_{B(z,(1+c)t^{1/m})^c} f\, ||_\infty \;\leq\; C_0\, ||\, f\, ||_\infty$$

On the other hand, the maximal estimate (1.3) [which is our substitute of (2.2) in order to treat non-integral operators] in the hypothesis of our Theorem 1.1 means in the integral case $(p, q_o) = (1,\infty)$ and for $n = 1$ the following:

$$(2.4)\quad ||P_{B(z,t^{1/m}/2)}\, (T\, DS_t)^*\, P_{B(z,4t^{1/m})^c} f\, ||_\infty \;\leq\; C_1 M_1 f(x)\,,\; x \in B(z, \tfrac{t^{1/m}}{2})\,.$$

Hence the global uniform estimate (2.3) is replaced by the local maximal estimate (2.4). We point out that our proof of Theorem 1.1 does not work if the hypothesis (1.3) is replaced by the seemingly natural adaptation of (2.3)

$$|| P_{B(z,t^{1/m}/2)} (T\,DS_t)^* P_{B(z,4t^{1/m})^c} f ||_{p'} \;\leq\; C\, || f ||_{p'}\;,$$

i.e. by the boundedness of $\tau := \{P_{B(z,4t^{1/m})^c} (T\,DS_t) P_{B(z,t^{1/m}/2)}\,;\, z\in\Omega, t>0\}$ on $L_p(\Omega)$. On the other hand, the hypothesis (1.3) can be replaced by the R_1-boundedness of τ on the weak Lebesgue space $L_p^w(\Omega)$ in the sense of Weis [W] :

$$|| \sum_j |T_j f_j| \,||_X \;\leq\; C\, || \sum_j |f_j| \,||_X$$

whenever $T_j \in \tau$ and $f_j \in X := L_p^w(\Omega)$. This yields the following modification of Theorem 1.1 which will be proved in Section 4.

Theorem 2.1 *Let (Ω_1, d, μ) be a space of homogeneous type and dimension $D \geq 1$. Let Ω be a measurable subset of Ω_1, let $1 \leq p < p_o < q \leq \infty$ and $(S_t)_{t\geq 0}$ a family in $\mathfrak{L}(L_{p_o}(\Omega))$ satisfying $S_0 = I$ and the weighted norm estimate*

$$|| P_{A(x,t^{1/m},k)}\, S_t\, P_{B(x,t^{1/m})} ||_{p\to q} \quad\leq\quad |B_{\Omega_1}(x,t^{1/m})|^{\frac{1}{q}-\frac{1}{p}}\; g_t(k)$$

for all $x \in \Omega_1$, $t > 0$, $k \in \mathbb{N}_0$, for some $m > 0$ and functions $g_t : \mathbb{R}_{\geq 0} \to \mathbb{R}_+$ such that $\sum_k (k+1)^{D-1} g_t(k) \leq C$ for all $t > 0$. Let $T \in \mathfrak{L}(L_{p_o}(\Omega), L_{p_o}^w(\Omega))$ be such that

(2.5) $\{P_{B(z,4t^{1/m})^c} (T D^n S_t) P_{B(z,t^{1/m}/2)}\,;\, z\in\Omega, t>0\}$ *is R_1-bounded on $L_p^w(\Omega)$*

for some $n \in \mathbb{N}$. Then we have $T \in \mathfrak{L}(L_p(\Omega), L_p^w(\Omega))$.

2.2. Exponential weights on $\mathbb{R}^D$

The weighted norm estimates in the hypotheses of our main results use the characteristic functions $\chi_{B(x,t^{1/m})}$ and $\chi_{A(x,t^{1/m},k)}$ as weight functions. A similar hypothesis can be given in terms of the widely used exponential weights $e^{\rho d(x,\cdot)}$ [with $\rho \in \mathbb{R}$] which involves the so-called Davies perturbations $e^{-\rho d(x,\cdot)}\, e^{-tA}\, e^{\rho d(x,\cdot)}$ of a given semigroup (e^{-tA}). Recently, norm estimates for the Davies perturbations were used to show spectral p-independence, [KV], [LSV], or maximal regularity [BK1] of the operator A. Here, for the question of an H^∞ calculus of A, they allow the following modification of our Theorem 1.2 which will be proved in Section 4.

Theorem 2.2 *Let Ω be a measurable subset of $\mathbb{R}^D$, let $1 \leq p < 2 < q \leq p'$ and $w \in [0, \frac{\pi}{2})$. Let (e^{-tA}) be a bounded analytic semigroup of angle $\frac{\pi}{2} - w$ on $L_2(\Omega)$ satisfying*

$$(2.6) \qquad || e^{-tA} ||_{p\to p} \ , \ || e^{-tA} ||_{p'\to p'} \ , \ t^{\frac{D}{m}(\frac{1}{2}-\frac{1}{q})} || e^{-tA} ||_{2\to q} \ \leq \ C$$

$$(2.7) \qquad || e^{-\rho d(x,\cdot)} e^{-tA} e^{\rho d(x,\cdot)} ||_{2\to 2} \ \leq \ C\, e^{c|\rho|^m t}$$

for all $x \in \mathbb{R}^D$, $t > 0$, $\rho \in \mathbb{R}$ and some $m > 1$, $c > 0$. Furthermore, for all $\nu > w$, let A have a bounded $H^\infty(\Sigma_\nu)$ calculus on $L_2(\Omega)$. Then, for all $\nu > w$ and $r \in (p, p')$, A has a bounded $H^\infty(\Sigma_\nu)$ calculus on $L_r(\Omega)$.

It can be seen from the proof that the exponent p' in Theorem 2.2 may be replaced by an arbitrary $q_0 > 2$. We have chosen p' for simplicity since this is the right choice for many (but not all) applications.

In the next section, our Theorem 2.2 is applied to obtain new results on the H^∞ calculus of certain classes of elliptic operators A on $\mathbb{R}^D$. Note that in this case an unweighted $L_2 \to L_q$-norm estimate as in (2.6) can be checked by standard Sobolev or Nash type arguments and the weighted $L_2 \to L_2$-norm estimate (2.7) is obtained by well-known ellipticity arguments.

2.3. H^∞ calculus of elliptic operators on $\mathbb{R}^D$

Let A be an elliptic divergence form operator on $L_2(\mathbb{R}^D)$ of order $2m$, $m \in \mathbb{N}$, which is one-one and satisfies

$$|\arg\langle Af, f\rangle| \ \leq \ w \quad \text{for all } \ f \in D(A) \text{ and some } w \in [0, \tfrac{\pi}{2})\,.$$

Then (e^{-tA}) is a contractive analytic semigroup of angle $\frac{\pi}{2} - w$ on $L_2(\mathbb{R}^D)$ and thus, as mentioned above, A has a bounded $H^\infty(\Sigma_\nu)$ calculus on $L_2(\mathbb{R}^D)$ for all $\nu > w$. Hence, by using our Theorem 2.2, it suffices to prove the (weighted) norm estimates (2.6),(2.7) for some p, q in order to establish a bounded $H^\infty(\Sigma_\nu)$ calculus of A on $L_r(\mathbb{R}^D)$ for all $\nu > w$ and $r \in (p, p')$.

We go into some detail for higher order operators with complex coefficients (cf. [D1], [AT, §1.7]), Schrödinger operators with singular potentials (cf. [BS], [ScV]) and second order operators with singular lower order terms (cf. [KS], [LSV]).

