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On independent times and positions for Brownian motions

Bernard de Meyer, Bernard Roynette, Pierre Vallois and Marc Yor

Abstract

Let $(B_t \ ; \ t \geq 0)$, $\big($resp. $((X_t, Y_t) \ ; \ t \geq 0)\big)$ be a one (resp. two) dimensional Brownian motion started at 0. Let T be a stopping time such that $(B_{t\wedge T} \ ; \ t \geq 0)$ $\big($resp. $(X_{t\wedge T} \ ; \ t \geq 0)$; $(Y_{t\wedge T} \ ; \ t \geq 0)\big)$ is uniformly integrable. The main results obtained in the paper are:

1) if T and B_T are independent and T has all exponential moments, then T is constant.
2) If X_T and Y_T are independent and have all exponential moments, then X_T and Y_T are Gaussian.

We also give a number of examples of stopping times T, with only some exponential moments, such that T and B_T are independent, and similarly for X_T and Y_T. We also exhibit bounded non-constant stopping times T such that X_T and Y_T are independent and Gaussian.

1. Introduction

1.1 Here is the general thema of this paper :
Consider $(B_t, \ t \geq 0)$ a one-dimensional Brownian motion starting at 0, with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$, *i.e.*:

(i) a.s., $B_0 = 0$ and $t \to B_t$ is continuous,

(ii) $B_t - B_s$, for $t > s \geq 0$ is Gaussian distributed with mean 0 and variance $t - s$, and is independent of $\mathcal{F}_s$.

We shall not assume a priori $(\mathcal{F}_t)_{t\geq 0}$ to be the natural filtration of $(B_t, t \geq 0)$.

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We also consider $(\mathcal{F}_t)$ stopping times T such that :

$$(B_{t\wedge T}\ ;\ t \geq 0) \text{ is uniformly integrable.} \tag{1.1}$$

Following Falkner ([15], Proposition 4.9, p. 386), we shall call such stopping times B-standard times.

As is well-known, the integrability condition :

$$E(\sqrt{T}) < \infty \tag{1.2}$$

implies (1.1), but not conversely.

Let us define the set

$$\mathcal{J}_T \stackrel{\text{def}}{=} \left\{ \lambda \in \mathbb{R} : \left(\exp\{\lambda B_{t\wedge T} - \frac{\lambda^2}{2}(t\wedge T)\},\ t \geq 0 \right) \begin{array}{c}\text{is a uniformly}\\ \text{integrable martingale}\end{array} \right\}$$

In particular, the well known Novikov's criterion implies that :

$$\text{if } E\left[\exp\left(\frac{a^2}{2}T\right)\right] < \infty, \text{ then } [-a,a] \subset \mathcal{J}_T\,.$$

In any case, for any $\lambda \in \mathcal{J}_T$, Wald's equation :

$$E\Big[\exp(\lambda B_T - \frac{\lambda^2}{2}T)\Big] = 1 \tag{1.3}$$

holds, which confers a "general" character to this equation.

Even if $\mathcal{J}_T = \mathbb{R}$ and if μ, the law of B_T, is given, equation (1.3) does not determine the law of T. Indeed, in the probabilistic literature, for a given μ, there are many different solutions T to Skorokhod's problem relative to μ, that is : B-standard times T such that the law of B_T is μ ; see, in particular, [2], [3], [4], [12], [39], [40].

This remark brings us naturally to look for some additional assumptions on the joint law of (B_T, T), under which one hopes that the law of T is determined from the law μ of B_T.

For instance, if we assume that

$$E[e^{\theta T}] < +\infty, \qquad \text{for some } \theta > 0, \tag{1.4}$$

and, furthermore :

$$B_T \text{ and } T \text{ are independent,} \tag{1.5}$$

then Wald's equation (1.3) shows that the law of T is determined from μ.

However it turns out that the conjunction of (1.5) and

$$T \text{ admits all exponential moments (then, } \mathcal{J}_T = \mathbb{R}) \tag{1.6}$$

leads to a very restricted class of laws, as the following theorem asserts.

Theorem 1 *If T is a $(\mathcal{F}_t)$ stopping time such that T and B_T are independent, and T admits all exponential moments, then T is a.s. constant (and consequently, B_T is Gaussian).*

The proof is given in Section 2.

It is not enough in Theorem 1 to assume that some positive exponential moment of T is finite, as shown by T_a^* with:

$$T_a^* \equiv \inf\big\{t : |B_t| = a\big\}.$$

It is well known (cf. [23]) that:

$$E[e^{\lambda T_a^*}] < +\infty \quad \text{iff} \quad \lambda < \frac{\pi^2}{8a^2},$$

and $B(T_a^*)$ and T_a^* are independent.

When some positive exponential moment of T is finite and T is independent of B_T, the distributions of T and B_T determine each other uniquely via (1.3). But as shown later there are infinitely many different T's corresponding to B_T with uniform distribution on $[-1, 1]$ (cf. Proposition 3.4; 1), Remark 3.3; 2) and Example 3 in Section 3.2).

1.2 We now drop the condition (1.6), but retain the independence hypothesis (1.5). The next theorem shows that this hypothesis alone has strong consequences concerning the law of B_T.

Theorem 2 *Suppose that T is B-standard, T and B_T are independent. Then*

i) B_T admits all exponential moments ;

ii) For every $\lambda \in \mathbb{R}$, $E\,(\exp \lambda B_T)\, E\,(\exp -\frac{\lambda^2}{2} T) = 1$. In particular $\mathcal{J}_T = \mathbb{R}$.

iii) a) The function $\varphi(z) = E\,(\exp zB_T)$ $(z \in \mathbb{C})$ is holomorphic on $\mathbb{C}$.

b) For every $z \in \mathbb{C}$, $\varphi(z) = \varphi(-z)$; consequently, the law of B_T is symmetric.

c) There exists $c > 0$ such that $\varphi(\lambda) \leq \exp c\lambda^2$ $(\lambda \in \mathbb{R})$.

d) φ has no zeros on the set $\{z = x + iy : |x| \geq |y|\}$.

iv) $E[e^{\lambda T}] < +\infty$ for all $\lambda < \lambda_0$, for some $\lambda_0 > 0$.

We consider this theorem to be a first step in the description of the laws of pairs (B_T, T), with B_T and T independent, about which, despite the present study, we still do not know very much.

1.3 We now discuss related questions which involve a two dimensional $(\mathcal{F}_t)$ Brownian motion : $Z_t = X_t + iY_t,\ t \geq 0$ $\big($again, we do not assume a priori that $(\mathcal{F}_t)$ is the natural filtration of $(Z_t)\big)$.

We first remark that, if S is a stopping time with respect to the filtration of X and if S and X_S are independent, then X_S and Y_S are independent.

More generally, this brings us to the study of $(\mathcal{F}_t)$ stopping times T such that X_T and Y_T are independent.

Theorem 3 *Let $Z_t = X_t + iY_t$ be a $2-$dimensional $(\mathcal{F}_t)$ Brownian motion started at 0, and T is assumed to be both a X-and Y-standard time. We assume:*

(1.7) *X_T and Y_T have all exponential moments,*

(1.8) *X_T and Y_T are independent.*

Then, X_T and Y_T are two independent centered Gaussian variables, with the same variance.

The proof will be given in Section 5.

However, a main difference with the conclusion of Theorem 1 is that, under the hypotheses of Theorem 3, there exist some T's which are not a.s. constant. We prove this by solving affirmatively the following related question which was asked by Tortrat (cf [24]), and is relative to a one-dimensional $(\mathcal{F}_t)$ Brownian motion (B_t): does there exist a bounded non constant $(\mathcal{F}_t)$ stopping time T such that B_T is Gaussian?

We generalize this question to $d-$dimensional Brownian motions and we construct in Section 5 (cf Theorems 5.1 and 5.6) a family of such bounded non constant stopping times. More precisely, we prove:

Theorem 4 *For each d, there exists a d-dimensional Brownian motion $(B_t\ ;\ t \geq 0)$ started at 0, a non constant and bounded stopping time T such that the law of B_T is $\mathcal{N}(0, I_d)$ (*). Moreover, if $d \geq 3$, T can be chosen as a stopping time with respect to the natural filtration of $(B_t\ ;\ t \geq 0)$.*

Related to Tortrat's question, here is an earlier question which was asked by Cantelli (1917), and discussed by Tricomi ([43]) and Dudley ([13]): let

(*) I_d denotes the identity on $\mathbb{R}^d$.

X, U, X' be three real valued r.v's. such that :

$$(X, X') \text{ is a reduced Gaussian variable } \mathcal{N}(0, I_2) \tag{1.9}$$

$$X' \text{ is independent from } (X, U) \tag{1.10}$$

$$U \geq 0. \tag{1.11}$$

Then, define $Y = X + UX'$.

Under which condition is Y Gaussian? Cantelli formulated the conjecture that, if $U = f(X)$, then Y is Gaussian iff U is a.s. constant. In fact, in Section 5, we construct a class of examples where U is not constant, and Y is Gaussian.

Let us explain how Cantelli's problem is related to Wald's equation : indeed, if Y is Gaussian with variance σ^2 and (1.9), (1.10), (1.11) are satisfied, then

$$\exp\Big(\frac{\lambda^2}{2}\sigma^2\Big) = E\Big[\exp\Big(\lambda X + \frac{\lambda^2}{2}U^2\Big)\Big] .$$

It is not difficult to deduce from this that $U^2 \leq \sigma^2$ a.s., and so, if we define $T = \sigma^2 - U^2$, T is a positive r.v. such that Wald's equation :

$$E\Big[\exp\Big(\lambda X - \frac{\lambda^2}{2}T\Big)\Big] = 1$$

is satisfied.

To conclude this introduction, we indicate how the rest of this paper is organized :

Section 2 consists in the proof of Theorem 1, presented in the framework of Brownian motion with drift. Section 3 presents a number of examples of pairs (B_T, T), with B_T and T independent and exploits several intertwinings between Brownian motion and a second Markov process. Section 4 consists in the proof of Theorem 2, and includes some remarks on the laws of (B_T, T) again in the independent case. Section 5 is devoted to the proof of Theorem 3, and Section 6 to that of Theorem 4.

We have gathered in two appendices:

a) a discussion of the Skorokhod embedding problem for the space-time Brownian motion $((B_t, t); t \geq 0)$, a question which pervades our whole study;

b) a generalization of Theorem 1 for the Ornstein-Uhlenbeck process.

After writing the present paper, we found that a similar discussion for the pairs (B_T, L_T), where $(L_t; t \geq 0)$ denotes the local time of $(B_t; t \geq 0)$ at 0, could be done, and in fact is considerably simpler (see [41]).

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2. A proof of Theorem 1

We need to show that if T is a $(\mathcal{F}_t)$ stopping time, having all exponential moments, and such that B_T and T are independent, then T is constant. Our approach allows also to prove similar results, when the Brownian motion $(B_t)_{t\geq 0}$ is replaced by Brownian motion with drift, Ornstein-Uhlenbeck or Bessel processes. We only give the full proof for Brownian motion with drift δ, including the case $\delta = 0$. The arguments for the Ornstein-Uhlenbeck and Bessel cases are postponed to the Appendix and to Corollary 3.7.

Let $\big(B(t);\ t \geq 0\big)$ be a $(\mathcal{F}_t)-$Brownian motion, taking its values in $\mathbb{R}$ and starting at 0. We do not suppose $(\mathcal{F}_t)_{t\geq 0}$ is the natural filtration of $(B_t)_{t\geq 0}$. The Brownian motion with drift δ is the process :

$$B_\delta(t) := B(t) + \delta t\ ;\ t \geq 0.$$

δ is a real number (which may be equal to 0).

We start with a preliminary result, which will be useful in the sequel.

Lemma 2.1 *Suppose T is a $(\mathcal{F}_t)_{t\geq 0}$ stopping time having all exponential moments. Then the characteristic function $\varphi(z) = E[e^{izB_\delta(T)}]$, $z \in \mathbb{C}$ is well defined, and holomorphic on the whole plane $\mathbb{C}$.*

Proof. The usual exponential (local) martingale and Fatou arguments lead easily to the following : for a complex number z, we set $\lambda = |z|$, and the following inequality holds

$$E\Big[\Big|\exp\big(zB_\delta(T)\big)\Big|\Big] \leq \Big(E\Big[\exp\big\{2(|\delta|\lambda + \lambda^2)T\big\}\Big]\Big)^{1/2}. \tag{2.1}$$

Using Cauchy-Schwarz, the function $z \to E\big[B_\delta(T)e^{zB_\delta(T)}\big]$ is locally bounded. A classical argument now shows that φ can be defined for any $z \in \mathbb{C}$, and is holomorphic. (cf [20] for some similar arguments). ■

We generalize now Theorem 1 to the case of Brownian motion with drift.

Theorem 2.2 *Let $\delta \in \mathbb{R}$ and $(B_t : t \geq 0)$ be a $(\mathcal{F}_t)_{t\geq 0}$ Brownian motion and T a $(\mathcal{F}_t)_{t\geq 0}$ stopping time, with all exponential moments. We assume that for any $z \in \mathbb{C}$,*

$$E\big[e^{zB_\delta(T)}e^{-(z\delta+z^2/2)T}\big] = E\big[e^{zB_\delta(T)}\big]E\big[e^{-(z\delta+z^2/2)T}\big]. \tag{2.2}$$

Then T is a.s. constant and $B_\delta(T)$ is a gaussian r.v.

Remark 2.3 1) Since T has all exponential moments, both sides of (2.2) are equal to 1.
2) If we suppose T is bounded, we can give a shorter proof of Theorem 2.2; see at the end of this section (alinea 2.2).

