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Slopes of hypergeometric systems of codimension one

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Abstract

We describe the slopes, with respect to the coordinates hyperplanes, of the hypergeometric systems of codimension one, that is when the toric ideal is generated by one element.

1. Introduction

The $\mathcal{D}$-module theory generalizes the concepts in the classic theory of ordinary differential equations with holomorphic coefficients for a complex variable x.

We consider the Weyl algebra:

$$\mathcal{A}_n = \mathbb{C}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \rangle$$

that is, the ring of differential operators with polynomial coefficients in n variables. This ring is not commutative, and the above elements verify the relations: $[x_i, x_j] = 0$, $[\partial_i, \partial_j] = 0$, and $[\partial_i, x_j] = \delta_{i,j}$. We can also consider $\mathcal{D}_n$:

$$\mathcal{D}_n = \mathbb{C}\{x_1, \ldots, x_n\}\langle \partial_1, \ldots, \partial_n \rangle,$$

with the same relations between the generators. We denote by $\langle M \rangle$ the left ideal generated by the set M.

If we take an element in $\mathcal{A}_1$:

$$P = a_m(x)\partial^m + a_{m-1}(x)\partial^{m-1} + \cdots + a_0(x), \quad \text{with} \quad a_m(x) \neq 0$$

it defines an ordinary differential equation.

Fuch's condition [6] states that the point $x = 0$ is a regular singular point if and only if we have the equality: $m - \mathrm{val}(a_m(x)) = \max_{j=0,\ldots,m}\{j - \mathrm{val}(a_j(x))\}$, where $\mathrm{val}(a_j(x))$ is the order of $a_j(x)$ at $x = 0$.

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We can define a combinatorial object related to P, the Newton polygon, defined as:

$$\mathcal{N}(P) = \text{convex hull}\Bigg(\bigcup_{j=0}^{m}(j, j - val(a_j(x)) + (-\mathbb{N})^2\Bigg).$$

Clearly P has a regular singular point in $x = 0$ if and only if $\mathcal{N}(P)$ is a quadrant. If not, we have a slope in $\mathcal{N}(P)$. The equation is regular if and only if all the singular points are regular, i.e. if there are no slopes.

The generalization of irregularity in several variables is given by the irregularity sheaf with respect to a hypersurface, which was introduced by Mebhkout (see [10]). We also have the concept of slope of a $\mathcal{D}$-module with respect to a hypersurface introduced by Laurent [7], which generalizes the analogous in one variable. Indeed the slopes of a module describe the jumps in the Gevrey filtration of that sheaf [8]. We have that a $\mathcal{D}$-module in several variables is regular if and only if it has no slopes for all the hypersurfaces ([8]). In the next section we describe briefly the notion of slopes of a $\mathcal{D}$-module with respect to a smooth hypersurface.

The cases we study are the $\mathcal{D}$-modules arising from the so-called hypergeometric systems ([4] and [11]). They are defined from integer matrices of maximal rank. Given $A = (a_{ij})$ an $d \times n$ integer matrix with rank d, we can define the toric ideal $I_A \subset \mathbb{C}[\partial]$ as the ideal generated by $\{\partial^u - \partial^v | u, v \in \mathbb{N}^n, Au^t = Av^t\}$. We take $\beta = (\beta_1, \dots, \beta_d) \in \mathbb{C}^d$ and we denote $\theta = (\theta_1, \dots, \theta_n)$, where $\theta_i = x_i\partial_i$ is an operator of the Weyl algebra. We denote by $A\theta^t - \beta$ the ideal generated by the operators $\sum_{i=1}^n a_{ij}\theta_i - \beta_j$, $j = 1, \dots, d$. Finally, we can define the hypergeometric system of Gelfand'-Kapranov-Zelevinski as the system defined by the ideal $H_A(\beta) = \langle I_A, A\theta^t - \beta\rangle$. The $\mathcal{A}_n$-module $\mathcal{H}_A(\beta) = \mathcal{A}_n/H_A(\beta)$ is a holonomic module ([4] and [1]).

Given two integer matrices A, A', such that there exists $G \in GL_d(\mathbb{Q})$ with $A' = GA$, we have $I_A = I_{A'}$ and $H_A(\beta) = H_{A'}(G\beta)$. So, to study the slopes of a hypergeometric system defined by an integer matrix we can always consider the row-reduced form matrix.

A result by Hotta [5] tells that a hypergeometric system is regular if the toric ideal is homogeneous with respect to the usual grading or, equivalently, if $(1, \dots, 1)$ is in the $\mathbb{Q}$-span of the rows of A. The aim of our work is calculate the slopes for an irregular hypergeometric system of codimension one. In first place, we consider the case when the semigroup is reduced. Hence, we shall see that there always exist slopes with respect to the coordinates hyperplanes $x_i = 0$ and they can be detected from a generator of I_A. Secondly, we treat the non-reduced semigroup case, and we prove that there are no slopes with respect to any coordinate plane at zero. Finally, we look for slopes at infinity, and we find that the system has slopes with respect to a coordinate hyperplane.

2. Slopes

There exists an algorithm ([2]) for the explicit calculation of slopes of a $\mathcal{D}$-module with respect to a smooth hypersurface. In this section we define what we call a slope with respect to $x_j = 0$ at the origin.

