Rev. Mat. Iberoamericana, **18** (2002), 325–354

Harnack's inequality for solutions of some degenerate elliptic equations

Ahmed Mohammed

Abstract

We prove a Harnack's inequality for non-negative solutions of some degenerate elliptic operators in divergence form with the lower order term coefficients satisfying a Kato type condition.

1. Introduction.

In this paper, we study the behavior of solutions of certain degenerate elliptic equations $Lu = 0$, where L is the operator

$$L := -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\,\frac{\partial}{\partial x_j}\Big) + \sum_{i=1}^{n} b_i(x)\,\frac{\partial}{\partial x_i} + V(x)\,.$$

The coefficients a_{ij} are real-valued measurable functions whose coefficient matrix $A(x) := (a_{ij}(x))$ is symmetric and satisfies

$$\omega(x)\,|\xi|^2 \le \langle A(x)\,\xi,\xi\rangle \le \upsilon(x)\,|\xi|^2\,. \tag{1.1}$$

Here $\langle\cdot,\cdot\rangle$ denotes the usual inner product on $\mathbb{R}^n$, and υ,ω are non-negative functions which will be described below.

Let us fix some notations that will be used throughout the paper. For functions f and g, we shall write $f \lesssim g$ to indicate that $f \le Cg$ for some

---

2000 *Mathematics Subject Classification*: 35B45, 35B65, 35J10, 35J15, 35J70.
Keywords: Kato class, Green's function, Harnack's inequality.

positive constant C. We write $f \approx g$ if $f \lesssim g$ and $g \lesssim f$. We shall use $B_t(x)$ to denote a ball of radius t centered at x. Also, tB will stand for the ball concentric with the ball B, but with radius t times as big. Given a locally integrable function f, we let $f(B)$ denote the Lebesgue integral of f over the set B. If $f \in L_{\mathrm{loc}}(d\mu)$, where $d\mu := \gamma(x)\,dx$ is a weighted measure, then we denote by

$$\fint_B f(x)\gamma(x)\,dx := \frac{1}{\gamma(B)}\int_B f(x)\gamma(x)\,dx\,,$$

the μ-average of f over B. This average shall also be denoted by f_B, γ.

A non-negative locally integrable functions ω on $\mathbb{R}^n$ is said to be in the class A_2 if there is a constant C such that for all balls B,

$$\Big(\fint_B \omega(x)\,dx\Big)\Big(\fint_B \frac{1}{\omega(x)}\,dx\Big) \leq C\,.$$

A non-negative locally integrable functions υ on $\mathbb{R}^n$ is said to satisfy a doubling condition if there is a constant C such that $\upsilon(2B) \leq C\,\upsilon(B)$ for all balls B. Here C is independent of the center and radius of B. We denote this by writing $\upsilon \in D_\infty$. It is known that $A_2 \subset D_\infty$.

It is also known (see [12]) that if υ satisfies a doubling condition, then it satisfies

$$\upsilon(tB) \leq C_1\, t^k \upsilon(B)\,, \qquad \text{and} \qquad \upsilon(B) \leq C_2\, t^{-m}\upsilon(tB)\,, \qquad t > 1\,,$$

for some positive constants C_1, C_2, k, and m. The latter condition is called a reverse doubling condition.

Throughout the paper, we will require that ω, and υ satisfy the assumptions stipulated below.

ω and υ are non-negative locally integrable functions on $\mathbb{R}^n$ that satisfy the following conditions.

$$\omega \in A_2\,,\ \upsilon \in D_\infty\,. \tag{1.2}$$

ω and υ are related by the existence of some $q > 2$ such that

$$\frac{s}{t}\Big(\frac{\upsilon(B_s(x))}{\upsilon(B_t(x))}\Big)^{1/q} \leq C\Big(\frac{\omega(B_s(x))}{\omega(B_t(x))}\Big)^{1/2}, \qquad 0 < s < t\,,\ x \in \mathbb{R}^n\,, \tag{1.3}$$

for some constant C independent of x, s and t.

