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Perturbing plane curve singularities

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Abstract

We describe the singularity of all but finitely-many germs in a pencil generated by two germs of plane curve sharing no tangent.

Introduction

Let $\xi : f = 0$, $f \in \mathbb{C}\{x, y\}$, be a germ of analytic curve at the origin of $\mathbb{C}^2$ and assume that $g \in \mathbb{C}\{x, y\}$ has $n = \operatorname{ord} g \geq \operatorname{ord} f$ and the initial forms of f and g share no factor. In this paper we describe the singularities of the germs of curve $\zeta^\lambda : f + \lambda g = 0$ for all but finitely-many $\lambda \in \mathbb{C}$, by giving their infinitely near singular points and multiplicities. This in particular determines their topological (or equisingularity) type in terms of n and the singularity of ξ (the topological type of ξ if it is reduced). As already well known, for ξ reduced, n big enough and no further hypothesis on g, all germs ζ^λ have the topological type of ξ (see [8] and [5], where the minimal n with this property is computed). Also a case with a non-reduced ξ and $n \gg 0$ has been treated in [6], chap. 5.

1. Free and satellite points. Clusters

In this section we briefly recall basic notions about infinitely near points. The reader is referred to [2], [3] or [4] for more details. Also, we introduce some new numerical invariants related to infinitely near points that are needed in the sequel.

Points infinitely near to a point O on a smooth analytic surface S being constructed by successive blowing-ups, each point p infinitely near to O lies on the exceptional divisor $E_p = \pi_p^{-1}(O)$ of the composition $\pi_p : S_p \longrightarrow S$

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of a finite sequence of blowing-ups. We write $<$ the ordering on infinitely near points induced by the blowing-ups, i.e. $p < q$ means that q is infinitely near to p. The point p is called a *satellite point* if it is a singular (double in fact) point of E_p, otherwise it is called a *free point*. Assume that p is equal or infinitely near to O. Points lying on the exceptional divisor of blowing up p or on any of its successive strict transforms by further blowing-ups are called *points proximate to p*. As it is easy to see, free points are proximate to just one point, while satellite points are proximate to exactly two points.

Let p be either O or a free point infinitely near to O and let p' be a point infinitely near to p with no free points between p and p'. If p' is free, then we will say that it is a point *next* p. Otherwise, if p' is satellite, it will be called a *satellite of p*.

For a point p infinitely near to O, we denote $\widetilde{\xi}_p$ (respectively, $\bar{\xi}_p$) the germ at p of the strict transform (respectively, total transform) of the germ of curve ξ by the composition π_p of the blowing-ups giving rise to p. We denote by $e_p(\xi)$ the multiplicity at p of $\widetilde{\xi}_p$, usually called the *(effective) multiplicity* of ξ at p. The point p is said to be a *non-singular* point of ξ if and only if it is simple on ξ (i.e., $e_p(\xi) = 1$) and ξ contains no satellite point equal or infinitely near to p. Equivalently, p is a non-singular point of ξ if and only if $\widetilde{\xi}_p$ and E_p are transverse at p.

A *cluster* with origin at O is a finite set K of points equal or infinitely near to O such that for each $p \in K$ it contains all points preceding p (by the ordering of the blowing-ups). A pair $\mathcal{K} = (K, \nu)$, where K is a cluster and $\nu : K \longrightarrow \mathbb{Z}$ an arbitrary map, will be called a *weighted cluster*. For each $p \in K$, $\nu_p = \nu(p)$ is called the virtual multiplicity of p in $\mathcal{K}$. *Consistent clusters* are the weighted clusters $\mathcal{K} = (K, \nu)$ such that

$$\nu_p - \sum_{q \text{ prox. to } p} \nu_q \geq 0, \quad \text{for all } p \in K.$$

We will say that a germ ξ at O *goes sharply through* the weighted cluster $\mathcal{K} = (K, \nu)$ if ξ goes through K with effective multiplicities equal to the virtual ones (i.e., for all $p \in K$, $e_p(\xi) = \nu_p$) and has no singular points outside of K. The reader may notice that if ξ goes sharply through $\mathcal{K}$, then the singularity of ξ, both regarding its topological or equisingularity type (see [10] or also [1] or [4]) and the position of singular points, is fully determined by $\mathcal{K}$.

If p is a free point on a germ of curve ξ, we will write $\mathcal{S}_p(\xi)$ for the set of points consisting of p and all satellite points of p on ξ. As it is well known $\mathcal{S}_p(\xi)$ is a finite set. Also if the free point p belongs to a cluster K, $\mathcal{S}_p(\mathcal{K})$ will denote the set of p and all satellite points of p in K.

Let $\mathcal{K} = (K, \nu)$ be a weighted cluster and $p \in K$ a free point. We define the set of *extremal satellites of* p *in* $\mathcal{K}$, $\mathcal{R}_p(\mathcal{K})$, as the set of all points $q \in \mathcal{S}_p(\mathcal{K})$ such that

$$\varepsilon_q(\mathcal{K}) = \nu_q - \sum_{p'} \nu_{p'} > 0,$$

summation running on the points $p' \in \mathcal{S}_p(\mathcal{K})$ proximate to q. Note that p may belong to $\mathcal{R}_p(\mathcal{K})$.

Let ξ be a germ of curve at O and p a free point on ξ. Similarly, the set of *extremal satellites of* p *on* ξ, $\mathcal{R}_p(\xi)$ is defined as the set of the points $q \in \mathcal{S}_p(\xi)$ for which

$$\varepsilon_q(\xi) = e_q(\xi) - \sum_{p'} e_{p'}(\xi) > 0,$$

summation running on the points $p' \in \mathcal{S}_p(\xi)$ proximate to q.

Remark 1.1 If ξ is a germ of curve going sharply through $\mathcal{K} = (K, \nu)$, then for any free $p \in K$, $\mathcal{S}_p(\mathcal{K}) = \mathcal{S}_p(\xi)$; for any $q \in \mathcal{S}_p(\xi)$, $\varepsilon_q(\mathcal{K}) = \varepsilon_q(\xi)$ and hence $\mathcal{R}_p(\mathcal{K}) = \mathcal{R}_p(\xi)$.

Remark 1.2 Since for any branch γ of a germ of curve ξ, $e_q(\gamma)$ equals the sum of the multiplicities of γ at points proximate to q (proximity equality, cf. [2], 1.4.1), one has

$$\varepsilon_q(\xi) = \sum_{\gamma} e_q(\gamma)\,,$$

where γ ranges over the set of branches of ξ with a free point in the first neighbourhood of q. In particular, $q \in \mathcal{R}_p(\xi)$ if and only if ξ has a point next p in the first neighbourhood of q. Clearly, $\mathcal{R}_p(\xi)$ is cofinal in $\mathcal{S}_p(\xi)$.

