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Integral Closure of Monomial Ideals on Regular Sequences

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Abstract

It is well known that the integral closure of a monomial ideal in a polynomial ring in a finite number of indeterminates over a field is a monomial ideal, again. Let R be a noetherian ring, and let $(x_1, \ldots, x_d)$ be a regular sequence in R which is contained in the Jacobson radical of R. An ideal $\mathfrak{a}$ of R is called a monomial ideal with respect to $(x_1, \ldots, x_d)$ if it can be generated by monomials $x_1^{i_1} \cdots x_d^{i_d}$. If $x_1R + \cdots + x_dR$ is a radical ideal of R, then we show that the integral closure of a monomial ideal of R is monomial, again. This result holds, in particular, for a regular local ring if $(x_1, \ldots, x_d)$ is a regular system of parameters of R.

1. Introduction

Let A be a polynomial ring over a field in a finite number of indeterminates. It is well known that the integral closure $\overline{\mathfrak{A}}$ of a monomial ideal $\mathfrak{A}$ of A is a monomial ideal, again: $\overline{\mathfrak{A}}$ is generated by all monomials m with $m^l \in \mathfrak{A}^l$ for some $l \in \mathbb{N}$ [cf. [12], section 6.6, Example 6.6.1]. While studying a particular class of ideals in two-dimensional regular local rings [cf. the example at the end of this paper], the following question arose naturally: Let R be a noetherian ring, and let $(x_1, \ldots, x_d)$ be a regular sequence in R such that $\mathfrak{q} := x_1R + \cdots + x_dR$ is contained in the Jacobson radical of R. Let $\mathfrak{a}$ be an ideal of R that is generated by monomials in $x_1, \ldots, x_d$; such ideals shall be called monomial ideals. Is the integral closure $\overline{\mathfrak{a}}$ of $\mathfrak{a}$ a monomial ideal, again?

In this paper the question is answered in the positive *under the assumption that $R/\mathfrak{q}$ is a reduced ring.*

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In section 2 we collect some useful results on monomial ideals; in particular, we show that the usual ideal-theoretic operations, applied to monomial ideals, lead again to monomial ideals. It is also shown that for a monomial ideal $\mathfrak{a}$ the ideal $\mathrm{gr}(\mathfrak{a})$ in the associated graded ring $\mathrm{gr}_{\mathfrak{q}}(R)$ which is a polynomial ring over $R/\mathfrak{q}$ is a monomial ideal.

In section 3 we introduce the notion of a monomial representation of an element of R and we show that, if R is complete, every element of R admits a monomial representation. In section 4 we associate with a monomial ideal $\mathfrak{a}$ the ideal $\widetilde{\mathfrak{a}}$ which is generated by all monomials m in R with $m^l \in \mathfrak{a}^l$ for some $l \in \mathbb{N}$. In section 5 we study monomial ideals in a polynomial ring over a reduced ring, and we show that for a monomial ideal $\mathfrak{A}$ we have $\overline{\mathfrak{A}} = \widetilde{\mathfrak{A}}$ where $\overline{\mathfrak{A}}$ denotes the integral closure of $\mathfrak{A}$. Let $\mathfrak{a}$ be a monomial ideal in R. Using the results of section 5 we show in section 6 that $\overline{\mathfrak{a}} = \widetilde{\mathfrak{a}}$ if R is complete and $\mathfrak{q}$ is a prime ideal. As a last step we show that this equality holds also if R is not necessarily complete, and if $R/\mathfrak{q}$ is a reduced ring.

2. Monomial Ideals

2.1. Basic Definitions

Notation 1 Let R be a ring. A sequence $\mathbf{x} := (x_1, \ldots, x_d)$ in R is called a weak regular sequence in R if

(a) x_i is regular for $R/(x_1, \ldots, x_{i-1})$ [i.e., the image of x_i in $R/(x_1, \ldots, x_{i-1})$ is a non-zero divisor] for every $i \in \{1, \ldots, d\}$,

and it is called a regular sequence in R if, in addition,

(b) $R \neq \mathbf{x}R$.

In the sequel, we consider regular sequences $\mathbf{x}$ in R with the following additional property:

(c) every permutation $(x_{\pi(1)}, \ldots, x_{\pi(d)})$ of $\mathbf{x}$ is a regular sequence in R.

Then every subsequence of $\mathbf{x}$ satisfies (a)-(c).

If R is noetherian, and if a regular sequence $\mathbf{x}$ in R is contained in the Jacobson radical [i.e., in the intersection of all maximal ideals] of R, then (a) implies (c) [cf. [2], Ch. X, § 9, no. 7, Th. 1 and Cor. 1], and for the ideal $\mathfrak{q}$ generated by $x_1, \ldots, x_d$ we have $\bigcap \mathfrak{q}^p = (0)$ [cf. [3], Ch. III, § 3, no. 3, Prop. 6].

If $\varphi \colon R \to S$ is a flat homomorphism of rings, and if $\varphi(\mathbf{x})S \neq S$, then the sequence $\varphi(\mathbf{x})$ in S satisfies (a)-(c) [cf. [4], Ch. I, Prop. 1.1.1].

(1) For every d-tuple $\mathbf{i} := (i_1, \ldots, i_d) \in \mathbb{N}_0^d$ we define $\deg(\mathbf{i}) := i_1 + \cdots + i_d$, the degree of $\mathbf{i}$, and we write

$$\mathbf{x}^{\mathbf{i}} := x_1^{i_1} \cdots x_d^{i_d}.$$

Since $\mathbf{x}$ is a regular sequence, we have, for $\mathbf{i}, \mathbf{j} \in \mathbb{N}_0^d$, $\mathbf{x}^{\mathbf{i}} = \mathbf{x}^{\mathbf{j}}$ iff $\mathbf{i} = \mathbf{j}$.

(2) An element $m \in R$ is called a monomial with respect to $\mathbf{x}$ if there exists $\mathbf{i} \in \mathbb{N}_0^d$ with $m = \mathbf{x}^{\mathbf{i}}$; $\mathbf{i}$ is determined uniquely by m. We call $\deg(m) := \deg(\mathbf{i})$ the degree of m.

(3) Let $\mathbf{x}^{\mathbf{i}} = x_1^{i_1} \cdots x_d^{i_d}$ be a monomial with respect to $\mathbf{x}$. The set

$$\mathrm{Supp}(\mathbf{x}^{\mathbf{i}}) := \{j \mid j \in \{1, \ldots, d\}, i_j \neq 0\}$$

is called the support of $\mathbf{x}^{\mathbf{i}}$.

(4) Let $M(\mathbf{x})$ be the set of all monomials of R with respect to $\mathbf{x}$. Clearly $M(\mathbf{x})$ is a commutative monoid with cancellation law, and $\deg\colon M(\mathbf{x}) \to \mathbb{N}_0$ is a surjective homomorphism of monoids.

(5) An ideal $\mathfrak{a}$ of R is called monomial with respect to $\mathbf{x}$ if it is generated by elements in $M(\mathbf{x})$. In particular, the zero ideal and R itself are monomial ideals.

Remark 1 Let $\mathbf{i} = (i_1, \ldots, i_d)$, $\mathbf{j} = (j_1, \ldots, j_d) \in \mathbb{N}_0^d$.

(1) If $\mathbf{x}^{\mathbf{i}} \in \mathbf{x}^{\mathbf{j}}R$, then we have $i_1 \geqslant j_1, \ldots, i_d \geq j_d$ and $\mathbf{x}^{\mathbf{i}} = \mathbf{x}^{\mathbf{j}}\mathbf{x}^{\mathbf{i}-\mathbf{j}}$. In this case we say that $\mathbf{x}^{\mathbf{j}}$ divides $\mathbf{x}^{\mathbf{i}}$, and we write $\mathbf{x}^{\mathbf{j}} \mid \mathbf{x}^{\mathbf{i}}$.

(2) We define

$$k_\tau := \min\{i_\tau, j_\tau\},\ l_\tau := \max\{i_\tau, j_\tau\} \quad \text{for } \tau \in \{1, \ldots, d\}$$

and

$$\mathbf{k} := (k_1, \ldots, k_d),\ \mathbf{l} := (l_1, \ldots, l_d);$$

then

$$\gcd(\mathbf{x}^{\mathbf{i}}, \mathbf{x}^{\mathbf{j}}) := \mathbf{x}^{\mathbf{k}},\ \mathrm{lcm}(\mathbf{x}^{\mathbf{i}}, \mathbf{x}^{\mathbf{j}}) := \mathbf{x}^{\mathbf{l}}$$

is the greatest common divisor resp. the least common multiple of $\mathbf{x}^{\mathbf{i}}$ and $\mathbf{x}^{\mathbf{j}}$. In particular, for monomials m, n we have $mR : nR = (\mathrm{lcm}(m, n)/n)R = (m/\gcd(m, n))R$.

Notation 2 For the rest of this paper let R be a noetherian ring, and let $\mathbf{x} = (x_1, \ldots, x_d)$ be a fixed sequence in R which satisfies (a)-(c) above; all monomials of R are monomials with respect to $\mathbf{x}$, and all monomial ideals of R are monomial ideals with respect to $\mathbf{x}$. The set of all monomials of R shall be denoted by M.

Definition 1 Let U be a subset of $\{1, \ldots, d\}$; we define

$$\mathfrak{q}_U := \sum_{i \in U} x_i R, \ \mathcal{P}_U := \mathrm{Ass}(R/\mathfrak{q}_U).$$

If $U = \{1, \ldots, d\}$, then we write

$$\mathfrak{q} := \mathfrak{q}_U = \sum_{i=1}^{d} x_i R, \ \mathcal{P} := \mathrm{Ass}(R/\mathfrak{q}).$$

Remark 2 (1) Note that $\mathrm{Ass}(R) = \mathcal{P}_\emptyset$.

(2) Let $U \subset \{1, \ldots, d\}$, $i \in \{1, \ldots, d\} \setminus U$. Then x_i is regular for $R/\mathfrak{q}_U$, hence, in particular, $x_i \notin \mathfrak{p}$ for every $\mathfrak{p} \in \mathcal{P}_U$.

Lemma 1 *Let $\mathfrak{a}$ be a monomial ideal of R, and let $\{m_1, \ldots, m_r\}$ be a system of generators of $\mathfrak{a}$ consisting of monomials. Then we have*

$$\mathrm{Ass}(R/\mathfrak{a}) \subset \bigcup_{U \subset \mathrm{Supp}(m_1) \cup \cdots \cup \mathrm{Supp}(m_r)} \mathcal{P}_U.$$

Proof: There is nothing to prove if $\mathfrak{a} = (0)$. We consider the case that $\mathfrak{a} \neq (0)$. We define $V := \mathrm{Supp}(m_1) \cup \cdots \cup \mathrm{Supp}(m_r)$. We prove the assertion by induction on $s := \deg(m_1) + \cdots + \deg(m_r) - r$. If $s = 0$, then we have $\mathfrak{a} = \mathfrak{q}_V$; in this case the assertion holds. Let $s > 0$, and assume that the assertion holds for all monomial ideals of R which admit a system of monomial generators $m'_1, \ldots, m'_{r'}$ with $\deg(m'_1) + \cdots + \deg(m'_{r'}) - r' < s$. Now let $\mathfrak{a}$ be a monomial ideal of R having a system of monomial generators $m_1, \ldots, m_r$ with $\deg(m_1) + \cdots + \deg(m_r) - r = s$. Then there exists $j \in \{1, \ldots, r\}$ with $\deg(m_j) \geq 2$; by relabelling, we may assume that $j = 1$.