We shall use the notation $\sigma = q/2$ so that $\sigma > 1$. Note that when υ and ω are positive constants, as in the strongly elliptic case, the value of q in (1.3) is $q = 2\,n/(n-2)$, so that $\sigma = n/(n-2)$.

Let now L_0 be the principal part of L; that is

$$L_0 := -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\,\frac{\partial}{\partial x_j}\Big)\,,$$

and let B_0 be a ball in $\mathbb{R}^n$ that will be fixed in the sequel. Under conditions (1.2) and (1.3), S. Chanillo and R. Wheeden have established, in [4] the existence and integrability properties of the Green function of L_0. Among several important properties, they have shown that if $G(x,y)$ is the Green function of L_0 on $2B_0$, then for $0 < p < \sigma$,

$$\sup_{y\in B_0}\int_{2B_0} G(x,y)^p\, \upsilon(x)\,dx < \infty\,. \tag{1.4}$$

Let $B \subset B_0$. In analogy with the way the usual Kato class is defined, we introduce a class of functions $K^n(B)$ as

$$K^n(B) := \{h \in L^1_{\text{loc}}(B) : \lim_{r\to 0^+} \eta(h)(r) = 0\}\,,$$

where

$$\eta(h)(r) := \sup_{x\in B}\int_{B_r(x)\cap B} G(y,x)\,|h(y)|\,dy\,.$$

If $L^p_\mu(B)$ denotes the usual L^p space with respect to a measure μ, and $B \subset B_0$, then

$$L^p_{\upsilon^{1-p}}(B) \subset K^n(B)\,,$$

provided that $p > \sigma/(\sigma-1)$ (see [10]).

For notational simplicity, we shall use K for the function space $K^n(B_0)$.

Remark 1.1. We should remark that when υ and ω are identically equal to positive constants, as in the strongly elliptic case, the class of functions K coincides with the usual Kato class (see [5] for definition). Also, if υ and ω are constant multiples of each other, then again K is the same as the one introduced in [7].

We will make the following assumptions on the lower order coefficients $\mathbf{b} = (b_1, b_2, \ldots, b_n)$, and V of the elliptic operator L.

$$|\mathbf{b}|^2\,\omega^{-1}\,, \qquad V \in K\,. \tag{1.5}$$

In their celebrated work [1], M. Aizenman and B. Simon used probablistic methods to prove that non-negative weak solutions of $-\Delta u + Vu = 0$

satisfy uniform Harnack's inequality, where V is a potential from the classical Kato class, and Δ is the Laplace operator. Later, F. Chiarenza, E. Fabes and N. Garofalo developed in [5], a real variable technique to prove Harnack's inequality when the Laplace operator is replaced by a uniformly elliptic operator in divergence form. Subsequently, these methods were used by several authors to derive Harnack's inequality for more general elliptic equations in divergence form. Using the techniques of [5], K. Kurata proved in [8], Harnack's inequality for non-negative solutions of $L_0 u + \mathbf{b} \cdot \nabla u + Vu = 0$, where L_0 is a uniformly elliptic operator in divergence form and $|\mathbf{b}|^2, V$ belong to the classical Kato class. Harnack's inequality has also been derived for degenerate elliptic equations by several authors. In the degenerate case, the following important works are worth mentioning. In the absence of lower order terms and the case when v and ω are constant multiples of each other, Harnack's inequality was derived in [6]. In [3], where the unequal weights case was considered, the authors obtain Harnack's inequality for non-negative solutions of degenerate equations in divergence form without lower order terms. In [7], C. Gutierrez considers the equal weights case with a potential V from the Kato class K. In this paper, the author sucessfully applies the methods of [5] to derive Harnack's inequality in the degenerate case. See also [9] for related results. Our work here is largely motivated by the papers [3], [4], and [7]. Our main result in this paper is Theorem 4.1 which establishes Harnack's inequality for functions naturally associated with non-negative solutions of the operator L. As the work here uses results obtained in [10], we will state these results for easy reference and the reader's convenience. The results in [10] were motivated by the important works of S. Chanillo and R. Wheeden in their papers [2], [3] and [4]. In Section 3, we will prove some mean-value inequalities involving weak solutions of the opeartor L. To obtain these inequalities, we adapt a combination of the methods developed in [5], and [3] (see also [11]). In Section 4, Harnack's inequality is proved. Here we follow the paper [3] closely.