Proof of Theorem 2.2. *(i)* Let $\lambda \in \mathbb{R}$. Property (2.2) and Remark 2.3, 1), imply that

$$E\Big[e^{\lambda B_\delta(T)}\Big] = \frac{1}{E\Big[\exp\big\{-(\lambda\delta + \lambda^2/2)T\big\}\Big]}.$$

We choose a such that $P(T < a) > 0$.
Since

$$e^{-(\lambda\delta+\lambda^2/2)T} \geq e^{-\big(|\lambda\delta|+\lambda^2/2\big)T} \geq e^{-\big(|\lambda\delta|+\lambda^2/2\big)a}\, 1_{\{T<a\}},$$

then

$$E\Big[e^{\lambda B_\delta(T)}\Big] \leq \frac{1}{P(T<a)} e^{\big(|\lambda\delta|+\lambda^2/2\big)a} \quad ; \forall \lambda \in \mathbb{R}.$$

Consequently, for any $z \in \mathbb{C}$,

$$\text{(2.3)} \qquad \Big|E\Big[e^{zB_\delta(T)}\Big]\Big| \leq E\Big[e^{|z||B_\delta(T)|}\Big] \leq E\Big[e^{|z|B_\delta(T)} + e^{-|z|B_\delta(T)}\Big] \leq \frac{2}{P(T>a)} e^{\big(|z||\delta|+|z|^2/2\big)a}.$$

(ii) The order of a holomorphic function $\psi : \mathbb{C} \to \mathbb{C}$, is the element of $\mathbb{R} \cup \{+\infty\}$ defined as follows :

$$\text{(2.4)} \qquad o(\psi) = \limsup_{r\to+\infty} \frac{\ln\Big(\ln\big(M(r,\psi)\big)\Big)}{\ln\ r},$$

where $M(r,\psi) = \sup_{|z|=r} \big|\psi(z)\big|$ (cf [44]).
Let ψ be the characteristic function of $B_\delta(T)$:

$$\psi(z) = E\Big[e^{izB_\delta(T)}\Big] \quad , \ z \in \mathbb{C}.$$

Lemma 2.1 tells us that ψ is defined and holomorphic on $\mathbb{C}$. Moreover inequality (2.3) implies that the order of ψ is less than or equal to 2.

Let us summarise the properties of ψ : ψ is holomorphic on $\mathbb{C}$, does not vanish and has a finite order. Thus, we may apply Hadamard's theorem $\big($[44], p. 429–433$\big)$: there is a polynomial P, with degree less than or equal to 2 such that $\psi(z) = \exp P(z) = \exp\{a + bz + cz^2\}$. $\psi(0)$ being equal to 1, then $a = 0$.

Now, relation (2.2) implies

$$E\big[\exp-(\delta z+z^2/2)T\big]=\exp\{-bz-cz^2\}. \tag{2.5}$$

For any $u\geq 0$, the second order equation (in the z variable) $u=\delta z+z^2/2$, has two real solutions $z=-\delta\pm\sqrt{\delta^2+2u}$. So (2.5) gives easily $E[e^{-uT}]=e^{-2cu}$, hence $T=2c$ a.s. ∎

2.2. Another proof of Theorem 2.2, for bounded T's.

(i) We assume that $T\leq a$, for a positive constant a , and that the r.v.'s $B_\delta(T)$ and T are independent. We introduce : $B'_\delta(s)=B_\delta(T+s)-B_\delta(T)$; $s\geq 0$. Then, $\big(B'_\delta(s)\ ;\ s\geq 0\big)$ is a Brownian motion with drift δ, starting at 0, and independent of $\mathcal{F}_T$.

T being smaller than a, we may write:

$$B_\delta(a)=B_\delta(T)+B'_\delta(a-T). \tag{2.6}$$

On one hand T and $B_\delta(T)$ are $\mathcal{F}_T-$measurable, on the other hand T and $B_\delta(T)$ are independent r.v.'s; consequently, $B_\delta(T)$ and $B'_\delta(a-T)$ are independent r.v's.

But since $B_\delta(a)$ has a Gaussian distribution, the Cramer-Lévy theorem (see for instance [25], p. 243) implies that $B_\delta(T)$ and $B'_\delta(a-T)$ are Gaussian r.v's.

$B_\delta(T)$ being normally distributed, using the relation (2.2), then T is constant.

(ii) In addition, we now give an even more direct proof, in the case $\delta=0$. Using the scaling property of Brownian motion $\big(B'_0(t)\big)_{t\geq 0}$, the following identity in law holds :

$$B'_0(a-T)\overset{(d)}{=}\sqrt{a-T}\,G\,,$$

G denoting a standard Gaussian r.v. (i.e. with zero mean and unit variance), independent of T.

Moreover $B_0(T)\sim\mathcal{N}\big(0,E(T)\big)$, then (2.6) tells us $B'_0(a-T)\sim\mathcal{N}\big(0,a-E(T)\big)$. Comparing the two results we have : $\sqrt{a-T}\,G\overset{(d)}{=}\sqrt{a-E(T)}\,G$. Therefore T is constant. ∎

Remark 2.4 When T is bounded, the above proof of Theorem 2.2 is based on the Cramer-Lévy theorem and the fact that Brownian motion with drift has independent increments.

We now present an extension of Theorem 2.2 for a linear combination of a Brownian and a Poisson process.

Let $\big(N(t)\ ;\ t \geq 0\big)$ be a Poisson process, independent of the Brownian motion $\big(B_0(t)\ ;\ t \geq 0\big)$. We set $X(t) = aN(t) + bB_0(t) + ct\ ;\ t \geq 0$, where a, b, c are three real numbers. $\big(X(t)\big)_{t\geq 0}$ is a Lévy process.

Linnick generalized the result of Cramer-Lévy to processes $\big(X(t);\ t \geq 0\big)$ of the previous type $\big($see for instance [25], p. 245$\big)$: if there exist $t > 0$, and two independent r.v's ξ_1 and ξ_2 such that $X(t) \overset{(d)}{=} \xi_1 + \xi_2$, then there exist four independent r.v.'s N_1, N_2, G_1, G_2, such that N_1 (resp. N_2) is Poisson distributed, and G_1 (resp. G_2) is Gaussian, and $(a_1, a_2, b_1, b_2, c_1, c_2) \in \mathbb{R}^6$ such that $\xi_1 \overset{(d)}{=} a_1N_1 + b_1G_1 + c_1$ and $\xi_2 \overset{(d)}{=} a_2N_2 + b_2G_2 + c_2$.

The proof of this remark is quite straightforward. When $b = c = 0$ and $a = 1$ (i.e. $X(t) = N(t)$) the result is known as Raïkov's theorem $\big($[25], p. 243$\big)$.

As a result we obtain:

Proposition 2.5 *Let $\big(N(t)\,;\, t \geq 0\big)$ be a Poisson process, independent of the Brownian motion $\big(B_0(t)\ ;\ t \geq 0\big)$, and $X(t) = aN(t)+bB_0(t)+ct\ ;\ t \geq 0$. Suppose T is a bounded stopping time such that $X(T)$ and T are independent ; then T is a.s. constant.*

Remark 2.6 Let $(X_t\,;\, t \geq 0)$ be either an Ornstein-Ulhenbeck process started at 0 with parameter $a \neq 0$, or a Bessel process with dimension $d > 0$, started at 0. We prove the following:
If T is a bounded stopping time such that X_T and T are independent, then T is a.s. constant (cf Theorem 7.8 below).

3. Examples of independent pairs $(T,\, B_T)$

In this section $(B_t,\ t \geq 0)$ will denote a one dimensional $(\mathcal{F}_t)_{t\geq 0}$−Brownian motion, starting at 0. We exhibit an easy procedure which allows to create a large class of examples of $(\mathcal{F}_t)_{t\geq 0}$− stopping times T such that B_T and T are independent r.v.'s. As usual T is assumed to be B-standard.

3.1. Examples obtained by iteration

Suppose that T_1 is a $(\mathcal{F}_t)_{t\geq 0}$−stopping time and the two r.v.'s B_{T_1} and T_1 are independent. Let $(B'_t)_{t\geq 0}$ be the Brownian motion : $B'_t = B_{t+T_1} - B_{T_1}$ $t \geq 0$. Consider T_2 a $(\mathcal{F}_{T_1+t})$ stopping time, such that B'_{T_2} and T_2 are independent and T_2 is independent of $\mathcal{F}_{T_1}$.

Lemma 3.1 *$B_{T_1+T_2}$ and $T_1 + T_2$ are independent.*

We begin with a class of examples based on the hitting times family $(T^*_a; a > 0)$, where

$$T^*_a = \inf\{t \geq 0,\ |B_t| = a\},\ a \geq 0. \tag{3.1}$$

As an easy consequence of Lemma 3.1, we obtain:

Proposition 3.2 *Let $(a_n)_{n\geq 1}$ be a sequence of positive numbers such that $\sum_{n\geq 1} a_n^2$ is finite. Consider $(U_k)_{k\geq 1}$ the sequence of stopping times defined by induction :*

$$U_1 = T^*_{a_1}, \quad U_{k+1} = \inf\{t \geq U_k;\ |B_t - B_{U_k}| = a_{k+1}\},\ k \geq 1.$$

1) $(U_k)_{k\geq 1}$ is an increasing sequence of stopping times, converging, as $k \to \infty$, to U, which is a.s. finite.

2) U and B_U are two independent r.v.'s.

3) $E(U) = \sum_{k\geq 1} a_k^2$ and $B_U \stackrel{(d)}{=} \sum_{k\geq 1} a_k \varepsilon_k$, where $(\varepsilon_k)_{k\geq 1}$ is a sequence of i.i.d. random variables such that $P(\varepsilon_k = \pm 1) = \dfrac{1}{2}$.

4) The Laplace transforms of B_U and U are :

$$\begin{aligned} E[e^{\lambda B_U}] &= \prod_{k\geq 1} \cosh(\lambda a_k) \\ E[e^{-\lambda^2 U/2}] &= \prod_{k\geq 1} \Big(\frac{1}{\cosh(\lambda a_k)}\Big), \end{aligned} \qquad \lambda \geq 0. \tag{3.2}$$

Remark 3.3 1) The distributions of r.v.'s of the form $\sum_{k\geq 1} a_k \varepsilon_k$ are of pure type (see [7] p. 49).
2) Obviously any permutation acting on $(a_k)_{k\geq 1}$ does not change the distribution of (U, B_U). Therefore there exists an uncountable number of the previous constructions leading to the same final distribution.

For some extensions of the independence property of $B_{T^*_a}$ and T^*_a to higher dimensional Brownian motions with drift, see [45], [46], [36] and ([38], vol 2, p. 84). More precisely Reuter proved (cf [38], vol 2, p. 84, theorem 39.6) the following theorem: let $\delta > 0$ and $B_\delta(t) = B(t) + \delta t$; let also $T = \inf\{t > 0; |B_\delta(t)| = 1\}$. Then T and $B_\delta(T)$ are independent.

Some interesting properties in the above iteration procedure are summarized, without proof, in the following proposition :

Proposition 3.4 *We denote by U the stopping time associated with the sequence $(a_k)_{k\geq 1}$ (cf Proposition 3.2).*

1) Suppose $a_k = 2^{-k}$, $k \geq 1$, then the law of B_U is uniform on $[-1, 1]$.

2) Let $(a_k)_{k\geq 1}$ be a sequence of positive numbers such that

$$(3.3) \qquad (i) \lim_{k\to\infty} 2^k a_k = 0 \ ; \quad (ii)\ a_k \geq \sum_{n>k} a_n, \quad \forall n \geq 1.$$

Then the distribution of B_U is singular with respect to the Lebesgue measure. The sequence $a_k = 3^{-k}$; $k \geq 1$, satisfies (3.3), and the law of $2B_U$ is the Cantor measure on $[-1, 1]$.

3.2. Examples obtained from intertwinings

The following set-up, which originates from [35] and [10], provides us very naturally with pairs of independent random variables (T, B_T), where :

$$\begin{cases} \text{a) } (B_t,\ t \geq 0) \text{ is a (one dimensional, say) Brownian motion with} \\ \text{respect to a given filtration } (\mathcal{F}_t) \ ; \\ \text{b) } T \text{ is a (particular) } (\mathcal{F}_t) \text{ stopping time.} \end{cases}$$

We first consider, more generally, on a given probability space, a pair $\big(\{(B_t), (\mathcal{F}_t)\}\,;\, \{(Y_t), (\mathcal{G}_t)\}\big)$ of "good" Markov processes valued in $\mathbb{R}$, such that

$$(3.4) \quad \begin{cases} \text{(i) for any } t,\ \mathcal{G}_t \subseteq \mathcal{F}_t, \\ \text{(ii) there exists a Fellerian Markov kernel } K : (\mathbb{R}, \mathcal{B}_{\mathbb{R}}) \to \\ (\mathbb{R}, \mathcal{B}_{\mathbb{R}}) \text{ (i.e., for any } f \in C_c(\mathbb{R}),\ Kf \text{ is continuous), such that} \\ \forall t > 0,\ \forall f \in C_c(\mathbb{R}), \quad E\big[f(B_t)|\mathcal{G}_t\big] = Kf(Y_t). \end{cases}$$

In this section, all our examples of independent pairs (T, B_T) will be obtained as consequences of the following

Proposition 3.5 *Assume that (B_t) and (Y_t) satisfy (3.4). Then the following holds :*

α) for any $(\mathcal{G}_t)$ stopping time T, one has :

$$(3.5) \quad \forall f \text{ Borel} : \mathbb{R} \to \mathbb{R}_+, \qquad E\big[f(B_T)|\mathcal{G}_T\big] = Kf(Y_T), \text{ on } (T < \infty).$$

β) Let $T_a = \inf\{t \geq 0 : Y_t > a\}$. If $(Y_t,\ t \geq 0)$ does not jump upwards, then conditionally on $(T_a < \infty)$, the r.v. B_{T_a} is independent from $\mathcal{G}_{T_a}$, hence independent from T_a, and its law is given by $K(a, dx)$.