Let $i \in \mathrm{Supp}(m_1)$; let us label the monomials $m_1, \ldots, m_r$ in such a way that $i \in \mathrm{Supp}(m_j)$ for $j \in \{1, \ldots, t\}$ and $i \notin \mathrm{Supp}(m_j)$ for $j \in \{t+1, \ldots, r\}$; here we have $t \in \{1, \ldots, r\}$. For $j \in \{1, \ldots, t\}$ we have $m_j = x_i m'_j$ where $m'_1, \ldots, m'_t$ are monomials. We put

$$\mathfrak{a}_1 := m'_1 R + \cdots + m'_t R, \ \mathfrak{a}_2 = m_{t+1} R + \cdots + m_r R, \ \mathfrak{b} := \mathfrak{a}_1 + \mathfrak{a}_2,$$

$$V_1 := \bigcup_{j=1}^{t} \mathrm{Supp}(m'_j), \quad V_2 := \bigcup_{j=t+1}^{r} \mathrm{Supp}(m_j).$$

If $\mathfrak{a}_2 = (0)$, then we have $\mathfrak{a} : x_i = \mathfrak{b}$. This is also true if $\mathfrak{a}_2 \neq (0)$. In fact, by our induction assumption we get $\mathrm{Ass}(R/\mathfrak{a}_2) \subset \bigcup_{U \subset V_2} \mathcal{P}_U$. Using $i \notin V_2$, we see that $V_2 \subset \{1, \ldots, d\} \setminus \{i\}$. From Remark 2 we get the following: If $U \subset V_2$, then $x_i \notin \mathfrak{p}$ for every prime ideal $\mathfrak{p} \in \mathcal{P}_U$, hence $x_i \notin \mathfrak{p}$ for every $\mathfrak{p} \in \mathrm{Ass}(R/\mathfrak{a}_2)$, hence x_i is regular for $R/\mathfrak{a}_2$. This implies that $\mathfrak{a} : x_i = \mathfrak{a}_1 + \mathfrak{a}_2 = \mathfrak{b}$ since $\mathfrak{a} = x_i\mathfrak{a}_1 + \mathfrak{a}_2$.

Therefore the sequence

$$0 \longrightarrow R/\mathfrak{b} \xrightarrow{x_i} R/\mathfrak{a} \longrightarrow R/(\mathfrak{a} + x_iR) \longrightarrow 0$$

is exact; note that

$$\mathrm{Ass}(R/\mathfrak{a}) \subset \mathrm{Ass}(R/\mathfrak{b}) \cup \mathrm{Ass}(R/(\mathfrak{a} + x_iR)). \qquad (*)$$

We have $\mathfrak{a} + x_iR = x_iR + m_{t+1}R + \cdots + m_rR$. Applying our induction assumption to $\mathfrak{b}$ and to $\mathfrak{a} + x_iR$ we obtain

$$\mathrm{Ass}(R/\mathfrak{b}) \subset \bigcup_{U \subset V_1 \cup V_2} \mathcal{P}_U \subset \bigcup_{U \subset V} \mathcal{P}_U,$$

$$\mathrm{Ass}(R/(\mathfrak{a} + x_iR)) \subset \bigcup_{U \subset \{i\} \cup V_2} \mathcal{P}_U \subset \bigcup_{U \subset V} \mathcal{P}_U.$$

Therefore we get, using $(*)$, that $\mathrm{Ass}(R/\mathfrak{a}) \subset \bigcup_{U \subset V} \mathcal{P}_U$. ■

Corollary 1 *If $i \notin \bigcup_{j=1}^{r} \mathrm{Supp}(m_j)$, then we have $\mathfrak{a} : x_i = \mathfrak{a}$.*

Proof: The element x_i is not contained in any of the prime ideals in $\mathrm{Ass}(R/\mathfrak{a})$ [cf. Lemma 1]. ■

2.2. Operations on Monomial Ideals

Lemma 2 *Let $\mathfrak{a} = m_1R + \cdots + m_rR$ with $m_1, \ldots, m_r \in M$ be a monomial ideal in R. For every $m \in M$ the ideal $\mathfrak{a} : m$ is monomial, again. More precisely, we have*

$$\mathfrak{a} : m = \sum_{j=1}^{r} \frac{\mathrm{lcm}(m_j, m)}{m} R.$$

Proof: We may assume that $\mathfrak{a} \neq (0)$. We prove the assertion by induction on $\deg(m)$. The case $\deg(m) = 0$, i.e., $m = 1$, is clear. Let $\deg(m) > 0$; then there exists $i \in \{1, \ldots, d\}$ with $x_i \mid m$, and we write $m = x_i m'$ with $m' \in M$. As in the proof of Lemma 1 we label the monomials $m_1, \ldots, m_r$ in such a way that $x_i \mid m_j$ for $j \in \{1, \ldots, t\}$, $x_i \nmid m_j$ for $j \in \{t+1, \ldots, r\}$

with $t \in \{0, \dots, r\}$, and we write, for $j \in \{1, \dots, t\}$, $m_j = x_i m'_j$ with monomials $m'_1, \dots, m'_j$. Then we have, as above,

$$\begin{aligned}\mathfrak{a} : m = (\mathfrak{a} : x_i) : m' &= \left(\sum_{j=1}^{t} m'_j R + \sum_{j=t+1}^{r} m_j R\right) : m' \\ &= \sum_{j=1}^{t} \frac{\operatorname{lcm}(m'_j, m')}{m'} R + \sum_{j=t+1}^{r} \frac{\operatorname{lcm}(m_j, m')}{m'} R = \sum_{j=1}^{r} \frac{\operatorname{lcm}(m_j, m)}{m} R.\end{aligned}$$

■

Corollary 2 *Let $\mathfrak{a} = m_1 R + \cdots + m_r R$ with $m_1, \dots, m_r \in M$ be a monomial ideal in R. Let $m \in M$; then we have*

$$\mathfrak{a} \cap mR = \sum_{j=1}^{r} \operatorname{lcm}(m_j, m) R.$$

Proof: We have $\mathfrak{a} \cap mR = (\mathfrak{a} : m)m$. ■

Lemma 3 *Let $\mathfrak{a} = m_1 R + \cdots + m_r R$, $\mathfrak{b} = n_1 R + \cdots + n_s R$ with $m_1, \dots, n_s \in M$ be monomial ideals in R. Then $\mathfrak{a} \cap \mathfrak{b}$ is a monomial ideal; more precisely, we have*

$$\mathfrak{a} \cap \mathfrak{b} = \sum_{i=1}^{r} \sum_{j=1}^{s} \operatorname{lcm}(m_i, n_j) R. \qquad (*)$$

Proof: It is clear that the right-hand side of $(*)$ is contained in the left-hand side. We prove that the left-hand side of $(*)$ is contained in the right hand side by induction on s. For $s = 0$ the assertion is clear, and for $s = 1$ the assertion follows from Cor. 2. Now we assume that $s \geq 2$, and we define $\mathfrak{b}' = n_1 R + \cdots + n_{s-1} R$. Let $z \in \mathfrak{a} \cap \mathfrak{b}$. We write $z = a_1 m_1 + \cdots + a_r m_r = b_1 n_1 + \cdots + b_s n_s$ with $a_1, \dots, b_s \in R$. Since $b_s n_s = a_1 m_1 + \cdots + a_r m_r - (b_1 n_1 + \cdots + b_{s-1} n_{s-1})$, we have $b_s n_s \in (\mathfrak{a} + \mathfrak{b}') \cap n_s R$, hence we can write [cf. Cor. 2]

$$b_s n_s = \sum_{i=1}^{r} c_i \operatorname{lcm}(m_i, n_s) + \sum_{j=1}^{s-1} d_j \operatorname{lcm}(n_j, n_s) \quad \text{with } c_1, \dots, d_{s-1} \in R.$$

We define

$$w := \sum_{j=1}^{s-1} \big(b_j n_j + d_j \operatorname{lcm}(n_j, n_s)\big).$$

Then we have $w \in \mathfrak{b}'$, and since $w = z - (c_1 \mathrm{lcm}(m_1, n_s) + \cdots + c_r \mathrm{lcm}(m_r, n_s)) \in \mathfrak{a}$, we have

$$w \in \mathfrak{a} \cap \mathfrak{b}' = \sum_{i=1}^{r} \sum_{j=1}^{s-1} \mathrm{lcm}(m_i, n_j)R$$

by our induction assumption. Then we get

$$z = w + \sum_{i=1}^{r} c_i \mathrm{lcm}(m_i, n_s) \in \sum_{i=1}^{r} \sum_{j=1}^{s} \mathrm{lcm}(m_i, n_j)R,$$

and therefore the left-hand side of $(*)$ lies in the right hand side. ■

Collection our results, we have

Proposition 1 *Let $\mathfrak{a}$, $\mathfrak{b}$ be monomial ideals in R. Then $\mathfrak{a} \cap \mathfrak{b}$, $\mathfrak{a} \cdot \mathfrak{b}$, $\mathfrak{a} : \mathfrak{b}$ are monomial ideals, again. More precisely, if $\mathfrak{a} = m_1 R + \cdots + m_r R$ and $\mathfrak{b} = n_1 R + \cdots + n_s R$ with monomials $m_1, \ldots, n_s \in M$, then we have*

$$\mathfrak{a} \cap \mathfrak{b} = \sum_{i=1}^{r} \sum_{j=1}^{s} \mathrm{lcm}(m_i, n_j)R, \tag{2.1}$$

$$\mathfrak{a} : \mathfrak{b} = \bigcap_{j=1}^{s} \sum_{i=1}^{r} \frac{\mathrm{lcm}(m_i, n_j)}{n_j} R. \tag{2.2}$$

If $\mathfrak{c}$ is another monomial ideal, then we have

$$(\mathfrak{a} + \mathfrak{b}) \cap \mathfrak{c} = (\mathfrak{a} \cap \mathfrak{c}) + (\mathfrak{b} \cap \mathfrak{c}). \tag{2.3}$$

Proof: (2.3) follows from (2.1), and (2.2) is a consequence of Lemma 2 since

$$\mathfrak{a} : \mathfrak{b} = \bigcap_{j=1}^{s} (\mathfrak{a} : n_j).$$ ■

Corollary 3 *Let $\mathfrak{a} = m_1 R + \cdots + m_r R$ with $m_1, \ldots, m_r \in M$ be a monomial ideal in R, and let $m \in M$. Then we have $m \in \mathfrak{a}$ iff $m_i \mid m$ for some $i \in \{1, \ldots, r\}$.*

Proof: We have $m \in \mathfrak{a}$ iff

$$1 \in \mathfrak{a} : m = (\mathrm{lcm}(m_1, m)/m)R + \cdots + (\mathrm{lcm}(m_r, m)/m)R\,,$$

hence iff $\mathrm{lcm}(m_i, m)/m = 1$ for some $i \in \{1, \ldots, r\}$, and this is the case iff $m_i \mid m$ for some $i \in \{1, \ldots, r\}$. ■

Corollary 4 *Let $\mathfrak{a}$ be a monomial ideal in R, and let $m_1, \ldots, m_r$, $n_1, \ldots, n_s$ be monomials with*

$$\mathfrak{a} = \sum_{i=1}^{r} m_i R = \sum_{j=1}^{s} n_j R.$$

(1) We assume that $m_i \nmid m_k$ for all i, $k \in \{1, \ldots, r\}$ with $i \neq k$. Then we have $\{m_1, \ldots, m_r\} \subset \{n_1, \ldots, n_s\}$.

(2) We assume, furthermore, that $n_j \nmid n_l$ for all j, $l \in \{1, \ldots, s\}$ with $j \neq l$. Then we have $r = s$ and $\{m_1, \ldots, m_r\} = \{n_1, \ldots, n_s\}$.

Proof: (1) Note that $\#\{m_1, \ldots, m_r\} = r$. Let $i \in \{1, \ldots, r\}$. Then, by Cor. 3, there exist $j \in \{1, \ldots, s\}$ and $k \in \{1, \ldots, r\}$ with $m_i \mid n_j$ and $n_j \mid m_k$, hence we have $m_i \mid m_k$. Therefore we have $i = k$ and $m_i = n_j \in \{n_1, \ldots, n_s\}$. This implies that $\{m_1, \ldots, m_r\} \subset \{n_1, \ldots, n_s\}$.