2. Preliminaries and background.

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Using a standard notation, let $\mathrm{Lip}(\overline{\Omega})$ denote the class of Lipschitz continuous functions on the closure $\overline{\Omega}$. We say that $\phi \in \mathrm{Lip}_0(\Omega)$ if $\phi \in \mathrm{Lip}(\overline{\Omega})$ and ϕ has compact support contained in Ω. The following two-weight Sobolev's and Poincaré inequalities have been proved in [2].

Let ω, v be non-negative locally integrable functions that satisfy (1.2), (1.3), and q be the constant that appears in (1.3). Then, for a ball B,

$$\left(\fint_B |f|^q\, v\, dx\right)^{1/q} \le C\,|B|^{1/n}\left(\fint_B |\nabla f|^2\, \omega\, dx\right)^{1/2}, \qquad f \in \mathrm{Lip}_0(B) \tag{2.1}$$

and

$$\left(\fint_B |f - f_{B,v}|^q\, v\, dx\right)^{1/q} \le C\,|B|^{1/n}\left(\fint_B |\nabla f|^2\, \omega\, dx\right)^{1/2}, \tag{2.2}$$

$f \in \mathrm{Lip}(\overline{B})$.

In (2.1), and (2.2) the constant C is independent of both the ball B and f.

Now let us consider the inner product

$$a(u,\varphi) := \int_\Omega \langle A\nabla u, \nabla\varphi\rangle + \int_\Omega u\,\varphi\, v\,, \qquad u,\varphi \in \mathrm{Lip}(\overline{\Omega})\,.$$

The completion of $\mathrm{Lip}(\overline{\Omega})$ with respect to the norm $\|u\| := a(u,u)^{1/2}$ is denoted by $H(\Omega)$. Thus $H(\Omega)$ is formed by adjoining to $\mathrm{Lip}(\overline{\Omega})$ elements $\{u_k\}$, $u_k \in \mathrm{Lip}(\overline{\Omega})$ such that $\{u_k\}$ is a Cauchy sequence with respect to the norm $\|\cdot\|$ on $\mathrm{Lip}_0(B)\Omega$. If $u,\varphi \in H(\Omega)$, with $u = \{u_k\}$, $\varphi = \{\varphi_k\}$, $u_k, \varphi_k \in \mathrm{Lip}(\overline{\Omega})$, then $a(u_k,\varphi_k)$ is convergent, and we define

$$a(u,\varphi) = \lim_k a_0(u_k,\varphi_k)\,.$$

This turns $H(\Omega)$ into a Hilbert space with inner product $a(u,\varphi)$, and norm $\|u\| := a(u,u)^{1/2}$. As a consequence of the inequality

$$\int_\Omega |\nabla u|^2\omega + \int_\Omega u^2 v \le \|u\|^2\,,$$

we see that, if $u := \{u_k\} \in H(\Omega)$, then $\{u_k\}$, and $\{|\nabla u_k|\}$ are Cauchy sequences in $L^2_v(\Omega)$, and $L^2_\omega(\Omega)$ respectively. Therefore $u_k \longrightarrow \widetilde{u}$ in $L^2_v(\Omega)$, and $\nabla u_k \longrightarrow \nabla\widetilde{u}$ in $L^2_\omega(\Omega)$. We shall refer to $\widetilde{u}$ as the element in $L^2_v(\Omega)$ associated with $u \in H(\Omega)$. (See [3], or [4] for details).