The ring $\mathcal{D} = \mathcal{D}_n$ admits several filtrations. First, we consider the filtration defined by the order of the differential operators. We denote this filtration by $F_k(\mathcal{D})$. Given a non-zero operator

$$P = \sum_{\beta} a_\beta(x)\partial^\beta$$

we consider

$$\mathrm{ord}_F(P) = \max\{|\beta|, a_\beta \neq 0\}, \quad F_k(\mathcal{D}) = \{P \in \mathcal{D}, \text{ such that} \, \mathrm{ord}_F(P) \leq k\}.$$

This defines a filtration in $\mathcal{D}$. In its associated graded ring, which is a polynomial ring in ξ, we consider the F-symbol:

$$\sigma_F(P) = \sum_{\beta,\, |\beta| = \mathrm{ord}_F(P)} a_\beta(x)\xi^\beta.$$

Given an ideal I in $\mathcal{D}$ we define the graded ideal:

$$\mathrm{gr}^F(I) = \langle \sigma_F(P),\, P \in I\rangle.$$

We can also consider the Malgrange-Kashiwara filtration with respect to $x_j = 0$, denoted $V_k(\mathcal{D})$. Given

$$P = \sum_{\alpha,\beta} a_{\alpha,\beta} x^\alpha \partial^\beta$$

we consider

$$\begin{aligned} \mathrm{ord}_V(P) &= \max\{\beta_j - \alpha_j,\, a_{\alpha,\beta} \neq 0\}, \\ V_k(\mathcal{D}) &= \{P \in \mathcal{D}, \text{ such that} \, \mathrm{ord}_V(P) \leq k\}. \end{aligned}$$

In its associated graded ring, which is isomorphic to a non commutative subring of $\mathcal{D}$, we consider the V-symbol:

$$\sigma_V(P) = \sum_{(\alpha,\beta),\, \beta_j - \alpha_j = \mathrm{ord}_V(P)} a_{\alpha,\beta} x^\alpha \xi^\beta.$$

Given an ideal I in $\mathcal{D}$ we define the graded ideal:

$$\mathrm{gr}^V(I) = \langle \sigma_V(P),\, P \in I\rangle.$$

Using the above filtrations we can define an ordered family of filtrations. Given $(p,q) \neq (0,0)$, non negative integers, we define the linear form over $\mathbb{Q}^2$, given by $L(a,b) = pa + qb$. Then, given an operator as before, we define

$$\mathrm{ord}_L(P) = \max\{L(|\beta|, \beta_j - \alpha_j),\, a_{\alpha,\beta} \neq 0\},$$

$$L_k(\mathcal{D}) = \{P \in \mathcal{D}, \text{ such that} \, \mathrm{ord}_L(P) \leq k\}.$$

In its associated graded ring, that is a polynomial ring in ξ if $L \neq V$, we consider the L-symbol:

$$\sigma_L(P) = \sum_{(\alpha,\beta),\, L(|\beta|,\beta_j-\alpha_j)=\mathrm{ord}_L(P)} a_{\alpha,\beta} x^\alpha \xi^\beta.$$

Given an ideal I in $\mathcal{D}$ we define the graded ideal:

$$\mathrm{gr}^L(I) = \langle \sigma_L(P),\, P \in I \rangle.$$

The L-filtration also describes the F and V filtrations. Indeed we can order these filtrations. Given L, L' defined by pairs (p,q) and (p',q'), we say that $L < L'$ if and only if $-p/q < -p'/q'$. Given a L-filtration we define its slope as the ratio $-p/q$.

If $L \neq V$ we can define de L-characteristic variety, noted $Ch^L(I)$, as the analytic variety in $\mathbb{C}^{2n}$ defined by the graded ideal $\mathrm{gr}^L(I)$.

Definition 1 *[7] Let I be an ideal of $\mathcal{D}$. The slopes of the $\mathcal{D}$-module $\mathcal{D}/I$ with respect to $x_j = 0$, are the slopes of the linear forms $L \neq F, V$ such that $\sqrt{\mathrm{gr}^L(I)}$ is not bihomogeneous for the F and V filtrations.*

Remark 1 *In the case when we have a holonomic $\mathcal{D}$-module $\mathcal{D}/I$, if we find L a filtration with respect to $x_j = 0$ such that $(x_1\xi_1, \ldots, x_n\xi_n) \subset \sqrt{\mathrm{gr}^L(I)}$, then L is not a slope with respect to $x_j = 0$ of this module, because all the components of the L-characteristic variety are bihomogeneous with respect to F and V.*

3. Slopes of hypergeometric systems of codimension one

Let A be a $n \times (n+1)$ integer matrix of rank n and let $\beta \in \mathbb{C}^n$. We add by now one condition: all the $n \times n$ minors of A are not zero. Using this fact we can ensure that a generator u of the kernel of A has all its coordinates different from zero. We denote $P \in \mathbb{C}[\partial]$ the generator of the toric ideal.