(2) This follows immediately from (1). ■

Remark 3 The result of Cor. 4 implies the following: Every monomial ideal of R admits a uniquely determined minimal set of monomial generators where "minimal" can be understood as "minimal with respect to number" or as "irredundant". We denote this number by $\nu(\mathfrak{a})$. But we can even say more:

Corollary 5 *Let $\mathfrak{a}$ be a monomial ideal in R, let $r := \nu(\mathfrak{a})$, and let $\{m_1, \ldots, m_r\} \subset M$ be a minimal set of monomial generators of $\mathfrak{a}$. Then we have*

$$\mu_{R_\mathfrak{p}}(\mathfrak{a}R_\mathfrak{p}) = r \quad \textit{for all } \mathfrak{p} \in V((x_1, \ldots, x_r)).$$

Moreover, every set of generators which generates $\mathfrak{a}$ contains at least r elements.

(In a local ring A we denote by $\mu_A(M)$ the minimal number of generators of a finitely generated A-module M.)

Proof: The second statement follows from the first one, and the first statement is obtained from Cor. 4 by replacing R by $R_\mathfrak{p}$. ■

2.3. The Associated Graded Ring

Remark 4 The associated graded ring

$$\mathrm{gr}(R) := \mathrm{gr}_\mathfrak{q}(R) = \bigoplus_{p \geq 0} \mathfrak{q}^p/\mathfrak{q}^{p+1} = R/\mathfrak{q}[\,\overline{x}_1, \ldots, \overline{x}_d\,]$$

is a polynomial ring over $R/\mathfrak{q}$ in $\overline{x}_1 := x_1 \bmod \mathfrak{q}^2, \ldots, \overline{x}_d := x_d \bmod \mathfrak{q}^2$ [cf. [2], Ch. X, § 9, no. 7, Th. 1]. Notice that the sequence $(\overline{x}_1, \ldots, \overline{x}_d)$ is a sequence in $\mathrm{gr}(R)$ which satisfies (a)-(c) above.

(1) Let $\overline{M} = \{\overline{\mathbf{x}}^{\mathbf{i}} := \overline{x}_1^{i_1} \cdots \overline{x}_d^{i_d} \mid \mathbf{i} \in \mathbb{N}_0^d\}$ be the set of monomials of the polynomial ring $R/\mathfrak{q}[\overline{x}_1, \ldots, \overline{x}_d]$; the map $\mathbf{x}^{\mathbf{i}} \mapsto \overline{\mathbf{x}}^{\mathbf{i}} : M \to \overline{M}$ is an isomorphism of monoids. An ideal $\mathfrak{A}$ of $\mathrm{gr}(R)$ is called a monomial ideal if it can be generated by elements in $\overline{M}$; such an ideal is a homogeneous ideal of the graded ring $\mathrm{gr}(R)$. Every non-zero element $z \in \mathrm{gr}(R)$ has a unique representation $z = \overline{e}_1\overline{m}_1 + \cdots + \overline{e}_r\overline{m}_r$ with pairwise distinct monomials $\overline{m}_1, \ldots, \overline{m}_r \in \overline{M}$ and non-zero elements $\overline{e}_1, \ldots, \overline{e}_r \in R/\mathfrak{q}$; we call this the monomial representation of z.

(2) For every $z \in R$ with $z \notin \bigcap \mathfrak{q}^p$ we define the order $\mathrm{ord}(z)$ to be the largest integer p with $z \in \mathfrak{q}^p$. Let $p := \mathrm{ord}(z)$; then we define the initial form of z as $\mathrm{In}(z) := z \bmod \mathfrak{q}^{p+1} \in \mathrm{gr}(R)_p$; note that $\mathrm{In}(z)$ is a homogeneous non-zero polynomial of degree p. In particular, for a monomial $m \in M$ $\mathrm{ord}(m)$ is defined, and we have $\mathrm{ord}(m) = \deg(m)$ and $\mathrm{In}(m) = \overline{m}$.

(3) For every ideal $\mathfrak{a}$ of R we define

$$\mathrm{gr}(\mathfrak{a}) := \bigoplus_{p \geq 0} (\mathfrak{a} \cap \mathfrak{q}^p + \mathfrak{q}^{p+1})/\mathfrak{q}^{p+1} \subset \mathrm{gr}(R);$$

$\mathrm{gr}(\mathfrak{a})$ is a homogeneous ideal in $\mathrm{gr}(R)$. If $\mathfrak{b}$ is another ideal in R, then we have $\mathrm{gr}(\mathfrak{a})\mathrm{gr}(\mathfrak{b}) \subset \mathrm{gr}(\mathfrak{a}\mathfrak{b})$.

(4) Let $\mathfrak{a} = m_1R + \cdots + m_rR$ with $m_1, \ldots, m_r \in M$ be a monomial ideal in R. Then we have $\mathrm{gr}(\mathfrak{a}) = \overline{m}_1\mathrm{gr}(R) + \cdots + \overline{m}_r\mathrm{gr}(R)$, hence, in particular, $\mathrm{gr}(\mathfrak{a})$ is a monomial ideal in $\mathrm{gr}(R)$ [note that, for $p \in \mathbb{N}_0$, $\mathfrak{a} \cap \mathfrak{q}^p$ is generated by the elements $m_{ij} := \mathrm{lcm}(m_i, n_j)$ where $n_j \in M$ is of degree p by Lemma 3, and that $m_{ij} \in \mathfrak{q}^{p+1}$ if $\deg(m_{ij}) > p$]. In particular, for monomial ideals $\mathfrak{a}, \mathfrak{b}$ in R we have $\mathrm{gr}(\mathfrak{a}\mathfrak{b}) = \mathrm{gr}(\mathfrak{a})\mathrm{gr}(\mathfrak{b})$ and $\mathrm{gr}(\mathfrak{a}^i) = (\mathrm{gr}(\mathfrak{a}))^i$ for every $i \in \mathbb{N}$.

Remark 5 Now we assume that $\mathfrak{q}$ *is a prime ideal of* R *which is contained in the Jacobson radical of* R and we equip R with the $\mathfrak{q}$-adic topology. Then $\bigcap \mathfrak{q}^p = (0)$ [cf. [3], Ch. III, § 3, no. 3, Prop. 6], $\mathrm{gr}(R)$ is a domain, hence R is a domain, also, and the order function is a valuation of the quotient field of R [cf. [13], vol. II, Ch. VIII, § 1, Th. 1]. Moreover, all the ideals $\mathfrak{q}_U$ for every $U \subset \{1, \ldots, d\}$ are prime ideals as is easily seen by considering the sequence $(x_i \bmod \mathfrak{q}_U)_{i \in \{1,\ldots,d\} \setminus U}$ in $R/\mathfrak{q}_U$. Therefore all the associated ideals of a monomial ideal $\mathfrak{a}$ of R are of the form $\mathfrak{q}_U$ for some $U \subset \{1, \ldots, d\}$ [cf. Lemma 1], and therefore, by considering a primary representation of $\mathfrak{a}$, we get: *if* $em \in \mathfrak{a}$ *with* $e \in R \setminus \mathfrak{q}$ *and* $m \in M$, *then we have* $m \in \mathfrak{a}$.

Let $\hat{R}$ be the $\mathfrak{q}$-adic completion of R. Then $\mathbf{x}$ is a sequence in $\hat{R}$ which satisfies (a)-(c), $\widehat{\mathfrak{q}} = \mathfrak{q}\hat{R}$ is a prime ideal in $\hat{R}$, and $\hat{R}$ is a faithfully flat R-module [cf. [3], Ch. III, § 3, no. 3, Prop. 6].

3. Monomial Representations

Assumption 1 In this section we assume that $\mathfrak{q}$ *is a prime ideal of* R *which is contained in the Jacobson radical of* R.

Notation 3 Let $w \in R$ be different from 0. Then $\mathrm{In}(w) \in \mathrm{gr}(R)$ is a homogeneous polynomial of degree $\mathrm{ord}(w)$; therefore there exist uniquely determined and pairwise distinct monomials $m_1, \dots, m_r \in M$ having degree $\mathrm{ord}(w)$ and elements $e_1, \dots, e_r \in R \setminus \mathfrak{q}$ such that $\mathrm{In}(w) = \mathrm{In}(e_1 m_1 + \cdots + e_r m_r)$; we define the set of terms of w by

$$\mathrm{Tm}(w) := \{m_1, \dots, m_r\}.$$

For $w = 0$ we put $\mathrm{In}(w) = 0$ and $\mathrm{Tm}(w) = \emptyset$.

Definition 2 We say that $w \in R$, $w \neq 0$, admits a monomial representation (with respect to $\mathbf{x}$), if there exist monomials $m_1, \dots, m_r \in M$ and elements $e_1, \dots, e_r \in R \setminus \mathfrak{q}$ such that

$$w = e_1 m_1 + \cdots + e_r m_r \text{ and } \nu(m_1 R + \cdots + m_r R) = r. \qquad (*)$$

In $(*)$ we have $m_i \nmid m_j$ for all $i, j \in \{1, \dots, r\}$ with $i \neq j$; in particular, the monomials $m_1, \dots, m_r$ are pairwise distinct. For every nonempty subset $U \subset \{1, \dots, r\}$ clearly $\sum_{i \in U} e_i m_i =: z$ is a monomial representation of z.

Lemma 4 *Let* $w \in R \setminus \{0\}$. *If* w *admits a monomial representation* $w = e_1 m_1 + \cdots + e_r m_r$, *then we have*

$$\begin{aligned}\mathrm{In}(w) &= \sum_{\substack{i=1 \\ \deg(m_i) = \mathrm{ord}(w)}}^{r} \mathrm{In}(e_i)\mathrm{In}(m_i), \\ \mathrm{ord}(w) &= \min\{\deg(m_i) \mid i \in \{1, \dots, r\}\}, \\ \mathrm{Tm}(w) &= \{m_i \mid i \in \{1, \dots, r\}, \deg(m_i) = \mathrm{ord}(w)\}.\end{aligned}$$

Proof: Let $s := \min\{\deg(m_i) \mid i \in \{1, \dots, r\}\}$. Then

$$\mathrm{In}\Bigg(\sum_{\substack{i=1 \\ \deg(m_i) = s}}^{r} e_i m_i \Bigg) = \sum_{\substack{i=1 \\ \deg(m_i) = s}}^{r} \mathrm{In}(e_i)\mathrm{In}(m_i),$$

and since $\mathrm{In}(e_i) \neq 0$ for $i \in \{1, \dots, r\}$, we obtain

$$\mathrm{ord}\Bigg(\sum_{\substack{i=1 \\ \deg(m_i) = s}}^{r} e_i m_i \Bigg) = s,$$

hence $\operatorname{ord}(w) = s$. Clearly we have

$$\operatorname{In}\Bigl(\sum_{i=1}^{r} e_i m_i\Bigr) = \operatorname{In}\Bigl(\sum_{\substack{i=1\\ \deg(m_i)=s}}^{r} e_i m_i\Bigr) = \operatorname{In}(w).$$

■

Proposition 2 *Let R be complete with respect to the $\mathfrak{q}$-adic topology. Every $w \in R$, $w \neq 0$, admits a monomial representation.*

Proof: (1) Let $w \in R$, $w \neq 0$. Let $\operatorname{Tm}(w) = \{m_1, \dots, m_r\}$. There exist elements $e_1, \dots, e_r \in R \setminus \mathfrak{q}$ such that

$$\operatorname{In}(w) = \operatorname{In}(e_1 m_1 + \cdots + e_r m_r);$$

let us put $\iota(w) := e_1 m_1 + \cdots + e_r m_r$. Then we have $\operatorname{ord}(w) = \operatorname{ord}(\iota(w))$ and $\operatorname{ord}(w - \iota(w)) > \operatorname{ord}(w)$. If $w = 0$, then we put $\iota(w) = 0$.