If $a_0(u,\varphi)$ is the inner product on $\mathrm{Lip}_0(\Omega)$ defined by

$$a_0(u,\varphi) := \int_\Omega \langle A\nabla u, \nabla\varphi\rangle\,, \qquad u,\varphi \in \mathrm{Lip}_0(\Omega)\,,$$

then the completion of $\mathrm{Lip}_0(\Omega)$ under the induced norm is denoted by $H_0(\Omega)$, and the inner product $a_0(\cdot,\cdot)$ extends to $H_0(\Omega)$ by the same procedure used above to extend $a(\cdot,\cdot)$ to $H(\Omega)$. The space then becomes a

Hilbert space under this inner product and the norm $\|u\|_0 := a_0(u,u)^{1/2}$, $u \in H_0(\Omega)$.

For future reference, we record the following inequality that can be easily verified using the Cauchy-Schwartz inequality.

$$\|\varphi\phi\|_0 \lesssim \|\varphi\|_\infty \, \|\phi\|_0 + \|\phi\|_\infty \, \|\varphi\|_0 \, , \qquad \varphi, \phi \in \mathrm{Lip}_0(\Omega) \, . \tag{2.3}$$

As a consequence of the Sobolev's inequality (2.1), the Hilbert space $H_0(\Omega)$ is seen to be continuously embedded in $H(\Omega)$.

For $u \in H(\Omega)$ we say that $u \geq 0$ on Ω, if $u_k \geq 0$ for all k and some $\{u_k\}$ representing u. If $u \geq 0$ on Ω, then $\widetilde{u} \geq 0$ almost everywhere on Ω.

We now recall some results that will be needed in this paper. The reader can find proofs of these results in [10]. We will use the numbering A.1, A.2, etc to label these results.

The first Lemma is a slight extension of Lemma (2.7) of [4], and we will use it repeatedly.

Lemma A.1. *Let $u = \{u_k\}$, $\varphi = \{\varphi_k\}$ be in $H(\Omega)$. If $\{\zeta_k\}$ is a bounded sequence in $L^\infty(\Omega)$ that converges pointwise almost everywhere to $\zeta \in L^\infty(\Omega)$, then*

$$\int_\Omega \langle A\nabla u_k, \nabla \varphi_k \rangle \, \zeta_k \longrightarrow \int_\Omega \langle A\nabla \widetilde{u}, \nabla \widetilde{\varphi} \rangle \, \zeta \, , \qquad \text{as } k \longrightarrow \infty \, .$$

As a consequence of this Lemma, we see that

$$a_0(u, \varphi) = \int_\Omega \langle A\nabla \widetilde{u}, \nabla \widetilde{\varphi} \rangle \, , \qquad u, \varphi \in H_0(\Omega) \, .$$

The following embedding lemma is useful in the subsequent development (see [10] for a proof).

Lemma A.2. *If $f \in K$, and $B \subset\subset B_0$ is a ball of radius r, then for any $u \in H_0(B)$ the following holds.*

$$\int_B |f| \, \widetilde{u}^{\,2} \, dx \lesssim \eta(f)(3\, r) \int_B \langle A\nabla \widetilde{u}, \nabla \widetilde{u} \rangle \, .$$

Let us now consider the general elliptic operator

$$Mu := -\mathrm{div}(A(x)\nabla u + \mathbf{c}(x)\, u) + \mathbf{b}(x) \cdot \nabla u + V(x)\, u \, ,$$

where, in addition to (1.5) we also assume that $|\mathbf{c}|^2\omega^{-1} \in K$. With M, and its adjoint operator

$$M^*u := -\mathrm{div}(A(x)\nabla u + \mathbf{b}(x)\, u) + \mathbf{c}(x)\cdot\nabla u + V(x)\, u\,,$$

we associate the bilinear forms $D(\cdot,\cdot)$ and $D_*(\cdot,\cdot)$ as follows. Fix a ball $B \subset\subset B_0$ of radius r, and let us define

$$D(u,\varphi) := \int_B \langle A\nabla u, \nabla\varphi\rangle + \mathbf{c}(x)\cdot\nabla\varphi\, u + \mathbf{b}(x)\cdot\nabla u\,\varphi + V u\,\varphi\,,$$