Our aim is to calculate the slopes of the $\mathcal{D}$-module $\mathcal{H}_A(\beta)$, so we assume that it is irregular. Then, P is not homogeneous with respect to the usual grading. We have that $P = \partial^{u^+} - \partial^{u^-}$, where u^+ are the positive coordinates of u and $-u^-$ are the negative ones.

We assume that $|u^+| \neq 0$ and $|u^-| \neq 0$. Without loss of generality $|u^+| > |u^-|$ and we order the variables to have the first k coordinates those different from zero in u^+.

Lemma 2 *Let A be an $n \times (n+1)$ integer matrix with all its $n \times n$ minors different from zero. We consider the variables ordered as before. $\mathcal{H}_A(\beta)$ has no slopes with respect to the k first variables.*

Proof. We shall prove that $\mathcal{H}_A(\beta)$ has no slopes with respect to the hyperplane $x_1 = 0$.

Using the condition about the minors we can, taking into account a row reduced form of A, obtain the operators in $H_A(\beta)$:

$$Q_1 = a_1\theta_2 + b_1\theta_1 - \beta_1',\ Q_2 = a_2\theta_3 + b_2\theta_1 - \beta_2', \ldots,\ Q_n = a_n\theta_{n+1} + b_n\theta_1 - \beta_n',$$

with $a_i, b_i \neq 0$, for all i and these elements generate $A\theta^t - \beta$. Using the remark 1, if we want to prove that a given L is not slope it is sufficient to prove that $x_1\xi_1 \in \sqrt{\mathrm{gr}^L(H_A(\beta))}$. Hence we need an operator $H \in H_A(\beta)$ such that $\sigma_L(H) = x_1^{k_1}\xi_1^{k_2}$.

Let L be any slope with respect to the hyperplane $x_1 = 0$. We have the following sequence of elements S_i in $H_A(\beta)$, with $\mathrm{ord}_L(R_i) < \mathrm{ord}_L(P) = \mathrm{ord}_L(S_i)$:

$$\begin{aligned}
S_1 &= \partial_1^{u_1}\partial_2^{u_2-1}\partial_3^{u_3}\cdots\partial_k^{u_k}Q_1 - a_1x_2P\\
&= b_1\theta_1\partial_1^{u_1}\partial_2^{u_2-1}\partial_3^{u_3}\cdots\partial_k^{u_k} + R_1.\\
S_2 &= b_1\theta_1\partial_1^{u_1}\partial_2^{u_2-2}\partial_3^{u_3}\cdots\partial_k^{u_k}Q_1 - a_1x_2S_1\\
&= b_1^2\theta_1^2\partial_1^{u_1}\partial_2^{u_2-2}\partial_3^{u_3}\cdots\partial_k^{u_k} + R_2.\\
&\ \vdots\\
S_{u_2} &= b_1^{u_2-1}\theta_1^{u_2-1}\partial_1^{u_1}\partial_3^{u_3}\cdots\partial_k^{u_k}Q_1 - a_1x_2S_{u_2-1}\\
&= b_1^{u_2}\theta_1^{u_2}\partial_1^{u_1}\partial_3^{u_3}\cdots\partial_k^{u_k} + R_{u_2}.\\
S_{u_2+1} &= b_1^{u_2}\theta_1^{u_2}\partial_1^{u_1}\partial_3^{u_3-1}\cdots\partial_k^{u_k}Q_2 - a_2x_3S_{u_2}\\
&= b_1^{u_2}b_2\theta_1^{u_2+1}\partial_1^{u_1}\partial_3^{u_3-1}\cdots\partial_k^{u_k} + R_{u_2+1}.\\
&\ \vdots\\
S_{u_2+u_3} &= b_1^{u_2}b_2^{u_3-1}\theta_1^{u_2+u_3-1}\partial_1^{u_1}\partial_4^{u_4}\cdots\partial_k^{u_k}Q_2 - a_2x_3S_{u_2+u_3-1}\\
&= b_1^{u_2}b_2^{u_3}\theta_1^{u_2+u_3}\partial_1^{u_1}\partial_4^{u_4}\cdots\partial_k^{u_k} + R_{u_2+u_3}.\\
&\ \vdots\\
S_{u_2+\cdots+u_k} &= b_1^{u_2}\cdots b_{k-1}^{u_k-1}\theta_1^{u_2+\cdots+u_k-1}\partial_1^{u_1}Q_{k-1} - a_{k-1}x_kS_{u_2+\cdots+u_k-1}\\
&= b_1^{u_2}\cdots b_{k-1}^{u_k}\theta_1^{u_2+\cdots+u_k}\partial_1^{u_1} + R_{u_2+\cdots+u_k}.
\end{aligned}$$

This finishes the proof. ∎

Theorem 3 *Let A be an $n \times (n+1)$ integer matrix with all its $n \times n$ minors different from zero. We consider the variables ordered as before. The only slope of $\mathcal{H}_A(\beta)$ with respect to $x_j = 0$, with $j > k$, is $L_j = u_j F + (|u^+| - |u^-|)V$.*

Proof. First, we prove that if we have a slope L with respect to $x_j = 0$, $L \neq L_j$, then L is not slope of $\mathcal{H}_A(\beta)$.