Proof. To prove α) we use (3.4) *(ii)*,and approximate a general $(\mathcal{G}_t)$ stopping time T by a decreasing sequence of $(\mathcal{G}_t)$ stopping times which take only a countable number of values.
Then the property α) follows from the right continuity of (Y_t) on one hand, and the Feller property of K on the other hand.
β) Since (Y_t) does not jump upwards, one has :

$$Y_{T_a} = a, \quad \text{on } \{T_a < \infty\}.$$

Hence, we deduce from α) that :

$$\forall \text{ Borel } f \geq 0, \quad E\big[f(B_{T_a})|\mathcal{G}_{T_a}\big] = Kf(a), \quad \text{on } \{T_a < \infty\}. \qquad \blacksquare$$

We now give a number of applications of the previous discussion made in [10] where the reader shall find a large number of examples of intertwinings; we also discuss a more recent example involving exponential functions of Brownian motion [29] ; we also draw on the paper [11] about affine decompositions of the stable (1/2) random variable.

Example 1. (Beta laws) We consider $(\rho_t,\ t \geq 0)$ a Bessel process independent from (B_t), starting from $\rho_0 = 0$, with dimension $\delta > 0$.
Then, $R_t = \sqrt{B_t^2 + \rho_t^2},\ t \geq 0$, is again a Bessel process starting from 0, with dimension $d = \delta + 1$; we denote by $(\mathcal{G}_t)$ its natural filtration, whereas $(\mathcal{F}_t)$ is the natural filtration of the two-dimensional process (B, ρ) $\big($or (B, R), which amounts to the same$\big)$. Applying the previous setting with $Y_t = R_t$, and $T_a = \inf\{t > 0; R_t = a\}$ then B_{T_a} is independent from $\mathcal{G}_{T_a}$, and in particular from T_a.
In this case, the intertwining kernel, which we shall denote as $K^{(\delta)}$, is given by :

$$K^{(\delta)}f(a) = \frac{1}{B(\frac{1}{2}, \frac{\delta}{2})} \int_{-1}^{1} (1-u^2)^{\frac{\delta}{2}-1} f(au)du \tag{3.6}$$

Since (cf [10])

$$B_{T_a} \overset{(law)}{=} a\varepsilon\sqrt{\beta\Big(\frac{1}{2}, \frac{\delta}{2}\Big)} \tag{3.7}$$

where ε is a symmetric Bernoulli variable, independent of $\beta\big(\frac{1}{2}, \frac{\delta}{2}\big)$, (3.6) follows from (3.7). (cf. the intertwining relation (3.f), Theorem 3.1 in [10]).
In the particular case where $d = \delta + 1$ (or δ !) is an integer, the independence of T_a and B_{T_a} is well known, because the $d-$dimensional Brownian motion is invariant by rotation.

This class of examples corresponds to family (8) in Newman's paper ([32]). In the same vein, Pitt [34], proved that, for $(B_m(t); t \geq 0)$ a Brownian motion with drift m, the exit time $T_A = \inf\{t \geq 0;\ B_m(t) \notin A\}$ of a bounded domain in $\mathbb{R}^d$, and the exit place $B_m(T_A)$ are independent if and only if A is essentially a ball centered at 0.

Example 2. (Rayleigh laws) Here, $(\mathcal{F}_t)$ denotes the natural filtration of $(B_t)_{t\geq 0}$, our real-valued Brownian motion, and we introduce $g_t = \sup\{s \leq t : B_s = 0\}$. The so-called age process $(A_t = t - g_t,\ t \geq 0)$ is Markovian with respect to its natural filtration $\mathcal{G}_t \equiv \mathcal{A}_t$ and if $T_a = \inf\{t : A_t = a\}$, then B_{T_a} and $\mathcal{G}_{T_a}$, (and in particular B_{T_a} and T_a) are independent.
$\big($The intertwining relationship between (B_t) and (A_t) is a particular case of the more general set-up in ([10], 2.4) ; it plays an important role in the study of Azéma's martingale made in [1]$\big)$.
The intertwining kernel is given by :

$$Kf(a) = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{|y|}{a} e^{\frac{-y^2}{2a}}\, f(y)dy\,.$$

Example 3. (The uniform law) This class of examples corresponds to family (7) in Newman's paper ([32]). A celebrated theorem due to Pitman $\big($[33], see e.g. [37] p. 242$\big)$ states that, if $R_t = 2M_t - B_t$, where $M_t = \sup_{s\leq t} B_s$, then $(R_t,\ t \geq 0)$ is a 3$-$dimensional Bessel process, with respect to its own filtration $\mathcal{G}_t \equiv \sigma\{R_s, s \leq t\}$.
Moreover, (R_t) and (B_t) are intertwined $\big($as discussed in 2.3 in [10] ; see, also, [35]) ; more precisely conditionally on $\mathcal{G}_t$, B_t is distributed uniformly on $[-R_t, R_t]$.$\big)$. Consequently, if $T_a = \inf\{t : R_t = a\}$, then B_{T_a} is distributed uniformly on $[-a, a]$, and is independent of $\mathcal{G}_{T_a}$, hence of T_a.
Thus, the intertwining kernel is, with our notation developed in Example 1, $K^{(2)}$. However, note that the joint laws of the processes $\big((R_t, B_t),\ t \geq 0\big)$, for $\delta = 2$ in Example 2 and Example 3 are different.
This example/remark appears as a particular case in the Skorokhod embedding construction of Azéma-Yor [2].

Example 4. The process :

$$\Big(Z_t = \exp(-B_t)\ \int_0^t ds\, \exp(2B_s)\ ; \mathcal{G}_t = \sigma(Z_s, s \leq t), t \geq 0\Big)$$

is intertwined with (B_t) ; this is discussed in ([30], [31]), and may be considered as a generalization of Pitman's theorem ; consequently, if

$$T_a = \inf\big\{t : \log\big(\int_0^t ds\, \exp(2B_s)\big) - B_t = a\big\}$$

then B_{T_a} is independent from $\mathcal{G}_{T_a}$, hence from T_a.

The expression of the intertwining kernel in terms of generalized inverse gaussian laws is discussed in detail in [30] and [31]. It is given by :

$$Kf(z) = \frac{1}{2K_0(\frac{1}{z})} \int_{-\infty}^{+\infty} f(x) \exp\Big(-\frac{\cosh x}{z}\Big) dx\,,$$

where K_0 denotes the modified Bessel function of the third kind, with index 0.

This class of examples corresponds to family (9) in Newman's paper ([32]).

3.3. A generalization of Theorem 1

We now exploit the intertwining hypothesis to generalize Theorem 1.

Theorem 3.6 *Let* $\{(B_t, \mathcal{F}_t); (Y_t, \mathcal{G}_t)\}$ *be a pair of processes satisfying condition (3.4). Let* T *be a* $(\mathcal{G}_t)$*-stopping time with all exponential moments, and such that* T *and* Y_T *are independent. Then* T *is a.s. constant.*

Proof. As a consequence of (3.5), for f, g positive Borel functions, we obtain:

$$\begin{aligned} E[f(B_T)g(T)] &= E[(Kf)(Y_T)g(T)] = E[(Kf)(Y_T)]E[g(T)] \\ &= E[f(B_T)]E[g(T)], \end{aligned}$$

hence B_T and T are independent, and by Theorem 1, T is constant. ■

We now apply this theorem in the frameworks of Examples 1 and 2.

Corollary 3.7 *Let* $(R_t; t \geq 0)$ *be a* d*-dimensional* $(d > 1)$ *Bessel process, started at* 0*, and* T *a stopping time for its natural filtration, with all exponential moments, and such that* T *and* R_T *are independent. Then* T *is a.s. constant.*

A similar statement holds when R is replaced by the age process defined in Example 2.

4. A proof of Theorem 2

4.1. Proof of Theorem 2

For the reader's convenience we write again the statement of Theorem 2.

Theorem 4.1 *Suppose that T is B-standard, T and B_T are independent. Then*

i) *B_T admits all exponential moments ;*

ii) *For every $\lambda\in\mathbb{R}$, $E\,(\exp \lambda B_T)\,E\,(\exp -\frac{\lambda^2}{2}T) = 1$. In particular $\mathcal{J}_T = \mathbb{R}$.*

iii) *a) The function $\varphi(z) = E\,(\exp z B_T)$ $(z \in \mathbb{C})$ is holomorphic on $\mathbb{C}$.*

b) For every $z \in \mathbb{C}$, $\varphi(z) = \varphi(-z)$; consequently, the law of B_T is symmetric.

c) There exists $c > 0$ such that $\varphi(\lambda) \leq \exp c\lambda^2$ $(\lambda \in \mathbb{R})$.

d) φ has no zeros on the set $\{z = x + iy \,:\, |x| \geq |y|\}$.

iv) *$E[e^{\lambda T}] < +\infty$ for all $\lambda < \lambda_0$, for some $\lambda_0 > 0$.*

Proof. *i)* $\big\{\exp\,(\lambda B_{t\wedge T} - \frac{\lambda^2}{2}\, t\wedge T)\ ;\ t \geq 0\big\}$ being a martingale, we have by Fatou's lemma :

$$1 \geq E\,(\exp \lambda B_T - \frac{\lambda^2}{2} T) = E\,(\exp \lambda B_T).\, E\,(\exp\ - \frac{\lambda^2}{2} T),$$

and the result follows since $E\,(\exp - \frac{\lambda^2}{2} T) > 0$.

ii) By Jensen's inequality and (1.1) :

$$\exp\,(\lambda B_{t\wedge T}) = \exp\,\big(\lambda E^{\mathcal{F}_{t\wedge T}}(B_T)\big) \leq E^{\mathcal{F}_{t\wedge T}}(\exp \lambda B_T),$$

hence, the martingale $\big(\exp\,(\lambda B_{t\wedge T} - \frac{\lambda^2}{2}\, t\wedge T)\ ;\ t \geq 0\big)$ is majorized by the uniformly integrable family $\big(E^{\mathcal{F}_{t\wedge T}}(\exp \lambda B_T)\ ;\ t \geq 0\big)$.

iii) a) is a consequence of $E\,(\exp \lambda B_T) < \infty$ for all $\lambda \in \mathbb{R}$;

b) $\varphi(\lambda) = \varphi(-\lambda)$ for all $\lambda \in \mathbb{R}$, hence for all $\lambda \in \mathbb{C}$;

c) for $\lambda \in \mathbb{R}$, we write $1 = E\,(\exp \lambda B_T)\, E\,(\exp - \frac{\lambda^2}{2} T)$, and the result follows from:

$$E\,(e^{-\frac{\lambda^2}{2} T}) \geq e^{-\frac{\lambda^2}{2} C}\, P(T \leq C),$$

with C such that $P(T \leq C) > 0$;

d) is a consequence of $E\,(\exp z B_T)\, E\,(\exp - \frac{z^2}{2} T) = 1$ when $\mathrm{Re}(z^2) = x^2 - y^2 \geq 0$, i.e. $|x| \geq |y|$.

iv) Since φ is holomorphic in $\mathbb{C}$, $\varphi(0) \neq 0$, there exists a ball A centered at 0 such that $\varphi(z) \neq 0, \forall z \in A$, hence $z \to E\,(\exp - \dfrac{z^2}{2} T)$ is holomorphic on A. *iv)* follows immediately. ■

4.2. Some remarks on the law of (T, B_T)

Remark 4.2 The aim of this remark is to prove that, if T and B_T are independent, then, under some suitable hypothesis, the law of T has a very particular form.

We suppose T and B_T are independent (and (1.1)).

i) For any r.v. $T > 0$ a.s., independent of the Brownian motion $(C_t; t \geq 0)$ we have:

$$E[e^{i\lambda C_T}] = E[e^{-\lambda^2 T/2}] = \int_{-\infty}^{+\infty} e^{i\lambda x} \Lambda(x) dx, \tag{4.1}$$

where $\Lambda(x)$ the density of C_T is equal to $E\big(p_T(x)\big)$, where p is the heat kernel $\big(p_t(x) = 1/(\sqrt{2\pi t})e^{-x^2/2t}\big)$.

ii) If $T > 0$ is a B-standard time such that B_T and T are independent, we have, by Theorem 2:

$$E[e^{-\lambda^2 T/2}] = \frac{1}{\psi(i\lambda)}, \qquad \text{where} \qquad \psi(\lambda) = E[e^{i\lambda B_T}] \tag{4.2}$$

is an entire function with order less than or equal to 2.

iii) By comparison of these two expressions (4.1) and (4.2) we obtain:

$$E[e^{-\lambda^2 T/2}] = \frac{1}{\psi(i\lambda)} = \int_{-\infty}^{+\infty} e^{i\lambda x} \Lambda(x) dx,$$

where $\Lambda(x) = E\big(p_T(x)\big)$.

From the classical results of ([42]) we obtain: The zeros of ψ are real iff the density Λ is a Polya frequency function, i.e.

a) $\Lambda(x) \geq 0$

b) $\displaystyle\int_{-\infty}^{+\infty} \Lambda(x)\, dx = 1$

c) $\forall n$, $\forall x_1 < x_2 < \cdots < x_n$, $\forall y_1 < y_2 < \cdots < y_n$, $\det\big(\Lambda(x_i - y_j)\big) \geq 0$

Remark 4.3 Suppose that T is a B-standard time such that :

i) T and B_T are independent ;

ii) B_T is Gaussian.

Then T is almost surely constant.