(2) Let $w \in R$, $w \neq 0$. We define a sequence $(w_p)_{p \in \mathbb{N}_0}$ in R: Let $w_0 := w$; if $p \in \mathbb{N}_0$, and if w_p is defined, then we define $w_{p+1} := w_p - \iota(w_p)$.

Note the following: If $w_p = 0$ for one $p \in \mathbb{N}_0$, then $w_q = 0$ for every $q \in \mathbb{N}_0$ with $q \geq p$, and if $w_p \neq 0$ for one $p \in \mathbb{N}_0$, then the elements $w_0, \dots, w_{p-1}$ are different from 0, and we have

$$\operatorname{ord}(w) = \operatorname{ord}(w_0) < \operatorname{ord}(w_1) < \cdots < \operatorname{ord}(w_p);$$

in particular, we have $\operatorname{ord}(w_p) \geq p$.

For every $p \in \mathbb{N}_0$ let $\mathfrak{a}_p$ be that monomial ideal of R which is generated by the monomials in $\operatorname{Tm}(w_0), \dots, \operatorname{Tm}(w_p)$. Then $(\mathfrak{a}_p)_{p \in \mathbb{N}_0}$ is an increasing sequence of ideals in R, and therefore it becomes stationary, i.e., there exists $q \in \mathbb{N}_0$ with $\mathfrak{a}_q = \mathfrak{a}_{q+1} = \cdots =: \mathfrak{a}$. We can write $\mathfrak{a} = m_1 R + \cdots + m_r R$ where $m_1, \dots, m_r \in M$ and $r := \nu(\mathfrak{a})$.

(3) We have

$$w = w_{p+1} + \sum_{j=0}^{p} \iota(w_j) \quad \text{for every } p \in \mathbb{N}_0;$$

note that $w_{p+1} = 0$ or $\operatorname{ord}(w_{p+1}) \geq p + 1$, hence $w_{p+1} \in \mathfrak{q}^{p+1}$.

Let $j \in \mathbb{N}_0$ with $w_j \neq 0$. Then we can write $\iota(w_j)$ as a sum

$$\iota(w_j) = \sum_{i=1}^{r} a_{ji} m_i$$

where the elements $a_{ji} \in R$ for $i \in \{1, \dots, r\}$ satisfy the following condition: If $\operatorname{ord}(w_j) < \deg(m_i)$, then $a_{ji} = 0$, and if $\operatorname{ord}(w_j) \geq \deg(m_i)$ and $a_{ji} \neq 0$, then a_{ji} is a linear combination of monomials of degree $\operatorname{ord}(w_j) - \deg(m_i)$ with coefficients which lie in $R \setminus \mathfrak{q}$ [note that the monomials in $\operatorname{Tm}(w_j)$ lie in $\mathfrak{a}$]. For $p \in \mathbb{N}_0$ we have

$$\sum_{j=0}^{p} \iota(w_j) = \sum_{i=1}^{r} e_{pi} m_i$$

with

$$e_{pi} := \sum_{j=0}^{p} a_{ji} \quad \text{for every } i \in \{1, \dots, r\}.$$

Let $i \in \{1, \dots, r\}$. There exists a unique $j_i \in \{0, \dots, q\}$ with $\operatorname{ord}(w_{j_i}) = \deg(m_i)$ [cf. (2) and note that $\{m_1, \dots, m_r\}$ is a minimal system of generators of $\mathfrak{a}$].

We consider any integer $p \geq q$. Then we have $a_{ji} = 0$ for $j \in \{0, \dots, j_i - 1\}$, $a_{j_i i} \in R \setminus \mathfrak{q}$, and $a_{ji} \in \mathfrak{q}^{j - \deg(m_i)}$ for $j \in \{j_i + 1, \dots, p\}$. In particular, $e_{pi} \in R \setminus \mathfrak{q}$. Furthermore, we have

$$e_{p+1,i} - e_{pi} = a_{p+1,i} \in \mathfrak{q}^{p+1-\deg(m_i)};$$

therefore, the sequence $(e_{pi})_{p \geq 0}$ is a Cauchy sequence in $R \setminus \mathfrak{q}$. Since $\mathfrak{q}$ is an open ideal in the $\mathfrak{q}$-adic topology, we have

$$e_i := \lim_{p \to \infty} e_{pi} \in R \setminus \mathfrak{q}.$$

From

$$\sum_{i=1}^{r} e_i m_i = \sum_{i=1}^{r} (\lim_{p \to \infty} e_{pi}) m_i = \lim_{p \to \infty} \left(\sum_{i=1}^{r} e_{pi} m_i \right)$$
$$= \lim_{p \to \infty} \left(\sum_{j=0}^{p} \iota(w_j) \right) = \lim_{p \to \infty} (w - w_{p+1})$$

and $w_{p+1} \in \mathfrak{q}^{p+1}$ for every $p \in \mathbb{N}_0$ we obtain

$$w = \sum_{i=1}^{r} e_i m_i.$$

■

Proposition 3 *Let* $\mathfrak{a} \neq (0)$ *be an ideal in* R. *The following statements are equivalent:*

(1) $\mathfrak{a}$ *is a monomial ideal.*

(2) *For every* $w \in \mathfrak{a}$, $w \neq 0$, *we have* $\mathrm{Tm}(w) \subset \mathfrak{a}$.

Now we assume, in addition, that R *is complete in the* $\mathfrak{q}$*-adic topology. Then the following statements are equivalent with* (1) *and* (2)*:*

(3) *Every* $w \in \mathfrak{a}$, $w \neq 0$, *admits a monomial representation* $w = e_1m_1 + \cdots + e_rm_r$ *with* $m_1, \ldots, m_r \in \mathfrak{a}$.

(4) *Let* $w \in \mathfrak{a}$, $w \neq 0$, *and let* $w = e_1m_1 + \cdots + e_rm_r$ *be a monomial representation of* w, *then* $m_1, \ldots, m_r \in \mathfrak{a}$.

Proof: (1) $\Rightarrow$ (2): Let $w \in \mathfrak{a}$, $w \neq 0$, and let $\mathrm{Tm}(w) = \{m_1, \ldots, m_r\}$; let $s := \mathrm{ord}(w)$, hence we have $\deg(m_1) = \cdots = \deg(m_r) = s$ [cf. Lemma 4]. There exist elements $e_1, \ldots, e_r \in R \setminus \mathfrak{q}$ with $\mathrm{ord}(w - (e_1m_1 + \cdots + e_rm_r)) > s$. Let $i \in \{1, \ldots, r\}$, and define

$$\mathfrak{b}_i := \mathfrak{a} + m_1R + \cdots + m_{i-1}R + m_{i+1}R + \cdots + m_rR + \mathfrak{q}^{s+1};$$

$\mathfrak{b}_i$ is a monomial ideal of R. Note that $e_im_i \in \mathfrak{b}_i$, and therefore we have $m_i \in \mathfrak{b}_i$ [cf. Remark 5]. For no monomial $m \in \mathfrak{q}^{s+1}$ we have $m \mid m_i$ [since $\deg(m_i) = s < \deg(m)$], and we have $m_j \nmid m_i$ for $j \in \{1, \ldots, r\}$, $j \neq i$. Therefore, by Cor. 3, there exists a monomial $m \in \mathfrak{a}$ with $m \mid m_i$, hence we have $m_i \in \mathfrak{a}$, and therefore we have shown that $\mathrm{Tm}(w) \subset \mathfrak{a}$.

(2) $\Rightarrow$ (1): Suppose that $\mathfrak{a}$ is not a monomial ideal. This means, in particular, that $\mathfrak{a} \neq R$. Let $\mathfrak{a}'$ be the monomial ideal which is generated by all the monomials which lie in $\mathfrak{a}$; then we have $\mathfrak{a}' \subsetneqq \mathfrak{a}$. By assumption we have $\mathrm{Tm}(w) \subset \mathfrak{a}'$ for every $w \in \mathfrak{a}$, $w \neq 0$. The prime ideals in $\mathrm{Ass}(R/\mathfrak{a}')$ are of the form $\mathfrak{q}_U$ for $U \subset \{1, \ldots, d\}$, hence are contained in $\mathfrak{q}$ [cf. Remark 5]. By Krull's intersection theorem [cf. [13], Vol. I, Ch. 4, § 7, Th. 12′] we have $\bigcap_{n \geq 0}(\mathfrak{a}' + \mathfrak{q}^n) = \mathfrak{a}'$. Therefore there exists $n \in \mathbb{N}_0$ with $\mathfrak{a} \subset \mathfrak{a}' + \mathfrak{q}^n$, $\mathfrak{a} \not\subset \mathfrak{a}' + \mathfrak{q}^{n+1}$. We choose $w \in \mathfrak{a}$, $w \notin \mathfrak{a}' + \mathfrak{q}^{n+1}$; we can write $w = w_1 + z$ with $w_1 \in \mathfrak{a}'$, $z \in \mathfrak{q}^n$ and $z \notin \mathfrak{q}^{n+1}$. This implies that $z = w - w_1 \in \mathfrak{a}$, $z \neq 0$, and, by assumption, we have $\mathrm{Tm}(z) \subset \mathfrak{a}$, hence $\mathrm{Tm}(z) \subset \mathfrak{a}'$. Let $\mathrm{Tm}(z) = \{m_1, \ldots, m_r\}$. Then there exist elements $e_1, \ldots, e_r \in R \setminus \mathfrak{q}$ such that, putting $z_1 := e_1m_1 + \cdots + e_rm_r$, we have $z_1 \in \mathfrak{a}'$ and $z - z_1 \in \mathfrak{q}^{n+1}$. This implies that $w = w_1 + z = w_1 + z_1 + (z - z_1) \in \mathfrak{a}' + \mathfrak{q}^{n+1}$, in contradiction with the choice of w.

Now we assume that R is complete; then every $w \in R$, $w \neq 0$, admits a monomial representation [cf. Prop. 2].

(2) ⇒ (4): Let $w \in \mathfrak{a}$, $w \neq 0$, and let $w = e_1m_1+\cdots+e_rm_r$ be a monomial representation of w. We show by induction on r that $\{m_1, \ldots, m_r\} \subset \mathfrak{a}$. Let $r = 1$, hence $\operatorname{Tm}(w) = \{m_1\} \subset \mathfrak{a}$. Now let $r > 1$. It is clear that $\operatorname{Tm}(w) \subset \{m_1, \ldots, m_r\}$. We label the elements $m_1, \ldots, m_r$ in such a way that $\operatorname{Tm}(w) = \{m_1, \ldots, m_q\}$ with $q \leq r$. We put $w_1 := e_1m_1 + \cdots + e_qm_q$. Now we have $w_1 \in \mathfrak{a}$ by assumption. If $q = r$, then the elements $m_1, \ldots, m_q$ lie in $\mathfrak{a}$. If $q < r$, then we have $w - w_1 = e_{q+1}m_{q+1} + \cdots + e_rm_r$, and since $w - w_1 \in \mathfrak{a}$, we get by our induction assumption that $m_{q+1}, \ldots, m_r \in \mathfrak{a}$.

(4) ⇒ (3) and (3) ⇒ (1) are trivial. ■

4. Integral Elements

Remark 6 Let S be a ring, and let $\mathfrak{a}$ be an ideal in S. The integral closure of the Rees ring

$$\mathcal{R}(\mathfrak{a}, S) = \bigoplus_{p\geq 0} \mathfrak{a}^p T^p \ \subset S[\,T\,]$$

in the polynomial ring $S[\,T\,]$ is the graded ring $\bigoplus_{p\geq 0} \overline{\mathfrak{a}^p}\, T^p$ where, for every $p \in \mathbb{N}$, $\overline{\mathfrak{a}^p}$ is the integral closure of $\mathfrak{a}^p$ in S [cf. [10], Ch. II, § 5]. In particular, an element $z \in S$ is integral over $\mathfrak{a}$ iff $zT \in S[\,T\,]$ is integral over $\bigoplus_{p\geq 0} \mathfrak{a}^p T^p$.