Let $L' < L_j$ be any slope with respect to $x_j = 0$. Then $\sigma_{L'}(P) = \xi_1^{u_1} \cdots \xi_k^{u_k}$, and following the proof of lemma 2 we have that $\mathrm{ord}_{L'}(R_i) < \mathrm{ord}_{L'}(P) = \mathrm{ord}_{L'}(S_i)$ and

$$\sigma_{L'}(S_{u_2+\cdots+u_k}) = c x_1^{u_2+\cdots+u_k} \xi_1^{u_1+\cdots+u_k}.$$

Then L' is not a slope of $\mathcal{H}_A(\beta)$ with respect to $x_j = 0$.

Let $L'' > L_j$ be any slope with respect to $x_j = 0$, then $\sigma_{L''}(P) = -\xi_{k+1}^{u_{k+1}} \cdots \xi_{n+1}^{u_{n+1}}$. As before, we can obtain operators in $H_A(\beta)$:

$$\begin{aligned}
&Q'_1 = c_1\theta_1 + d_1\theta_j - \beta''_1,\ Q'_2 = c_2\theta_2 + d_2\theta_j - \beta''_2, \ldots,\\
&\ldots\ Q'_{j-1} = c_{j-1}\theta_{j-1} + d_{j-1}\theta_j - \beta''_{j-1},\ Q'_j = c_j\theta_{j+1} + d_j\theta_j - \beta''_j,\ \ldots,\\
&\ldots\ Q'_n = c_n\theta_{n+1} + d_n\theta_j - \beta''_n.
\end{aligned}$$

with $c_i, d_i \neq 0$, and we can consider the following sequence of operators S'_i in $H_A(\beta)$, such that $\mathrm{ord}_{L''}(R'_i) < \mathrm{ord}_{L''}(P) = \mathrm{ord}_{L''}(S'_i)$:

$$\begin{aligned}
S'_1 &= \partial_{k+1}^{u_{k+1}-1}\partial_{k+2}^{u_{k+2}} \cdots \partial_{n+1}^{u_{n+1}} Q'_{k+1} + c_{k+1}x_{k+1}P\\
&= d_{k+1}\theta_j \partial_{k+1}^{u_{k+1}-1}\partial_{k+2}^{u_{k+2}} \cdots \partial_{n+1}^{u_{n+1}} + R'_1.\\
S'_2 &= d_{k+1}\theta_j \partial_{k+1}^{u_{k+1}-2}\partial_{k+2}^{u_{k+2}} \cdots \partial_{n+1}^{u_{n+1}} Q'_{k+1} - c_{k+1}x_{k+1}S'_1\\
&= d_{k+1}^2\theta_j^2 \partial_{k+1}^{u_{k+1}-2}\partial_{k+2}^{u_{k+2}} \cdots \partial_{n+1}^{u_{n+1}} + R'_2.\\
&\ \vdots\\
S'_{u_{k+1}} &= d_{k+1}^{u_{k+1}-1}\theta_j^{u_{k+1}-1}\partial_{k+2}^{u_{k+2}} \cdots \partial_{n+1}^{u_{n+1}} Q'_{k+1} - c_{k+1}x_{k+1}S'_{u_{k+1}-1}\\
&= d_{k+1}^{u_{k+1}}\theta_j^{u_{k+1}}\partial_{k+2}^{u_{k+2}} \cdots \partial_{n+1}^{u_{n+1}} + R'_{u_{k+1}}.\\
&\ \vdots\\
S'_{|u^-|-u_j} &= d_{k+1}^{u_{k+1}} \cdots d_{j-1}^{u_{j-1}} d_j^{u_{j+1}} \cdots d_n^{u_{n+1}-1}\theta_j^{|u^-|-u_j-1}\partial_j^{u_j} Q'_n - c_n x_{n+1} S'_{|u^-|-u_j-1}\\
&= d_{k+1}^{u_{k+1}} \cdots d_{j-1}^{u_{j-1}} d_j^{u_{j+1}} \cdots d_n^{u_{n+1}}\theta_j^{|u^-|-u_j}\partial_j^{u_j} + R'_{|u^-|-u_j}.
\end{aligned}$$

Thus, L'' is not slope of $\mathcal{H}_A(\beta)$ with respect to $x_j = 0$. Then the only possible slope is L_j.

Our next idea is similar to one in the work of Castro-Jiménez and Takayama [3].

Now suppose that L_j is not a slope. Then there is no slope, which implies that the L-characteristic variety $Ch^L(\mathcal{H}_A(\beta))$ is invariant for all $L = pF + qV_j$.

First, we are going to prove that:

$$\mathrm{gr}^F(H_A(\beta)) = \langle \sigma_F(P), \mathrm{gr}^F(A\theta^t - \beta)\rangle. \tag{3.1}$$

The right hand side of the above equality is called in [11] the fake initial ideal, noted $\mathrm{fin}_F(H_A(\beta))$. We consider as in [11] the following exact sequence of modules over the algebra $\mathrm{gr}^F(\mathcal{D})/\mathrm{gr}^F(I_A)$:

$$\bigoplus_{i=1}^{n} \big(\mathrm{gr}^F(\mathcal{D})/\mathrm{gr}^F(I_A)\big)\cdot e_i \xrightarrow{\bar{d_1}} \mathrm{gr}^F(\mathcal{D})/\mathrm{gr}^F(I_A) \longrightarrow \mathrm{gr}^F(\mathcal{D})/\mathrm{fin}_F(H_A(\beta)) \longrightarrow 0$$

where $\bar{d_1}(\sum_{i=1}^n P_i e_i) = \sum_{i=1}^n P_i \sigma_F((A\theta^t - \beta)_i)$.