Remark 1.3 Let $\mathcal{K} = (K, \nu)$ be a weighted cluster and $p \in K$ a free point. The integers $\varepsilon_q(\mathcal{K})$, for $q \in \mathcal{S}_p(\mathcal{K})$, determine (and are of course determined by) the virtual multiplicities ν_q. Indeed if q is maximal in $\mathcal{S}_p(\mathcal{K})$, then $\varepsilon_q(\mathcal{K}) = \nu_q$ after which the multiplicities ν_q are inductively determined by the equalities defining the $\varepsilon_q(\mathcal{K})$. Similarly, if p is a free point and lies on a germ of curve ξ, the effective multiplicities of ξ at the points $q \in \mathcal{S}_p(\xi)$ are determined by their corresponding $\varepsilon_q(\xi)$. The inductive procedure that determines the multiplicities being in both cases the same, if $\mathcal{S}_p(\mathcal{K}) = \mathcal{S}_p(\xi)$ and $\varepsilon_q(\mathcal{K}) = \varepsilon_q(\xi)$ for all $q \in \mathcal{S}_p(\mathcal{K})$, then $e_q(\xi) = \nu_q$ for all $q \in \mathcal{S}_p(\mathcal{K})$.

Let p be a free point infinitely near to O. Let q be either p or a satellite of p. Write $p = q_1, q_2, \dots, q_h = q$ the ordered sequence of points between p and q. One may decompose $h = h_1 + \cdots + h_r$, all $h_i > 0$

and $h_r > 1$, in such a way that $q_1, \dots, q_{h_1+1}$ are proximate to the point just preceding p, $q_{h_1+1}, \dots, q_{h_1+h_2+1}$ are proximate to q_{h_1}, and so on, till $q_{h_1+\cdots+h_{r-1}+1}, \dots, q_{h_1+\cdots+h_r}$ that are proximate to $q_{h_1+\cdots+h_{r-1}}$. Then, we define the *slope* of the satellite point q as

$$s(q) = \cfrac{1}{h_1 + \cfrac{1}{h_2 + \cfrac{1}{\ddots \; \cfrac{1}{h_r}}}}.$$

Since satellite points are quite determined by the points they are proximate to, it easily follows

Lemma 1.4 a) $s(q) \leq 1$ *and the equality holds if and only if* $q = p$.

b) $s(q) = s(q')$ *if and only if* $q = q'$.

Let ξ be a germ of curve at O, p a free point on ξ and $q \in \mathcal{R}_p(\xi)$. Fix a branch θ_p^q with origin at p, having multiplicity one at q and such that all its points after q are non-singular and do not belong to ξ: the integer $I(p,q)$ is defined as

$$I(p,q) = [\theta_p^q \cdot \widetilde{\xi}_p] \,,$$

where $[\,\cdot\,]$ stands for intersection multiplicity of germs at p.

The multiplicities $e_{p'}(\theta_q^p)$, $p' < q$, being all determined by the proximity equalities from the fact that q is simple and followed by non-singular points, it easily follows from the Noether formula ([2], 1.3.1) that $I(p,q)$ does not depend on θ_q^p, but only on ξ, p and q. Moreover, $I(p,q)$ may be easily computed from a weighted Enriques diagram of ξ.

2. Virtual and total transforms

For any point p equal or infinitely near to O, denote by $\mathcal{O}_p$ its local ring on the surface $\mathcal{S}_p$ it is lying as a proper point, $\mathcal{O}_p \simeq \mathbb{C}\{x,y\}$ if x, y are local coordinates on S_p at p. Let $\mathcal{K} = (K, \nu)$ be a weighted cluster and η a germ of curve, both with origin at O. *Going through* $\mathcal{K}$ (or through the points $p \in K$ with the virtual multiplicities ν_p) is defined using induction on $\#K$ in the following way

a) If $K = \{O\}$, then η goes through $\mathcal{K}$ if and only if $e_O(\eta) \geq \nu_O$.

In such a case, for each q in the first neighbourhood of O we define the virtual transform $\widehat{\eta}_q$ of η (relative to ν_O) as $\widetilde{\eta}_q + (e_O(\eta) - \nu_O)\mathcal{E}_q$, where $\mathcal{E}_q$ is the germ at q of the exceptional divisor of blowing up O.

b) If $K \neq \{O\}$, let $q_1, \ldots, q_s$ be the points of K in the first neighbourhood of O and denote by $\mathcal{K}_i$ the weighted cluster consisting of q_i and the points infinitely near to it in K, and the restriction of ν. Then, η goes through $\mathcal{K}$ if and only if η goes through (O, ν_O) and the virtual transforms $\widehat{\eta}_{q_i}$, relative to ν_O, go through $\mathcal{K}_i$ for $i = 1, \ldots, s$.

Assume that η goes through $\mathcal{K}$ and let q be a point in the first neighbourhood of any $p \in K$. The virtual transform $\widehat{\eta}_q$ of η with origin at q and relative to the multiplicities $\nu_{p'}$, $p' < q$ has been already defined if $p = O$. Otherwise and using induction on the order of the neighbourhood, $\widehat{\eta}_q$ is the virtual transform of $\widehat{\eta}_p$ relative to ν_p. If needed we will take $\widehat{\eta}_O = \eta$.

We will make use of the following result, see [2], (2.4) or [4], chap. 4 for its proof.

Proposition 2.1 *The equations of the germs going through a weighted cluster $\mathcal{K}$ describe the set of non-zero elements of a finite codimensional ideal $H_{\mathcal{K}}$ of $\mathcal{O}_O$. Furthermore, for each $p \in K$ there is a morphism of $\mathcal{O}_O$-modules $\psi_p : H_{\mathcal{K}} \longrightarrow \mathcal{O}_p$ such that for any $f \in H_{\mathcal{K}}$, $\psi_p(f)$ is an equation of the virtual transform $\widehat{\eta}_p$ of $\eta : f = 0$.*

Let $p \in K$. The exceptional divisor E_p decomposes into a sum of components, $E_p = \sum_{q<p} F_p^q$, each F_p^q being the strict transform of the exceptional divisor of blowing up the point q.