Notation 4 Let $\mathfrak{a}$, $\mathfrak{b}$ be monomial ideals in R.

(1) We define

$$\widetilde{\mathfrak{a}} := (\{m \in M \mid \text{ there exists } l \in \mathbb{N} \text{ with } m^l \in \mathfrak{a}^l\});$$

$\widetilde{\mathfrak{a}}$ is a monomial ideal of R. Since the monomials which generate $\widetilde{\mathfrak{a}}$ are integral over $\mathfrak{a}$, $\widetilde{\mathfrak{a}}$ is an ideal which is integral over $\mathfrak{a}$, and therefore $\widetilde{\mathfrak{a}}$ is contained in the integral closure $\overline{\mathfrak{a}}$ of $\mathfrak{a}$ in R, and we have

$$\mathfrak{a} \subset \widetilde{\mathfrak{a}} \subset \overline{\mathfrak{a}}.$$

It is clear that $\widetilde{\mathfrak{a}}\,\widetilde{\mathfrak{b}} \subset \widetilde{\mathfrak{a}\mathfrak{b}}$, and if $\mathfrak{a} \subset \mathfrak{b}$, then we have $\widetilde{\mathfrak{a}} \subset \widetilde{\mathfrak{b}}$.

(2) We show that

$$\widetilde{\widetilde{\mathfrak{a}}} = \widetilde{\mathfrak{a}}.$$

In fact, let $\widetilde{\mathfrak{a}} = m_1R+\cdots+m_rR$. For every $i \in \{1, \ldots, r\}$ there exists $l_i \in \mathbb{N}$ with $m_i^{l_i} \in \mathfrak{a}^{l_i}$. Let m be a monomial in $\widetilde{\widetilde{\mathfrak{a}}}$. Then there exists $l \in \mathbb{N}$ with $m^l \in \widetilde{\mathfrak{a}}^l$. This implies that there exist $(i_1, \ldots, i_r) \in \mathbb{N}_0^r$ with $i_1 + \cdots + i_r = l$ and such that $m_1^{i_1} \cdots m_r^{i_r}$ divides m^l [cf. Cor. 3]. Since $(m_1^{i_1} \cdots m_r^{i_r})^{l_1\cdots l_r}$ lies in $\mathfrak{a}^{ll_1\cdots l_r}$, we see that $m^{ll_1\cdots l_r}$ lies in $\mathfrak{a}^{ll_1\cdots l_r}$, also, and this means that $m \in \widetilde{\mathfrak{a}}$.

(3) By (1) we get $\widetilde{\mathfrak{a}^p}\widetilde{\mathfrak{a}^q} \subset \widetilde{\mathfrak{a}^{p+q}}$ for all $p, q \in \mathbb{N}_0$. Therefore

$$\widetilde{\mathcal{R}(\mathfrak{a}, R)} := \bigoplus_{p\geq 0} \widetilde{\mathfrak{a}^p} T^p \subset R[\,T\,]$$

is a graded R-algebra and a graded R-subalgebra of $R[\,T\,]$, and it contains the Rees ring $\mathcal{R}(\mathfrak{a}, R) := \bigoplus_{p\geq 0} \mathfrak{a}^p T^p$ of $\mathfrak{a}$ as a graded R-subalgebra.

(4) Since $\widetilde{\mathfrak{a}^p} \subset \overline{\mathfrak{a}^p}$ for every $p \in \mathbb{N}$, the integral closure of $\widetilde{\mathcal{R}(\mathfrak{a}, R)}$ in $R[\,T\,]$ is the ring $\bigoplus_{p\geq 0} \overline{\mathfrak{a}^p}\, T^p$ [cf. Remark 6].

(5) Just as in [8], Prop. 4.6, one may prove, using (4): For $z \in R$ we have $z \in \overline{\mathfrak{a}}$ iff there exist $p \in \mathbb{N}$ and elements $a_i \in \widetilde{\mathfrak{a}^i}$, $i \in \{1, \dots, p\}$, such that

$$z^p + a_1 z^{p-1} + \cdots + a_p = 0.$$

Assumption 2 For the rest of this section we again assume that $\mathfrak{q}$ *is a prime ideal of* R *which is contained in the Jacobson radical of* R. The $\mathfrak{q}$-adic completion of R shall be denoted by $\hat{R}$.

Proposition 4 *Let* $\mathfrak{a}$ *be a monomial ideal of* R*, and let* $m = x_1^{j_1} \cdots x_d^{j_d} \in M$*. The following statements are equivalent:*

(1) *m is integral over $\mathfrak{a}$.*

(2) *m is integral over $\mathfrak{a}\hat{R}$.*

(3) *There exists $l \in \mathbb{N}$ with $m^l \in \mathfrak{a}^l$.*

(4) *$(j_1, \dots, j_d)$ lies in the convex hull of $\Gamma + \mathbb{R}^d_{\geq 0}$ where $\Gamma \subset \mathbb{N}_0^d$ is the set of exponents of monomials appearing in $\mathfrak{a}$.*

In particular, every monomial in $\overline{\mathfrak{a}}$ lies in $\widetilde{\mathfrak{a}}$.

Proof: (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1) hold trivially.

(2) $\Rightarrow$ (3): Let $T^p + a_1 T^{p-1} + \cdots + a_p \in \hat{R}[\,T\,]$ with $a_i \in (\mathfrak{a}\hat{R})^i = \mathfrak{a}^i\hat{R}$ for $i \in \{1, \dots, p\}$ be an equation of integral dependence for m over $\mathfrak{a}\hat{R}$. Let $i \in \{1, \dots, p\}$. Since $\mathfrak{a}^i$ is a monomial ideal of R, the ideal $\mathfrak{a}^i\hat{R}$ is a monomial ideal of $\hat{R}$, and, by Prop. 2, there exist elements $e_{i1}, \dots, e_{ir_i} \in \hat{R} \setminus \mathfrak{q}\hat{R}$ and monomials $m_{i1}, \dots, m_{ir_i} \in M$ with

$$a_i = \sum_{j=1}^{r_i} e_{ij} m_{ij}.$$

From Prop. 3 we obtain $m_{ij} \in \mathfrak{a}^i\hat{R} \cap R = \mathfrak{a}^i$ for $i \in \{1, \dots, p\}$, $j \in \{1, \dots, r_i\}$ [note that $\hat{R}$ is a faithfully flat extension of R]. Therefore the monomial m^p

lies in the $\hat{R}$-ideal which is generated by the set $\{m_{ij}m^{p-i} \mid i \in \{1,\dots,p\}, j \in \{1,\dots,r_i\}\}$. Using Cor. 3 we find $i \in \{1,\dots,p\}$ and $j \in \{1,\dots,r_i\}$ with $m_{ij}m^{p-i} \mid m^p$, hence $m_{ij} \mid m^i$. Thus, we have shown that $m^i \in m_{ij}R \subset \mathfrak{a}^i$.

(3) $\Longleftrightarrow$ (4) This is an easy consequence of Cor. 3 and Carathéodory's theorem [for Carathéodory's theorem cf. [11], Th. 17.1]. ■

Corollary 6 *Let* $\mathfrak{a}$ *be a monomial ideal of* R.

(1) *We have* $\widetilde{\mathfrak{a}}\hat{R} = \widetilde{\mathfrak{a}\hat{R}}$ *and* $\overline{\mathfrak{a}}\hat{R} \subset \overline{\mathfrak{a}\hat{R}}$.

(2) *We have* $\widetilde{\operatorname{gr}(\mathfrak{a})} = \operatorname{gr}(\widetilde{\mathfrak{a}})$.

Proof: (1) The first assertion is an easy consequence of Prop. 4, and the second assertion is clear.

(2) Let $\widetilde{\mathfrak{a}}$ be generated by the monomials $m_1,\dots,m_r$. Then $\operatorname{gr}(\widetilde{\mathfrak{a}})$ is generated by the monomials $\overline{m}_1,\dots,\overline{m}_r$ [cf. (4) in Remark 4]. For every $i \in \{1,\dots,r\}$ there exists $l_i \in \mathbb{N}$ with $m_i^{l_i} \in \mathfrak{a}^{l_i}$, hence $\overline{m}_i^{l_i} \in \operatorname{gr}(\mathfrak{a}^{l_i}) = \operatorname{gr}(\mathfrak{a})^{l_i}$, and therefore we have $\overline{m}_i \in \widetilde{\operatorname{gr}(\mathfrak{a})}$. Conversely, let $m \in M$ be a monomial with $\overline{m} \in \widetilde{\operatorname{gr}(\mathfrak{a})}$. Then there exists $l \in \mathbb{N}$ with $\overline{m}^l \in (\operatorname{gr}(\mathfrak{a}))^l = \operatorname{gr}(\mathfrak{a}^l)$, hence $m^l \in \mathfrak{a}^l$, and therefore $m \in \widetilde{\mathfrak{a}}$, hence $\overline{m} \in \operatorname{gr}(\widetilde{\mathfrak{a}})$. ■

5. Monomial Ideals in Polynomial Rings

The following result in Prop. 5 should be known, but we could not find a source for it.

Notation 5 Let $(\Gamma, \prec)$ be a totally ordered commutative monoid with neutral element 0 satisfying the following condition:

Every non-empty subset of Γ has a smallest element.

This condition is satisfied if $\prec$ is a well-ordering; in particular, a monomial ordering on $\mathbb{N}_0^d$ satisfies this condition.

Let $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ be a Γ-graded ring. For $z \in R$ let $z_\gamma \in R_\gamma$ be the homogeneous component of z of degree γ, and if $z \neq 0$, then define

$$\operatorname{Supp}(z) := \{\gamma\in\Gamma \mid z_\gamma \neq 0\},\ \deg(z) := \max\nolimits_\prec\{\gamma \mid \gamma\in\operatorname{Supp}(z)\},\ z^* := z_{\deg(z)}.$$

Let z, $w \in R\setminus\{0\}$; then we have $\deg(zw) \preceq \deg(z) + \deg(w)$ if $zw \neq 0$ and $\deg(z+w) \preceq \max_\prec\{\deg(z), \deg(w)\}$ if $z + w \neq 0$. Notice that, if z is not homogeneous, then we have $\deg(z - z^*) \prec \deg(z)$.

Proposition 5 *Let S be a Γ-graded ring, and let R be a Γ-graded subring of S. Then the integral closure $\overline{R}$ of R in S is a Γ-graded subring of S.*

Proof: (1) Firstly, we consider the case that every homogeneous element of S which is integral over R already lies in R. Then we have to show that $\overline{R} = R$. Suppose that $R \subsetneqq \overline{R}$, and choose $z \in \overline{R} \setminus R$ in such a way that $\#(\mathrm{Supp}(z)) \leq \#(\mathrm{Supp}(w))$ for every $w \in \overline{R} \setminus R$. Now z is not homogeneous by our assumption on R. If $z^* \in \overline{R}$, then we would have $z^* \in R$ since z^* is homogeneous, hence $z - z^* \in \overline{R}$, and therefore $z - z^* \in R$ by the choice of z [note that $\#(\mathrm{Supp}(z - z^*)) < \#(\mathrm{Supp}(z))$]. Therefore we have $z^* \notin \overline{R}$. In particular, we have $(z^*)^i \neq 0$ for every $i \in \mathbb{N}$, hence $(z^i)^* = (z^*)^i$ and $\deg(z^i) = i \deg(z)$ for every $i \in \mathbb{N}$.