Since $\sigma_F((A\theta^t-\beta)_j)e_i - \sigma_F((A\theta^t-\beta)_i)e_j$ clearly belongs to the kernel of $\bar{d_1}$, we can extend the exact sequence to the Koszul complex $K_\bullet^\beta(\mathrm{gr}^F(\mathcal{D}/I_A))$:

$$\cdots \xrightarrow{\bar{d_2}} K_1^\beta(\mathrm{gr}^F(\mathcal{D}/I_A)) \xrightarrow{\bar{d_1}} K_0^\beta(\mathrm{gr}^F(\mathcal{D}/I_A)) \longrightarrow 0$$

where

$$K_p^\beta(\mathrm{gr}^F(\mathcal{D}/I_A)) = \bigoplus_{1\le i_1<\cdots<i_p\le n} \mathrm{gr}^F(\mathcal{D}/I_A) e_{i_1\cdots i_p},$$

and

$$\bar{d_p}(e_{i_1\cdots i_p}) = \sum_{r=1}^{p} (-1)^{r-1}\sigma_F((A\theta^t - \beta)_{i_r}) e_{i_1\cdots \widehat{i_r}\cdots i_p}.$$

We can also define the Koszul complex $K_\bullet^\beta(\mathcal{D}/I_A)$ as:

$$\cdots \longrightarrow K_2^\beta(\mathcal{D}/I_A) \xrightarrow{d_2} K_1^\beta(\mathcal{D}/I_A) \xrightarrow{d_1} K_0^\beta(\mathcal{D}/I_A) \longrightarrow 0.$$

where

$$K_p^\beta(\mathcal{D}/I_A) = \bigoplus_{1\le i_1<\cdots<i_p\le n} \mathcal{D}/I_A e_{i_1\cdots i_p}$$

and

$$d_p(e_{i_1\cdots i_p}) = \sum_{r=1}^{p} (-1)^{r-1}((A\theta^t - \beta)_{i_r}) e_{i_1\cdots \widehat{i_r}\cdots i_p}.$$

This complex can be filtered by:

$$F_q(K_p^\beta(\mathcal{D}/I_A)) = \bigoplus_{1\le i_1<\cdots<i_p\le n} F_{q-p}\mathcal{D}/I_A e_{i_1\cdots i_p}.$$

Clearly

$$K_\bullet^\beta(\mathrm{gr}^F(\mathcal{D}/I_A)) = \mathrm{gr}^F(K_\bullet^\beta(\mathcal{D}/I_A)).$$

Using this filtration it is easy to see that the sequence:

$$K_1^\beta(\mathrm{gr}^F(\mathcal{D}/I_A)) \longrightarrow K_0^\beta(\mathrm{gr}^F(\mathcal{D}/I_A)) \longrightarrow \mathrm{gr}^F(\mathcal{H}_A(\beta)) \longrightarrow 0$$

is exact if $H_1(K_\bullet^\beta(\mathrm{gr}^F(\mathcal{D}/I_A))) = 0$.

If this sequence is exact we have proved statement 3.1, so all we need to prove is $H_1(K_\bullet^\beta(\mathrm{gr}^F(\mathcal{D}/I_A))) = 0$, or equivalently that $\{\sigma_F(P), \sigma_F((A\theta^t - \beta)_1), \ldots, \sigma_F((A\theta^t - \beta)_n)\}$ form a regular sequence in the commutative graded ring $\mathrm{gr}^F(\mathcal{D})$.

It is sufficient to prove that $\{\sigma_F(P), \sigma_F((A\theta^t - \beta)_1), \ldots, \sigma_F((A\theta^t - \beta)_n)\}$ form a regular sequence in the commutative graded ring $\mathbb{C}(x)[\xi]$. Those elements are homogeneous so if we have that:

$$\sqrt{\sigma_F(P), \sigma_F((A\theta^t - \beta)_1), \ldots, \sigma_F((A\theta^t - \beta)_n)}$$

is the maximal ideal of the graded ring, then they form a regular sequence. Given A in row-reduced form, as in the proof of lemma 2, it suffices to prove that

$$x_1^{k_1}\xi_1^{k_2} \in \langle \sigma_F(P), \sigma_F((A\theta^t - \beta)_1), \ldots, \sigma_F((A\theta^t - \beta)_n)\rangle.$$