2.3.1. Higher order operators with complex coefficients

These operators are given by forms $\mathfrak{a} : H^m(\mathbb{R}^D) \times H^m(\mathbb{R}^D) \to \mathbb{C}$ of the type

$$(2.8) \qquad \mathfrak{a}(u, v) = \int_{\mathbb{R}^D} \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}\, \partial^\alpha u\, \overline{\partial^\beta v}\, dx \ ,$$

where the $a_{\alpha,\beta} : \mathbb{R}^D \to \mathbb{C}$ are bounded measurable functions and $\mathfrak{a}$ satisfies Garding's inequality

$$\text{(2.9)} \qquad Re\, \mathfrak{a}(u) \;\geq\; \delta\, \|\nabla^m u\|_2^2 \quad \text{for all } \; u \in H^m(\mathbb{R}^D)$$

and for some $\delta > 0$, where $\|\nabla^m u\|_2^2 := \sum_{|\alpha|=m} \|\partial^\alpha u\|_2^2$. Then $\mathfrak{a}$ is a closed sectorial form and the associated operator A is given by $u \in D(A)$ and $Au = g$ if and only if $u \in H^m$ and $\langle g, v\rangle = \mathfrak{a}(u,v)$ for all $v \in H^m$. We denote the sectoriality angle of the form $\mathfrak{a}$ by $w \in [0, \frac{\pi}{2})$. In this situation, Theorem 2.2 is applicable and yields

Proposition 2.3 *Let $p_0 := \frac{2D}{2m+D} \vee 1$. Then, for all $r \in (p_0, p_0')$ and $\nu > w$, A has a bounded $H^\infty(\Sigma_\nu)$ calculus on $L_r(\mathbb{R}^D)$.*

If the semigroup (e^{-tA}) is bounded on $L_p(\mathbb{R}^D)$ and $L_{p'}(\mathbb{R}^D)$ for some $p \in [1, p_0)$ then, for all $r \in (p, p')$ and $\nu > w$, A has a bounded $H^\infty(\Sigma_\nu)$ calculus on $L_r(\mathbb{R}^D)$.

For $p = 1$ the last assertion can be derived from [DR] , [DM]. Note that boundedness of (e^{-tA}) on L_1 and L_∞ can be obtained by imposing suitable regularity assumptions on the coefficients; cf. [D1] , [AT] , [AQ] and the references given there.

Whenever (e^{-tA}) is not bounded on L_1 and L_∞ the result on boundedness of the H^∞-calculus is new. Note in this context that [D2] shows optimality of p_0 for the given class of operators.

If we denote the inf over all p such that (e^{-tA}) is bounded on L_p and $L_{p'}$ by p_{opt} we obtain an H^∞ calculus of optimal angle in L_r for all $r \in (p_{\text{opt}}, p'_{\text{opt}})$. This shows how close to optimal our results are.

2.3.2. Schrödinger operators with singular potentials

We now turn to another class of operators for which Theorem 2.2 yields new results. We study Schrödinger operators $H = -\Delta + V$ on $\mathbb{R}^D$, $D \geq 3$, where $V = V_+ - V_-$, $V_\pm \geq 0$ are locally integrable, and V_+ is bounded for simplicity (for the general case, see [BS] , [ScV]). We assume

$$\text{(2.10)} \quad \langle V_- \phi, \phi\rangle \leq \gamma\Big(\|\nabla\phi\|_2^2 + \langle V_+\phi, \phi\rangle \Big) + c(\gamma)\|\phi\|_2^2 \quad \text{for all } \; \phi \in H^1(\mathbb{R}^D)$$

and some $0 \leq \gamma < 1$. Then the form sum $H := -\Delta + V = (-\Delta + V_+) - V_-$ is defined and the associated form is closed and symmetric with form domain $H^1(\mathbb{R}^D)$. For $A := H + c(\gamma)$, we obtain the last estimate in (2.6) for $q := \frac{2D}{D-2}$ by Sobolev embedding. By [BS], the first two estimates in (2.6) hold for $p = t(\gamma) := \frac{2}{1+\sqrt{1-\gamma}}$, and the interval $r \in [t(\gamma), t(\gamma)']$ is optimal for quasi-contractivity of (e^{-tH}) on $L_r(\mathbb{R}^D)$.

In the case $\gamma > 0$, however, (e^{-tH}) extends to a C_0-semigroup on $L_r(\mathbb{R}^D)$ for all $r \in (p(\gamma), p(\gamma)')$, where $p(\gamma)' := \frac{D}{D-2} t(\gamma)'$, i.e. $p(\gamma) = 2D/(D(1 + \sqrt{1-\gamma}) + 2(1-\sqrt{1-\gamma}))$; cf. the remark on [BS, p. 542] and [LSV] . Hence our Theorem 2.2 yields

Proposition 2.4 *For all $r \in (p(\gamma), p(\gamma)')$ there exists a constant $c_r \geq 0$ such that, for all $\nu > 0$, the operator $H + c_r = -\Delta + V + c_r$ has a bounded $H^\infty(\Sigma_\nu)$ calculus on $L_r(\mathbb{R}^D)$.*

For $r \in [t(\gamma), t(\gamma)']$ one might use transference and Stein interpolation to obtain a bounded H^∞ calculus on L_r but this would not give the optimal angle. If the semigroup (e^{-tH}) is not quasi-contractive in L_r then transference cannot be used.

2.3.3. Second order operators with singular lower order terms

More generally than Schrödinger operators, we consider uniformly elliptic second order operators with real coefficients and with unbounded first order terms as studied in [KS] [L] [LSV]. These operators are no longer symmetric. Except for [LSV] which is based on a new method for the construction of semigroups (more general than the form method) the L_p-scale for existence of the semigroup depends on form bounds associated to the lower order terms.

To give an example, assume that $b = (b_1, \ldots, b_D) : \mathbb{R}^D \to \mathbb{R}^D$ is measurable and such that $B := \sum_j b_j^2$ is locally integrable and

$$\langle B\phi, \phi \rangle \quad \leq \quad \beta \, \|\nabla \phi\|_2^2 \; + \; c(\beta) \, \|\phi\|_2^2 \qquad \text{for all} \;\; \phi \in C_c^\infty(\mathbb{R}^D)$$

and for some $0 \leq \beta < 1$. Moreover, assume that V is as above and such that (2.10) and $\sqrt{\beta} + \gamma < 1$ hold. Let the form $\mathfrak{a}$ on $H^1(\mathbb{R}^D)$ be given by

$$\mathfrak{a}(u, v) := \int \overline{\nabla v^t} \, \nabla u + \overline{v}(b^t \, \nabla u + V v) \, dx \; .$$

Then $\mathfrak{a}$ is closed and quasi-sectorial, hence the associated operator A is formally given by $A = -\Delta + b\nabla + V$. It is shown in [L] that (e^{-tA}) acts as a quasi-contractive semigroup on $L_r(\mathbb{R}^D)$ for all

$$r \in \left[\frac{4}{2 - \sqrt{\beta} + \sqrt{(2-\sqrt{\beta})^2 - 4\gamma}} \, , \, \frac{4}{2 - \sqrt{\beta} - \sqrt{(2-\sqrt{\beta})^2 - 4\gamma}} \right] .$$

Denoting this interval by $[p_-, p_+]$ it is shown in [LSV] that (e^{-tA}) acts as a C_0-semigroup on $L_r(\mathbb{R}^D)$ for all $r \in (p_{\min}, p_{\max})$, where $p_{\max} := \frac{D}{D-2} p_+$ and $p_{\min} := (\frac{D}{D-2} p_-')'$. Hence Theorem 2.2 yields

Proposition 2.5 *For all $r \in (p_{min}, p_{max})$ there exists a constant $c_r \geq 0$ such that, for all $\nu > w$, the operator $A + c_r$ has a bounded $H^\infty(\Sigma_\nu)$ calculus on $L_r(\mathbb{R}^D)$, where $w \in [0, \frac{\pi}{2})$ denotes the sectoriality angle of the form $\mathfrak{a} + c_r$.*

We want to remark that the most general results on weighted norm estimates for operators of this type are contained in [LSV] and that Theorem 2.2 also yields boundedness of H^∞ calculi for the operators studied there.