Let

$$\mathcal{V} := \{\mathbf{a} = (a_1, \ldots, a_p) \mid a_1, \ldots, a_p \in R, z^p + a_1 z^{p-1} + \cdots + a_p = 0\}.$$

Obviously $\mathcal{V}$ is not empty. For every $\mathbf{a} = (a_1, \ldots, a_p) \in \mathcal{V}$ we define

$$\gamma(\mathbf{a}) := \max\nolimits_{\prec}\{\deg(a_i) - i \deg(z) \mid a_i \neq 0, i \in \{0, 1, \ldots, p\}\} \in \Gamma,$$

$$s(\mathbf{a}) := \min\{i \in \{0, \ldots, p\} \mid a_i \neq 0, \deg(a_i) - i \deg(z) = \gamma(\mathbf{a})\} \in \{0, \ldots, p\}$$

[we define $a_0 := 1$]. Then we have $\gamma(\mathbf{a}) \succeq 0$ [since $a_0 = 1 \in R_0$]. Suppose that there exists $\mathbf{a} = (a_1, \ldots, a_p) \in \mathcal{V}$ with $\gamma(\mathbf{a}) = 0$. Then we have for every $i \in \{1, \ldots, p\}$ with $a_i z^{p-i} \neq 0$

$$\begin{aligned} \deg(a_i z^{p-i}) &\preceq \deg(a_i) + \deg(z^{p-i}) = \deg(a_i) + (p-i)\deg(z) \\ &\preceq p \deg(z) + \gamma(\mathbf{a}) = p \deg(z). \end{aligned}$$

In $z^p + a_1 z^{p-1} + \cdots + a_p = 0$ we consider the homogeneous component of degree $p \deg(z) = \deg(z^p)$. Then we get $(z^*)^p + a'_1 (z^*)^{p-1} + \cdots + a'_p = 0$ with

$$a'_i := \begin{cases} a_i^* & \text{if } a_i z^{p-i} \neq 0 \text{ and } \deg(a_i z^{p-i}) = p \deg(z), \\ 0 & \text{else} \end{cases} \qquad \text{for } i \in \{1, \ldots, p\}.$$

But this would imply that $z^* \in \overline{R}$, in contradiction with our observation above.

Therefore we have $\gamma(\mathbf{a}) \succ 0$ for every $\mathbf{a} \in \mathcal{V}$. This implies that $s(\mathbf{a}) > 0$; moreover, we have $s(\mathbf{a}) \leq p - 1$ since otherwise $a_p^* = 0$.

Let

$$\gamma_0 := \min\nolimits_{\prec}\{\gamma(\mathbf{a}) \mid \mathbf{a} \in \mathcal{V}\}, \ \mathcal{V}_0 := \{\mathbf{a} \in \mathcal{V} \mid \gamma(\mathbf{a}) = \gamma_0\}.$$

Then we have $\gamma_0 \succ 0$. We choose $\mathbf{a} = (a_1, \ldots, a_p) \in \mathcal{V}_0$ with $s(\mathbf{b}) \le s(\mathbf{a})$ for every $\mathbf{b} \in \mathcal{V}_0$. We define

$$a'_j := \begin{cases} a_j^* & \text{if } a_j \neq 0, \deg(a_j) - j\deg(z) = \gamma_0, \\ 0 & \text{else} \end{cases} \quad \text{for } j \in \{1, \ldots, p\}.$$

By the choice of s we have $a'_1 = \cdots = a'_{s-1} = 0$, $a'_s = a^*_s \neq 0$, and

$$a'_s (z^*)^{p-s} + a'_{s+1} (z^*)^{p-s-1} + \cdots + a'_p = 0 \qquad (*)$$

[consider in $z^p + a_1 z^{p-1} + \cdots + a_p = 0$ the homogeneous component of degree $\gamma_0 + p\deg(z)$]. We multiply $(*)$ by ${a'_s}^{p-s-1}$ and obtain

$$(a'_s z^*)^{p-s} + a'_{s+1} (a'_s z^*)^{p-s-1} + \cdots + a'_p {a'_s}^{p-s-1} = 0.$$

Therefore the homogeneous element $a'_s z^*$ is integral over R, hence lies in R. Since $a'_s z - a'_s z^*$ is integral over R, and since either $a'_s z = a'_s z^*$ or $\#(\mathrm{Supp}(a'_s z - a'_s z^*)) < \#(\mathrm{Supp}(a'_s z))$, we have $a'_s z - a'_s z^* \in R$ by the choice of z, hence $a'_s z \in R$. We define

$$\overline{a}_i := \begin{cases} a_i & \text{if } i \neq s, s+1, \\ a_s - a'_s & \text{if } i = s, \\ a_{s+1} + a'_s z & \text{if } i = s+1 \end{cases} \qquad \text{for } i \in \{1, \ldots, p\}.$$

Then we have $\overline{\mathbf{a}} = (\overline{a}_1, \ldots, \overline{a}_p) \in R^p$, and since $z^p + \overline{a}_1 z^{p-1} + \cdots + \overline{a}_p = 0$, we have $\overline{\mathbf{a}} \in \mathcal{V}$. We show that we even have $\overline{\mathbf{a}} \in \mathcal{V}_0$. We have $\overline{a}_s = 0$ or $\deg(a_s - a'_s) - s\deg(z) \prec \deg(a_s) - s\deg(z) \preceq \gamma_0$, and we have $\overline{a}_{s+1} = 0$ or $\deg(a_{s+1} + a'_s z) - (s+1)\deg(z) \preceq \gamma_0$, and therefore we have $\gamma(\overline{\mathbf{a}}) = \gamma_0$. Obviously we have $s(\overline{\mathbf{a}}) \ge s+1$, in contradiction with the choice of $\mathbf{a}$. Therefore we have $\overline{R} = R$.

(2) Now we consider the general case. Let $R' := R[\Sigma]$ where Σ is the set of homogeneous elements of S which are integral over R; then R' is a Γ-graded subring of S. We have $R \subset R' \subset \overline{R}$, hence $\overline{R} = \overline{R'}$. Since $\overline{R'} = R'$ by (1), we have $\overline{R} = R'$. ■

Corollary 7 *Let R be a Γ-graded ring, and let $\mathfrak{a}$ be a Γ-homogeneous ideal of R. Then the integral closure of $\mathfrak{a}$ in R is a Γ-homogeneous ideal of R, again.*

Proof: We equip the polynomial ring $R[T]$ in a natural way with a $\Gamma \times \mathbb{N}_0$-grading; then we can consider the Rees ring $\mathcal{R}(\mathfrak{a}, R)$ as a $\Gamma \times \mathbb{N}_0$-graded subring of $R[T]$. The integral closure of $\mathcal{R}(\mathfrak{a}, R)$ in $R[T]$ is a $\Gamma \times \mathbb{N}_0$-graded subring by Prop. 5, and $w \in R$ is integral over $\mathfrak{a}$ iff $wT \in R[T]$ lies in

$$\overline{\mathcal{R}(\mathfrak{a}, R)} = \bigoplus_{p \geq 0} \overline{\mathfrak{a}^p}\, T^p$$

[cf. Remark 6]. ■

Notation 6 For the rest of this section let k be a ring, and let $A = k[x_1, \dots, x_d]$ be the polynomial ring over k in d variables $x_1, \dots, x_d$. Then $(x_1, \dots, x_d)$ is a regular sequence in A which satisfies (a)-(c) above; let M be the set of monomials $\mathbf{x}^{\mathbf{i}} = x_1^{i_1} \cdots x_d^{i_d}$, $\mathbf{i} \in \mathbb{N}_0^d$. Every non-zero $z \in A$ has a unique representation $z = c_1 m_1 + \cdots + c_r m_r$ with non-zero elements $c_1, \dots, c_r \in k$ and pairwise distinct monomials $m_1, \dots, m_r \in M$; we call this the monomial representation of z.

An ideal $\mathfrak{A}$ of A is called a monomial ideal if it is generated by a set of monomials. Let $\mathfrak{A}$ be a monomial ideal in A; then $\mathfrak{A}$ is generated by a *finite set of monomials* [Dickson's Lemma, cf. [1], Ch. 4, Cor. 4.48 and Th. 5.2 or [5], Ch. II, § 4, in particular Exercise 7] and a monomial $m \in M$ belongs to $\mathfrak{A}$ iff it is a multiple of a monomial in $\mathfrak{A}$. Moreover, if $cm \in \mathfrak{A}$ with $c \in k \setminus \{0\}$ and $m \in M$, then $m \in \mathfrak{A}$.

Corollary 8 *Let $\mathfrak{A}$ be a monomial ideal in A. Then we have*

$$\overline{\mathfrak{A}} = \mathrm{rad}_k(0)A + \widetilde{\mathfrak{A}}\,.$$

Proof: Clearly we have $\mathrm{rad}_k(0) \subset \overline{\mathfrak{A}}$ and $\widetilde{\mathfrak{A}} \subset \overline{\mathfrak{A}}$. Let $z \in \overline{\mathfrak{A}}$, $z \neq 0$; since $\overline{\mathfrak{A}}$ is an $\mathbb{N}_0^d$-homogeneous ideal of A [cf. Cor. 7], there exist $s \in \mathbb{N}$, non-zero elements $c_1, \dots, c_s \in k$ and monomials $n_1, \dots, n_s \in M$ with $z = c_1 n_1 + \cdots + c_s n_s$ and such that $c_i n_i$ is integral over $\mathfrak{A}$ for $i \in \{1, \dots, s\}$. Let $i \in \{1, \dots, s\}$. Then there exist $p \in \mathbb{N}$, elements $d_1, \dots, d_p \in k$ and monomials $m_1 \in \mathfrak{A}, \dots, m_p \in \mathfrak{A}^p$ such that

$$(c_i n_i)^p + d_1 m_1 (c_i n_i)^{p-1} + \cdots + d_p m_p = 0.$$

If $d_1 = \cdots = d_p = 0$, then we have $c_i^p = 0$, hence $c_i \in \mathrm{rad}_k(0)$. Otherwise, there exists $l \in \{1, \dots, p\}$ with $n_i^p = m_l\, n_i^{p-l}$, hence $n_i^l = m_l \in \mathfrak{A}^l$, hence $n_i \in \widetilde{\mathfrak{A}}$. Therefore we have $z \in \mathrm{rad}_k(0)A + \widetilde{\mathfrak{A}}$. ■

Corollary 9 *The following statements are equivalent:*

(1) *k is a reduced ring.*

(2) *There exists a monomial ideal $\mathfrak{A}$ in A such that $\overline{\mathfrak{A}} = \widetilde{\mathfrak{A}}$.*

(3) *For every monomial ideal $\mathfrak{A}$ of A we have $\overline{\mathfrak{A}} = \widetilde{\mathfrak{A}}$.*

6. The Main Theorem

We keep the notations and assumptions introduced in section 2.

Notation 7 (1) A monomial ordering $\prec$ of $\mathbb{N}_0^d$ is said to be degree-compatible if it satisfies the following condition: for any $\mathbf{i}$, $\mathbf{j} \in \mathbb{N}_0^d$ with $\deg(\mathbf{i}) < \deg(\mathbf{j})$ we have $\mathbf{i} \prec \mathbf{j}$.

(2) Let $\prec$ be a degree-compatible ordering on $\mathbb{N}_0^d$. Then every subset of $\mathbb{N}_0^d$ which is bounded above is finite.

(3) Let $\prec$ be a monomial ordering on $\mathbb{N}_0^d$. Let $\mathbf{i} \neq \mathbf{j}$ be in $\mathbb{N}_0^d$. We define $\mathbf{i} \prec_g \mathbf{j}$ if $\deg(\mathbf{i}) < \deg(\mathbf{j})$ or if $\deg(\mathbf{i}) = \deg(\mathbf{j})$ and $\mathbf{i} \prec \mathbf{j}$. Then $\prec_g$ is a degree-compatible monomial ordering on $\mathbb{N}_0^d$.

(4) If $\prec$ is the lexicographical ordering lex on $\mathbb{N}_0^d$, then $\prec_g$ is the degree-lexicographical ordering deglex on $\mathbb{N}_0^d$.

(5) Every monomial ordering $\prec$ on $\mathbb{N}_0^d$ induces an ordering on M which will be denoted by $\prec$, again.