Proof of Remark 4.3 By Theorem 4.1:

$$1 = E(e^{\lambda B_T - \frac{\lambda^2}{2}T}) = E(e^{\lambda B_T})E(e^{-\frac{\lambda^2}{2}T}) = e^{\sigma^2\lambda^2/2}E(e^{-\frac{\lambda^2}{2}T}). \qquad \blacksquare$$

Theorem 4.4 *Let T be a B-standard time such that B_T and T are independent. Then :*

$$\frac{1}{3}E(B_T^4) \le \big(E(B_T^2)\big)^2 \le E(B_T^4). \tag{4.3}$$

In case equality holds on the LHS, i.e. : $\frac{1}{3}E(B_T^4) = \big(E(B_T^2)\big)^2$ then T is constant almost surely (and B_T is Gaussian and centered).
We also have:

$$E[T^2] \le 2\big(E[T]\big)^2. \tag{4.4}$$

Moreover if $E[T^2] = 2\Big(E[T]\Big)^2$ then $T = 0$ a.s.

Remark 4.5 The constants $\frac{1}{3}$ and 1 in (4.3) are optimal. Indeed, for any $\gamma \in]\frac{1}{3}, 1[$, there exists a non constant stopping time T such that $\big(E(B_T^2)\big)^2 = \gamma\, E(B_T^4)$, with T and B_T independent (cf Example 1, section 3.2), for which we have : $B_T^2 \sim \beta\big(\frac{1}{2}, \frac{\delta}{2}\big)$ (i.e. with density $cx^{-1/2}(1-x)^{\frac{\delta}{2}-1}1_{[0,1]}$) and so :

$$E(B_T^2) = \frac{1}{1+\delta}, \quad E(B_T^4) = \frac{3}{(3+\delta)\,(1+\delta)}, \quad \text{and} \quad \gamma := \frac{1+\frac{\delta}{3}}{1+\delta}$$

is a decreasing function of $\delta \in]0, \infty[$ which takes its values in $]\frac{1}{3}, 1[\big)$.

Proof of Theorem 4.4. Theorem 4.1 shows all moments of T are finite. These moments determine those of B_T^2 and vice versa by the sequence of identities obtained by equating coefficients of λ^{2n} in the identity between analytic functions of λ displayed in Theorem 4.1. In particular, the coefficients of λ^2 and λ^4 give

$$E[B_T^2] - E[T] = 0\,, \quad E[B_T^4] - 6E[B_T^2]E[T] + 3E[T^2] = 0. \tag{4.5}$$

Combine these identities to see that

$$E[B_T^4] = 3\Big(E[B_T^2]\Big)^2 - 3\mathrm{Var}T.$$

Thus (4.3) holds with equality iff T is a.s. constant, that is iff B_T is Gaussian (by Remark 4.3).

(4.4) is a consequence of (4.5):

$$6E[B_T^2]E[T] - 3E[T^2] = 6\big(E[T]\big)^2 - 3E[T^2] = E[B_T^4] \geq 0,$$

and if $E[T^2] = 2\big(E[T]\big)^2$, then $E[B_T^4] = 0$, i.e. $B_T = 0$ and $T = 0$ a.s. ■

As a final remark, we note that (4.3) limits the possible distributions of B_T^2.

5. On the independence of X_T and Y_T

5.1. A preliminary result

Theorem 5.1 *Let X and Y be two real r.v. with all exponential moments such that:*

$$E[\exp(zX + izY)] = E[\exp(zX)]\ E[\exp(izY)] = 1, \quad \forall z \in \mathbb{C}. \tag{5.1}$$

Then X and Y are independent, centered with the same Gaussian distribution.

Our proof of Theorem 5.1 is based on the study of the characteristic functions:

$$\varphi(z) = E\big[e^{izX}\big],\ \psi(z) = E\big[e^{izY}\big]\ ;\ z \in \mathbb{C}. \tag{5.2}$$

It is clear that the functions φ and ψ are holomorphic in $\mathbb{C}$ and by (5.1) :

$$\varphi(z)\psi(iz) = 1 \quad ; \quad \forall z \in \mathbb{C}. \tag{5.3}$$

The goal is to show $\varphi(z) = e^{az^2}$.

In a first step we suppose that X and Y have the same distribution, in other words $\varphi = \psi$. Consequently

$$\varphi(z)\varphi(iz) = 1 \quad ; \quad \forall z \in \mathbb{C}. \tag{5.4}$$

In the next lemma, we characterize holomorphic functions which satisfy (5.4). Then, in Lemma 5.3, using the additional property that φ is a characteristic function, we prove that $\varphi(z) = e^{az^2}$.

In a second step we reduce the problem to the symmetric one, i.e. when $\varphi = \psi$.

Lemma 5.2 *1) Any entire function φ on $\mathbb{C}$ verifying (5.4) is given by $\varphi(z) = \exp\{g(z)\}$; $z \in \mathbb{C}$, where*

$$g(z) = \sum_{k\geq 0} a_k z^{2+4k}, \tag{5.5}$$

2) If, moreover, φ is a characteristic function, then the (a_k) are real numbers.

Proof of Lemma 5.2. 1) Replacing z by iz in (5.4) we have $\varphi(z) = \varphi(-z)$; $\forall z \in \mathbb{C}$.
Since, from the identity (5.4), $\varphi(z)$ is never equal to 0, we may write $\varphi(z) = \exp\big\{g(z)\big\}$, with $g(0) = 0$, and in fact $g(z) = \sum_{k\geq 0} b_k z^{2k}$.
The relation (5.4) is then equivalent to

$$g(z) + g(iz) = \sum_{k=1}^{\infty} b_k \big(1 + (-1)^k\big) z^{2k} = 0 \quad , \quad \forall z \in \mathbb{C}.$$

Therefore $b_{2n} = 0$, for any $n \geq 1$, (5.5) follows immediately.
2) φ being an even characteristic function is automatically real valued for z belonging to $\mathbb{R}$. Hence $a_k = b_{2k+1} \in \mathbb{R}$. ■

Lemma 5.3 *Suppose that $\varphi(z) = \exp\big\{g(z)\big\}$ is the characteristic function of a real valued r.v., and g is given by (5.5), then $g(z) = -\sigma^2 z^2/2$.*

Proof of Lemma 5.3. 1) Let $h(z)$, $z \in \mathbb{C}$ be the characteristic function of a real valued r.v. ξ which admits all exponential moments then

$$\big|h(x+iy)\big| = \Big|E\big[\exp\{-y\xi + ix\xi\}\big]\Big| \leq E\big[e^{-y\xi}\big] = h(iy). \tag{5.6}$$

2) Suppose $\varphi(z) = \exp\big\{g(z)\big\}$, g being defined by (5.5).
Applying (5.6) we get

$$Re\Big(\sum_{k\geq 0} a_k (x+iy)^{2+4k}\Big) \leq -\sum_{k\geq 0} a_k y^{2+4k} \quad , \quad \forall (x,y) \in \mathbb{R}^2.$$

Write $x + iy = te^{i\theta}(t \geq 0, \theta \in \mathbb{R})$, hence denoting $\rho = t^4 \geq 0$, we get

$$\text{for all} \ \ \rho \geq 0, \quad \sum_{k\geq 0} a_k \rho^k \Big(\cos\big((2+4k)\theta\big) + (\sin\theta)^{2+4k}\Big) \leq 0. \tag{5.7}$$

The next lemma implies the result. ■

Lemma 5.4 *The inequality (5.7) implies that* $a_0 \leq 0$ *and* $a_k = 0$ *for every* $k \geq 1$.

Proof of Lemma 5.4. Our approach is based on the existence of a function $Q \geq 0$ on $[0, \pi/4]$, which will play an essential role in the proof.

a) As a first step, we introduce the sequence of reals :

$$\beta_k = \frac{\operatorname{sgn}(a_k)}{(1+k)^3},$$

with the convention sgn $(0) = 0$ (we use $\dfrac{1}{(1+k)^3}$ as a “slowly” convergent series).

Consider the function Q_0 :

$$Q_0(\theta) = \sum_{k\geq 0} \beta_k \cos\big((2+4k)\theta\big) \quad ; \quad \theta \in \mathbb{R}.$$

Q_0 is well defined, since the series is uniformly convergent :

$$\big|Q_0(\theta)\big| \leq \sum_{k\geq 0} |\beta_k| = \sum_{k\geq 0} \frac{1}{(1+k)^3} < +\infty. \tag{5.8}$$

Moreover Q_0 is differentiable and

$$\big|Q_0'(\theta)\big| \leq \sum_{k\geq 0} (2+4k)|\beta_k| = B < \infty,$$

with $B = \displaystyle\sum_{k\geq 0} \frac{2+4k}{(1+k)^3} < \infty$.

We set

$$Q(\theta) = Q_0(\theta) + \sqrt{2}B\cos(2\theta); \quad \theta \in \mathbb{R}.$$

Taking derivatives on both sides we obtain

$$Q'(\theta) = Q_0'(\theta) - 2\sqrt{2}B\sin(2\theta).$$

Consequently if $\theta \in \left[\dfrac{\pi}{8}, \dfrac{\pi}{4}\right]$,

$$Q'(\theta) \leq B - 2\sqrt{2}B\sin\left(\frac{\pi}{4}\right) = -B < 0\,.$$

We remark that $Q(\pi/4) = 0$; Q being a decreasing function on $\big[\pi/8,\ \pi/4\big]$, then $Q(\theta)$ is positive for any θ in $\big[\pi/8,\ \pi/4\big]$.

By (5.8), $\big|Q_0(\theta)\big|$ is less than $B/2$.
Suppose $\theta \in \big[0, \pi/8\big]$, then

$$Q(\theta) \geq -\frac{B}{2} + \sqrt{2}B\cos(\pi/2) = \frac{B}{2} > 0.$$

Finally, $Q(\theta) \geq 0$, for θ in $[0,\ \pi/4]$.

b) By a straightforward calculation, we easily verify that for any $k, l \in \mathbb{N}$,

$$\text{(5.9)} \qquad \begin{cases} \displaystyle\int_0^{\pi/4} \cos\big((2+4k)\theta\big)\cos\big((2+4l)\theta\big)d\theta &= 0 \quad \text{if } k \neq l, \\ \displaystyle\int_0^{\pi/4} \cos^2\big((2+4k)\theta\big)d\theta &= \pi/8, \end{cases}$$

$$\text{(5.10)} \qquad \int_0^{\pi/4} \sin^{2+4k}(\theta)d\theta \leq \frac{\pi}{4}\Big(\frac{1}{2}\Big)^{1+2k}.$$

We now come back to (5.7), where we multiply both sides by $Q(\theta)$, and we integrate with respect to Lebesgue measure on $[0, \pi/4]$; $Q(\theta)$ being positive, we have,

$$\int_0^{\pi/4} Q(\theta)\Big\{\sum_{k\geq 0} a_k\rho^k\Big(\cos\big((2+4k)\theta\big) + \big(\sin\theta\big)^{2+4k}\Big)\Big\}d\theta \leq 0.$$

Since g (defined by (5.5)) is analytic, we may exchange $\displaystyle\int_0^{\pi/4}$ and $\sum$, hence

$$\text{(5.11)} \qquad \sum_{k\geq 0} \alpha_k\rho^k \leq 0\ ; \quad \forall \rho \in \mathbb{R}_+,$$

with $\alpha_k = a_k \displaystyle\int_0^{\pi/4} Q(\theta)\Big\{\cos\big((2+4k)\theta\big) + \big(\sin\theta\big)^{2+4k}\Big\}d\theta.$

Using both definitions of Q, Q_0 and (5.9) we have

$$\alpha_k = a_k\Big(\beta_k\frac{\pi}{8} + R_k\Big) \quad ; \quad k \geq 1,$$

where $R_k = \displaystyle\int_0^{\pi/4} Q(\theta)\big(\sin\theta\big)^{2+4k}d\theta.$

Using the definition of β_k, we obtain

$$\text{(5.12)} \qquad \alpha_k = |a_k|\frac{\pi}{8}\Big(\frac{1}{(1+k)^3} + R'_k\Big), \quad k \geq 1$$

$$R'_k = (\operatorname{sgn} a_k)\frac{8}{\pi}R_k.$$

Since we have proved that $\big|Q_0(\theta)\big| \le B/2$, then $\big|Q(\theta)\big| \le B/2 + \sqrt{2}B$, for any $\theta \in [0, \pi/4]$.
But this inequality yields :

$$|R'_k| \le B(\frac{1}{2} + \sqrt{2})\big(\frac{1}{2}\big)^{2k} \tag{5.13}$$

Consequently (5.12) implies that $\alpha_k \ge 0$, for k large enough. Combining this and (5.11) we conclude that $\alpha_k = 0$, for k sufficiently large, which is equivalent to $a_k = 0$, k large. As a result, $\varphi(z) = \exp A(z)$, with A a polynomial.

c) Thus to finish our proof of Lemma 5.4 it suffices to prove that, if for some $N \in \mathbb{N}$

$$\sum_{n=1}^{N} a_n \rho^n \big(\cos\,(2n\theta) + (\sin\theta)^{2n}\big) \le 0 \tag{5.14}$$

for every $\rho \ge 0, \theta \in \mathbb{R}$, then $a_n = 0$ for $n > 1$ and $a_1 \le 0$.
Indeed, dividing by ρ^N, and letting $\rho \to \infty$, we obtain

$$a_N(\cos(2N\theta) + (\sin\theta)^{2N}) \le 0 \tag{5.15}$$

for all $\theta \in \mathbb{R}$, which, if $N > 1$, ensures $a_N = 0$ (take $\theta = 0$, and $\theta = \frac{\pi}{2N}$)$\cdot$ ■

Remark 5.5 We note that the last step c) in our proof obviously provides a proof of a weak formulation of Marcinkiewicz' Theorem ([27]), namely : if $\varphi(z) = \exp(A(z))$, $(z \in \mathbb{C})$, with A an even polynomial such that $A(0) = 0$, is the characteristic function of a real valued r.v., then $A(z) = -\delta^2 z^2$ $(\delta \in \mathbb{R})$.