3. Tools

In this section we present our main tools which are the Calderón-Zygmund decomposition of L_p-functions as well as the interplay of weighted norm estimates and maximal functions [BK1].

3.1. Calderón-Zygmund decomposition of L_p-functions

The following is a straight forward generalization of the usual Calderón-Zygmund decomposition of L_1-functions. It can, however, not be obtained by applying the usual Calderón-Zygmund decomposition to $|f|^p$. One has to examine the proof and we give full details here for convenience.

Theorem 3.1 *Let Ω be a space of homogeneous type and $p \in [1, \infty)$. There exists a constant M such that, for all $f \in L_p(\Omega)$ and $\alpha > 0$, we find a function g and sequences (b_j) of functions and (B_j^*) of balls such that*

(i) $f = g + \sum_j b_j$;

(ii) $\|g\|_\infty \leq C\alpha$;

(iii) *supp*$(b_j) \subset B_j^*$ *and* $\#\{j; x \in B_j^*\} \leq M$ *for all* $x \in \Omega$;

(iv) $\|b_j\|_p \leq C\alpha |B_j^*|^{1/p}$;

(v) $(\sum_j |B_j^*|)^{1/p} \leq C\alpha^{-1}\|f\|_p$.

Proof. We follow [St, I 4.1] in arguments and notation. Let $E_\alpha := \{x \in \Omega; \tilde{M}(|f|^p) > \alpha\}$, where $\tilde{M}$ is the uncentered maximal operator. By [St, I 3.2], we find sequences (B_j) of balls and (Q_j) of mutually disjoint “cubes” (which may recursively obtained from the balls by $Q_k := B_k^* \cap (\bigcup_{j<k} Q_j)^c \cap (\bigcup_{j>k} B_j)^c$) such that

$$B_j \subset Q_j \subset B_j^* \subset E_\alpha \ , \quad \bigcup_j B_j^* = E_\alpha \ , \quad B_j^{**} \not\subset E_\alpha \ .$$

If $B_j = B(x_j, r_j)$ then $B_j^* = B(x_j, c^* r_j)$ and $B_j^{**} = B(x_j, c^{**} r_j)$ where c^*, c^{**} are the constants from [St, I 3.2].

Now define g and the sequence (b_j) by

$$g(x) := \begin{cases} f(x), & x \notin E_\alpha \\ (|Q_j|^{-1} \int_{Q_j} |f(y)|^p \, dy)^{1/p}, & x \in Q_j \end{cases}$$

$$b_j(x) := 1_{Q_j}(x) \Big[f(x) - \Big(|Q_j|^{-1} \int_{Q_j} |f(y)|^p \, dy \Big)^{1/p} \Big]. \tag{3.1}$$

Then (i) holds. For each j we have $B_j^{**} \cap E_\alpha \neq \emptyset$, hence, by the definition of $\tilde{M}$,

$$|B_j^{**}|^{-1} \int_{B_j^{**}} |f(y)|^p \, dy \leq \alpha^p$$

which leads to

$$\int_{Q_j} |f(y)|^p \, dy \; \leq \; \int_{B_j^{**}} |f(y)|^p \, dy \; \leq \; \alpha^p |B_j^{**}| \; \leq \; c_1 \alpha^p |B_j| \; \leq \; c_1 \alpha^p |Q_j|$$

by the doubling property. Hence $|g(x)| \leq c_2^{1/p} \alpha$ for $x \in Q_j$ and clearly $|g(x)| \leq \alpha$ for $x \notin E_\alpha$. Since [St, I 3.2] gives also the bounded intersection property for the B_j^* we have proved (ii) and (iii). To prove (iv) notice that $(a+b)^p \leq 2^{p-1}(a^p + b^p)$ for $a, b \geq 0$, hence

$$\|b_j\|_p^p \; \leq \; 2^p \int_{Q_j} |f(x)|^p \, dx \; \leq \; 2^p \alpha^p c_2 |B_j^*|.$$

Now (v) follows from the weak $(1,1)$ boundedness of $\tilde{M}$:

$$\sum_j |B_j^*| \; \leq \; c_3 \sum_j |B_j| \; \leq \; c_3 \sum_j |Q_j| = c_3 |E_\alpha| \; \leq \; c_4 \alpha^{-p} \| |f|^p \|_1 = \frac{c_4}{\alpha^p} \|f\|_p^p.$$

■

Remark 3.2 Recall from the proof that the b_j have disjoint supports, hence

$$\| \sum_j b_j \|_p^p \; = \; \sum_j \|b_j\|_p^p \; \leq \; C \alpha^p \sum_j |B_j^*| \; \leq \; C' \, \|f\|_p^p$$

by (iv) and (v), which by (i) implies the following additional property:

(vi) $\|g\|_p \; \leq \; C \, \|f\|_p$.

3.2. Weighted norm estimates and maximal functions

Here we continue the development of techniques from [BK1] relating weighted norm estimates of the type appearing in our main results and the Hardy-Littlewood p-maximal operator M_p defined by

$$M_p f(x) := \sup_{r>0} N_{p,r} f(x)\,, \quad \text{where} \quad N_{p,r} f(x) := |B(x,r)|^{-1/p}\, \|\, f\,\|_{L_p(B(x,r))}$$

with obvious modification for the case $p = \infty$.

Lemma 3.3 *Let $p \in [1,\infty)$, $q \in [1,\infty]$ and Ω be a space of dimension $D \geq 1$:*

$$|B(x,\lambda s)| \;\leq\; C_0\, \lambda^D\, |B(x,s)| \quad \text{for all } \; x \in \Omega\,,\; \lambda \geq 1\,,\; s > 0\,.$$

Let S be a linear operator such that for some $r > 0$ we have

$$\text{(3.2)} \quad \|\, P_{B(x,r)}\, S\, P_{A(x,r,k)}\, \|_{p\to q} \leq |B(x,r)|^{\frac{1}{q}-\frac{1}{p}}\; g(k) \quad \text{for all } \; x \in \Omega\,,\; k \in \mathbb{N}_0$$

and a function $g : \mathbb{N}_0 \to \mathbb{R}_+$ satisfying $K := \sum_{k=0}^{\infty} (k+1)^{D-1}\, g(k) < \infty$.
(a) We have for all $s > 0$ and $x \in \Omega$, $z \in B(x,s)$:

$$N_{q,s}(S\, P_{B(z,5s)^c} f\,)(z) \leq C_1\, K \,\Big(\sum_{k \geq 4sr^{-1}} k^{D-1} g(k) \Big)^{1/p}\, (1+s^{-1}r)^{D/q}\, M_p f(x)\,.$$

(b) We have for all $x \in \Omega$:

$$N_{q,r} S f(x) \;\leq\; C_1\, K\, M_p f(x)\,.$$

Here the constant C_1 depends on p, q, C_0, D and on nothing else.

Note that part (b) corresponds to [BK1, Lemma 2.6]. Before giving the proof of Lemma 3.3, we collect some simple but important properties of the operators $N_{p,r}$.