Let η be a germ of curve with origin at O. We will assign to each $p \in K$ integers $u_p^{\mathcal{K}}(\eta)$, $v_p(\eta)$ defined using induction on the order of the neighbourhood p is belonging to. If $p = O$, $u_O^{\mathcal{K}}(\eta) = e_O(\eta) - \nu_O$, $v_O(\eta) = e_O(\eta)$. Let $p \in K$ be infinitely near to O. The points p is proximate to belong to K and we may define

$$\begin{aligned} u_p^{\mathcal{K}}(\eta) &= e_p(\eta) - \nu_p + \sum_{p \text{ prox. to } q} u_q^{\mathcal{K}}(\eta)\,, \\ v_p(\eta) &= e_p(\eta) + \sum_{p \text{ prox. to } q} v_q(\eta)\,. \end{aligned}$$

Remark 2.2 a) The integer $u_p^{\mathcal{K}}(\eta)$ depends only on p and the points preceding p, their virtual multiplicities and the multiplicities of η at these points.

b) The integer $v_p(\eta)$ depends only on p and the points preceding p and the multiplicities of η at these points.

Proposition 2.3 *Let $\mathcal{K} = (K, \nu)$ be a weighted cluster with origin at O and denote by p' any point in the first neighbourhood of some $p \in K$. Let η be a germ of curve with origin at O.*

a) *η goes through $\mathcal{K}$ if and only if $u_p^{\mathcal{K}}(\eta) \geq 0$ for all $p \in K$. In such a case the $u_q^{\mathcal{K}}(\eta)$, $q < p'$, are the multiplicities of the germs of the components $F_{p'}^q$ of the exceptional divisor in the virtual transform $\widehat{\eta}_{p'}$.*

b) *The multiplicities of the germs of the components $F_{p'}^q$ of the exceptional divisor in the total transform $\bar{\eta}_{p'}$ are the $v_q(\eta)$, $q < p'$.*

c) *The difference $v_p(\eta) - u_p^{\mathcal{K}}(\eta)$ does not depend on η. In particular, $v_p(\eta) - u_p^{\mathcal{K}}(\eta) = v_p(\xi)$ for any germ ξ going through $\mathcal{K}$ with effective multiplicities equal to the virtual ones.*

Proof: Parts a), b) and c) follow from the definitions by an easy induction (see [4] chap. 4 for details). ■

3. Newton polygon

Let ξ be a germ of curve at O, fix a free point p on ξ (hence $p \neq O$) and take local coordinates x, y at p so that the y-axis is the germ of the exceptional divisor at p and the x-axis is not tangent to $\widetilde{\xi}_p$. Next we will show how $s(q)$, $\varepsilon_q(\xi)$ and $I(p, q)$, for $q \in \mathcal{R}_p(\xi)$, are related to the Newton polygon of $\widetilde{\xi}_p$.

Remark 3.1 Assume that $\widetilde{\xi}_p$ has equation $f = \sum a_{i,j} x^i y^j$ and denote by $\mathbf{N}(f)$ its Newton polygon. Let $\Gamma_1, \ldots, \Gamma_k$ be the sides of $\mathbf{N}(f)$, ordered so that, for each i, Γ_i has ends $(\alpha_{i-1}, \beta_{i-1})$ and (α_i, β_i), and $\beta_{i-1} > \beta_i$. For each of these sides write

$$\Omega_i(z) = \sum_{(\alpha,\beta)\in\Gamma_i} a_{\alpha,\beta} z^{\beta-\beta_i} ,$$

which is currently called the equation associated to Γ_i.

Then, as it is well known ([7], appendix B, for instance), the branches of $\widetilde{\xi}_p$ (or the branches of ξ through p) correspond to the sides of $\mathbf{N}(f)$ so that the branches corresponding to the side Γ_i have a Puiseux series

$$y = b x^{m_i/n_i} + \cdots , \tag{1}$$

$-n_i/m_i$ being the slope of Γ_i and b a root of Ω_i. Furthermore, for any side of $\mathbf{N}(f)$ and any root b of its associated equation, there is at least one such branch. Notice that $m_i/n_i \leq 1$, for $i = 1, \ldots k$, as, by hypothesis, there are no branches of $\widetilde{\xi}_p$ tangent to the x-axis. Assume that γ is a branch of ξ whose strict transform $\widetilde{\gamma}_p$ has the Puiseux series (1) above and let p' be the point on γ next p. We will take coordinates at p' according to next lemma (proved in [2], 10.2).

Lemma 3.2 *Denote $\bar{x}$, $\bar{y}$ the inverse images at p' of the local coordinates x, y at p. There are local coordinates $\tilde{x}$, $\tilde{y}$ at p' related to $\bar{x}$, $\bar{y}$ by the equalities*

$$\begin{aligned} \bar{x} &= \tilde{x}^{n_i} \\ \bar{y} &= \tilde{x}^{m_i}(b+\tilde{y}) \end{aligned}$$

and so that $\tilde{x}$ is an equation of the germ of the exceptional divisor at p'.

Remark 3.3 It follows from an easy computation using the above lemma that p' is a non-singular point of ξ if and only if b is a simple root of Ω_i. In the sequel we will assume that $\gcd(n_i, m_i) = 1$.

By the Enriques theorem (see [4], 5.5.1 or [1], III.8.4, th. 12), all irreducible germs θ with origin at p and Puiseux series

$$y = ax^{m_i/n_i} + \cdots ,$$

$a \neq 0$, and so in particular all branches corresponding to Γ_i go through the same sequence of satellite points of p, the last of them q_i having $s(q_i) = m_i/n_i$ (if $m_i/n_i = 1$, then $i = k$, the sequence is empty and we take $q_k = p$). Furthermore, the germ θ above shares a further point (hence a point next p) with one of the branches of $\widetilde{\xi}_p$ if and only if $\Omega_i(a) = 0$.

It follows from (1.2) that the extremal satellites of p on ξ are one for each side of $\mathbf{N}(f)$, more precisely $\mathcal{R}^p(\xi) = \{q_1, \ldots, q_k\}$.

Lemma 3.4 *For $i = 1, \ldots, k$,*

a) $I(p, q_i) = n_i\alpha_i + m_i\beta_i$.

b) $\beta_{i-1} - \beta_i = \varepsilon_{q_i}(\xi)n_i$, $\alpha_i - \alpha_{i-1} = \varepsilon_{q_i}(\xi)m_i$. *In particular,* $\varepsilon_{q_i}(\xi) = \gcd(\beta_{i-1} - \beta_i, \alpha_i - \alpha_{i-1})$.

Proof: a) By (3.3), $\theta_p^{q_i}$ has a Puiseux parameterization of the form

$$\begin{aligned} x &= t^{n_i} \\ y &= at^{m_i} + \cdots \end{aligned} \tag{2}$$

with $\Omega_i(a) \neq 0$, because $\theta_p^{q_i}$ goes through no point on ξ in the first neighbourhood of q_i. By substituting (2) in the equation of $\widetilde{\xi}_p$ and computing the initial term, one easily gets $[\theta_p^{q_i} \cdot \widetilde{\xi}_p] = n_i\alpha_i + m_i\beta_i$, as wanted.

b) Since the side Γ_i has slope $-n_i/m_i$ and ends $(\alpha_{i-1},\beta_{i-1})$, (α_i,β_i) it is enough to check that $\beta_{i-1}-\beta_i=\varepsilon_{q_i}(\xi)n_i$.