End of the proof of Theorem 5.1. Suppose that X and Y have all exponential moments and (5.1) holds. We recall that φ (resp. ψ) is the characteristic function of X (resp. Y), and that φ and ψ are related by (5.3).
We introduce four independent r.v.'s : U, U', V and V' such that :

$$U \overset{(d)}{=} U' \overset{(d)}{=} X \quad , \quad V \overset{(d)}{=} V' \overset{(d)}{=} Y.$$

We set $\xi = U - U' + V - V'$ and we denote by h the characteristic function of ξ.
The independence property of the four variables implies that the characteristic function of ξ is:

$$h(z) = \varphi(z)\varphi(-z)\psi(z)\psi(-z) \; ; \quad z \in \mathbb{C}$$

and it follows from (5.3) that :

$$h(z)h(iz) = 1 \quad , \quad \forall z \in \mathbb{C}. \tag{5.16}$$

Lemma 5.3 tells us that h is the characteristic function of a centered Gaussian r.v.

Let us summarize : ξ is a Gaussian r.v., $\xi = U - U' + V - V'$ the r.v.'s U, U', V and V' being independent ; hence the Cramer-Lévy theorem implies that $U \overset{(d)}{=} X$ and $V \overset{(d)}{=} Y$ have a Gaussian distribution. ■

5.2. On the independence of X_T and Y_T

In this section $\big((X_t, Y_t);\ t \geq 0\big)$ will denote a $(\mathcal{F}_t)_{t\geq 0}$ Brownian motion, starting at 0, taking its values in $\mathbb{R}^2$. We first exhibit a large class of $(\mathcal{F}_t)_{t\geq 0}-$stopping times T, such that

$$X_T \text{ and } Y_T \text{ are independent r.v}'\text{s}. \tag{5.17}$$

From a family of stopping times which satisfies (5.17) we can generate a new family which also satisfies (5.17). The scheme is the following :

a) Let T_1 be a $(\mathcal{F}_t)_{t\geq 0}$ stopping time such that X_{T_1} and Y_{T_1} are independent r.v.'s,

b) We set $X'_t = X_{t+T_1} - X_{T_1}$, $Y'_t = Y_{t+T_1} - Y_{T_1}$; $t \geq 0$, let T_2 be a $(\mathcal{F}_{t+T_1})_{t\geq 0}$-stopping time such that X'_{T_2} and Y'_{T_2} are independent r.v.'s.

Then $X_{T_1+T_2}$ and $Y_{T_1+T_2}$ are independent r.v.'s.

For instance we can choose T_1 a stopping time with respect to the natural filtration of $(X_t,\ t \geq 0)$ and T_2 a stopping time with respect to the filtration generated by $(Y'_t;\ t \geq 0)$.

We now state the following theorem.

Theorem 5.6 *Let T be a $(\mathcal{F}_t)_{t\geq 0}-$stopping time such that*

i) is both a X- and Y- standard time,

ii) X_T and Y_T have all exponential moments,

iii) $E[\exp(zX_T + izY_T)] = E[\exp(zX_T)]\ E[\exp(izY_T)]$ for any $z \in \mathbb{C}$.

Then X_T and Y_T are independent, centered with the same Gaussian distribution $\mathcal{N}(0, E[T])$.

Proof of Theorem 5.6. i) It is a classical result due to P. Lévy that if $f : \mathbb{C} \to \mathbb{C}$ is an holomorphic function, then $\big(f(X_t + iY_t);\ t \geq 0\big)$ is a continuous local martingale. In particular $\big(\exp\{zX_t + izY_t\};\ t \geq 0\big)$ is a continuous martingale.
ii) For $\lambda \in \mathbb{R}$, we have, by Jensen's inequality and i) :

$$E^{\mathcal{F}_{t\wedge T}}\left(\exp \lambda X_T\right) \geq \exp\left(\lambda\, E^{\mathcal{F}_{t\wedge T}}(X_T)\right) = \exp\left(\lambda X_{t\wedge T}\right)$$

(and the same relation with Y instead of X), and it then follows, from Doob's L^p inequality ($p > 1$), that, for every $z \in \mathbb{C}$, $\big(\exp z\left(X_{t\wedge T} + iY_{t\wedge T}\right)\ ;\ t \geq 0\big)$ is a uniformly integrable martingale. So :

$$1 = E(\exp z\, X_T + izY_T) = E(\exp z\, X_T)\, E\left(\exp izY_T\right)$$

and the proof of Theorem 5.6 follows as a direct consequence of Theorem 5.1. ■

5.3. Proof of Theorem 3

For clarity we first recall the statement of Theorem 3.

Theorem 5.7 *Let T be a $(\mathcal{F}_t)_{t\geq 0}-$stopping time having all exponential moments. Then*

(5.18) $E[\exp(\lambda X_T)] < +\infty$ *and* $E[\exp(\lambda Y_T)] < +\infty$, *for every* $\lambda \in \mathbb{R}$.

We assume that X_T and Y_T satisfy :

$$E\Big[\exp\big\{zX_T + izY_T\big\}\Big] = E\big[e^{zX_T}\big]E\big[e^{izY_T}\big]\ ;\ \forall z \in \mathbb{C}. \tag{5.19}$$

Then X_T and Y_T are independent, centered with the same Gaussian distribution $\mathcal{N}(0, E[T])$.

Proof of Theorem 5.7. By (2.1) we have:

$$E[\exp(\lambda X_T)] \leq 2\Big(E[\exp(2\lambda^2 T)]\Big)^{1/2}, \quad \forall \lambda \in \mathbb{R}, \tag{5.20}$$

and the same with Y_T instead of X_T. Theorem 5.7 is then an obvious corollary of Theorem 5.6. ■

Remark 5.8 1) The assumption that T has all exponential moments is optimal. Recall that there exist stopping times T, with respect to the filtration of $(X_t;\ t \geq 0)$, such that:

$$X_T \text{ and } T \text{ are independent r.v.'s,} \tag{5.21}$$

but

(5.22) T has only "small" exponential moments.

As we noticed in the introduction of this section, the two r.v.'s X_T and Y_T are independent. Moreover since $Y_T \overset{(d)}{=} \sqrt{T}Y_1$, one has:

$$E\big[e^{i\lambda Y_T}\big] = E\big[e^{i\lambda\sqrt{T}Y_1}\big] = E\big[e^{-\lambda^2 T/2}\big]\ ,\ \lambda \in \mathbb{R}.$$

Consequently, if Y_T is Gaussian distributed, then it is symmetric, so that: $E\big[e^{i\lambda Y_T}\big] = e^{-\lambda^2\sigma^2/2}$; hence: $T = \sigma^2$.
In conclusion for stopping times T satisfying (5.21) and (5.22), if T is not constant, X_T and Y_T are independent r.v.'s but the distribution of Y_T is not Gaussian.

2) Theorem 2.2 may be obtained as a consequence of Theorem 5.7. Indeed, let $(B_t^{(1)}, \mathcal{F}_t^{(1)};\ t \geq 0)$ be a linear Brownian motion and T a $(\mathcal{F}_t^{(1)})$ stopping time with all exponential moments independent of $B_T^{(1)}$. Let $(B_t^{(2)};\ t \geq 0)$ another Brownian motion, independent of $\mathcal{F}_\infty^{(1)}$. Then $B_T^{(1)}$ and $B_T^{(2)}$ are independent and by Theorem 5.7, $B_T^{(2)}$ is Gaussian. But :

$$e^{-\frac{\lambda^2}{2}\sigma^2} = E(e^{i\lambda B_T^{(2)}}) = E(e^{i\lambda\sqrt{T}B_1^{(2)}}) = E(e^{-\frac{\lambda^2}{2}T})$$

which implies T is constant.

6. A conjecture of Tortrat

Recall that Theorem 5.7 says that if X_T and Y_T are independent and T is a stopping time having all exponential moments, then X_T and Y_T are Gaussian distributed. In this section, we show that nonetheless this does not imply that T is constant. Indeed we exhibit a class of bounded non constant stopping times T such that, if $(W_t,\ t \geq 0)$ is a $n-$dimensional Brownian motion, then W_T is distributed as $\mathcal{N}(0, I_n)$. In particular, this answers negatively a conjecture of Tortrat (cf : [24]) which asserted that for $n = 1$ such T's are necessarily constant. However if we strenghten the assumptions the conjecture is true (see Remark 6.8 and Proposition 6.9).

At the end of this section, we also say a few words on a similar conjecture of Cantelli (see [13]).

Theorem 6.1 *There exist a filtration $(\mathcal{G}_t)$, a $(\mathcal{G}_t)$ linear Brownian motion $(B_t)_{t\geq 0}$, $B_0 = 0$, and a bounded, non-constant $(\mathcal{G}_t)$ stopping time T such that B_T has a Gaussian distribution with mean zero.*

The proof of Theorem 6.1 will be given after Remark 6.4.

We say that a probability measure μ on $\mathbb{R}^n$ has a bounded Brownian representation if there exist a $\mathbb{R}^n-$valued $(\mathcal{F}_t)$ Brownian motion $(B_t)_{t\geq 0}$, and a bounded $(\mathcal{F}_t)$ stopping time T such that the law of $B_T + c$ is μ, for some c in $\mathbb{R}^n$.

Obviously $c = (c_1, \ldots, c_n)$ with $c_i = \displaystyle\int_{\mathbb{R}^n} t_i d\mu(t)$.

Proposition 6.2 *Let $X_1, \ldots, X_n$ be independent, each X_k being Gaussian $\mathcal{N}(0,1)$-distributed, and $f : \mathbb{R}^n \to \mathbb{R}$ such that :*

(6.1) *f is of C^2 class and the partial derivatives $\dfrac{\partial f}{\partial x_i}$ are bounded for any $1 \leq i \leq n$.*

Then the law of $f(X_1, \ldots, X_n)$ has a bounded Brownian representation. Moreover if f is non linear the stopping time T is not constant.

Remark 6.3 1) In our construction, we do not know whether we can choose $(\mathcal{F}_t)$ as the natural filtration of $(B_t; t \geq 0)$ (cf. [22] for a discussion of this type of question).

2) Bass $\big([3]\big)$ proves a similar result when $n = 1$. However he does not require that the derivative of f is bounded. This assumption is crucial in our approach.

Proof of Proposition 6.2 1) Let $(B_t;\ t \geq 0)$ be a linear Brownian motion started at 0 and $(\mathcal{F}_t; t \geq 0)$ its natural filtration.

We set $Y = f(B_1, B_2 - B_1, \ldots, B_n - B_{n-1})$. Obviously Y and $f(X_1, \ldots, X_n)$ have the same distribution.

We first compute Itô's representation of the martingale $Y_t \stackrel{\text{def}}{=} E[Y|\mathcal{F}_t],\ t \leq n$.

By the independence of the increments of Brownian motion,

$$Y_t = g_k(B_1, \ldots, B_{k-1} - B_{k-2}, B_t - B_{k-1}, t)\ ; \quad k-1 \leq t \leq k,$$

where

$$g_k(x_1, \ldots, x_k, t)$$
$$= E\Big[f\big(x_1, \ldots, x_{k-1}, x_k + (\sqrt{k-t})(B_k - B_{k-1}), B_{k+1} - B_k, \ldots, B_n - B_{n-1}\big)\Big]$$

with $(x_1, \ldots, x_k) \in \mathbb{R}^k,\ 0 \leq t \leq k$, and $1 \leq k \leq n$. g_k is of class C^2 on $\mathbb{R}^k \times [0, k[$.

2) Applying Itô's formula to represent the martingale $\big(Y_t; k-1 \leq t \leq k\big)$ we obtain:

$$(6.2)\ \ Y_t - Y_{k-1} = \int_0^t 1_{[k-1,k]}(s)\frac{\partial g_k}{\partial x_k}(B_1, \dots, B_{k-1} - B_{k-2}, B_s - B_{k-1}, s)dB_s,$$

$t \in [k-1, k]$. We set

$$(6.3)\qquad a(s,x) = \sum_{k=1}^{n} 1_{[k-1,k[}(s)\frac{\partial g_k}{\partial x_k}(B_1, \dots, B_{k-1} - B_{k-2}, x - B_{k-1}, s),$$

$0 \leq s \leq n,\ x \in \mathbb{R}$.

Taking $t = n$, and adding the terms $Y_k - Y_{k-1}, 1 \leq k \leq n$, in (6.2), we obtain:

$$(6.4)\qquad Y = Y_n = E[Y] + \int_0^n a(s, B_s)dB_s.$$

3) Let $\big(A(t);\ t \geq 0\big)$ be the continuous, and non-decreasing process :

$$A(t) = \int_0^t a(s, B_s)^2 ds, \quad t \geq 0,$$

where we define $a(s,x) \equiv 1$, for $s \geq n$.

We then define $\mathcal{G}_u = \mathcal{F}_{\tau_u},\ u \geq 0$, with $\tau_u = \inf\big\{s;\ A_s > u\big\}$. It follows from the Dubins-Schwarz representation theorem that there exists a $(\mathcal{G}_u)$ Brownian motion $(\beta_u;\ u \geq 0)$ such that :

$$(6.5)\qquad \int_0^t a(s, B_s)dB_s = \beta\big(A(t)\big) \quad ; \quad t \geq 0.$$

Assumption (6.1) implies that the function a is bounded. Hence $U = A(n)$ is bounded. Moreover U is a non constant $(\mathcal{G}_u)$ stopping time if f is not linear.

4) Here is another proof of Proposition 6.2. Let $(B_t^{(n)}, t \geq 0)$ a n-dimensional Brownian motion. We have as a basic example of Clark's representation ([37], chap. V ; [9]) :

$$f(X_1, \dots, X_n) \overset{law}{=} f(B_1^{(n)}) = E(f(B_1^{(n)})) + \int_0^1 dB_s^{(n)}.P_{1-s}(\nabla f)(B_s^{(n)}).$$

But ∇f is a bounded function, and we conclude as in 3) above. ■

Remark 6.4 Suppose that the assumptions of Proposition 6.2 are satisfied, and let $(B_t)_{t\geq 0}$ be a real valued Brownian motion. Thanks to this proposition it is possible to define a bounded stopping time such that

$$f(X_1, \dots, X_n) \stackrel{(d)}{=} B_T.$$

We note that, concerning Proposition 6.2, if we followed uniquely Bass's arguments [3], we could only prove the Proposition under the additional assumption that f is separately, in each of the variables, a non-decreasing function.