Lemma 3.4 *Let $p \in [1,\infty]$ and Ω be a space of dimension D :*

$$|B(x,\lambda r)| \;\leq\; C_0\, \lambda^D\, |B(x,r)| \quad \text{for all } \; x \in \Omega\,,\; \lambda \geq 1\,,\; r > 0\,.$$

(a) We have for all $R \geq 2r > 0$, $x \in \Omega$, $y \in B(x,r)$, $f \in L_p(\Omega)$:

$$N_{p,r} f(y) \;\leq\; (C_0^2 2^D)^{1/p} \left(\tfrac{R}{r}\right)^{D/p} N_{p,R}\, f(x)$$

(b) We have for all $r > 0$, $f \in L_p(\Omega)$:

$$\|\, f\,\|_p \;\leq\; (C_0 2^D)^{1/p}\, \|\, N_{p,r} f\,\|_p$$

Proof. (a) Using the evident inclusions $B(x,r) \subset B(y,2r)$ and $B(y,r) \subset B(x,R)$ we can estimate as follows :

$$\begin{aligned} N_{p,r}f(y) &= |B(y,r))|^{-1/p} \, \| \, P_{B(y,r)} f \, \|_p \\ &\leq (C_0 2^D)^{1/p} \, |B(x,r)|^{-1/p} \, \| \, P_{B(x,R)} f \, \|_p \\ &\leq (C_0^2 2^D)^{1/p} \left(\tfrac{R}{r}\right)^{D/p} |B(x,R)|^{-1/p} \, \| \, P_{B(x,R)} f \, \|_p \end{aligned}$$

(b) A simple calculation using Fubini's Theorem shows $\| \, f \, \|_p = \| \, \widetilde{N}_{p,r} f \, \|_p$ for the

$$\widetilde{N}_{p,r}f(x) \;:=\; \left(\int_{B(x,r)} |f(y)|^p \, \tfrac{d\mu(y)}{|B(y,r)|} \right)^{1/p} .$$

But reasoning as in the proof of (a) yields $\widetilde{N}_{p,r}f(x) \leq (C_0 2^D)^{1/p} N_{p,r}f(x)$. ■

Proof of Lemma 3.3. We use the symbol $\preceq$ to indicate domination up to constants depending only on p, q, C_0, D. Define $b_k := k^D(g(k-1) - g(k))$. At first we will show that we have for all $s, t \geq 0$ and $x, z \in \Omega$:

$$N_{q,r}(P_{B(z,s)} \, S \, P_{B(z,t)^c} f \,)(x) \;\preceq\; K \left(\sum_{k > (t-s)r^{-1}} b_k \, N_{p,kr} f(x)^p \right)^{1/p} \tag{3.3}$$

For the proof of (3.3) we can assume $supp(f) \subset B(z,t)^c$ and $d(x,z) \leq s + r$. The latter implies:

$$k \;\leq\; (t-s)r^{-1} - 1 \quad \Longrightarrow \quad B(x,kr) \cap B(z,t)^c \;=\; \emptyset \, .$$

Therefore we deduce line (3.3) as follows :

$$\begin{aligned} &N_{q,r}(P_{B(z,s)} \, S \, P_{B(z,t)^c} f)(x) \;\leq\; |B(x,r)|^{-1/q} \, \| \, P_{B(x,r)} \, Sf \, \|_q \\ &\leq\; |B(x,r)|^{-1/p} \sum_{k=0}^{\infty} g(k) \, \| \, P_{A(x,r,k)} \, f \, \|_p \\ &\leq\; K^{1/p'} \left(|B(x,r)|^{-1} \sum_{k > (t-s)r^{-1}} \tfrac{b_k}{k^D} \, \| \, P_{B(x,kr)} f \, \|_p^p \right)^{1/p} \\ &\preceq\; K \left(\sum_{k > (t-s)r^{-1}} b_k \, N_{p,kr} f(x)^p \right)^{1/p} \end{aligned}$$

(b) follows now from (3.3) for $s := d(x,z) + r$ and $t = 0$ since one obviously has $N_{p,kr}f(x) \leq M_p f(x)$.

(a) The case $r \geq 2s$ can be handled directly using part (b) and Lemma 3.4(a):

$$N_{q,s}(S\,P_{B(z,5s)^c}f\,)(z) \preceq \left(\tfrac{r}{s}\right)^{D/q} N_{q,r}(S\,P_{B(z,5s)^c}f\,)(x) \preceq K\,\left(\tfrac{r}{s}\right)^{D/q} M_p f(x)$$

Hence we assume for the rest of the proof that $r \leq 2s$. Lemma 3.4(a) then shows

$$(3.4)\quad N_{p,kr}f(y) \preceq N_{p,2kr}f(x) \leq M_p f(x) \quad \text{for all } \ y \in B(x,4s)\,,\ k \geq 4sr^{-1}.$$

Moreover, since $z \in B(x,s)$, it is easy to see that

$$(3.5)\qquad N_{q,r}(P_{B(z,s)}f\,)(y) \;=\; 0 \quad \text{for all } \ y \in B(x,4s)^c\,.$$

Now we can finish the proof as follows :

$$\begin{aligned} N_{q,s}(S\,P_{B(z,5s)^c}f\,)(z) \;&=\; |B(z,s)|^{-1/q}\,||\,P_{B(z,s)}\,S\,P_{B(z,5s)^c}f\,||_q \\ \preceq\;& |B(x,s)|^{-1/q}\,||\,N_{q,r}(P_{B(z,s)}\,S\,P_{B(z,5s)^c}f\,)\,||_{L_q(B(x,4s))} \quad [\text{by } (3.5)\,, 3.4(\text{b})] \\ \preceq\;& \sup_{y\in B(x,4s)} N_{q,r}(P_{B(z,s)}\,S\,P_{B(z,5s)^c}f\,)(y) \\ \preceq\;& K \sup_{y\in B(x,4s)} \Big(\sum_{k>4sr^{-1}} b_k\,N_{p,kr}f(y)^p\Big)^{1/p} \qquad [\text{by } (3.3) \text{ for } t=5s] \\ \preceq\;& K\Big(\sum_{k>4sr^{-1}} b_k\Big)^{1/p} M_p f(x) \qquad [\text{by } (3.4)] \end{aligned}$$

■

In the special case $(p,q) = (1,\infty)$, the weighted norm estimates appearing in our main results characterize the fact that the corresponding operators have integral kernels satisfying certain Poisson upper bounds. This can be seen from the following observation which is proved as e.g. [BK1, Prop. 2.9] .

Proposition 3.5 *Let (Ω_1, d, μ) be a space of homogenous type :*

$$|B_{\Omega_1}(x,2s)| \;\leq\; C_d\,|B_{\Omega_1}(x,s)| \quad \textit{for all } \ x \in \Omega_1\,,\ s > 0\,.$$

Let Ω be a measurable subset of Ω_1 and let $S \in \mathfrak{L}(L_1(\Omega), L_\infty(\Omega))$ have the integral kernel $K \in L_\infty(\Omega^2)$. Furthermore, let $g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a decreasing function and $r > 0$. Then the following are equivalent :

(a) For all $x \in \Omega_1$, $k \in \mathbb{R}_{\geq 0}$ we have

$$||\,P_{B(x,r)}\,S\,P_{A(x,r,k)}\,||_{L_1(\Omega)\to L_\infty(\Omega)} \;\;\leq\;\; |B_{\Omega_1}(x,r)|^{-1}\,g(k)\,.$$

(b) For all $x, y \in \Omega$ we have

$$|K(x,y)| \quad\leq\quad |B_{\Omega_1}(x,r)|^{-1}\,g(d(x,y)r^{-1})\,.$$

Here the statement is written modulo identification of g and $\widetilde{g}$, where $\widetilde{g}(s) := C_d\,g((s-1)_+)$ and C_d is the doubling constant from above.