If we consider the elements:

$$Q_1'' = \sigma_F(Q_1) = a_1x_2\xi_2 + b_1x_1\xi_1,\ Q_2'' = \sigma_F(Q_2) = a_2x_3\xi_3 + b_2x_1\xi_1,$$

$$\ldots,\ Q_n'' = \sigma_F(Q_n) = a_nx_n\xi_{n+1} + b_nx_n\xi_1,$$

$$\begin{aligned}
S_1'' &= \xi_1^{u_1}\xi_2^{u_2-1}\xi_3^{u_3}\cdots\xi_k^{u_k}Q_1'' - a_1x_2\sigma_F(P)\\
&= b_1x_1\xi_1^{u_1+1}\xi_2^{u_2-1}\xi_3^{u_3}\cdots\xi_k^{u_k}.\\
S_2'' &= b_1x_1\xi_1^{u_1+1}\xi_2^{u_2-2}\xi_3^{u_3}\cdots\xi_k^{u_k}Q_1'' - a_1x_2S_1''\\
&= b_1^2x_1^2\xi_1^{u_1+2}\xi_2^{u_2-2}\xi_3^{u_3}\cdots\xi_k^{u_k}.\\
&\vdots\\
S_{u_2}'' &= b_1^{u_2-1}x_1^{u_2-1}\xi_1^{u_1+u_2-1}\xi_3^{u_3}\cdots\xi_k^{u_k}Q_1'' - a_1x_2S_{u_2-1}''\\
&= b_1^{u_2}x_1^{u_2}\xi_1^{u_1+u_2}\xi_3^{u_3}\cdots\xi_k^{u_k}.\\
&\vdots\\
S_{u_2+\cdots+u_k}'' &= b_1^{u_2}\cdots b_{k-1}^{u_k-1}x_1^{u_2+\cdots+u_k-1}\xi_1^{|u^+|-1}Q_{k-1}'' - a_{k-1}x_kS_{u_2+\cdots+u_k-1}''\\
&= b_1^{u_2}\cdots b_{k-1}^{u_k}x_1^{u_2+\cdots+u_k}\xi_1^{|u^+|}.
\end{aligned}$$

Then we have that:

$$\mathrm{gr}^F(H_A(\beta)) = \langle \xi_1^{u_1} \cdots \xi_k^{u_k}, a_1x_2\xi_2 + b_1x_1\xi_1, \dots, a_nx_{n+1}\xi_{n+1} + b_nx_1\xi_1 \rangle.$$

It is clear that

$$\mathrm{gr}^F(H_A(\beta)) \subset \langle \xi_1 \cdots \xi_k, x_1\xi_1, \dots, x_{n+1}\xi_{n+1} \rangle,$$

and

$$\langle \xi_1 \cdots \xi_k, x_1\xi_1, \dots, x_{n+1}\xi_{n+1} \rangle \subset \sqrt{\mathrm{gr}^F(H_A(\beta))}.$$

So the characteristic variety is

$$\langle \xi_1 \cdots \xi_k, x_1\xi_1, \dots, x_{n+1}\xi_{n+1} \rangle = \sqrt{\mathrm{gr}^F(H_A(\beta))}.$$

We are supposing that the characteristic variety is invariant, but if we take $L'' > L_j$, then using the operators $S'_1, \dots S'_{|u^-|-u_j}$ we have that:

$$\langle \xi_{k+1} \cdots \xi_{n+1}, x_1\xi_1, \dots, x_{n+1}\xi_{n+1} \rangle \subset \sqrt{\mathrm{gr}^{L''}(H_A(\beta))}$$

and the component of the characteristic variety

$$T^*_{(x_{k+1}=\cdots=x_{n+1}=0)}\mathbb{C}^{n+1}$$

is not contained in any of $Ch^{L''}(\mathcal{H}_A(\beta))$. ■

At the begining of this section we added the condition $|u^+| \neq 0$ and $|u^-| \neq 0$. If we have $|u^-| = 0$ this means that the generator of the toric ideal is of the form $P = \partial^u - 1$. We now study this case.

Lemma 4 *Let A be an $n \times (n+1)$ integer matrix, with all the $n \times n$ minors different from zero and $I_A = \langle \partial^u - 1 \rangle$. Then, $\mathcal{H}_A(\beta)$ has no slopes at the origin with respect to any $x_j = 0$.*

Proof. We shall prove that $\mathcal{H}_A(\beta)$ has no slopes with respect to the hyperplane $x_1 = 0$.

Using the condition about the minors we can, with a row reduced form of A, obtain the following operators in $H_A(\beta)$:

$$Q_1 = a_1\theta_2 + b_1\theta_1 - \beta'_1,\ Q_2 = a_2\theta_3 + b_2\theta_1 - \beta'_2, \dots,\ Q_n = a_n\theta_{n+1} + b_n\theta_1 - \beta'_n,$$

with $a_i, b_i \neq 0$, for all i and these elements generate $A\theta^t - \beta$. Using the remark 1, if we want to prove that a given L is not a slope, it is sufficient to prove that $x_1\xi_1 \in \sqrt{\mathrm{gr}^L(H_A(\beta))}$. Hence we need an operator $H \in H_A(\beta)$ such that $\sigma_L(H) = x_1^{k_1}\xi_1^{k_2}$.