Proof of Theorem 6.1. Thanks to Proposition 6.2 it suffices to be able to represent a $\mathcal{N}(0,1)$ variable G as, say $f(X_1, X_2)$, where X_1 and X_2 are two independent $\mathcal{N}(0,1)$ variables, and f satisfies (6.1).
Using the independence of $R = \sqrt{X_1^2 + X_2^2}$, and $\theta := \arg(Z) \quad (\in]0, 2\pi])$, where $Z = X_1 + iX_2$, it is easily seen that :

$$Z_g := Z \exp(ig(R)) \stackrel{law}{=} Z,$$

where $g : [0, \infty[\to [0, \infty[$ is of class C^∞, $g \neq 0$ and g has compact support in $[\alpha, \beta]$, with $0 < \alpha < \beta < 2\pi$.
Moreover, the function

$$\begin{aligned} f_g(x,y) &= \mathrm{Re}((x+iy)e^{ig(\sqrt{x^2+y^2})}) \\ &= x \cos(g(\sqrt{x^2+y^2})) - y \sin(g(\sqrt{x^2+y^2})) \end{aligned} \tag{6.6}$$

satisfies (6.1). ∎

Theorem 6.1 admits an extension to the $d-$dimensional case.

Theorem 6.5 *There exist a d-dimensional Brownian motion $\big((B_t, \mathcal{F}_t); t \geq 0\big)$, started at 0, and a non-constant and bounded stopping time T, such that the law of B_T is $\mathcal{N}(0, I_d)$. Moreover, if $d \geq 3$, T can be chosen as a stopping time with respect to the natural filtration of $(B_t \ ; \ t \geq 0)$.*

Our main idea to prove Theorem 6.5 is to define a $\mathbb{R}^d-$valued diffusion process $(Z_t;\ t \geq 0)$ which is a time changed Brownian motion. The density of Z_t solves the Fokker-Planck equation. In the next lemma, we choose the diffusion matrix such that Z_2, i.e. Z taken at time 2, has the required distribution $\mathcal{N}(0, I_d)$.

Lemma 6.6 *Suppose $d \geq 3$. There exist :*

a) a bounded and non-constant function $c : [0,+\infty[\times\mathbb{R}^d \to]0,+\infty[$, c being of class C^∞,

b) a family of density functions $\big(u(t,.);\ t \geq 1\big)$ defined on $\mathbb{R}^d$, solving

$$(6.7) \qquad \begin{cases} (i) & \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\Delta(c^2 u) \\ (ii) & u(1,.) = p(1,.) \quad ; \quad u(2,.) = p(2,.), \end{cases}$$

where $p(t,x) = \dfrac{1}{(2\pi t)^{d/2}} \exp(-\dfrac{||x||^2}{2t})$, $x \in \mathbb{R}^d$, $t>0$, *and* $\Delta = \displaystyle\sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$.

Proof of Lemma 6.6. 1) Let $\varepsilon : [1,2] \to [0,+\infty[$. We suppose that the support of ε is included in $]1,2[$, ε is of class C^∞ and

$$\varepsilon(t) \leq \frac{1}{2}\frac{1}{2^{d/2}-1}, \quad \big|\varepsilon'(t)\big| \leq 2^{-(1+d/2)} \ ;\ \forall t \in [1,2].$$

We set

$$(6.8) \qquad u(t,x) = p(t,x) + \varepsilon(t)\big(p(t,x) - p(1,x)\big) \quad ; \quad t \in [1,2].$$

t belonging to $[1,2]$,

$$p(t,x) \geq 2^{-d/2} p(1,x) \quad , \quad \forall x \in \mathbb{R}^d,$$

then

$$u(t,x) \geq 2^{-d/2}\big(1 - \varepsilon(t)(2^{d/2}-1)\big)p(1,x) > 2^{-(1+d)/2} p(1,x) \quad , t \in [1,2].$$

Since $p(t,.)$ is a density function, it follows from (6.8) that the integral of $u(t,x)$ over x, is equal to 1.

Finally $u(t,.)$ is a density function ; ε cancels at $t = 1$ and 2, therefore (6.7) *(ii)* holds.

2) Before we prove (6.7), *(i)*, we recall two important facts. If q is the Newtonian potential kernel in $\mathbb{R}^d$:

$$q(z) := \frac{C_d}{|z|^{d-2}} \quad , z \in \mathbb{R}^d,\ z \neq 0,$$

with $C_d = \dfrac{\Gamma(d/2)}{2(2-d)\pi^{d/2}} < 0$, we have :

a) if f is a "nice" function of class C^2 :

$$f(z) = \int q(z-y)\Delta f(y)dy, \quad z \in \mathbb{R}^d. \tag{6.9}$$

(cf [18], formula (2.17), p. 18).

b) p solves the heat equation, hence :

$$\frac{1}{2}p(t,.) = \frac{\partial p}{\partial t}(t,.) * q, \tag{6.10}$$

where $*$ denotes the convolution product.

3) It remains to verify (6.7) *(i)*.
By (6.9), (6.7) *(i)* is equivalent to :

$$2\frac{\partial u}{\partial t}(t,.) * q = c^2 u(t,.).$$

Consequently, let us introduce

$$f(t,.) = \frac{2}{u(t,.)}\frac{\partial u}{\partial t}(t,.) * q \quad ; \quad 1 \leq t \leq 2. \tag{6.11}$$

We have to prove that f is positive, bounded and of class C^∞. The last point is clear since $p(t,.)$ is of class C^∞.
We calculate $\frac{\partial u}{\partial t}(t,.)$ via (6.10) and we replace it in (6.11) :

$$f(t,.) = \frac{g(t,.)}{u(t,.)},$$

with $g(t,.) = 2\Big\{\frac{\partial p}{\partial t}(t,.) + \varepsilon(t)\frac{\partial p}{\partial t}(t,.) + \varepsilon'(t)\big(p(t,.) - p(1,.)\big)\Big\} * q$.
By (6.9), we have

$$\begin{aligned} g(t,.) &= \big(1+\varepsilon(t))p(t,.) + 2\varepsilon'(t)\Big(\int_1^t \frac{\partial p}{\partial s}(s,.)ds\big) * q \\ g(t,.) &= \big(1+\varepsilon(t)\big)p(t,.) + \varepsilon'(t)\int_1^t p(s,.)ds. \end{aligned}$$

We easily verify

$$p(s,.) \leq 2^{d/2}p(t,.) \quad ; \quad 1 \leq s \leq t \leq 2. \tag{6.12}$$

Then $\int_1^t p(s,.)ds \leq 2^{d/2}p(t,.)$ and

$$\begin{aligned} g(t,.) &\geq p(t,.) - |\varepsilon'(t)|2^{d/2}p(t,.) = p(t,.)\big(1-|\varepsilon'(t)|2^{d/2}\big) > \frac{1}{2}p(t,.). \\ g(t,.) &\leq c_1 p(t,.). \end{aligned}$$

Since $\varepsilon(t)$ is bounded, $u(t,.) \leq c_2 p(t,.)$.
$\varepsilon(t)$ belonging to $[0, 2^{-(d+2)/2}]$, (6.12) implies that

$$u(t,.) \geq p(t,.) - \varepsilon(t)p(1,.) \geq p(t,.) - \varepsilon(t)2^{d/2}p(t,.) \geq c_3 p(t,.).$$

Consequently $f(t,.) \in [c_4, c_5],\ c_4 > 0$. ■

Proof of Theorem 6.5 1) Let c be the function defined by Lemma 6.6. c being smooth, there exists a unique solution $(X_t)_{t\geq 0}$ to the following stochastic differential equation :

$$X_t^i = X_1^i + \int_1^t c(s, X_s)dB_s^i \quad ; \quad 1 \leq i \leq d,\ 1 \leq t \leq 2\,,$$

where $(B_t;\ t \geq 0)$ is a $d-$dimensional Brownian motion, with components $B^1, B^2, ..., B^d$.
It is supposed that $(X_1^1, \ldots, X_1^d)$ is independent of $(B_t)_{t\geq 0}$, and is $\mathcal{N}(0, I_d)$ distributed.
2) We set $M_t^i = X_t^i - X_1^i \quad ; \quad 1 \leq t \leq 2,\, 1 \leq i \leq d$.
The processes $(M_t^i;\ 1 \leq t \leq 2),\ i \in \{1, \ldots, d\}$ are continuous martingales which satisfy:

$$(\star) \qquad < M^i, M^j >= 0\ ;\ < M^i >=< M^j > \qquad (i \neq j).$$

The Dubins-Schwarz representation theorem for continuous (local) martingales easily extends to $d-$dimensional continuous martingales which satisfy $(\star)$; this is in particular the case for conformal martingales (see [17], [14]); this extension differs from and is easier than Knight's more general result ([22]) on orthogonal continuous martingales (cf [37], Theorem 1.6, p. 173).
Then there exists a $d-$dimensional Brownian motion $(\beta_u;\ u \geq 0)$ such that,

$$X_t = \beta\big(1 + \int_1^t c^2(s, X_s)ds\big) \quad ; \quad 1 \leq t \leq 2.$$

The r.v. $T = 1 + \int_1^2 c^2(s, X_s)ds$ is a non-constant and bounded stopping time with respect to the natural filtration $(\mathcal{G}_u)_{u\geq 0}$ of the Brownian motion $(\beta_u;\ u \geq 0)$.

Indeed let (α_u) be the right inverse of $(\int_1^t c^2(s,X_s)ds; t\geq 1)$, by a straightforward calculation we have:

$$\alpha_u = 1 + \int_0^u \frac{dh}{c^2(\alpha_h,\beta_h)}.$$

Since c is a smooth function (cf Lemma 6.6), the above identity implies that (α_u) is $(\mathcal{G}_u)$-adapted and $\{T_t < u\} = \{t < \alpha_u\} \in \mathcal{G}_u$.
Since $|c|$ is bounded from below by a positive constant and it is a smooth function, the law of Z_t, for any $t \in [1,2]$, admits a density function $v(t,.)$. v solves the Fokker-Planck equation :

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,.) &= \dfrac{1}{2}\Delta(c^2 v)(t,.) \\ v(1,.) &= p(1,.). \end{cases}$$

But $\big(u(t,.);\ t\in[1,2]\big)$ defined in Lemma 6.6, solves the previous P.D.E. As a result $v(t,.) = u(t,.)$. In particular $v(2,.) = u(2,.) = p(2,.)$, this means that $Z_2 = \beta(T)$ is $\mathcal{N}(0,2I_2)-$distributed. ■

Remark 6.7 1) For $d \geq 3$ we have defined a Brownian motion $(\beta_t;\ t\geq 0)$ and a bounded and non-constant stopping time T such that $\beta(T) \sim \mathcal{N}(0,I_d)$. When $d \leq 2$, this result is still true but we need to add a Brownian motion independent of the initial one. The stopping time T is measurable with respect to the enlarged filtration.
2) Recall that as discussed in the above proof if $(B_t)_{t\geq 0}$ is a planar Brownian motion and $f:\mathbb{C}\to\mathbb{C}$ is holomorphic, then $\big(f(B_t)\big)_{t\geq 0}$ is the time change of a two-dimensional-Brownian motion. Then if $d=2$, a natural approach to prove Theorem 6.5 would be to look for an entire function $f:\mathbb{C}\to\mathbb{C}$ such that :

$$f(Z) \overset{law}{=} Z, \tag{6.13}$$

where Z is a two-dimensional centered Gaussian random variable with variance equal to I_2.

However, as we now show, the only such functions f are $f(z) = cz$, where $|c| = 1$.

Proof (Brossard [8]). *i)* We first show that $f(0) = 0$.
f is holomorphic :

$$f(0) = \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})d\theta.$$

But, the modulus of Z and its angle are independent r.v.'s, and the law of the angle is uniform on $[0, 2\pi]$:

$$0 = E(Z) = E\big[f(Z)\big] = \int_0^\infty \Big(\frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})d\theta\Big)P\big(|Z| \in dr\big) = f(0).$$

ii) We set $g(z) = f(z)/z$ and m the Lebesgue measure in $\mathbb{R}^2$.
(6.13) implies that $E\big[\exp\{\frac{1}{4}|f(Z)|^2\}\big] = 2$. Then

$$\begin{aligned} 4\pi &= \int_{\mathbb{R}^2} \exp\Big\{\frac{1}{4}|f(z)|^2 - \frac{1}{2}|z|^2\Big\}m(dz), \\ &\geq \int_{\{|g|\geq\alpha\}} \exp\Big\{\big(\frac{\alpha^2}{4} - \frac{1}{2}\big)|z|^2\Big\}m(dz) \geq e^{\alpha^2/8}m(|g| \geq \alpha), \end{aligned}$$

as soon as $\alpha \geq \alpha_0$, where α_0 verifies :

$$\frac{\alpha_0^2}{4} - \frac{1}{2} \geq \frac{\alpha_0^2}{8}$$

and $\big\{|g| > \alpha_0\big\} \cap \big\{|z| < 1\big\} = \emptyset$.
In particular

$$\int_{|g|\geq\alpha_0} |g(z)|m(dz) = \int_{\alpha_0}^\infty m(|g| \geq \alpha)d\alpha \leq 4\pi \int_{\alpha_0}^\infty e^{-\alpha^2/8}d\alpha < \infty.$$

iii) Since f is an holomorphic function with $f(0) = 0$ then g is also an holomorphic function. Hence

$$g(z) = \int_{D(z)} g(y)m(dy) = \int_{D(z)\cap\{|g|<\alpha_0\}} g(y)m(dy) + \int_{D(z)\cap\{|g|\geq\alpha_0\}} g(y)m(dy),$$

where $D(z) = \big\{y \in \mathbb{C}; \quad |y - z| \leq 1\big\}$. This shows that g is bounded. Hence g is constant. ■

A similar result holds in the one dimensional case. More precisely : if $f : \mathbb{R} \to \mathbb{R}$ is C^1 and $Z \overset{law}{=} f(Z)$, where Z is a one dimensional centered Gaussian r.v., then $f(z) = \pm z$. The proof is left to the reader.