3.3. A lemma on the inverse Laplace transform

We will need a result on the inversion of the Laplace transform

$$\mathcal{L}F(z) \;=\; \int_0^\infty e^{-sz}\, F(s)\, ds$$

on sectors $\Sigma_\nu := \{z; |\arg(z)| < \nu\}$. For the whole section we fix $0 < \theta < \mu < \nu < \frac{\pi}{2}$. For all $F \in H^\infty(\Sigma_\nu)$, the space of all bounded holomorphic functions on Σ_ν, and all $z \in \partial\Sigma_{\pi/2-\theta}$, we define

$$\mathcal{L}^{-1}F(z) \;:=\; \begin{cases} (2\pi i)^{-1} \int_{e^{i\mu}\mathbb{R}_+} e^{\lambda z}\, F(\lambda)\, d\lambda & \arg(z) = \pi/2 - \theta \\ (2\pi i)^{-1} \int_{e^{-i\mu}\mathbb{R}_+} e^{\lambda z}\, F(\lambda)\, d\lambda & \arg(z) = \theta - \pi/2 \end{cases} .$$

Notice that $\mathcal{L}^{-1}F(z)$ is well defined since we have for $b := -\cos(\frac{\pi}{2}+\mu-\theta) > 0$:

$$Re(\lambda z) \;=\; |\lambda|\,|z|\, Re(e^{\pm i(\frac{\pi}{2}+\mu-\theta)}) \;=\; -b\,|\lambda|\,|z| \;.$$

Standard Cauchy formula arguments show that the following inversion formula for the Laplace transform holds for all $F \in H^\infty(\Sigma_\mu)$ satisfying suitable decay conditions:

$$F(y) \;=\; \int_{\partial\Sigma_{\pi/2-\theta}} e^{-zy}\mathcal{L}^{-1}F(z)\, dz \quad , \quad y \in \Sigma_\theta \;.$$

We will employ the following boundedness property of $\mathcal{L}^{-1}$.

Lemma 3.6 *Let θ, μ, ν as before and $\alpha \in \mathbb{R}$, $\alpha - 1 < \beta, \gamma < n$ with $\beta > 0$. Then*

$$\mathcal{L}^{-1}\mathcal{E} \;:\; H^\infty(\Sigma_\nu) \;\to\; L_1(\partial\Sigma_{\pi/2-\theta}, w(z)|dz|\,) \;.$$

Here $\mathcal{E}$ is the operator of multiplication by

$$E(z) \;:=\; z^{-\alpha} \sum_{k=0}^{n} \binom{n}{k} (-1)^k\, e^{-kz} \;,$$

and we consider the weight function $w(z) := |z|^{\beta-\alpha}\,(1+|z|\,)^{\gamma-\beta}$.

Proof. Observe that $E \in H^\infty(\Sigma_\nu)$ and $|E(\lambda)| \le C_0\, \frac{1\wedge|\lambda|^n}{|\lambda|^\alpha}$ whenever $|\arg(\lambda)| = \mu$. By definition of $\mathcal{L}^{-1}$, this implies for $b := -\cos(\frac{\pi}{2}+\mu-\theta) > 0$:

$$|\mathcal{L}^{-1}(EF)(z)| \;\le\; C_0\,(2\pi)^{-1} \int_0^\infty e^{-bv|z|}\, \frac{1\wedge v^n}{v^\alpha}\, dv\; ||\, F\, ||_{H^\infty(\Sigma_\nu)} \;.$$

Hence we can finish the proof as follows, denoting $\Gamma := \partial\Sigma_{\pi/2-\theta}$:

$$\begin{aligned} \|\,\mathcal{L}^{-1}(EF)\,\|_{L_1(\Gamma,w(z)|dz|)} &\le C_0\,\pi^{-1}\int_0^\infty w(s)\int_0^\infty e^{-bvs\,\frac{1\wedge v^n}{v^\alpha}}\,dv\,ds\,\|\,F\,\|_{H^\infty(\Sigma_\nu)} \\ &= C_1\,\|\,F\,\|_{H^\infty(\Sigma_\nu)}\,. \end{aligned}$$

Here we used the subsequent lemma for the last step. ■

Lemma 3.7 *Let $\alpha \in \mathbb{R}$, $\alpha - 1 < \beta, \gamma < \delta$ with $\beta > 0$ and $b > 0$. Then*

$$\int_0^\infty s^{\beta-\alpha}(1+s)^{\gamma-\beta}\int_0^\infty e^{-bsv\,\frac{1\wedge v^\delta}{v^\alpha}}\,dv\,ds \quad < \quad \infty\,.$$

Proof. We split the integral as follows :

$$\begin{aligned} &\int_0^\infty s^{\beta-\alpha}(1+s)^{\gamma-\beta}\int_0^\infty e^{-bsv\,\frac{1\wedge v^\delta}{v^\alpha}}\,dv\,ds \\ &= \quad \int_0^\infty s^{\beta-\alpha}(1+s)^{\gamma-\beta}\int_0^1 e^{-bsv}\,v^{\delta-\alpha}\,dv\,ds \qquad =: \quad I_1 \\ &\quad + \int_0^\infty s^{\beta-\alpha}(1+s)^{\gamma-\beta}\int_1^\infty e^{-bsv}\,v^{-\alpha}\,dv\,ds \qquad =: \quad I_2 \end{aligned}$$

Hence it remains to show $I_1 < \infty$ and $I_2 < \infty$. For I_1, we observe

$$\begin{aligned} I_1 \;&=\; \int_0^1 v^{\delta-\beta-1}\int_0^\infty r^{\beta-\alpha}\,(1+\tfrac{r}{v})^{\gamma-\beta}\,e^{-br}\,dr\,dv \\ &\le\; C\int_0^1 v^{\delta-\beta-1}\int_0^\infty r^{\beta-\alpha}\,e^{-br}\,dr\,dv + C\int_0^1 v^{\delta-\gamma-1}\int_0^\infty r^{\gamma-\alpha}\,e^{-br}\,dr\,dv \end{aligned}$$

which is finite since $\alpha - 1 < \beta, \gamma < \delta$. For I_2, the estimate is similar:

$$I_2 \;=\; \int_0^\infty e^{-bw}w^{-\alpha}\int_0^w s^{\beta-1}\,(1+s)^{\gamma-\beta}\,ds\,dw \;<\; \infty\,.$$

■

These calculations are taken from [DM, p. 259] where the case $\gamma = 0$ is considered.

4. Proofs of the main results

Proof of Theorem 1.1. We can assume $\Omega = \Omega_1$ by otherwise applying this case for the following 0-extensions to Ω_1:

$$\begin{aligned} \widetilde{S}_t f(x) \;&:=\; \begin{cases} S_t(f|_\Omega)(x) & x \in \Omega \\ 0 & x \in \Omega_1 \setminus \Omega \end{cases} \qquad \text{for } t > 0 \qquad ; \qquad \widetilde{S}_0 := I \\ \widetilde{T} f(x) \;&:=\; \begin{cases} T(f|_\Omega)(x) & x \in \Omega \\ 0 & x \in \Omega_1 \setminus \Omega \end{cases} \,. \end{aligned}$$

Indeed, it is straightforward to check that the hypotheses (1.2) and (1.3) for the the (S_t) and T on Ω imply the corresponding hypotheses for the $(\widetilde{S}_t)$ and $\widetilde{T}$ on Ω_1. Hence the special case $\Omega = \Omega_1$ shows $T \in \mathfrak{L}(L_p(\Omega_1), L_p^w(\Omega_1))$ and it only remains to observe that $\|\widetilde{T}\|_{L_p(\Omega_1)\to L_p^w(\Omega_1)} = \|T\|_{L_p(\Omega)\to L_p^w(\Omega)}$.