Let $\gamma_1^{(i)},\dots,\gamma_{\ell_i}^{(i)}$ be the branches of ξ through q_i with a free point in the first neighbourhood of q_i. If g_i is the product of the equations of all branches of $\widetilde{\xi}_p$ corresponding to the side Γ_i, then g decomposes into factors $g_1,\dots,g_k$ and the Newton polygon of g_i has as single side a translated of Γ_i ([9]). In particular, $\deg_y g_i=\beta_{i-1}-\beta_i$ while

$$g_i=\prod_{j=1}^{\ell_i}(y^{d_jn_i}-a_jx^{d_jm_i}+\dots)$$

and $\gamma_j^{(i)}: y^{d_jn_i}-a_jx^{d_jm_i}+\dots=0$ are the branches of $\widetilde{\xi}_p$ corresponding to Γ_i. Then, by the Enriques theorem, $e_{q_i}(\gamma_j^{(i)})=\gcd(d_jn_i,d_jm_i)=d_j$ and so

$$\sum_{j=1}^{\ell_i}e_{q_i}(\gamma_j^{(i)})=\sum_{j=1}^{\ell_i}d_j=\deg_y g_i/n_i=(\beta_{i-1}-\beta_i)/n_i\,.$$

Since, by (1.2), $\varepsilon_{q_i}(\xi)=\sum_{j=1}^{\ell_i}e_{q_i}(\gamma_j^{(i)})$, the claim follows. ■

Remark 3.5 Let p be a free point infinitely near to O and assume there is given a set $\{(q_1,\varepsilon_1),\dots,(q_k,\varepsilon_k)\}$, where each q_i is either p or a satellite of p and each ε_i is a strictly positive integer. We associate to them a weighted cluster $\mathcal{A}=(A,\mu)$ with origin at p, by taking p and all its infinitely near points that precede or are equal to one of the q_i and the virtual multiplicities determined (cf. (1.3)) by taking $\varepsilon_{\mathcal{A}}(q_i)=\varepsilon_i$, $\varepsilon_{\mathcal{A}}(q)=0$ if $q\in A$, $q\neq q_i$, $i=1,\dots,k$.

Assume that the points q_i are ordered so that $s(q_1)<\dots<s(q_k)$. Clearly there is a single Newton polygon in $\mathbb{R}^2$, $\mathbf{N}_{\mathcal{A}}$, with both ends on the axis and sides $\Gamma_1,\dots,\Gamma_k$ such that for each i, $i=1,\dots,k$, Γ_i contains ε_i+1 integral points and its slope is $-1/s(q_i)$. If we write the ends of Γ_i, $(\alpha_{i-1},\beta_{i-1})$, $(\alpha_i,\beta_i)\in\mathbb{Z}^2$ with $\beta_{i-1}>\beta_i$, then $\alpha_{i-1}<\alpha_i$, $\gcd(\alpha_i-\alpha_{i-1},\beta_{i-1}-\beta_i)=\varepsilon_i$.

Take local coordinates x, y at p so that $x=0$ is the germ of the exceptional divisor at p.

Proposition 3.6 a) *Let ξ be a germ of curve with origin at O and assume that $\widetilde{\xi}_p$ is $f=0$, $f\in\mathbb{C}\{x,y\}$. If $\mathbf{N}(f)=\mathbf{N}_{\mathcal{A}}$ then, $\mathcal{S}_p(\xi)=A$ and $e_q(\xi)=\mu_q$ for all $q\in A$.*
b) *Let $\eta: g=0$, $g\in\mathbb{C}\{x,y\}$, be a germ of curve with origin at p. If $\mathbf{N}(g)$ has no vertex below $\mathbf{N}_{\mathcal{A}}$, then η goes through $\mathcal{A}$.*

Proof: a) Since $\mathbf{N}(f) = \mathbf{N}_{\mathcal{A}}$, by (3.3), the extremal satellites of p on ξ are $q_1, \ldots, q_k$ and therefore $\mathcal{S}_p(\xi) = A$. Moreover, by (3.4), $\varepsilon_{q_i}(\xi) = \varepsilon_i$ so, by (1.3), $e_q(\xi) = \mu_q$ for all $q \in A$, as wanted.

b) By (2.1), it is enough to prove that for any (α, β) not below $\mathbf{N}_{\mathcal{A}}$, the germ $x^\alpha y^\beta = 0$ goes through $\mathcal{A}$.

Choose any $h \in \mathbb{C}\{x, y\}$ such that $\mathbf{N}(h) = \mathbf{N}_{\mathcal{A}}$. We claim that $\zeta : h = 0$ goes through $\mathcal{A}$. Indeed, since $\mathbf{N}_{\mathcal{A}}$ has its ends on the axis, h has no factor x, so $\zeta : h = 0$ does not contain the germ of the exceptional divisor and therefore $\zeta = \widetilde{\xi}_p$ for some germ of curve ξ with origin at O. Thus, part a) applies, $e_q(\zeta) = \mu_q$ for all $q \in A$ and hence, ζ goes through $\mathcal{A}$ as claimed.

Since (α, β) does not lie below $\mathbf{N}_{\mathcal{A}}$ one may clearly choose $\lambda \in \mathbb{C} \setminus \{0\}$ so that $\mathbf{N}(h + \lambda x^\alpha y^\beta) = \mathbf{N}_{\mathcal{A}}$. Arguing as above for $h = 0$, also the germ $h + \lambda x^\alpha y^\beta = 0$ goes through $\mathcal{A}$ and thus, by (2.1), so does

$$x^\alpha y^\beta = (h^\lambda - h)/\lambda = 0\,.$$

■

Let $g = \sum_{i,j \geq 0} a_{i,j} x^i y^j \in \mathbb{C}\{x, y\}$ and $(n, m) \in \mathbb{N}^2$. We define

$$\deg_{(n,m)}(g) = \min\{ni + mj \mid a_{ij} \neq 0\}\,.$$

Proposition 3.7 *Let $\eta : g = 0$ be a germ of curve with origin at p so that $\mathbf{N}(g) = \mathbf{N}_{\mathcal{A}}$. Assume that $\zeta : f = 0$ is any germ with origin at p. Then,*

a) $v_{q_\ell}(\zeta) = \deg_{(n_\ell, m_\ell)}(f)$.

b) $u^{\mathcal{A}}_{q_\ell}(\zeta) = \deg_{(n_\ell, m_\ell)}(f) - \deg_{(n_\ell, m_\ell)}(g)$.