Proposition 6 *We assume that $R/\mathfrak{q}$ is a reduced ring. Let $\mathfrak{a}$ be a monomial ideal of R; then $\mathrm{gr}(\widetilde{\mathfrak{a}})$ is the integral closure of the monomial ideal $\mathrm{gr}(\mathfrak{a})$ in $\mathrm{gr}(R)$.*

Proof: Since $\widetilde{\mathfrak{a}}$ is integral over $\mathfrak{a}$, obviously $\widetilde{\mathrm{gr}(\mathfrak{a})} = \mathrm{gr}(\widetilde{\mathfrak{a}})$ [cf. Cor. 9(2)] is integral over $\mathrm{gr}(\mathfrak{a})$. Let $m \in M$ be a monomial, and assume that $\mathrm{In}(m) = \overline{m}$ is integral over $\mathrm{gr}(\mathfrak{a})$. Then there exists $h \in \mathbb{N}$ with $\mathrm{In}(m)^h \in (\mathrm{gr}(\mathfrak{a}))^h = \mathrm{gr}(\mathfrak{a}^h)$ [cf. Cor. 9], hence we see that $m^h \in \mathfrak{a}^h \cap \mathfrak{q}^{h \deg(m)} \subset \mathfrak{a}^h$, hence $m \in \widetilde{\mathfrak{a}}$, and therefore we obtain that $\mathrm{In}(m) \in \mathrm{gr}(\widetilde{\mathfrak{a}})$. ■

Remark 7 We assume that R is complete, and that $\mathfrak{q}$ is a prime ideal which is contained in the Jacobson radical of R. Let $\prec$ be a degree-compatible monomial ordering on M, and let $z \in R \setminus \{0\}$; we define

$$\mathrm{lm}(z) := \min_{\prec}\{\mathrm{Tm}(z)\}.$$

Let

$$z = e_1 m_1 + \cdots + e_r m_r$$

be a monomial representation of z, then we have $\mathrm{lm}(z) \preccurlyeq m_j$ for every $j \in \{1, \ldots, r\}$ [cf. Lemma 4 and note that $\prec$ is a degree-compatible ordering], hence we even have

$$\mathrm{lm}(z) = \min_{\prec}\{m_i \mid i \in \{1, \ldots, r\}\}.$$

For z, $w \in R \setminus \{0\}$ we obviously have

$$\mathrm{lm}(zw) = \mathrm{lm}(z)\mathrm{lm}(w).$$

Proposition 7 *We assume that R is complete, and that $\mathfrak{q}$ is a prime ideal which is contained in the Jacobson radical of R. For every monomial ideal $\mathfrak{a}$ of R we have $\overline{\mathfrak{a}} = \widetilde{\mathfrak{a}}$.*

Proof: (1) We have $\widetilde{\mathfrak{a}} \subset \overline{\mathfrak{a}}$ for every monomial ideal $\mathfrak{a}$ of R [cf. (1) in Notation 4]. Suppose that the proposition does not hold. Then the family

$$\mathcal{I} := \{\mathfrak{a} \mid \mathfrak{a} \text{ monomial ideal of } R,\ \widetilde{\mathfrak{a}} \subsetneqq \overline{\mathfrak{a}}\}$$

is not empty. For every $\mathfrak{a} \in \mathcal{I}$ we define $r(\mathfrak{a}) \in \mathbb{N}$ in the following way: If $y \in \overline{\mathfrak{a}} \setminus \widetilde{\mathfrak{a}}$, and if $y = e_1m_1 + \cdots + e_rm_r$ is a monomial representation of y [cf. Prop. 2], then we have $r \geq r(\mathfrak{a})$. Now we choose $\mathfrak{a} \in \mathcal{I}$ in such a way that $r(\mathfrak{a}) \leq r(\mathfrak{b})$ for every $\mathfrak{b} \in \mathcal{I}$. We define $r := r(\mathfrak{a})$, and we choose $y \in \overline{\mathfrak{a}} \setminus \widetilde{\mathfrak{a}}$ such that y admits a monomial representation $y = e_1m_1 + \cdots + e_rm_r$ having r terms. By Prop. 4 we have $r \geq 2$. By (5) in Notation 4 there exist $p \in \mathbb{N}$ and $a_i \in \widetilde{\mathfrak{a}^i}$ for $i \in \{1, \ldots, p\}$ with

$$y^p + a_1y^{p-1} + \cdots + a_p = 0.$$

(2) Let $\prec$ be a degree-compatible monomial ordering on M. Without loss of generality we may assume that in the monomial representation of y we have $m_1 \prec m_2 \prec \cdots \prec m_r$, hence that $\operatorname{lm}(y) = m_1$, and that $\deg(m_1) \leq \deg(m_2) \leq \cdots \leq \deg(m_r)$. We choose $t \in \{1, \ldots, r\}$ with $\deg(m_1) = \deg(m_2) = \cdots = \deg(m_t) < \deg(m_{t+1})$, and we define $y_1 := e_1m_1 + \cdots + e_tm_t$; then we have $\operatorname{In}(y) = \operatorname{In}(y_1)$.

(3) Let

$$\mathcal{S} := \{\mathbf{b} = (b_1, \ldots, b_p) \mid b_i \in \widetilde{\mathfrak{a}^i} \text{ for } i \in \{1, \ldots, p\},\ y^p + b_1y^{p-1} + \cdots + b_p = 0\}.$$

The set $\mathcal{S}$ is not empty [cf. (1)]; we define for $\mathbf{b} \in \mathcal{S}$

$$\rho(\mathbf{b}) := \min\nolimits_{\prec}\{\operatorname{lm}(b_iy^{p-i}) \mid i \in \{1, \ldots, p\},\ b_i \neq 0\} \in M,$$

$$s(\mathbf{b}) := \min\{i \in \{1, \ldots, p\} \mid b_i \neq 0,\ \operatorname{lm}(b_iy^{p-i}) = \rho(\mathbf{b})\} \in \{1, \ldots, p\}.$$

(4) There exists $\mathbf{b} \in \mathcal{S}$ with

$$\rho(\mathbf{b}) \succcurlyeq \operatorname{lm}(y^p).$$

Proof: Let us suppose, on the contrary, that

$$\rho(\mathbf{b}) \prec \operatorname{lm}(y^p) \quad \text{for every } \mathbf{b} \in \mathcal{S}.$$

This implies that $s(\mathbf{b}) \leq p-1$ for every $\mathbf{b} \in \mathcal{S}$. The set $\{\rho(\mathbf{b}) \mid \mathbf{b} \in \mathcal{S}\}$ is bounded above, hence finite; we define

$$\rho := \max_{\prec}\{\rho(\mathbf{b}) \mid \mathbf{b} \in \mathcal{S}\} \in M.$$

Furthermore, we define

$$\mathcal{S}' := \{\mathbf{b} \in \mathcal{S} \mid \rho(\mathbf{b}) = \rho\}.$$

We choose $\mathbf{b}' = (b'_1, \dots, b'_p) \in \mathcal{S}'$ in such a way that $s(\mathbf{b}) \leq s(\mathbf{b}')$ for every $\mathbf{b} \in \mathcal{S}'$, and we define $s := s(\mathbf{b}')$; note that $1 \leq s \leq p-1$.

Let $i \in \{1, \dots, p\}$ with $b'_i \neq 0$. We consider a monomial representation

$$b'_i = e_{i1}m_{i1} + \dots + e_{i,r_i}m_{i,r_i}.$$

Since $\widetilde{\mathfrak{a}^i}$ is a monomial ideal, we have $m_{i1}, \dots, m_{i,r_i} \in \widetilde{\mathfrak{a}^i}$ [cf. Prop. 3]. Without loss of generality we may assume that $m_{i1} \prec m_{i2} \prec \dots \prec m_{i,r_i}$. We choose $t_i \in \{1, \dots, r_i\}$ with $\deg(m_{i1}) = \dots = \deg(m_{i,t_i}) < \deg(m_{i,t_i+1})$, and we define $b''_i := e_{i1}m_{i1} + \dots + e_{i,t_i}m_{i,t_i}$; then we have $\operatorname{In}(b'_i) = \operatorname{In}(b''_i)$ in $\operatorname{gr}(R)$.

For $i \in \{1, \dots, p\}$ we define

$$d_i := \begin{cases} 0 & \text{if } b'_i = 0 \text{ or if } b'_i \neq 0 \text{ and } \operatorname{lm}(b'_i y^{p-i}) \succ \rho, \\ b''_i & \text{if } b'_i \neq 0 \text{ and } \operatorname{lm}(b'_i y^{p-i}) = \rho. \end{cases}$$

Then we have $d_i \in \widetilde{\mathfrak{a}^i}$ for every $i \in \{1, \dots, p\}$.

We consider the equation

$$y^p + b'_1 y^{p-1} + \dots + b'_p = 0. \qquad (*)$$

For $i \in \{1, \dots, p\}$ we replace b'_i by d_i, and we replace y by y_1; using the inequality $\rho \prec \operatorname{lm}(y^p)$, we obtain the following equation in $\operatorname{gr}(R)$

$$\operatorname{In}(d_s)\operatorname{In}(y_1^{p-s}) + \operatorname{In}(d_{s+1})\operatorname{In}(y_1^{p-s-1}) + \dots + \operatorname{In}(d_p) = 0. \qquad (**)$$

We multiply $(**)$ with $\operatorname{In}(d_s^{p-s-1})$, and we obtain

$$(\operatorname{In}(d_s y_1))^{p-s} + \operatorname{In}(d_{s+1})(\operatorname{In}(d_s y_1))^{p-s-1} + \operatorname{In}(d_{s+2}d_s)(\operatorname{In}(d_s y_1))^{p-s-2} + \\ \dots + \operatorname{In}(d_p d_s^{p-s-1}) = 0.$$

We have

$$d_{s+l}\, d_s^{l-1} \in \widetilde{\mathfrak{a}^{s+l}}(\widetilde{\mathfrak{a}^s})^{l-1} \subset \widetilde{\mathfrak{a}^{(s+1)l}} \quad \text{for } l \in \{1, \dots, p-s\}.$$

Therefore we have $\operatorname{In}(d_{s+l}d_s^{l-1}) \in \operatorname{gr}(\widetilde{\mathfrak{a}^{(s+1)l}}) = (\operatorname{gr}(\widetilde{\mathfrak{a}^{s+1}})^l)$ [cf. Cor. 6(2) and (4) in Remark 4] for $l \in \{1, \dots, p-s\}$, hence $\operatorname{In}(d_s y_1)$ is integral over $(\operatorname{gr}(\mathfrak{a}))^{s+1}$ [cf. (5) in Notation 4], $\operatorname{In}(m_{s1}m_1)$ is integral over $(\operatorname{gr}(\mathfrak{a}))^{s+1}$, also [cf. Cor. 9], and therefore $e_{s1}e_1m_{s1}m_1$ is an element of $\widetilde{\mathfrak{a}^{s+1}}$. We multiply $(*)$ with $(e_{s1}m_{s1})^p$ and we obtain

$$(e_{s1}m_{s1}y)^p + b_1' e_{s1}m_{s1}(e_{s1}m_{s1}y)^{p-1} + \cdots + b_p'(e_{s1}m_{s1})^p = 0.$$

Note that

$$b_l'(e_{s1}m_{s1})^l \in \widetilde{\mathfrak{a}}^l(\widetilde{\mathfrak{a}^s})^l \subset \widetilde{(\mathfrak{a}^{s+1})^l} \quad \text{for } l \in \{1, \dots, p\},$$

and therefore $e_{s1}m_{s1}y$ is integral over $\mathfrak{a}^{s+1}$ [cf. (5) in Notation 4]. Let $y' := y - e_1m_1$; then $e_{s1}m_{s1}y'$ is integral over $\mathfrak{a}^{s+1}$, and $e_{s1}m_{s1}y' = \sum_{i=2}^r e_i e_{s1} m_{s1} m_i$ admits a monomial representation having only $r-1$ terms.