and $D_*(u,\varphi) := D(\varphi, u)$ for all $u,\varphi \in \mathrm{Lip}_0(B)$. Observe that by Hölder inequality and Lemma A.2, it follows that

$$(2.4) \qquad |D(u,\varphi) - a_0(u,\varphi)| \lesssim \vartheta(r)\,\|u\|_0\,\|\varphi\|_0\,, \qquad \varphi,\, u \in \mathrm{Lip}_0(B)\,,$$

where

$$\vartheta(r) := (\eta\,(|\mathbf{c}|^2\,\omega^{-1})(3\,r))^{1/2} + (\eta\,(|\mathbf{b}|^2\,\omega^{-1})(3\,r))^{1/2} + \eta(V)(3\,r)\,.$$

Therefore, we get

$$(2.5) \qquad |D(u,\varphi)| \lesssim (1+\vartheta(r))\,\|u\|_0\,\|\varphi\|_0\,, \qquad \varphi, u \in \mathrm{Lip}_0(B)\,.$$

Thus if $u = \{u_k\}$, $\varphi = \{\varphi_k\}$, $u_k, \varphi_k \in \mathrm{Lip}_0(B)$ are elements of $H_0(B)$ then the above inequality shows that $\{D(u_k,\varphi_k)\}$ is a Cauchy sequence and hence $\lim_k D(u_k,\varphi_k)$ exists. Therefore we define

$$D(u,\varphi) := \lim_k D(u_k,\varphi_k)\,.$$

Having defined $D(u,\varphi)$ for $u,\varphi \in H_0(B)$, the inequality (2.5) still holds for any $u,\varphi \in H_0(B)$. As a result of this inequality we see that for a fixed $u \in H_0(B)$, the map $\varphi \longmapsto D(u,\varphi)$ is a continuous linear functional on $H_0(B)$.

By Lemma A.2, it can also be shown along similar lines that

$$|D(u,\varphi)| \le C\,\|u\|\,\|\varphi\|_0\,,$$

for any $\varphi \in \mathrm{Lip}_0(B)$, and $u \in \mathrm{Lip}(\overline{B}_0)$. The constant C here depends on the distance of ∂B to ∂B_0. Consequently $D(u,\varphi)$ can be defined as the limit of $D(u_k,\varphi_k)$ whenever $u = \{u_k\} \in H(B_0)$, and $\varphi = \{\varphi_k\} \in H_0(B)$. Furthermore the inequality $|D(u,\varphi)| \le C\,\|u\|\,\|\varphi\|_0$ holds for $u \in H(B_0)$, and $\varphi \in H_0(B)$.

Using (2.4) one obtains $(1 - C\,\vartheta(r))\,\|u\|_0^2 \leq D(u,u)$ for $u \in \mathrm{Lip}_0(B)$, and some constant C. Therefore for sufficiently small r_0, and all $0 < r \leq r_0$ we have

$$\|u\|_0^2 \lesssim D(u,u)\, u \in H_0(B)\,,$$

so that $D(\cdot,\cdot)$ is a coercive bilinear form on $H_0(B)$.

Given $f \in K$, we shall say that $u = \{u_k\} \in H(B_0)$ is a weak solution of $Mu = f$ in B if

$$D(u,\varphi) = \int_B f\widetilde{\varphi}\,, \qquad \text{for all } \varphi = \{\varphi_k\} \in H_0(B)\,.$$

Similar statements and definitions hold for the adjoint operator M^* and the associated bilinear form $D_*(\cdot,\cdot)$.

The following two Remarks will be useful at several stages in our subsequent proofs.

Remark 2.1. If $f \in K$, and $B \subset\subset B_0$ is a ball, then by Lemma A.2, the map

$$\varphi \longmapsto \int_B f\widetilde{\varphi}$$

is a continuous linear functional on $H_0(B)$. Therefore, by the Lax-Milgram theorem there is a unique element $u \in H_0(B)$ such that

$$D(u,\varphi) = \int_B f\widetilde{\varphi}\,, \qquad \varphi \in H_0(B)\,.$$

The same remark holds for the bilinear form $D_*(\cdot,\cdot)$.