Now fix $f \in L_p(\Omega)$ and $\alpha > 0$. In the following we use the symbol $\preceq$ to indicate domination up to constants independent of f and α.

Consider the Calderón-Zygmund decomposition $f = g + \sum b_j$ at height α according to Theorem 3.1. We write $B_j = B(x_j, r_j)$ instead of B_j^* and let $t_j := (2r_j)^m$. We then decompose $\sum_j b_j = h_1 + h_2$, where

$$h_1 = \sum_j (I - D^n S_{t_j}) b_j \quad \text{and} \quad h_2 = \sum_j (D^n S_{t_j}) b_j \,.$$

Then

$$\begin{aligned} &|\{\, x \in \Omega \,;\, |Tf(x)| > \alpha \,\}| \\ &\qquad \le |\{\, x \in \Omega \,;\, |Tg(x)| > \alpha/3 \,\}| \;+\; \sum_{k=1}^{2} |\{\, x \in \Omega \,;\, |Th_k(x)| > \alpha/3 \,\}| \end{aligned}$$

and we shall estimate the three terms separately where we write α instead of $\alpha/3$. We start with the first term. The assumption $T \in \mathfrak{L}(L_{p_o}(\Omega), L_{p_o}^w(\Omega))$, the properties (ii), (vi) in Theorem 3.1 and the hypothesis $p \le p_o$ imply

$$|\{\, x \in \Omega \,;\, |Tg(x)| > \alpha \,\}| \;\preceq\; \alpha^{-p_0} \, \|g\|_{p_0}^{p_0} \;\preceq\; \alpha^{-p} \, \|f\|_p^p \,.$$

We turn to the second term, i.e. the term involving h_1. We have

$$|\{\, x \in \Omega \,;\, |Th_1(x)| > \alpha \,\}| \;\preceq\; \alpha^{-p_0} \, \|\sum_j (I - D^n S_{t_j}) b_j\|_{p_0}^{p_0} \,,$$

where we used $T \in \mathfrak{L}(L_{p_o}(\Omega), L_{p_o}^w(\Omega))$ again. We shall show

$$\|\sum_j (I - D^n S_{t_j}) b_j\|_{p_0} \;\preceq\; \alpha \, \|\sum_j 1_{B_j}\|_{p_0} \,. \tag{4.1}$$

Indeed, by the properties (iii), (v) this implies

$$\|\sum_j (I - D^n S_{t_j}) b_j\|_{p_0} \;\preceq\; \alpha \, (\sum_j |B_j|\,)^{1/p_0} \;\preceq\; \alpha^{1-p/p_0} \, \|f\|_p^{p/p_0}$$

which leads to the desired bound for the second term :

$$|\{\, x \in \Omega \,;\, |Th_1(x)| > \alpha \,\}| \;\preceq\; \alpha^{-p} \, \|f\|_p^p \,.$$

Now for the proof of (4.1). For any $\phi \in L_{p'_o}$ and any j we have by property (iv)

$$\begin{aligned}|\langle \phi\,,\,(I-D^nS_{t_j})b_j\rangle| &= |\langle (I-D^nS_{t_j})^*\phi\,,\,b_j\rangle| \le \|1_{B_j}(I-D^nS_{t_j})^*\phi\|_{p'}\,\|b_j\|_p \\ &\preceq \alpha\,|B_j|\,N_{p',r_j}\Big((I-D^nS_{t_j})^*\phi\Big)(x_j) \\ &\preceq \alpha\int_{B_j} N_{p',t_j^{1/m}}\Big((I-D^nS_{t_j})^*\phi\Big)\,.\end{aligned}$$

Here we used Lemma 3.4(a) in the last step, recall $2r_j = t_j^{1/m}$. Observe that

$$I - D^nS_t \;=\; -\sum_{k=1}^{n}\binom{n}{k}(-1)^k S_{kt} \qquad \text{for all } \; t>0$$

where we used the assumption $S_0 = I$. Therefore, we obtain by hypothesis (1.2) and Lemma 3.3(b) :

$$N_{p',t^{1/m}}\Big((I-D^nS_t)^*f\Big)(x) \;\preceq\; M_{q'}f(x) \qquad \text{for all } \; t>0\,.$$

Hence, since $M_{q'}$ is bounded on $L_{p'_o}$ (here we use the hypothesis $1 \le p_o < q\,!$):

$$\begin{aligned}|\langle \phi\,,\,\sum_j (I-D^nS_{t_j})b_j\rangle| &\preceq \alpha\int (M_{q'}\phi)\sum_j 1_{B_j} \le \alpha\,\|M_{q'}\phi\|_{p'_0}\,\|\sum_j 1_{B_j}\|_{p_0} \\ &\preceq \alpha\,\|\phi\|_{p'_0}\,\|\sum_j 1_{B_j}\|_{p_0}.\end{aligned}$$

Since $\phi \in L_{p'_0}$ was arbitrary, this is (4.1). We turn to the third term, i.e. the term which involves h_2. Denoting $B_j^* := B(x_j, 8r_j)$ we have

$$\text{(4.2)}\quad |\{x\in\Omega\,;\,|Th_2(x)|>\alpha\}| \le \sum_j |B_j^*| + |\{x\in\Omega\backslash\cup_j B_j^*\,;\,|Th_2(x)|>\alpha\}|.$$

For the first term on the RHS, (v) together with the volume doubling property yields

$$\sum_j |B_j^*| \;\preceq\; \sum_j |B_j| \;\preceq\; \alpha^{-p}\,\|f\|_p^p\,.$$

The second term is estimated as follows by using (iii) :

$$\begin{aligned}|\{\,x\in\Omega\backslash\bigcup_j B_j^*\,;\,|Th_2(x)|>\alpha\,\}| &\le \alpha^{-p}\,\|\sum_j T(D^nS_{t_j})b_j\|^p_{p,\Omega\backslash\bigcup B_j^*} \\ &= \alpha^{-p}\,\|\sum_j 1_{(B_j^*)^c}T(D^nS_{t_j})1_{B_j}b_j\|^p_{p,\Omega\backslash\bigcup B_j^*} \le \alpha^{-p}\,\|\sum_j G_jb_j\|_p^p\end{aligned}$$

where $G_j := 1_{(B_j^*)^c}T(D^nS_{t_j})1_{B_j}$.

We shall establish

$$\| \sum_j G_j b_j \|_p \ \preceq \ \alpha \, \| \sum_j 1_{B_j} \|_p \tag{4.3}$$

and can then argue as before to obtain the desired bound

$$|\{ \, x \in \Omega \setminus \bigcup_j B_j^* \, ; \, |Th_2(x)| > \alpha \, \}| \ \preceq \ \alpha^{-p} \, \|f\|_p^p \, .$$

But the proof of (4.3) follows the lines of the proof of (4.1) :

$$\begin{aligned} |\langle \phi \, , \, G_j b_j \rangle| \ &= \ |\langle G_j^* \phi \, , \, b_j \rangle| \leq \|1_{B_j} G_j^* \phi\|_{p'} \, \|b_j\|_p \preceq \alpha \, |B_j| \, N_{p', r_j}(G_j^* \phi)(x_j) \\ &\leq \ \alpha \int_{B_j} N_{p', 2r_j}(G_j^* \phi) \preceq \alpha \int_{B_j} M_{q_o'} \phi \end{aligned}$$

where we used hypothesis (1.3) in the last step. Now we may finish as before since $M_{q_o'}$ is bounded on $L_{p'}$ due to $1 \leq p < q_o$. ■