This brings us naturally to look for functions $f : \mathbb{R}^d \to \mathbb{R}^d$, $d \geq 3$, such that $f(Z) \overset{law}{=} Z$, $Z \sim \mathcal{N}(0, I_d)$ implies f belongs to the orthogonal group on $\mathbb{R}^d$.

As said at the beginning of Section 6, if we reinforce the assumptions, the conjecture of Tortrat is true : in Remark 6.8 and in Proposition 6.9, we assume that $B_T + aT$ is Gaussian for several drifts a.

Remark 6.8 Let $\big(B(t);\ t \geq 0\big)$ be a one dimensional Brownian motion started at 0, and T a random time, i.e. simply a positive r.v. We assume that $B(T) + a_n T$ is a Gaussian r.v. for a sequence $(a_n)_{n\geq 0}$ with $|a_n| \to \infty$ as n goes to ∞. Then T is a.s. constant.

Proof of Remark 6.8. Let $G_n = B_T + a_n T$, with G_n Gaussian. Then, G_n/a_n converges in law to T, and so T is Gaussian, possibly degenerate (cf e.g. [37] p. 12). But $T \geq 0$, and T is constant.

Let $\big(\Omega, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, P_0\big)$ be the canonical space : Ω is the set of continuous functions defined on $[0, +\infty[$, vanishing at 0, $(X_t)_{t\geq 0}$ the process of coordinates, $(\mathcal{F}_t)_{t\geq 0}$ its natural filtration, P_0 the Wiener measure $\big($i.e. $(X_t)_{t\geq 0}$ is a one-dimensional P_0-Brownian motion vanishing at $0\big)$.
For any α in $\mathbb{R}$, P_α denotes the unique probability on Ω such that for any $t \geq 0$,

$$P_\alpha = \exp\big\{\alpha X(t) - \frac{\alpha^2}{2} t\big\} P_0 \quad \text{on } \mathcal{F}_t. \tag{6.14}$$

Recall that $\big(X(s) - \alpha s;\ 0 \leq s \leq t\big)$ is a $P_\alpha-$Brownian motion.

Proposition 6.9 *Let T be a stopping time having all exponential moments under P_0. We suppose there exists $\alpha \neq 0$, such that $X(T)$ and $X(T) - \alpha T$ are Gaussian r.v. under P_0, resp. P_α. Then for any $\lambda \in \mathbb{R}$, T is $P_\lambda-$almost surely constant.*

Proof. 1) Again, Novikov's criterion ensures that, for any ν,

$$\Big(\exp\big\{\nu X(T \wedge t) - \frac{\nu^2}{2} T \wedge t\big\};\ t \geq 0\Big) \quad \text{is } P_0\text{-uniformly integrable.}$$

Consequently, for any $\nu \in \mathbb{R}$,

$$P_\nu = \exp\big\{\nu X(T) - \frac{\nu^2}{2} T\big\} P_0 \quad ,\text{on } \mathcal{F}_T. \tag{6.15}$$

2) Suppose there exists $\lambda \in \mathbb{R}$ such that $X(T) - \lambda T$ is a $P_\lambda-$Gaussian r.v. Automatically $X(T) - \lambda T$ is $P_\lambda-$centered and its second moment is $E_\lambda(T)$, hence

$$\exp\big\{\frac{\mu^2}{2} E_\lambda(T)\big\} = E_\lambda\Big[\exp\big\{\mu\big(X(T) - \lambda T\big)\big\}\Big];\ \mu \in \mathbb{R}. \tag{6.16}$$

Let ρ be the right hand-side of (6.16). Applying successively the relation (6.15) with $\nu = \lambda$ and $\nu = \lambda + \mu$ we obtain :

$$\rho = E_0\Big[\exp\big\{\mu\big(X(T) - \lambda T\big) + \lambda X(T) - \frac{\lambda^2}{2} T\big\}\Big] = E_{\lambda+\mu}\Big[\exp\big(\frac{\mu^2}{2} T\big)\Big].$$

But $x \to \exp\big(\frac{\mu^2}{2}x\big)$ is convex, therefore,

$$\exp\Big\{\frac{\mu^2}{2}E_{\lambda+\mu}(T)\Big\} \leq E_{\lambda+\mu}\Big[\exp\Big\{\frac{\mu^2}{2}T\Big\}\Big]. \tag{6.17}$$

Comparing with (6.16), we have : $E_\lambda(T) \geq E_{\lambda+\mu}(T)$.
In other words, if $X(T)-\lambda T$ is a $P_\lambda-$Gaussian r.v., then $\mu \to E_\mu(T)$ realizes its maximum at $\mu = \lambda$.
3) Suppose the assumptions of Proposition 6.9 are satisfied. The above second step implies that $E_0(T) \leq E_\alpha(T)$ and $E_\alpha(T) \leq E_0(T)$, hence : $E_0(T) = E_\alpha(T)$. This equality implies that we have an equality in Jensen's inequality (6.17). This is possible if and only if T is a.s. constant. ■

Before ending this paper we would like to add a few words about the conjecture of Cantelli $\big($[13], [43]$\big)$. Let X and X' be real valued, independent r.v.'s, $X \overset{(d)}{=} X'$ having the $\mathcal{N}(0,1)$ distribution, $f : \mathbb{R} \to [0, +\infty[$ a Borel function. If $Y = X + f(X)X'$ is Gaussian, then f is constant.

Let us consider more generally the following question.
Let $(B_t;\ t \geq 0)$ be a one dimensional $(\mathcal{F}_t)_{t\geq 0}-$Brownian motion started at 0, is it possible to find a r.v. Z being bounded non-constant and $\mathcal{F}_1$-measurable, such that $B(1) + Z\big(B(2) - B(1)\big)$ has a Gaussian distribution? The family of stopping times introduced in Theorem 6.1 allows to give a positive answer.
Let T be a bounded and non-constant stopping time such that $B(T)$ is a Gaussian r.v. For simplicity we suppose $T \leq 1$.
We set $Y = B(1) + \sqrt{T}\big(B(2) - B(1)\big)$. We claim that Y is a Gaussian r.v. We compute the characteristic function ϕ of Y :

$$\phi(\lambda) = E[e^{i\lambda Y}] = E[e^{i\lambda B(1)}e^{i\lambda\sqrt{T}(B(2)-B(1))}];\ \lambda \in \mathbb{R}.$$

$B(2) - B(1)$ being independent of $\mathcal{F}_1$, and $\mathcal{N}(0,1)-$distributed,

$$\phi(\lambda) = E\Big[\exp\Big\{i\lambda B(1) - \frac{\lambda^2}{2}T\Big\}\Big] = E\Big[\exp\Big\{i\lambda B(1) + \frac{\lambda^2}{2}\Big\}\exp\Big\{-\frac{\lambda^2}{2}(1+T)\Big\}\Big]$$

T being bounded by 1, using the martingale property, we get

$$\phi(\lambda) = E\Big[\exp\Big\{i\lambda B(T) + \frac{\lambda^2}{2}T\Big\}\exp\Big\{-\frac{\lambda^2}{2}(1+T)\Big\}\Big] = e^{-\lambda^2/2}E[e^{i\lambda B(T)}].$$

But $B(T)$ is a centered Gaussian r.v., consequently Y is also a centered Gaussian r.v. However $Z = \sqrt{T}$ is bounded and non-constant.

Obviously this result does not contradict the conjecture of Cantelli, because $Z = \sqrt{T}$ is $\mathcal{F}_1-$measurable and cannot be written as $f(B_1)$. ■

7. Appendix

7.1. The Skorokhod problem for $\big((t, B_t)\ ;\ t \geq 0\big)$.

Let $(B_t;\ t \geq 0)$ be a $(\mathcal{F}_t)_{t\geq 0}-$ Brownian motion started at 0 and let $((t,\ B_t)\ ;\ t \geq 0)$ be the process which is often called space-time Brownian motion. It is a continuous Markov process, taking its values in $[0, +\infty[\times\mathbb{R}$.
Suppose μ is a probability measure on $[0, +\infty[\times\mathbb{R}$. Does there exist a stopping time T such that the distribution of (T, B_T) is μ ?
Although it would be natural to rely upon Rost's [40] solution of Skorokhod's problem for a general Markov process $(X_t;\ t \geq 0)$, we shall in fact use a criterium stated by Falkner and Fitzsimmons ([16]) which is more convenient in our context.

Proposition 7.1 ([16]) *Let $(X_t)_{t\geq 0}$, $P_x(x \in E)$ be a "good" $E-$valued Markov process, μ_1, μ_2 two positive measures on E, and U the potential kernel of $(X_t)_{t\geq 0}$. We suppose $\mu_1.\ U$ is a $\sigma-$finite measure on E. The following are equivalent :*

(i) there exists a stopping time T such that $P_{\mu_1}(X_T \in .) = \mu_2$,

(ii) $\mu_1.\ U(f) \geq \mu_2.\ U(f)$, for any Borel, positive function f.

We shall apply further this proposition to the case where X is the space time Brownian motion $(t, B_t;\ t \geq 0)$.

We set

$$p(t, x) = \frac{1}{\sqrt{2\pi t}} \exp\big(-\frac{x^2}{2t}\big)\ ;\ x \in \mathbb{R},\ t > 0, \tag{7.1}$$

$$(f * \mu)(t, x) = \int_{[0,t]\times\mathbb{R}} f(t - s, x - y)\mu(ds, dy), \tag{7.2}$$

where $f : [0, +\infty[\times\mathbb{R} \to \mathbb{R}$ is a Borel function, and μ is a positive measure on $[0, +\infty[\times\mathbb{R}$.
The following proposition is a direct application of Proposition 7.1 to the case where $(X_t,\ t \geq 0) = ((t,\ B_t)\ ;\ t \geq 0)$.

Proposition 7.2 *Let μ be a probability measure on $[0, +\infty[\times\mathbb{R}$. There exists a $(\mathcal{F}_t)_{t\geq 0}-$stopping time T such that the distribution of (T, B_T) is μ if and only if,*

$$p * \mu\ \leq p \qquad , \text{ a.e.} \tag{7.3}$$

Remark 7.3 1) Suppose that μ is the law of $(1, B_1)$, i.e. :

$$\mu = \delta_1 \otimes \big(p(1,x)dx\big)$$

and

$$p * \mu(t,x) = \begin{cases} 0 & \text{if } t < 1, \\ \displaystyle\int_{\mathbb{R}} p(t-1, x-y)p(1,y)dy = p(t,x) & \text{if } t > 1. \end{cases} \tag{7.4}$$

This implies (7.3).

2) If $\mu = P\big((T, B_T) \in .\big)$, it is easy to check (7.3).

For any $\varepsilon > 0$, $\big(p(t+\varepsilon-s, x+B_s)\ ;\ 0 \le s \le t\big)$ is a continuous martingale. Therefore the optional stopping theorem implies

$$E\big[p(t+\varepsilon - T\wedge t, x + B_{t\wedge T})\big] = p(t+\varepsilon, x). \tag{7.5}$$

But $p(t+\varepsilon-T\wedge t, B_{T\wedge t}+x) \ge p(t+\varepsilon-T, B_T+x)1_{\{T\le t\}}$. If we take the limit in (7.5), as $\varepsilon \to 0_+$ Fatou's lemma gives (7.3) directly.

We present an explicit resolution of the space-time Skorokhod problem for some absolutely continuous probability measure μ.

Proposition 7.4 *Let $\mu(dt,dx) = \varphi(t,x)dtdx$, be a probability measure on $[0,+\infty[\times\mathbb{R}$ verifying (7.3). We assume φ is continuous and :*

$$\{t, x; (p - p*\mu)(t,x) = 0\} \subset \{t, x; \varphi(t,x) = 0\} \tag{7.6}$$

Let h be the function :

$$h = \begin{cases} \dfrac{\varphi}{p - p*\mu} & \textit{if } p - p*\mu > 0, \\ 0 & \textit{otherwise} \end{cases} \tag{7.7}$$

Consider ξ a r.v. with standard exponential distribution, independent of the Brownian motion $(B_t;\ t \ge 0)$ and

$$T_h = \inf\Big\{t \ge 0\ ;\ \int_0^t h(s, B_s)ds \ge \xi\Big\} \tag{7.8}$$

Then the distribution of (T_h, B_{T_h}) is μ.

Remark 7.5 Bourekh stated this result in his thesis ([6]) as he gave an explicit solution to the space-time Skorokhod problem with a target probability measure μ of the form :

$$\mu(dt, dx) = \Big\{\sum_i \varphi(t_i, x)\delta_{t_i}(dt) + \psi(t,x)dt\Big\}dx. \tag{7.9}$$

However, our proof is completely different.

We ask the following question: is it possible to find φ, h being given ?