Remark 2.2. Let $f \in K$, and $u = \{u_k\} \in H(B_0)$ be a weak solution of $Mu = f$ in B. If $\{v_k\}$ is a bounded, weakly convergent sequence in $H_0(B)$, then

$$\lim_{k\to\infty}\Big(D(u_k,v_k) - \int_B f\,\widetilde{v}_k\Big) = 0\,.$$

To see this, suppose that $v \in H_0(B)$ is the weak limit of $\{v_k\}$ in $H_0(B)$. From the inequality $|D(u_k - u, v_k)| \leq C\,\|u_k - u\|\,\|v_k\|_0$, we observe that

$$\lim_k D(u_k,v_k) = \lim_k (D(u_k - u, v_k) + D(u,v_k)) = \lim_k D(u,v_k)\,.$$

Thus the assertion follows from this limit, and the fact that the linear functionals

$$\varphi \longmapsto D(u,\varphi)\,, \qquad \text{and} \qquad \varphi \longmapsto \int_B f\,\widetilde{\varphi}$$

are continuous on $H_0(B)$.

Henceforth, when we consider the bilinear forms $D(,\cdot)$, and $D_*(\cdot,\cdot)$, we will assume that $\mathbf{c} \equiv 0$.

Remark 2.3. Remark 2.1 shows that given $y \in B$, and a ball $B_\rho(y) \subset B$ there is a unique $G^\rho \in H_0(B)$ such that

$$D_*(G^\rho, \varphi) = \fint_{B_\rho(y)} \widetilde{\varphi}\, \upsilon\,, \qquad \varphi \in H_0(B)\,.$$

The associated function $\widetilde{G}^\rho$ in $L^2_\upsilon(B)$ is called the approximate Green function of L on B with pole y. It was shown in [10, Lemma 3.5, Lemma 3.6] that G^ρ is non-negative, and that G^ρ has a representative $G^\rho = \{G^\rho_k\}$, $G^\rho_k \in \mathrm{Lip}_0(B)$ such that $0 \le G^\rho_k \le C$ for some constant C independent of k.

The following two results about the approximate Green function $\widetilde{G}^\rho$ of L were proved in [10]. To obtain these results, in addition to conditions (1.5), the following was also assumed on the coefficient $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ of L.

$$\operatorname{div} \mathbf{b} \in K\,. \tag{2.6}$$

Conditions (1.5) and (2.6) were used in [8] to derive a reverse Hölder inequality for the Green function of L in the uniformly elliptic case.

Lemma A.3. *Let B be a ball of radius r with $2B \subset B_0$, and $\widetilde{G}^\rho$ be the approximate Green function of L on B. If the coefficients of L satisfy the conditions* (1.5), *and* (2.6), *then there is a constant C, independent of ρ and the pole of G^ρ such that*

$$\int_B (|\mathbf{b}|^2\omega^{-1} + |V|)\, \widetilde{G}^\rho \le C\,\eta\,(|\mathbf{b}|^2\omega^{-1} + |V|)(2\,r)\,,$$

for sufficiently small r.

The following Theorem on the uniform integrability of the approximate Green functions of L will be useful in obtaining mean-value inequalities for weak solutions of L.

Theorem A.1. *Let B be a ball of radius r with $2B \subset B_0$. Suppose $\widetilde{G}^\rho$ is the approximate Green function of L on B, where we assume that the coefficients of L satisfy the conditions* (1.5) *and* (2.6). *Then for $1 < p < \sigma$ there is a positive constant C, independent of ρ and the pole, such that*

$$\left(\fint_B (\widetilde{G}^\rho)^p v \right)^{1/p} \le C \frac{r^2}{\omega(B)} \, ,$$

when r is sufficiently small.