Proof: Let p' be any free point in the first neighbourhood of q_ℓ. Using at p' the coordinates of (3.2), an equation of the total transform $\bar{\eta}_{p'}$ is

$$\bar{g} = \tilde{x}^{k_\ell} \left(\sum_{(i,j) \in \Gamma_\ell} a_{ij} (b + \tilde{y})^j \right) + \sum_{n_\ell i + m_\ell j > k_\ell} a_{ij} \tilde{x}^{n_\ell i + m_\ell j} (b + \tilde{y})^j\,.$$

Thus, $\bar{g} = \tilde{x}^{k_\ell} \widetilde{g}$ and since $a_{ij} \neq 0$ for some $(i, j) \in \Gamma_\ell$, $\tilde{g}$ has no further factor $\tilde{x}$. By (2.3.b), $v_{q_\ell}(\eta) = k_\ell$. Computing as above, one also gets that the total transform of $\zeta : f = 0$ contains exactly $\deg_{(n_\ell, m_\ell)}(f)$ times the germ of $E_{p'}$, that is, by (2.3.b), $v_p(\zeta) = \deg_{(n_\ell, m_\ell)}(f)$. So, by (2.3.c), $u^{\mathcal{A}}_{q_\ell}(\zeta) = v_{q_\ell}(\zeta) - v_{q_\ell}(\eta) = \deg_{(n_\ell, m_\ell)}(g) - \deg_{(n_\ell, m_\ell)}(f)$, as claimed. ■

4. Behaviour of ζ^λ

Let O be the origin of $\mathbb{C}^2$ (or a point on a smooth surface, there is no difference from the local viewpoint). Let $\xi : f = 0$, $\eta : g = 0$ be (non-necessarily reduced) germs of curve at O. Assume that $e_O(\xi) \leq e_O(\eta)$ and that ξ and η share no tangent.

Consider the germs of curve $\zeta^\lambda : f + \lambda g = 0$, $\lambda \in \mathbb{C}$. For all but at most a finite number of λ, the germs ζ^λ go sharply through a weighted cluster $\mathcal{T} = (T, \tau)$ that we will describe in terms of the infinitely near points and multiplicities of ξ.

First we will assign to each p on ξ an integer u_p, defined using induction on the order of the neighbourhood p is belonging to:

If $p = O$, we take $u_O = e_O(\eta) - e_O(\xi)$ and for p on ξ and infinitely near to O,

$$u_p = \sum_{p \text{ prox. to } q} u_q - e_p(\xi)\,.$$

Remark 4.1 Let $\mathcal{K}_p = (K_p, \nu)$ be the weighted cluster consisting of all points q on ξ that precede or equal p with virtual multiplicities $\nu_q = e_q(\xi)$. Since ξ and η have no common tangent, $e_q(\eta) = 0$ for all $q \in K_p$ infinitely near to O, and so $u_p = u_p^{\mathcal{K}_p}(\eta)$, as defined in §2.

The weighted cluster $\mathcal{T} = (T, \tau)$ will be defined inductively. After taking $O \in T$ and assuming that either $p = O$ or p is a free point already in T, we will define

(1) The satellites of p in T, or equivalently $\mathcal{S}_p(\mathcal{T})$.

(2) The integers $\varepsilon_q(\mathcal{T})$ for $q \in \mathcal{S}_p(\mathcal{T})$.

(3) The points next p in T, all taken on ξ.

Once it is proved that such inductive procedure involves finitely many points only, it clearly defines the weighted cluster $\mathcal{T} = (T, \tau)$, the virtual multiplicities τ_p being determined by the $\varepsilon_q(\mathcal{T})$, by (1.3).

For $p = O$ we take

(1) $S_O(\mathcal{T}) = \{O\}$,

(2) $\varepsilon_O(\mathcal{T}) = e_O(\xi)$,

(3) either no point next O in T if $e_O(\xi) = e_O(\eta)$, or all points in the first neighbourhood of O on ξ if $e_O(\xi) < e_O(\eta)$.

Obviously, in case $e_O(\xi) = e_O(\eta)$ the definition is complete and $\mathcal{T} = (O, e_O(\xi))$. Otherwise assume that p is a free point on ξ already taken in T. Write $\mathcal{R}_p(\xi) = \{q_1, \ldots, q_k\}$ and

$$s(q_i) = \frac{m_i}{n_i}, \quad i = 1, \ldots, k \quad \left(\gcd(m_i, n_i) = 1, \frac{m_1}{n_1} < \cdots < \frac{m_k}{n_k}\right).$$

Put $w_p = u_p + e_p(\xi)$ and

$$\begin{aligned} r_p &= \max\{\{i \mid n_i w_p > I(p, q_i)\} \cup \{0\}\} \\ \alpha_k &= I(p, q_k)/n_k, \quad \beta_k = 0 \\ \alpha_{\ell-1} &= \alpha_\ell - \varepsilon_{q_\ell}(\xi) m_\ell \qquad \ell = 1, \ldots, k \\ \beta_{\ell-1} &= \beta_\ell + \varepsilon_{q_\ell}(\xi) n_\ell \qquad \ell = 1, \ldots, k\,. \end{aligned} \tag{4.2}$$

Then the definition of $\mathcal{T}$ continues as follows:

(1) The satellites of p are

  (a) the points $q_1, \ldots, q_{r_p}$ and all points infinitely near to p preceding one of them, and

  (b) in case $r_p < k$ and $w_p > 0$, the satellite $\bar{q}$ of p with slope $s(\bar{q}) = (w_p - \alpha_{r_p})/\beta_{r_p}$ and all points infinitely near to p preceding it.

(2) For $q \in \mathcal{S}_p(\mathcal{T}) \setminus \{q_1, \ldots, q_{r_p}, \bar{q}\}$, $\varepsilon_q(\mathcal{T}) = 0$, $\varepsilon_{q_i}(\mathcal{T}) = \varepsilon_{q_i}(\xi)$ for $i = 1, \ldots, r_p$ and, if $\bar{q}$ is defined, $\varepsilon_{\bar{q}}(\mathcal{T}) = \gcd(\beta_{r_p}, w_p - \alpha_{r_p})$.

(3) The points next p in T are the points next p on ξ lying in the first neighbourhood of some q_i, $i = 1, \ldots, r_p$.

Remark 4.3 By (3.4), (α_i, β_i), $i = 0, \ldots, k$ are the vertices of the Newton polygon of $\widetilde{\xi}_p$ relative to coordinates whose first axis is not tangent to $\widetilde{\xi}_p$ and whose second axis is the exceptional divisor.