Proposition 7.6 *Let $h : [0,+\infty[\times\mathbb{R} \to [0,+\infty[$, be a positive Borel function, and T_h be the stopping time defined by (7.8). Then the r.v. (T_h, B_{T_h}) on $\{T_h < \infty\}$ has a density h.p. ψ_h where*

$$\psi_h(t,x) = E\Big[\exp\big(-\int_0^t h(s,B_s)ds\big)|B_t = x\Big]. \tag{7.10}$$

Proof. We set $A_t = \int_0^t h(s,B_s)ds$. Therefore $t \to A_t$ is continuous and increasing. Let us denote its right inverse as A^{-1} $\big($i.e. $A_t^{-1} = \inf\{s \geq 0, A_s > t\}\big)$. Hence T_h can be expressed as :

$$T_h = A_\xi^{-1}.$$

Let f be a positive Borel function with compact support defined on $[0,+\infty[\times\mathbb{R}$ and

$$\Delta = E\big[f(T_h, B_{T_h})\big].$$

ξ being independent of $(B_t)_{t\geq 0}$, we have,

$$\Delta = E[\big[f(A_\xi^{-1}, B_{A_\xi^{-1}})\big] = E\big[\int_0^\infty e^{-t} f(A_t^{-1}, B_{A_t^{-1}})dt\big].$$

We set $s = A_t^{-1}$, we obtain

$$\Delta = \int_0^\infty E\big[e^{-A_s} f(s,B_s)h(s,B_s)\big]ds.$$

Conditioning by B_s, we check that $h.p.\ \psi_h$ is the density of (T_h, B_{T_h}). ■

Remark. Note that Proposition 7.4 gives the definition of the application $H : \varphi \to g \equiv g_\varphi$, whereas Proposition 7.6 describes its inverse $H^{-1} : g \to \varphi = g \cdot p \cdot \psi_g$.

Proof of Proposition 7.4. 1) Suppose the probability measure μ has φ as a density.

1) Let ω be the solution of :

$$\begin{cases} \dfrac{\partial\omega}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2\omega}{\partial x^2} - \varphi & \text{in }]0,+\infty[\times\mathbb{R}, \\ \omega(0,x)dx = \delta_0(dx) \end{cases} \tag{7.11}$$

This linear P.D.E. can be solved explicitly (see for instance [14], Chap. 8) :

$$\omega(t,x) = p(t,x) - E\Big[\int_0^t \varphi(t-s, x+B_s)ds\Big]$$

Since φ is the density of μ, by (7.6) and (7.7), we have :

$$h\omega = h(p - p*\mu) = \varphi\,.$$

Consequently, ω is a solution of

$$\begin{cases} \dfrac{\partial \omega}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 \omega}{\partial x^2} - h\omega & \text{in }]0,+\infty[\times\mathbb{R} \\ \omega(0,x)dx = \delta_0 \end{cases}$$

2) Let $\omega_h = \psi_h.\ p$, ψ_h being the function defined by (7.10). Then for any test function f,

$$\int_{\mathbb{R}} f(y)\omega_h(t,y)dy = E\Big[f(B_t)\exp\big(-\int_0^t h(s,B_s)ds\big)\Big].$$

It is well known that ω_h is the unique solution of :

$$\begin{cases} \dfrac{\partial \omega}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 \omega}{\partial x^2} - h\omega \\ \omega(0,x)dx = \delta_0(dx) \end{cases}$$

Using Proposition 7.6, $h\cdot\ p\cdot\ \psi_h = h\cdot\ \omega_h = h\cdot\ \omega = \varphi$ is the density of (T_h, B_{T_h}), (this proves, a posteriori, that $T_h < \infty$ a.s.) ■

Remark 7.7 1) Let f be a Borel and bounded function and $u(t,x) = E\big[f(x+B_t)\big]$, $t \geq 0$, $x \in \mathbb{R}$. It is well known that $\big(u(t - s\wedge T, B_{s\wedge T})\ ;\ 0 \leq s \leq t\big)$ is a bounded martingale.
Using the optimal stopping theorem we obtain

$$E\big[f(B_t)1_{\{t<T\}}\big] + E\big[u(t-T, B_T)1_{\{T\leq t\}}\big] = \int f(y)p(t,y)dy.$$

By a straightforward calculation we get

$$(p - p*\mu)(t,x) = P(T > t|B_t = x)p(t,x).$$

And finally,

$$(p*\mu)(t,x) = P(T \leq t|B_t = x)p(t,x), \quad t > 0,\ x \in \mathbb{R}.$$

2) Here is an explicit example : take $g(s,x) = \frac{b^2}{2} \cdot x^2$; then, the one-dimensional variant of Lévy's stochastic area formula (e.g. Yor ([47]), formula 2.5, p. 18) asserts that :

$$\begin{aligned}\psi_g(t,x) :&= E(\exp - \frac{b^2}{2}\int_0^t ds B_s^2 \,|B_t = x) \\ &= (\frac{bt}{\sinh bt})^{1/2} \exp - \Big(\frac{x^2}{2t}(bt\coth(bt) - 1)\Big)\end{aligned}$$

Consequently, by Lemma 7.6, we obtain :

$$\begin{aligned}P\big((T_g, B_{T_g}) \in (dt, dx)\big) &= \varphi(t,x)dtdx \\ &= \frac{b^2x^2}{2}(\frac{b}{2\pi \sinh bt})^{1/2} \exp(-\frac{x^2b^2}{2}\coth\ bt)\, dt\, dx\end{aligned}$$

and, in particular :

$$P(T_g \in dt) = \frac{b}{2}\, \frac{\sinh bt}{(\cosh bt)^{3/2}}\, dt.$$

We may as well develop similar computations in $\mathbb{R}^n$ with $g(s,x) = \dfrac{b^2}{2}|x|^2$. This yields, in particular :

$$P(T_g \geq t) = \frac{1}{\big(\cosh(bt)\big)^{n/2}}.$$

7.2. The case of the Ornstein-Uhlenbeck process

In this section, $\big(X(t);\ t \geq 0\big)$ will denote an Ornstein-Uhlenbeck process started at 0, with parameter $a \neq 0$. We recall that it solves the stochastic differential equation :

$$X(t) = B(t) + a\int_0^t X(s)ds \quad ;\ t \geq 0,$$

where $\big(B(t)\ ;\ t \geq 0\big)$ is a $\mathbb{R}-$valued Brownian motion, $B(0) = 0$.

Theorem 7.8 *Let T be a bounded stopping time such that $X(T)$ and T are independent r.v.'s., then T is a.s. constant.*

Remark 7.9 1) Recall that in the context of the Brownian motion with drift (i.e. Theorem 2.2), we only assumed that T has all exponential moments.

2) We know that for any $t > 0$, the law of $X(t)$ is Gaussian. Therefore we can add in the conclusion of Theorem 7.8, that $X(T)$ is a Gaussian r.v.

Our approach is based on a martingale technique closely connected to the proof given in the Brownian setting. Let us briefly describe the procedure. Let $\lambda \in \mathbb{R}$ and φ_λ be a "good" eigenfunction, with respect to the infinitesimal generator L of $\big(X(t)\big)_{t\geq 0}$ (i.e. $Lf = \lambda f$). It is well known that $\big(e^{-\lambda t}\varphi_\lambda\big(X(t)\big)$ $t \geq 0\big)$ is a continuous local martingale. Provided the optional stopping theorem applies, we have

$$E\Big[e^{-\lambda T}\varphi_\lambda\big(X(T)\big)\Big] = 1\,.$$

φ_λ having an holomorphic extension to the whole plane, we can conclude that T is constant.

Three preliminary steps for the proof of Theorem 7.8 are needed : Lemmata 7.10-7.12. For simplicity we suppose $a = -1/2$.

Lemma 7.10 *Let $\big(a_{2k}(\lambda)\big)_{k\geq 0}$ be the sequence of $\mathbb{C}$ valued polynomials defined on $\mathbb{C}$:*

$$a_0(\lambda) = 1 \quad ; \; a_{2k}(\lambda) = \frac{1}{(2k)!}\prod_{p=0}^{k-1}(2p + 2\lambda). \tag{7.12}$$

Then, define :

$$\varphi_\lambda(x) = \sum_{k\geq 0} a_{2k}(\lambda)x^{2k} \quad ; \; x \in \mathbb{R}, \; \lambda \in \mathbb{C}. \tag{7.13}$$

Then the radius of convergence of this series is infinite, $\varphi_\lambda(0) = 1$, $L\varphi_\lambda = \lambda\varphi_\lambda$, and

$$\big|a_{2k}(\lambda)\big| \leq a_{2k}\big(|\lambda|\big), \tag{7.14}$$

$$\big|\varphi_\lambda(x)\big| \leq \varphi_{|\lambda|}(x) \quad ; \; \forall\lambda \in \mathbb{C}, \; \forall x \in \mathbb{R}, \tag{7.15}$$

$$\Big|\frac{\partial}{\partial\lambda}a_{2k}(\lambda)\Big| \leq \frac{\partial}{\partial\lambda}a_{2k}\big(|\lambda|\big) \leq a_{2k}\big(1 + |\lambda|\big). \tag{7.16}$$

Moreover for any $\lambda \geq 0$, there exists a polynomial P_λ with non negative coefficients such that,

$$\big|\varphi_\lambda(x)\big| \leq e^{x^2/2}P_\lambda(x^2) \quad ; \; \lambda \geq 0, \; x \in \mathbb{R}. \tag{7.17}$$

The proof of this lemma is left to the reader; here, is our second lemma:

Lemma 7.11 *For any $\lambda \in \mathbb{C}$, we set $M^{(\lambda)}(t) = e^{-\lambda t}\varphi_\lambda\big(X(t)\big)$; $t \geq 0$. Then for any $a > 0$, $\big(M^{(\lambda)}(t);\ 0 \leq t \leq a\big)$ is a uniformly integrable martingale.*

Proof of Lemma 7.11. Since $L\varphi_\lambda = \lambda\varphi_\lambda$, Ito's formula implies that $\big(M^{(\lambda)}(t);\ t \geq 0\big)$ is a continuous local martingale.

We shall prove that $(M^{(\lambda)}(t),\ t \geq 0)$ is of class (DL) ([37], p. 117) which implies our claim. For a family $(Y_i,\ i \in I)$ of random variables to be uniformly integrable it is sufficient that :

$$\sup_{i \in I} E(|Y_i| \log_+ |Y_i|) < \infty.$$

Applying this criterion, we only need, by (7.17), to prove :

$$\sup_{T \leq a} E\big\{X_T^{2n} \exp\frac{X_T^2}{2}\big\} < \infty\,, \tag{7.18}$$

where T is (of course) assumed to be a stopping time.

Ito's formula tells us :

$$E\big\{(1 + X_{t\wedge T}^{2n})\exp\frac{X_{t\wedge T}^2}{2}\big\} \leq C_n E\int_0^t (1 + X_{s\wedge T}^{2n})\exp\frac{X_{s\wedge T}^2}{2} ds$$

and so, (7.18) follows by Gronwall's lemma. ∎

Finally the next lemma, whose proof is left to the reader, will be important in our application of Hadamard's theorem in the following proof of Theorem 7.8.

Lemma 7.12 *Let T be a bounded stopping time. Then for any $\lambda \in \mathbb{C}$, the r.v. $\varphi_\lambda\big(X(T)\big)$ is integrable. Moreover $\lambda \to E\Big[\varphi_\lambda\big(X(T)\big)\Big]$ is holomorphic.*

Proof of Theorem 7.8. We suppose that T is a stopping time bounded by a, and that the two r.v.'s X_T and T are independent. Let $\lambda \in \mathbb{C}$.

T being bounded, we may apply the stopping theorem to the martingale $\big(M^{(\lambda)}(t); 0 \leq t \leq a\big)$ (Lemma 7.11) :

$$E\big[M^{(\lambda)}(T)\big] = E\Big[\varphi_\lambda\big(X(T)\big)e^{-\lambda T}\Big] = E\big[M^{(\lambda)}(0)\big] = 1.$$

$X(T)$ and T being independent, the previous identity is equivalent to :

$$h(\lambda)E\big[e^{-\lambda T}\big] = 1\ ; \quad \forall \lambda \in \mathbb{C}, \tag{7.19}$$

where $h(\lambda) = E\Big[\varphi_\lambda\big(X(T)\big)\Big]$.

Using both $E\left[e^{-sT}\right] \geq e^{-sa} P(T < a)$ if $s \geq 0$, (7.14) and (7.15), we obtain

$$\left|h(\lambda)\right| \leq E\Big[\varphi_{|\lambda|}\big(X(T)\big)\Big] = h\big(|\lambda|\big) = \frac{1}{E\left[e^{-|\lambda| T}\right]} \leq \frac{e^{|\lambda| a}}{P(T < a)}, \quad \lambda \in \mathbb{C}.$$

As a result, h is a holomorphic function (Lemma 7.12), which does not vanish (as a consequence of (7.19)) and its order is smaller than or equal to 1 $\big($cf (2.4)$\big)$. Hadamard's theorem tells us

$$h(\lambda) = E\Big[\varphi_{\lambda}\big(X(T)\big)\Big] = \exp\{\alpha\lambda + \beta\}.$$

But $h(0) = 1$, hence $\beta = 0$ and

$$E\left[e^{-sT}\right] = \frac{1}{h(s)} = e^{-\alpha s} \quad , \; s \in \mathbb{R}.$$

This implies $T = \alpha$. ■

Remark 7.13 The methodology developed for the Ornstein-Ulhenbeck process applies equally well to Bessel processes with any dimension d. We have already developed otherwise (see Corollary 3.7) this study for $d > 1$.

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Bernard de Meyer
ESSTIN
Parc Robert Bentz, Rue Jean Lamour
54500 Vandœuvre-lès-Nancy, France
demeyer@iecn.u-nancy.fr

Bernard Roynette
Département de Mathématiques
Institut Elie Cartan BP 239, F
54506 Vandœuvre-lès-Nancy Cedex, France
roynette@iecn.u-nancy.fr

Pierre Vallois
Département de Mathématiques
Institut Elie Cartan BP 239, F
54506 Vandœuvre-lès-Nancy Cedex, France
vallois@iecn.u-nancy.fr

Marc Yor
Laboratoire de Probabilités et Modèles Aléatoires
Université Paris VI et VII, Tour 56, 3ème étage
75252 Paris Cedex 05, France
deaproba@proba.jussieu.fr