Proof of Theorem 2.1. We repeat the proof of Theorem 1.1 and modify only the estimation of the second term in (4.2). We denote again $G_j := 1_{(B_j^*)^c} T(D^n S_{t_j}) 1_{B_j}$. Moreover, we recall that, by definition of the b_j in (3.1), we have $\sum_j |b_j| \leq |f| + \widetilde{M}_p f$, where $\widetilde{M}_p$ denotes the uncentered p-maximal operator which is of weak type (p,p). Hence we can estimate as follows, using the R_1-boundedness hypothesis (2.5) in the third step :

$$\begin{aligned} |\{ \, x \in \Omega \setminus \bigcup_j B_j^* \, ; \, |Th_2(x)| > \alpha \, \}| \ &\leq \ |\{ \, x \in \Omega \, ; \, \sum_j |G_j b_j(x)| > \alpha \, \}| \\ &\leq \ \alpha^{-p} \, || \sum_j |G_j b_j| \, ||_{L_p^w(\Omega)}^p \ \preceq \ \alpha^{-p} \, || \sum_j |b_j| \, ||_{L_p^w(\Omega)}^p \\ &\leq \ \alpha^{-p} \, || \, |f| + \widetilde{M}_p f \, ||_{L_p^w(\Omega)}^p \ \preceq \ \alpha^{-p} \, || \, f \, ||_p^p \, . \end{aligned}$$

■

Before we prove Theorem 1.2 as an application of Theorem 1.1, we recall some definitions. For $\nu \in (0, \pi]$, we denote by Σ_ν the open sector $\Sigma_\nu := \{ \, z \, ; \, |\arg(z)| < \nu \, \}$ and by $H^\infty(\Sigma_\nu)$ the set of all bounded holomorphic functions on Σ_ν. Finally we say that an operator A *has a bounded* $H^\infty(\Sigma_\nu)$ *calculus on* L_r if there is an algebra homomorphism $H^\infty(\Sigma_\nu) \to \mathfrak{L}(L_r), F \mapsto F(A)$ such that

$$|| \, F(A) \, ||_{\mathfrak{L}(L_r)} \ \leq \ C \, || \, F \, ||_{H^\infty(\Sigma_\nu)} \quad \text{for all} \ \ F \in H^\infty(\Sigma_\nu)$$
$$F(A) = (A + \lambda)^{-1} \quad \text{if } F(z) = (z + \lambda)^{-1} \, ,$$

and the following approximation property is satisfied: If (F_n) is a bounded sequence in $H^\infty(\Sigma_\nu)$ which converges, uniformly on compact subsets of Σ_ν, to $F \in H^\infty(\Sigma_\nu)$, then $F_n(A)$ converges strongly to $F(A)$. If $\mu \in (\nu, \pi)$ and $\psi \in H^\infty(\Sigma_\mu)$ with

$$|\psi(z)| \;\le\; C\,|z|^s(1+|z|)^{-2s} \qquad \text{for all } z \in \Sigma_\mu \text{ and some } s > 0$$

then $\psi(A)$ can be computed by the absolutely convergent Cauchy integral

$$\psi(A) \;=\; \frac{1}{2\pi i}\int_\gamma \psi(z)(z-A)^{-1}\,dz,$$

where the path γ consists of two rays $re^{\pm i\theta}$, $r \ge 0$ and $\nu < \theta < \mu$, described counterclockwise. We denote the class of those functions ψ by $\Psi(\Sigma_\mu)$. By the approximation property the values on such functions ψ define the map $F \mapsto F(A)$ uniquely. We refer to [M] for details.

Proof of Theorem 1.2. As before, we can assume $\Omega = \Omega_1$. We fix $\nu \in (w, \pi/2)$. By applying our weak type (p,p) criterion Theorem 1.1 for $S_t := e^{-tA}$, $p_o := 2$, $q_o := q$, we shall establish a weak (p,p) estimate for the operator $T = F(A)$ where the constant does not depend on $F \in \Psi(\Sigma_\nu)$ with $\|F\|_{H^\infty(\Sigma_\nu)} \le 1$. Applying the argument in the dual situation leads to a weak (q,q) estimate. By interpolation we hence obtain a bounded $H^\infty(\Sigma_\nu)$ calculus for A on each $L_r(\Omega)$, $r \in (p,q)$.

By hypothesis (1.4), the condition (1.2) in Theorem 1.1 is satisfied. We fix $\mu, \theta \in (w,\nu)$ with $\mu > \theta$ and $n \in \mathbb{N}$ with $n > \frac{\kappa_\theta - D}{q'm} \vee \frac{D}{p'm}$ and are then left to check that

$$N_{p',t^{1/m}/2}\Big(\,(F(A)\,D^n S_t)^*\,P_{B(y,4t^{1/m})^c} f\,\Big)(y) \;\le\; C\,M_{q'}f(x) \tag{4.4}$$

whenever $y \in B(x, t^{1/m}/2)$. Indeed, in this case the desired weak (p,p) estimate for $F(A)$ follows from Theorem 1.1. Claim (4.4) will be established by combining estimates for

$$N_{p',t^{1/m}/2}\Big(\,(e^{-zA})^*\,P_{B(y,4t^{1/m})^c} f\,\Big)(y) \tag{4.5}$$

obtained from hypothesis (1.4) and Lemma 3.3(a) on the one hand, on the other hand a Laplace integral representation for $F(A)D^nS_t$ in terms of the e^{-zA} and the Laplace inversion Lemma 3.6. In the following we use the symbol $\preceq$ to indicate domination up to constants independent of $t > 0$, $x \in \Omega$, $y \in B(x, t^{1/m}/2)$ and the functions f, F. We follow the notation of Section 3.3 (for $\alpha = 0$) and set $\beta := \frac{\kappa_\theta - D}{q'm}$, $\gamma := \frac{D}{p'm}$ as well as

$$w(z) \;:=\; |z|^\beta\,(1+|z|)^{\gamma-\beta} \quad \text{and} \quad E(z) \;:=\; \sum_{k=0}^n \binom{n}{k}\,(-1)^k\,e^{-kz}\,.$$

Observe that $0 < \gamma, \beta < n$ by hypothesis $\kappa_\theta > D$ and choice of n. The weighted norm estimate (1.4) yields by Lemma 3.3(a) the following bound for the term in (4.5):

$$\begin{aligned}
& N_{p', t^{1/m}/2}\Big((e^{-zA})^* \, P_{B(y,4t^{1/m})^c} f \Big)(y) \\
& \preceq \Big(\sum_{k \geq 2t^{1/m}|z|^{-1/m}} (1+k)^{D-1-\kappa_\theta} \Big)^{1/q'} (1+|z|^{1/m} t^{-1/m})^{D/p'} \, M_{q'} f(x) \\
& \preceq (1+t|z|^{-1})^{-\beta} \, (1+t^{-1}|z|)^{\gamma} \, M_{q'} f(x) = w(t^{-1}z) \, M_{q'} f(x)
\end{aligned}$$

For the second part of the proof of (4.4) we observe first that $D^n S_t = (\delta_t E)(A)$, where δ_t denotes the dilation operator $\delta_t g(z) := g(tz)$. Now we apply the inverse Laplace transform $\mathcal{L}^{-1}$ as in Section 3.3 and obtain the following integral representation of the operator appearing in our claim (4.4):

$$\begin{aligned}
F(A) D^n S_t &= (F \, \delta_t E)(A) = \int_\Gamma e^{-zA} \mathcal{L}^{-1}(F \delta_t E)(z) \, dz \\
&= t^{-1} \int_\Gamma e^{-zA} \mathcal{L}^{-1}(E \delta_{t^{-1}} F)(t^{-1} z) \, dz
\end{aligned}$$