3. Mean-value inequalities.

We start this section by deriving a Caccioppoli-type estimate. To this end we need to consider a twice continuously differentiable function h such that

$$\begin{aligned} & h(\tau) h''(\tau) \ge 0 \, , \\ & |\tau^2 h''(\tau)| \lesssim |\tau h'(\tau)| \lesssim |h(\tau)| \, , \\ & \text{and } |h''(\tau)| + |h'(\tau)| + |h(\tau)| \lesssim 1 \, . \end{aligned} \tag{3.1}$$

Lemma 3.1 (Caccioppoli-Type Estimate)**.** *Let h be a twice continuously differentiable function that satisfies* (3.1). *Let $B \subset\subset B_0$ be a ball of radius r, and suppose $u \in H(B_0)$ is a weak solution of $Mu = 0$ in B which has a representative $u = \{u_k\}$ such that $h(u_k)$, $h'(u_k)$, and $h''(u_k)$ are all defined for all k. Then, given $0 < s < t < 1$ there is a constant $C > 0$ such that*

$$\int_{sB} \langle A \nabla h(\widetilde{u}), \nabla h(\widetilde{u}) \rangle \le \frac{C}{r^2 \, (t-s)^2} \int_{tB} h(\widetilde{u})^2 \, v \, ,$$

provided that r is sufficiently small.

Proof. Take $\varphi \in C_c^\infty(tB)$ such that $0 \le \varphi \le 1$, $\varphi \equiv 1$ on sB and $\|\nabla \varphi\|_\infty \lesssim ((t-s)\, r)^{-1}$. For each k, let $\psi_k := h'(u_k)\, h(u_k)\, \varphi^2$. Then $\psi_k \in \mathrm{Lip}_0(B)$, and because of conditions (3.1), one can use (2.3) to show that $\{\psi_k\}$ is bounded in $H_0(B)$. We thus pick a subsequence, still denoted by $\{\psi_k\}$ that converges weakly in $H_0(B)$. By taking a further subsequence if necessary, we can assume that $u_k \longrightarrow \widetilde{u}$ pointwise almost everywhere on B. Recalling (3.1), we have the following

$$\begin{aligned}
&\langle A\nabla h(u_k), \nabla h(u_k)\rangle\, \varphi^2 \\
&\qquad = \langle A\nabla h(u_k), \nabla(h(u_k)\,\varphi^2)\rangle - 2\,\langle A\nabla h(u_k), \nabla\varphi\rangle h(u_k)\,\varphi \\
&\qquad \le \langle A\nabla u_k, \nabla\psi_k\rangle - \langle A\nabla u_k, \nabla u_k\rangle h''(u_k) h(u_k)\,\varphi^2 \\
&\qquad\quad + \frac{1}{4}\,\langle A\nabla h(u_k), \nabla h(u_k)\rangle\,\varphi^2 + 4\,\langle A\nabla\varphi, \nabla\varphi\rangle h^2(u_k) \\
&\qquad \le T_k - \mathbf{c}\cdot\nabla\psi_k\, u_k - \mathbf{b}\cdot\nabla u_k\,\psi_k - V u_k\,\psi_k \\
&\qquad\quad + 4\,\langle A\nabla\varphi, \nabla\varphi\rangle h^2(u_k) + \frac{1}{4}\,\langle A\nabla h(u_k), \nabla h(u_k)\rangle\,\varphi^2\,,
\end{aligned} \tag{3.2}$$

where

$$T_k = \langle A\nabla u_k, \nabla\psi_k\rangle + \mathbf{c}\cdot\nabla\psi_k\, u_k + \mathbf{b}\cdot\nabla u_k\,\psi_k + V u_k\,\psi_k\ .$$