In particular, if $u_p \geq 0$, then $w_p \geq e_p(\xi) = n_k I(p, q_k)$, so in this case $r_p = k$ and therefore $\mathcal{S}_p(\mathcal{T}) = \mathcal{S}_p(\xi)$ and $\tau_q = e_q(\xi)$ for $q \in \mathcal{S}_p(\mathcal{T})$.

Remark 4.4 It easily follows from the definition of r_p, the above remark and (3.4.a) that in case $r_p > 0$, $w_p > I(p, q_{r_p})/n_{r_p} \geq \alpha_{r_p}$. Since $\alpha_0 = 0$ and we are assuming $w_p > 0$, in all cases $w_p - \alpha_{r_p} > 0$ and the definition of $\bar{q}$ makes sense.

It will turn out in the proof of next theorem that w_p is positive for all free points $p \in T$ and therefore the condition $w_p > 0$ in 1.b) above is in fact a redundant one.

Let us prove that T is actually a finite set.

Lemma 4.5 *The set T is finite.*

Proof: Since satellite points on a germ of curve ξ are always finitely many (they are among the singular points of $\xi_{\rm red}$) we take j_0 so that any point on ξ from the j_0-th neighbourhood onwards is free and, hence, proximate to just the point preceding it. Clearly the function u_p is strictly decreasing on these points (i.e. $u_p < u_{p'}$ if $p > p'$) and so p is free and $u_p \leq 0$ for all but finitely many points on ξ. Assume now that $p \in T$ is free (hence, it lies on ξ) and has $u_p \leq 0$. Then, clearly $\mathcal{S}_p(\xi) = \{p\}$, $s(p) = 1$, $I(p,p) = e_p(\xi) = \varepsilon_p(\xi) \geq w_p$, so $r_p = 0$ and there are no points next p in T. Thus, T is finite as claimed. ■

Theorem 4.6 *There exists a finite set $M \subset \mathbb{C}$ such that for $\lambda \in \mathbb{C} \setminus M$ the germs $\zeta^\lambda : f + \lambda g = 0$ go sharply through $\mathcal{T}$ and no two of them share any point outside of T.*

Proof: Unless otherwise stated all virtual transforms will be taken relative to the virtual multiplicities τ_q and denoted by the sign $\widehat{\ }$. If $p \in T$, we will write $\mathcal{E}_p$ for the germ at p of the exceptional divisor E_p.

Let $p \in T$, either $p = O$ or p a free point. We will use induction on the order of the neighbourhood p is belonging to for proving the following claim:

Claim. There exists a finite subset $M_p \subset \mathbb{C}$ so that for any $\lambda \in \mathbb{C} \setminus M_p$

a) $\mathcal{S}_p(\zeta^\lambda) = \mathcal{S}_p(\mathcal{T})$ and $e_q(\zeta^\lambda) = \tau_q$ for all $q \in \mathcal{S}_p(\mathcal{T})$.

b) Any point next p in T lies on ζ^λ.

c) For any point p' next p in T, both ξ and η go through all points q preceding p' with the virtual multiplicities τ_q and $\widehat{\xi}_{p'} = \widetilde{\xi}_{p'}$, $\widehat{\eta}_{p'} = w_{p'}\mathcal{E}_{p'}$ with $w_{p'} > 0$.

d) ζ^λ has no singular point next p outside of T and any two different germs ζ^λ share no point next p outside T.

It is clear that theorem (4.6), with $M = \bigcup_{p \in T} M_p$, follows from parts a) and d) of the above claim once it has been proved for all $p \in T$.

First we deal with the point O. Obviously $\mathcal{S}_O(\zeta^\lambda) = \mathcal{S}_O(\mathcal{T}) = \{O\}$ because O has no satellite points. Since $e_O(\xi) \leq e_O(\eta)$ there is at most one $\lambda_0 \in \mathbb{C}$ such that $e_O(\zeta^\lambda) = e_O(\xi)$ for $\lambda \neq \lambda_0$, as claimed in a).

If $e_O(\xi) = e_O(\eta)$, then $(T, \tau) = (\{O\}, e_O(\xi))$ and so there are no points next O in T. In this case, it is straightforward to check that for all but at most a finite number of λ the germs ζ^λ have $e_O(\xi)$ different tangents at O and no two of them have a common tangent, from which d) follows.

Assume now that $e_O(\xi) < e_O(\eta)$. Then, η goes through the points in the first neighbourhood of O on ξ. On the other hand, since we are assuming that η and ξ share no tangent, the effective multiplicity of η at the points infinitely near to O on ξ is zero, so $\widehat{\eta}_{p'} = (e_O(\eta) - e_O(\xi))\mathcal{E}_{p'}$, p' any point in the first neighbourhood of O on ξ. From the definition of $w_{p'}$ it follows that $w_{p'} = e_O(\eta) - e_O(\xi)$, which gives part c). Finally, since $e_O(\eta) > e_O(\xi)$, the tangent cone to the germs ζ^λ is the tangent cone to ξ for all $\lambda \in \mathbb{C}$, so part d) follows.

Let $p \in T$ be a free point infinitely near to O and assume, by induction, that a), b), c) and d) are satisfied for all free points in T preceding p. Next we will prove them for p.

Take local coordinates x, y at p so that the y-axis is the germ of the exceptional divisor at p and the x-axis is not tangent to $\widetilde{\xi}_p$.

Since $\zeta^\lambda : f + \lambda g = 0$, $\xi : f = 0$, $\eta : g = 0$, by (2.1), $\widehat{(\zeta^\lambda)}_p : \widetilde{f} + \lambda \widetilde{g} = 0$ where $\widetilde{f}$ is an equation of $\widetilde{\xi}_p = \widehat{\xi}_p$ and $\widetilde{g}$ is an equation of $\widehat{\eta}_p$. Since, by c) of the induction hypothesis, $\widehat{\eta}_p$ has equation $x^{w_p} = 0$, we may assume without restriction $\widetilde{g} = x^{w_p}$. For $\lambda \notin \bigcup_{q<p} M_q = M'_p$, by the induction hypothesis a), ζ^λ goes through the points preceding p with effective multiplicities equal to the virtual ones and so, $\widetilde{(\zeta^\lambda)}_p = \widehat{(\zeta^\lambda)}_p$.