Let L be any slope with respect to the hyperplane $x_1 = 0$. We have the following sequence of elements S_i in $H_A(\beta)$, with $\mathrm{ord}_L(R_i) < \mathrm{ord}_L(P) = \mathrm{ord}_L(S_i)$:

$$\begin{array}{rcl} S_1 & = & \partial_1^{u_1}\partial_2^{u_2-1}\partial_3^{u_3}\cdots\partial_{n+1}^{u_{n+1}}Q_1 - a_1x_2P \\ & = & b_1\theta_1\partial_1^{u_1}\partial_2^{u_2-1}\partial_3^{u_3}\cdots\partial_{n+1}^{u_{n+1}} + R_1. \\ S_2 & = & b_1\theta_1\partial_1^{u_1}\partial_2^{u_2-2}\partial_3^{u_3}\cdots\partial_{n+1}^{u_{n+1}}Q_1 - a_1x_2S_1 \\ & = & b_1^2\theta_1^2\partial_1^{u_1}\partial_2^{u_2-2}\partial_3^{u_3}\cdots\partial_{n+1}^{u_{n+1}} + R_2. \\ & \vdots & \\ S_{u_2} & = & b_1^{u_2-1}\theta_1^{u_2-1}\partial_1^{u_1}\partial_3^{u_3}\cdots\partial_{n+1}^{u_{n+1}}Q_1 - a_1x_2S_{u_2-1} \\ & = & b_1^{u_2}\theta_1^{u_2}\partial_1^{u_1}\partial_3^{u_3}\cdots\partial_{n+1}^{u_{n+1}} + R_{u_2}. \\ & \vdots & \\ S_{u_2+\cdots+u_{n+1}} & = & b_1^{u_2}\cdots b_n^{u_{n+1}-1}\theta_1^{u_2+\cdots+u_{n+1}-1}\partial_1^{u_1}Q_n - a_nx_{n+1}S_{u_2+\cdots+u_{n+1}-1} \\ & = & b_1^{u_2}\cdots b_n^{u_{n+1}}\theta_1^{u_2+\cdots+u_{n+1}}\partial_1^{u_1} + R_{u_2+\cdots+u_{n+1}}. \end{array}$$

This finishes the proof. ■

In this case the slopes are at infinity, so we must perform the following change of variables:

$$x_1' = -\frac{1}{x_1},\ x_2' = x_2,\ \ldots,\ x_{n+1}' = x_{n+1},$$

$$\partial_1 = x_1'^2\partial_1',\ \partial_2 = \partial_2',\ \ldots,\ \partial_{n+1} = \partial_{n+1}'.$$

To simplify the notation we change x_i' by x_i and we note:

$$[\theta]^i = \theta(\theta+1)\cdots(\theta+i-1).$$

The new system obtained is given by:

$$H_A(\beta)' = \langle x_1^{u_1}[\theta_1]^{u_1}\partial_2^{u_2}\cdots\partial_{n+1}^{u_{n+1}} - 1, a_1x_2\partial_2 - b_1x_1\partial_1 - \beta_1, \ldots$$
$$\ldots a_nx_{n+1}\partial_{n+1} - b_nx_1\partial_1 - \beta_n\rangle.$$

Theorem 5 *Let A be an $n\times(n+1)$ integer matrix with all its $n\times n$ minors different from zero and $I_A = \langle\partial^u - 1\rangle$. If we do the change of coordinates as before, $\mathcal{H}_A(\beta)'$ has only the slope $L_1 = u_1F + |u|V$ with respect to $x_1 = 0$.*

Proof. First, we prove that if $L \neq L_1$, then L is not a slope of $\mathcal{H}_A(\beta)'$.

Let be $L' < L_1$ any slope with respect to $x_1 = 0$. Then $\sigma_{L'}(P') = x_1^{2u_1}\xi_1^{u_1}\cdots\xi_{n+1}^{u_{n+1}}$, we can consider the sequence of operators S_i' in $H_A(\beta)'$, such that $\mathrm{ord}_{L'}(R'_i) < \mathrm{ord}_{L'}(P') = \mathrm{ord}_{L'}(S'_i)$:

$$\begin{array}{rcl}
S'_1 & = & -x_1^{u_1}[\theta_1]^{u_1}\partial_2^{u_2-1}\partial_3^{u_3}\cdots\partial_{n+1}^{u_{n+1}}Q'_1 + a_1x_2P' \\
& = & b_1x_1^{u_1}\theta_1[\theta_1]^{u_1}\partial_2^{u_2-1}\partial_3^{u_3}\cdots\partial_{n+1}^{u_{n+1}} + R'_1. \\
S'_2 & = & -b_1x_1^{u_1}\theta_1[\theta_1]^{u_1}\partial_2^{u_2-2}\partial_3^{u_3}\cdots\partial_{n+1}^{u_{n+1}}Q'_1 + a_1x_2S'_1 \\
& = & b_1^2x_1^{u_1}\theta_1^2[\theta_1]^{u_1}\partial_2^{u_2-2}\partial_3^{u_3}\cdots\partial_{n+1}^{u_{n+1}} + R'_2. \\
& \vdots & \\
S'_{u_2} & = & -b_1^{u_2-1}x_1^{u_1}\theta_1^{u_2-1}[\theta_1]^{u_1}\partial_3^{u_3}\cdots\partial_{n+1}^{u_{n+1}}Q'_1 + a_1x_2S'_{u_2-1} \\
& = & b_1^{u_2}x_1^{u_1}\theta_1^{u_2}[\theta_1]^{u_1}\partial_3^{u_3}\cdots\partial_{n+1}^{u_{n+1}} + R'_{u_2}. \\
& \vdots & \\
S'_{u_2+\cdots+u_{n+1}} & = & -b_1^{u_2}\cdots b_n^{u_{n+1}-1}x_1^{u_1}\theta_1^{u_2+\cdots+u_{n+1}-1}[\theta_1]^{u_1}Q'_n \\
& + & a_nx_{n+1}S'_{u_2+\cdots+u_{n+1}-1} \\
& = & b_1^{u_2}\cdots b_n^{u_{n+1}}x_1^{u_1}\theta_1^{u_2+\cdots+u_{n+1}}[\theta_1]^{u_1} + R'_{u_2+\cdots+u_{n+1}}.
\end{array}$$

Therefore L' is not a slope.