Here we denote $\Gamma := \partial \Sigma_{\pi/2-\theta}$ and use the fact that $\mathcal{L}^{-1}(\delta_t g) = t^{-1} \delta_{t^{-1}} (\mathcal{L}^{-1} g)$. Now we finish the proof of (4.4) and thus of Theorem 1.2 as follows by using Lemma 3.6 :

$$\begin{aligned}
& N_{p', t^{1/m}/2}\Big((F(A) \, D^n S_t)^* \, P_{B(y,4t^{1/m})^c} f \Big)(y) \\
& \leq t^{-1} \int_\Gamma N_{p', t^{1/m}/2}\Big((e^{-zA})^* \, P_{B(y,4t^{1/m})^c} f \Big)(y) \, |\mathcal{L}^{-1}(E \delta_{t^{-1}} F)(t^{-1} z)| \, |dz| \\
& \preceq t^{-1} \int_\Gamma w(t^{-1}z) \, |\mathcal{L}^{-1}(E \delta_{t^{-1}} F)(t^{-1} z)| \, |dz| \, M_{q'} f(x) \\
& = || \, \mathcal{L}^{-1}(E \delta_{t^{-1}} F) \, ||_{L_1(\Gamma, w(z)|dz|)} \, M_{q'} f(x) \preceq || \, \delta_{t^{-1}} F \, ||_{H^\infty(\Sigma_\nu)} \, M_{q'} f(x) \\
& = || \, F \, ||_{H^\infty(\Sigma_\nu)} \, M_{q'} f(x) \leq M_{q'} f(x) \, .
\end{aligned}$$

■

Proof of Theorem 2.2. By extrapolation as in [C, § 1], our hypothesis (2.6) yields

$$|| \, e^{-tA} \, ||_{p \to p'} \leq C \, t^{\frac{D}{m}(\frac{1}{p'} - \frac{1}{p})} \quad \text{for all } t > 0 \, . \tag{4.6}$$

Now fix $r \in (p, p')$ and then s such that $p < s < 2, r < s'$. By interpolation between the unweighted $L_p \to L_{p'}$-norm estimate (4.6), the weighted $L_2 \to$

L_2-norm estimate (2.7) and the L_2-analyticity of (e^{-tA}) as in the proof of [BK1, Thm. 1.6], we obtain for all $\theta > w$:

$$|| \, e^{-\rho d(x,\cdot)} \, e^{-zA} \, e^{\rho d(x,\cdot)} \, ||_{s\to s'} \quad \le \quad C_\theta \, |z|^{\frac{D}{m}(\frac{1}{s'}-\frac{1}{s})} \, e^{c_\theta |\rho|^m \, |z|} \qquad \text{for all} \;\; z \in \Sigma_{\frac{\pi}{2}-\theta}$$

and all $x \in \Omega_1$, $\rho \in \mathbb{R}$. By [BK1, Prop. 1.5], this yields for all $\theta > w$:

$$(4.7) \qquad || \, P_{B(x,|z|^{1/m})} \, e^{-zA} \, P_{A(x,|z|^{1/m},k)} \, ||_{s\to s'} \quad \le \quad 3C_\theta \, |z|^{\frac{D}{m}(\frac{1}{s'}-\frac{1}{s})} \, e^{-b_\theta k^{m/(m-1)}}$$

for all $x \in \Omega_1$, $k \in \mathbb{N}_0$, $z \in \Sigma_{\frac{\pi}{2}-\theta}$ and some $b_\theta > 0$. Since, by duality, the (weighted) norm estimates (4.6),(2.7) and the L_2-analyticity hold also for A^* instead of A, we obtain (4.7) also for A^* instead of A. In other words, we have for all $\theta > w$:

$$(4.8) \quad || \, P_{A(x,|z|^{1/m},k)} \, e^{-zA} \, P_{B(x,|z|^{1/m})} \, ||_{s\to s'} \quad \le \quad 3C_\theta \, |z|^{\frac{D}{m}(\frac{1}{s'}-\frac{1}{s})} \, e^{-b_\theta k^{m/(m-1)}}$$

for all $x \in \Omega_1$, $k \in \mathbb{N}_0$, $z \in \Sigma_{\frac{\pi}{2}-\theta}$. Hence, in view of (4.7) and (4.8), the hypotheses of our main result Theorem 1.2 are satisfied, and A has a bounded $H^\infty(\Sigma_\nu)$ calculus on $L_r(\Omega)$ for all $\nu > w$. ■

We now prove the applications of Theorem 2.2 in Section 2.3.

Proof of Proposition 2.3. Following Davies [D1] one defines the twisted forms

$$\mathfrak{a}_{\lambda\phi}(u,v) := \mathfrak{a}(e^{\lambda\phi}u, e^{-\lambda\phi}v) \quad , \quad u,v \in H^m(\mathbb{R}^D) \, ,$$

where $\lambda \in \mathbb{R}$ and ϕ is a real-valued C^∞-function with compact support satisfying $\|\partial^\alpha \phi\|_\infty \le 1$ for all $1 \le |\alpha| \le m$. The space of all such functions is denoted by $\mathcal{E}_m$. Observe that the functions $e^{\lambda\phi}$ are pointwise multipliers on $H^m(\mathbb{R}^D)$. Then

$$|\mathfrak{a}_{\lambda\phi}(u,u) - \mathfrak{a}(u,u)| \;\le\; \varepsilon \, \|\nabla^m u\|_2^2 \;+\; C(\varepsilon)(1+\lambda^{2m})\|u\|_2^2$$

for all $u \in H^m$, $\lambda \in \mathbb{R}$, $\phi \in \mathcal{E}_m$ and for each $\varepsilon > 0$. Following the lines of [D1] (for the case $D = 2m$ one has to argue as in [AT, p. 59]) one can show the following:

If $p_0 := \frac{2D}{2m+D} \vee 1$ then there are constants $M, \omega > 0$ such that

$$(4.9) \qquad \|e^{-\lambda\phi} e^{-tA} e^{\lambda\phi}\|_{p_0 \to p_0'} \le M \, t^{-D/(2m)\cdot(1/p_0 - 1/p_0')} \, e^{\omega\lambda^{2m} t}$$

for all $t > 0$, $\lambda \in \mathbb{R}$, and $\phi \in \mathcal{E}_m$. By arguments as in [D1] or [LSV] this yields that the semigroup (e^{-tA}) is bounded in L_{p_0} and $L_{p_0'}$. The last estimate in (2.6) and the estimate (2.7) can be obtained from (4.9) by interpolation, [D1, Lemma 4] and Sections 3 and 4 in [BK1]. (Actually, these estimates may be obtained as intermediate steps when proving (4.9).) The assertions follow by application of Theorem 2.2. ■

Proof of Proposition 2.4. Given $r \in (p(\gamma), p(\gamma)')$ we fix $p \in (p(\gamma), 2)$ such that $r \in (p, p')$. We choose c_r such that $(e^{-t(H+c_r)})$ is bounded in L_p and $L_{p'}$. The (weighted) estimates (2.6) , (2.7) we need for an application of Theorem 2.2 are proved in [LSV]. ■

Proof of Proposition 2.5. Given $r \in (p_{\min}, p_{\max})$ we fix $p, q_0 \in (p_{\min}, p_{\max})$ such that $r, 2 \in (p, q_0)$. The (weighted) norm estimates (2.6), (2.7) are proved in [LSV]. Hence the assertion follows by application of (a variant of) Theorem 2.2. ■

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