We have $e_{s1}m_{s1}y' \in \widetilde{\mathfrak{a}^{s+1}}$ [this is clear if $\overline{\mathfrak{a}^{s+1}} = \widetilde{\mathfrak{a}^{s+1}}$, and if $\overline{\mathfrak{a}^{s+1}} \supsetneqq \widetilde{\mathfrak{a}^{s+1}}$, then $\mathfrak{a}^{s+1}$ lies in $\mathcal{I}$, and by the choice of r [cf. (1)] we get $e_{s1}m_{s1}y' \in \widetilde{\mathfrak{a}^{s+1}}$ in this case, also]. Since $e_{s1}m_{s1}y'$ and $e_1e_{s1}m_1m_{s1}$ lie in $\widetilde{\mathfrak{a}^{s+1}}$, the element $e_{s1}m_{s1}y$ lies in $\widetilde{\mathfrak{a}^{s+1}}$, also.

We define [note that $s \le p-1$]

$$\widetilde{b}_i := \begin{cases} b_i' & \text{if } i \ne s, s+1, \\ b_s' - e_{s1}m_{s1} & \text{if } i = s, \\ b_{s+1}' + e_{s1}m_{s1}y & \text{if } i = s+1 \end{cases} \qquad \text{for } i \in \{1, \dots, p\}.$$

We have $\mathbf{b}' \in \mathcal{S}$, $e_{s1}m_{s1} \in \widetilde{\mathfrak{a}^s}$ and $e_{s1}m_{s1}y \in \widetilde{\mathfrak{a}^{s+1}}$, hence we have $\widetilde{b}_i \in \widetilde{\mathfrak{a}^i}$ for $i \in \{1, \dots, p\}$. Clearly we have

$$y^p + \widetilde{b}_1 y^{p-1} + \cdots + \widetilde{b}_p = 0,$$

and therefore $\widetilde{\mathbf{b}} := (\widetilde{b}_1, \dots, \widetilde{b}_p)$ lies in $\mathcal{S}$, and this implies that $\rho(\widetilde{\mathbf{b}}) \preccurlyeq \rho$ by the choice of ρ. We show that $\widetilde{\mathbf{b}}$ even lies in $\mathcal{S}'$.

We have $\widetilde{b}_s = 0$ or $\widetilde{b}_s = e_{s2}m_{s2} + \cdots + e_{s,r_s}m_{s,r_s}$ and $\operatorname{lm}(\widetilde{b}_s) = m_{s2} \succ m_{s1} = \operatorname{lm}(b_s') = \rho$. We have $\operatorname{lm}(e_{s1}m_{s1}y^{p-s}) = \rho$, and if $b_{s+1}' \ne 0$, then we have $\operatorname{lm}(b_{s+1}'y^{p-s-1}) \succcurlyeq \rho$. Therefore we have $\operatorname{lm}(\widetilde{b}_{s+1}y^{p-s-1}) \succcurlyeq \rho$, and since $\rho(\mathbf{b}') = \rho$, we obtain $\rho(\widetilde{\mathbf{b}}) \succcurlyeq \rho$. This implies that $\rho(\widetilde{\mathbf{b}}) = \rho$, hence we get, in fact, that $\widetilde{\mathbf{b}} \in \mathcal{S}'$.

Now we have $\widetilde{b}_s = 0$ or $\operatorname{lm}(\widetilde{b}_s) \succ \rho$ and $\widetilde{b}_i = b_i'$ for $i \in \{1, \dots, s-1\}$, and this implies $s(\widetilde{\mathbf{b}}) > s(\mathbf{b}') = s$, in contradiction with the choice of $\mathbf{b}'$.

(5) By (4) there exists $\mathbf{b} \in \mathcal{S}$ with $\operatorname{lm}(b_i y^{p-i}) \succcurlyeq \operatorname{lm}(y^p)$ for every $i \in \{1, \dots, p\}$ with $b_i \neq 0$.

Let $i \in \{1, \dots, p\}$ with $b_i \neq 0$, and let $b_i = e_{i1}m_{i1} + \cdots + e_{i,r_i}m_{i,r_i} \in \widetilde{\mathfrak{a}^i}$ be a monomial representation of b_i; without loss of generality we may assume that $m_{i1} \prec m_{i2} \prec \cdots \prec m_{i,r_i}$, which implies that $m_{i1} = \operatorname{lm}(b_i)$. We choose $t_i \in \{1, \dots, r_i\}$ with $\deg(m_{i1}) = \cdots = \deg(m_{i,t_i}) < \deg(m_{i,t_i+1})$, and we define

$$b'_i := e_{i1}m_{i1} + \cdots + e_{i,t_i}m_{i,t_i};$$

note that $\operatorname{In}(b_i) = \operatorname{In}(b'_i)$. We have $m_{ij} \in \widetilde{\mathfrak{a}^i}$ for $j \in \{1, \dots, r_i\}$ [cf. Prop. 3], hence, in particular, $b'_i \in \widetilde{\mathfrak{a}^i}$.

Now let $i \in \{1, \dots, p\}$; we define

$$c_i := \begin{cases} 0 & \text{if } b_i = 0 \text{ or if } b_i \neq 0 \text{ and } \operatorname{lm}(b_i y^{p-i}) \succ \operatorname{lm}(y^p), \\ b'_i & \text{if } b_i \neq 0 \text{ and } \operatorname{lm}(b_i y^{p-i}) = \operatorname{lm}(y^p). \end{cases}$$

Clearly we have $c_i \in \widetilde{\mathfrak{a}^i}$. From $y^p + b_1 y^{p-1} + \cdots + b_p = 0$ we obtain the following equation in $\operatorname{gr}(R)$

$$\operatorname{In}(y_1)^p + \operatorname{In}(c_1)\operatorname{In}(y_1)^{p-1} + \cdots + \operatorname{In}(c_p) = 0.$$

Now we have $\operatorname{In}(c_i) \in \operatorname{gr}(\widetilde{\mathfrak{a}^i})$ for every $i \in \{1, \dots, p\}$. Just as in (4) we see that $\operatorname{In}(y_1)$ is integral over $\operatorname{gr}(\mathfrak{a})$ and that therefore $\operatorname{In}(m_1)$ is integral over $\operatorname{gr}(\mathfrak{a})$, hence we have $m_1 \in \widetilde{\mathfrak{a}}$, hence $e_1 m_1 \in \widetilde{\mathfrak{a}}$. Now $y' := y - e_1 m_1$ lies in $\overline{\mathfrak{a}}$, and therefore y' lies in $\widetilde{\mathfrak{a}}$ by the choice of r. From this we get that $y = y' + e_1 m_1$ lies in $\widetilde{\mathfrak{a}}$, in contradiction with the choice of y. ■

Theorem 1 *Let R be a noetherian ring, let $\mathbf{x} = (x_1, \dots, x_d)$ be a regular sequence in R, and assume that $\mathfrak{q} := \mathbf{x}R$ is contained in the Jacobson radical of R and that $R/\mathfrak{q}$ is a reduced ring. For every monomial ideal $\mathfrak{a}$ of R we have $\overline{\mathfrak{a}} = \widetilde{\mathfrak{a}}$; in particular, $\overline{\mathfrak{a}}$ is a monomial ideal, also.*

Proof: (1) Firstly, let $\mathfrak{q}$ be a prime ideal. Let $y \in \overline{\mathfrak{a}}$. We have $\overline{\mathfrak{a}}\hat{R} \subset \overline{(\mathfrak{a}\hat{R})}$ and $\widetilde{\mathfrak{a}\hat{R}} = \widetilde{\mathfrak{a}}\hat{R}$ [cf. Cor. 6], hence $y \in \overline{\mathfrak{a}\hat{R}} = \widetilde{\mathfrak{a}\hat{R}} = \widetilde{\mathfrak{a}}\hat{R}$ [cf. Prop. 7], and since $\widetilde{\mathfrak{a}}\hat{R} \cap R = \widetilde{\mathfrak{a}}$ we obtain $y \in \widetilde{\mathfrak{a}}$. Thus, we have shown that $\overline{\mathfrak{a}} = \widetilde{\mathfrak{a}}$.

(2) Now we consider the case that $R/\mathfrak{q}$ is reduced.

(a) Let $\mathfrak{p} \in \operatorname{Ass}(R/\mathfrak{q})$. Then $\mathfrak{q}R_\mathfrak{p}$ is the maximal ideal of $R_\mathfrak{p}$, hence we have $\widetilde{\mathfrak{a}R_\mathfrak{p}} = \overline{\mathfrak{a}R_\mathfrak{p}}$ by (1). Obviously we have $\widetilde{\mathfrak{a}R_\mathfrak{p}} = \widetilde{\mathfrak{a}}R_\mathfrak{p}$ and $\overline{\mathfrak{a}}R_\mathfrak{p} \subset \overline{\mathfrak{a}R_\mathfrak{p}}$. Therefore we have $\overline{\mathfrak{a}}R_\mathfrak{p} \subset \widetilde{\mathfrak{a}}R_\mathfrak{p}$.

(b) For every $\mathfrak{p} \in \operatorname{Ass}(R/\mathfrak{q})$ there exists, by (a), an element $s_\mathfrak{p} \in R \setminus \mathfrak{p}$ with $\overline{\mathfrak{a}} \subset \widetilde{\mathfrak{a}} : s_\mathfrak{p}$. Let $\mathfrak{b}$ be the ideal generated by the elements $s_\mathfrak{p}$; then we have $\overline{\mathfrak{a}} \subset \widetilde{\mathfrak{a}} : \mathfrak{b}$. Let $\mathfrak{p}' \in \mathrm{Ass}(R/\widetilde{\mathfrak{a}})$. Since $\widetilde{\mathfrak{a}}$ is a monomial ideal, there exists $U \subset \{1, \dots, d\}$ with $\mathfrak{p}' \in \mathrm{Ass}(R/\mathfrak{q}_U)$ [cf. Lemma 1]. Repeated application of Lemma 1 in [13], vol. II, Appendix 6, shows that there exists a prime ideal $\mathfrak{p} \in \mathrm{Ass}(R/\mathfrak{q})$ with $\mathfrak{p}' \subset \mathfrak{p}$. Therefore $\mathfrak{b}$ is not contained in any prime ideal in $\mathrm{Ass}(R/\widetilde{\mathfrak{a}})$, hence $\widetilde{\mathfrak{a}} : \mathfrak{b} = \widetilde{\mathfrak{a}}$, hence $\overline{\mathfrak{a}} \subset \widetilde{\mathfrak{a}}$. The inclusion $\widetilde{\mathfrak{a}} \subset \overline{\mathfrak{a}}$ was noticed in (1) of Notation 4, and therefore we have $\overline{\mathfrak{a}} = \widetilde{\mathfrak{a}}$. ■

Example 1 Let R be a regular local two-dimensional ring, and let $\{x, y\}$ be a regular system of parameters of R. Let $m > n > 1$ be coprime integers, and write $m = s_1 n + n_1$ with $1 \le n_1 < n$. Let $\mathfrak{a}$ be the ideal of R generated by x^m and y^n. Then $\mathfrak{a}$ is a monomial ideal. It can be shown [cf. [7]] that the integral closure $\wp$ of $\mathfrak{a}$ has a minimal system of generators $\{x^{m-\sigma_{m,n}(i)} y^i \mid i \in \{0, \dots, n\}\}$ where $\sigma_{m,n} \colon \{0, \dots, n\} \to \{0, \dots, m\}$ is a strictly increasing function; in particular, one has

$$\sigma_{m,n}(0) = 0, \sigma_{m,n}(1) = s_1, \sigma_{m,n}(n-1) = m - (s_1 + 1), \sigma_{m,n}(n) = m,$$

and

$$\sigma_{m,n}(i+j) \ge \sigma_{m,n}(i) + \sigma_{m,n}(j) \quad \text{for } i, j \in \{0, \dots, n\} \text{ with } i + j \le n.$$

Moreover, the polar ideal $\mathfrak{P}_\wp$ of $\wp$ has

$$\{x^{m-\sigma_{m,n}(i+1)} y^i \mid i \in \{0, \dots, n-1\}\}$$

as minimal set of generators.

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