Let be $L'' > L_1$ any slope with respect to $x_1 = 0$, then $\sigma_{L''}(P') = 1$, and it is clear that it is bihomogenous, so L'' is not a slope.

The only possible slope is L_1. Suppose that L_1 is not a slope. Then the L characteristic variety is invariant for all $L = pF + qV$. We have that:

$$\mathrm{gr}^V(H_A(\beta)') = 1 \quad \text{and} \quad \sqrt{\mathrm{gr}^F(H_A(\beta)')} \neq 1.$$

■

Remark 2 *We have supposed that all the $n \times n$ minors of our matrix were different from zero. If the matrix A has an $n \times n$ minor equal to zero, this means that there exists an i such that $H_A(\beta) = \langle x_i\partial_i - \beta_i, H_{A'}(\beta')\rangle$. Where A' denotes the matrix obtained from A taking out the i-th row and the i-th column, and β' is the vector β without the i-th element.*

We can obtain, finally, a matrix B with all its minors different from zero such that:

$$H_A(\beta) = \langle x_1\partial_1 - \beta_1, \ldots, x_j\partial_j - \beta_j, H_B(\beta'')\rangle,$$

after renaming the variables, where $\beta'' = (\beta_{j+1}, \ldots, \beta_{n+1})$.

If we take L a slope with respect to $x_i = 0$, with $i < j$ it is easy to see:

$$\mathrm{gr}^L(H_A(\beta)) = \langle \sigma_L(x_1\partial_1 - \beta_1), \ldots, \sigma_L(x_j\partial_j - \beta_j), \mathrm{gr}^F(H_B(\beta''))\rangle,$$

and L is not a slope for $\mathcal{H}_A(\beta)$.

If we take L a slope with respect to $x_i = 0$, with $i > j$ we have:

$$\mathrm{gr}^L(H_A(\beta)) = \langle x_1\xi_1, \ldots, x_j\xi_j, \mathrm{gr}^L(H_B(\beta''))\rangle,$$

and L is a slope for $\mathcal{H}_A(\beta)$ if and only if L is a slope for $\mathcal{H}_B(\beta'')$.

References

[1] Adolphson, A.: Hypergeometric functions and rings generated by monomials. *Duke Math. J.* **73** (1994), 269–290.

[2] Assi, A., Castro-Jiménez, F. J. and Granger, J. M.: How to calculate the slopes of a $\mathcal{D}$-module. *Compositio Math.* **104** (1996), 107–123.

[3] Castro-Jiménez, F. J. and Takayama, N.: Slopes of a hypergeometric System Associated to a Monomial Curve. To appear in *Trans. Amer. Math. Soc.* Also available at `math.AG/0107046`.

[4] Gel'fand, I. M., Zelevinski, A. V. and Kapranov, M. M.: Hypergeometric functions and toric varieties. *Funktsional. Anal. i Prilozhen.* **23** (1989), no. 2, 12–26; translation in *Funct. Anal. Appl.* **23** (1989), no. 2, 94–106.

[5] Hotta, R.: Equivariant $\mathcal{D}$-modules. In *Proceedings of ICPAM Spring School in Wuhan*, edited by P. Torasso. To appear in Travaux en Cours, Hermann, Paris. Also available at `math.RT/9805021`.

[6] Ince, E. L.: *Ordinary Differential Equations.* Dover Publications, New York, 1956.

[7] Laurent, Y.: Polygone de Newton et b-fonctions pour les modules microdifférentiels. *Ann. Sci. École Norm. Sup. (4)* **20** (1987), 391–441.

[8] Laurent, Y. and Mebkhout, Z.: Pentes algébriques et pentes analytiques d'un $\mathcal{D}$-module. *Ann. Sci. École Norm. Sup. (4)* **32** (1999), 39–69.

[9] Mebkhout, Z.: Le formalisme des six opérations de Grothendieck pour les $\mathcal{D}_X$-modules cohérent. Travaux en cours **35**, Hermann, Paris, 1989.

[10] Mebkhout, Z.: Le théorème de positivité de l'irregularité pour le $\mathcal{D}$-modules. In *The Grothendieck Festschrift, III*, 84–131. Progr. Math. **88**, Birkhäuser, Boston, MA, 1990.

[11] Saito, M., Sturmfels, B. and Takayama, N.: *Gröbner deformations of hypergeometric differential equations.* Algorithms and Computations in Mathematics **6**, Springer-Verlag, Berlin, 2000.

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