Let $\mathcal{R}_p(\xi) = \{q_1, \ldots, q_k\}$ be the extremal satellites of p on ξ. Let $\Gamma_1, \ldots, \Gamma_k$ be the sides of $\mathbf{N}(\tilde{f})$ and $\Omega_1, \ldots, \Omega_k$ their associated equations. By (3.4), each Γ_i, $i = 1, \ldots, k$, has ends (α_i, β_i), $(\alpha_{i-1}, \beta_{i-1})$, $\beta_{i-1} > \beta_i$, given by the formulas (4.2), slope $-n_i/m_i$, with $s(q_i) = m_i/n_i$ ($\gcd(m_i, n_i) = 1$) and $I(p, q_i) = n_i\alpha_i + m_i\beta_i$.

By induction $w_p > 0$, so in case $r_p < k$, let $\bar{q}$ be the satellite of p with slope $s(\bar{q}) = (w_p - \alpha_{r_p})/\beta_{r_p}$ and let $\bar{\varepsilon} = \gcd(w_p - \alpha_{r_p}, \beta_{r_p})$. We define the set Λ in the following way

$$\Lambda = \begin{cases} \{(q_i, \varepsilon_{q_i}(\xi))\}_{i=1,\ldots,r_p} \cup \{(\bar{q}, \bar{\varepsilon})\} & \text{if } r_p < k \\ \{(q_i, \varepsilon_{q_i}(\xi))\}_{i=1,\ldots,r_p} & \text{if } r_p = k\,. \end{cases}$$

We associate to Λ the consistent cluster $\mathcal{A} = (A, \mu)$ as in (3.5). Notice that $A = \mathcal{S}_p(\mathcal{T})$ and since $\varepsilon_q(\mathcal{A}) = \varepsilon_q(\mathcal{T})$ for all $q \in A$, by (1.3), $\tau_q = \mu_q$ for all $q \in A$. So, we write $\mathcal{A} = (A, \tau)$.

The polygonal line $\mathbf{N}_{\mathcal{A}}$ has sides Γ_i, $i = 1, \ldots r_p$, with slope $-1/s(q_i)$, and, in case $r_p < k$, a further side $\bar{\Gamma}$ with slope $-\beta_{r_p}/(w_p - \alpha_{r_p})$ and $\bar{\varepsilon} + 1$ integral points. Clearly, for all but finitely-many $\lambda \in \mathbb{C}$, $\mathbf{N}(\widetilde{f} + \lambda x^{w_p}) = \mathbf{N}_{\mathcal{A}}$.

Thus, after enlarging M'_p to a still finite set M''_p, for $\lambda \in \mathbb{C}\setminus M''_p$, $\widetilde{(\zeta^\lambda)}_p = \widehat{(\zeta^\lambda)}_p$ and $\mathbf{N}(\widetilde{f} + \lambda x^{w_p}) = \mathbf{N}_{\mathcal{A}}$. Therefore, by (3.6.a), for $\lambda \notin M''_p$, $\mathcal{S}_p(\mathcal{T}) = \mathcal{S}_p(\zeta^\lambda)$ and $e_q(\zeta^\lambda) = \tau_q$ for all $q \in \mathcal{A}$, as claimed in a).

Now we prove part b). For $\lambda \notin M''_p$, the Newton polygons $\mathbf{N}(\widetilde{f})$ and $\mathbf{N}(\widetilde{f} + \lambda x^{w_p})$ have in common the sides $\Gamma_1, \dots, \Gamma_{r_p}$ with the same associated equations so, by (3.1), the germs $\widetilde{(\zeta^\lambda)}_p : \widetilde{f} + \lambda x^{w_p} = 0$ and $\widetilde{\xi}_p : \widetilde{f} = 0$ go through the same points next p in the first neighbourhood of $q_1, \dots, q_{r_p}$, that is, the points next p in T, as wanted.

Next we will prove part c). Let p' be a point next p in T, so p' is in the first neighbourhood of q_i for some $i = 1, \dots, r_p$. First we deal with $\widetilde{\xi}_p$. By (3.6.b), $\widetilde{\xi}_p$ goes through $\mathcal{A}$ because $\mathbf{N}(\widetilde{f})$ has no vertex below $\mathbf{N}_{\mathbf{A}}$. Since, by induction, $\widetilde{\xi}_p = \widehat{\xi}_p$, then the virtual transform $\widehat{\xi}_{p'}$ is the virtual transform of $\widetilde{\xi}_p$ relative to the virtual multiplicities τ_q, $p \le q < p'$.

On the other hand, for $\lambda \notin M''_p$, $\mathbf{N}(\tilde{f} + \lambda x^{w_p}) = \mathbf{N}_{\mathcal{A}}$, so, by (3.7.b), $u^{\mathcal{A}}_{q_i}(\widetilde{\xi}_p) = \deg_{(n_i,m_i)}(\widetilde{f}) - \deg_{(n_i,m_i)}(\widetilde{f} + \lambda x^{w_p})$. That is, by (2.3.a), $\widehat{\xi}_{p'}$ contains $\deg_{(n_i,m_i)}(\widetilde{f}) - \deg_{(n_i,m_i)}(\widetilde{f} + \lambda x^{w_p})$ times $\mathcal{E}_{p'}$. Since $\mathbf{N}(\widetilde{f})$ and $\mathbf{N}(\widetilde{f} + \lambda x^{w_p})$ have in common the side Γ_i of slope $-1/s(q_i) = -n_i/m_i$, then $\deg_{(n_i,m_i)}(\widetilde{f}) = \deg_{(n_i,m_i)}(\widetilde{f} + \lambda x^{w_p})$ and therefore $\widehat{\xi}_{p'}$ does not contain $\mathcal{E}_{p'}$. Hence, $\widetilde{\xi}_{p'} = \widehat{\xi}_{p'}$ as claimed.

Now we deal with $\widehat{\eta}_p$. Since we have shown that $\widetilde{\xi}_{p'} = \widehat{\xi}_{p'}$, by (2.3.a), $u^{\mathcal{T}}_{q_i}(\xi) = 0$ and, by (2.3.c),

$$v_{q_i}(\eta) - u^{\mathcal{T}}_{q_i}(\eta) = v_{q_i}(\xi)\,. \tag{3}$$

Let $\mathcal{K}_{p'}$ be as in (4.1). Since ξ goes through $\mathcal{K}_{p'}$ with effective multiplicities equal to the virtual ones, by (2.3.c),

$$v_{q_i}(\eta) - u^{\mathcal{K}_{p'}}_{q_i}(\eta) = v_{q_i}(\xi)\,. \tag{4}$$

Thus, by (3) and (4), $u^{\mathcal{K}_{p'}}_{q_i}(\eta) = u^{\mathcal{T}}_{q_i}(\eta)$ and, by (4.1), $u^{\mathcal{K}_{p'}}_{q_i}(\eta) = u_{q_i}$. Since, by induction, $\widehat{\eta}_p : x^{w_p} = 0$, by (3.6.b) $\widehat{\eta}_p$ goes through $\mathcal{A}$. Thus, by definition of going through, η goes through the points q preceding p' with the virtual multiplicities τ_q and $\widehat{\eta}_{p'}$ is the virtual transform of $\widehat{\eta}_p = w_p\mathcal{E}_p$ relative to the virtual multiplicities τ_q, $p \le q < p'$.

Hence, by (2.3.a), $u^{\mathcal{T}}_{q_i}(\eta) = u^{\mathcal{A}}_{q_i}(\widehat{\eta}_p)$ and so, by (3.7.b),

$$u^{\mathcal{T}}_{q_i}(\eta) = \deg_{(n_i,m_i)}(x^{w_p}) - \deg_{(n_i,m_i)}(\widetilde{f} + \lambda x^{w_p}) = w_p n_i - I(p, q_i).$$

Since, by definition, $u_{q_i} = w_{p'}$, then $w_{p'} = w_p n_i - I(p, q_i)$ and so, as $i \le r_p$, by (4.2), $w_{p'} > 0$ as claimed.

Finally we show part d). We have just proved that for $\lambda \notin M''_p$, $\widetilde{\xi}_p$ and $\widetilde{(\zeta^\lambda)}_p$ share the sides $\Gamma_1, \dots, \Gamma_{r_p}$ of their Newton polygons and also have the same associated equations $\Omega_1, \dots, \Omega_{r_p}$; therefore, for $\lambda \notin M''_p$, the points next p on ζ^λ and not belonging to T must be proximate to $\bar{q}$, the extremal satellite of p corresponding to the last side $\bar{\Gamma}$ of $\mathbf{N}(\widetilde{f} + \lambda x^{w_p})$.

Since the equation associated to this side is

$$\bar{\Omega} = \sum_{(\alpha,\beta)\in\bar{\Gamma}} a_{\alpha\beta} z^\beta + \lambda,$$

there is a finite set $M_p \subset \mathbb{C}$, $M''_p \subset M_p$, such that for all $\lambda \notin M_p$, all roots of $\bar{\Omega}$ are simple. So, by (3.3), all points on ζ^λ in the first neighbourhood of $\bar{q}$ are non-singular. Moreover, since different values of λ give different roots of $\bar{\Omega}$, no two germs ζ^λ share any point next p in the first neighbourhood of $\bar{q}$, as claimed. So, the claim is satisfied. ■

5. An example

Under the hypothesis of 4, let ξ be irreducible with characteristic exponents $\{10/6, 15/6\}$ (see figure 1) and write $e_O(\eta) = n$.

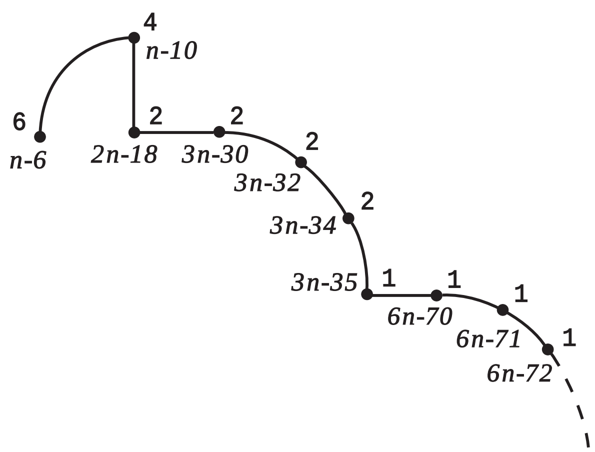

FIGURE 1: Enriques diagram of the points on ξ up to the 9-th neighbourhood. Besides each point p there is shown its multiplicity $e_p(\xi)$ and the corresponding value of u_p as a function of n.

The singularities of ζ^λ may be described, according to the values of n, as follows (cf. figure 2):

- $\mathbf{n = 6:}$ ζ^λ has an ordinary singular point of multiplicity six.
- $\mathbf{n = 7:}$ ζ^λ is irreducible with single characteristic exponent $7/6$ and tangent to ξ.
- $\mathbf{n = 8:}$ ζ^λ has two branches both tangent to ξ, with characteristic exponent $4/3$ and sharing all their singular points.
- $\mathbf{n = 9:}$ ζ^λ has three branches both tangent to ξ, with characteristic exponent $3/2$ and sharing all their singular points.
- $\mathbf{n = 10:}$ As in case $n = 8$ but with characteristic exponent $5/3$.
- $\mathbf{n = 11:}$ ζ^λ is irreducible with two characteristic exponents $\{10/6, 13/6\}$. All its singular points but the last one lie on ξ.
- $\mathbf{n \geq 12:}$ ζ^λ is equisingular to ξ, ζ^λ and ξ share all their singular points and $6n - 70$ non-singular points (C^0-sufficiency degree of ξ is 12).

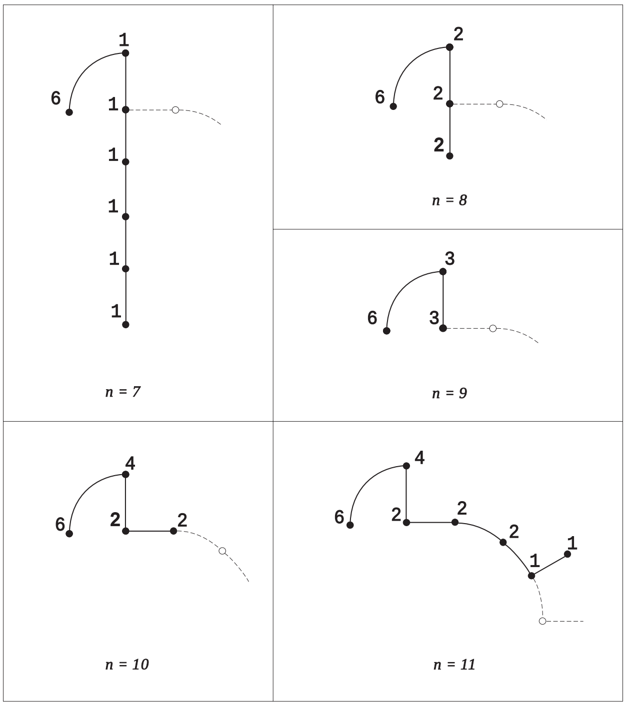

FIGURE 2: Enriques diagrams of the weighted clusters $\mathcal{T}$ for $n = 7, \ldots, 11$. Some points on ξ not in $\mathcal{T}$ are represented by unlabelled points on dotted lines in order to show relative position of infinitely near points.

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