Again taking (3.1) into account, and using the Cauchy-Schwartz inequality we can estimate

$$\begin{aligned}
&|\mathbf{c}|\,|\nabla\psi_k|\,|u_k| + |\mathbf{b}|\,|\nabla u_k|\,|\psi_k| + |V|\,|u_k\,\psi_k| \\
&\quad \le \frac{1}{4}\,\langle A\nabla h(u_k), \nabla h(u_k)\rangle\,\varphi^2 \\
&\qquad + C_1\,(|\mathbf{c}|^2\omega^{-1} + |\mathbf{b}|^2\omega^{-1} + |V|)\,(h(u_k)\,\varphi)^2 + C_2\,\langle A\nabla\varphi, \nabla\varphi\rangle h^2(u_k)\,,
\end{aligned}$$

for some constants C_1, and C_2. Using this last inequality in (3.2), and integrating over tB we obtain

$$\begin{aligned}
&\int_{tB} \langle A\nabla h(u_k), \nabla h(u_k)\rangle\,\varphi^2 \\
&\qquad\qquad \le 2\,\delta_k + C_3 \int_{tB} (|\mathbf{c}|^2\omega^{-1} + |\mathbf{b}|^2\omega^{-1} + |V|)\,(h(u_k)\,\varphi)^2 \\
&\qquad\qquad\quad + C_4 \int_{tB} \langle A\nabla\varphi, \nabla\varphi\rangle h^2(u_k)\,,
\end{aligned}$$

where $\delta_k := D(u_k, \psi_k)$, and C_3, C_4 are some positive constants. Thus by Lemma A.2, and noting that $\varphi \equiv 1$ on sB we see that for sufficiently small r,

$$\int_{sB} \langle A\nabla h(u_k), \nabla h(u_k)\rangle\,\varphi^2 \lesssim \delta_k + \int_{tB} \langle A\nabla\varphi, \nabla\varphi\rangle h^2(u_k)\,.$$

By Remark 2.2, we note that $\delta_k \longrightarrow 0$ as $k \longrightarrow \infty$. We now let $k \longrightarrow \infty$. By the continuity of h', we notice that $(h'(u_k))^2 \longrightarrow (h'(\widetilde{u}))^2$ pointwise almost everywhere on B. Furthermore, the sequence $h(u_k)$ converges to $h(\widetilde{u})$ in $L^2_v(B)$ as a result of the inequality $|h(t)-h(s)| \lesssim |t-s|$. Therefore, by (1.1), Lemma A.1, and these observations we get the desired result provided the radius of B is sufficiently small.

We will need the following technical Lemma in some of our proofs. As the statement is a slight generalization of a known Lemma, we have included the short proof for completeness.

Lemma 3.2. *Let ϑ, ϱ be functions such that ϑ is bounded on every closed subinterval of (a, b), and ϱ is an almost increasing function; that is $\varrho(s) \leq C\,\varrho(t)$ for some positive constant C and all $a < s < t < b$. Suppose there is $0 < \varepsilon < 1$ and a non-negative function γ defined on $(0, \infty)$ such that*

$$\vartheta(s) \leq \varepsilon\,\vartheta(t) + \gamma(t-s)\,\varrho(t)\,,$$

for all $a < s < t < b$. We assume that γ satisfies either of the following conditions for $x, y \in (0, \infty)$.

$$\gamma(x\,y) \leq \gamma(x)\,\gamma(y), \text{ and } \varepsilon\,\gamma(\tau) < 1 \text{ for some } 0 < \tau < 1\,. \tag{3.3}$$

$$\gamma(x\,y) \leq \gamma(x) + \gamma(y)\,. \tag{3.4}$$

Then,

$$\vartheta(s) \leq C\,\frac{\gamma(1-\tau)}{1-\varepsilon\,\gamma(\tau)}\,\gamma(t-s)\,\varrho(t)\,, \qquad \text{if } \gamma \text{ satisfies } (3.3)\,, \tag{1}$$

and

$$\vartheta(s) \leq C\,\frac{1}{1-\varepsilon}\Big(\gamma(t-s) + \frac{\gamma(1/2)}{1-\varepsilon}\Big)\varrho(t)\,, \qquad \text{if } \gamma \text{ satisfies } (3.4)\,. \tag{2}$$

Proof. Let $s_0 = s$, $s_{k+1} = s_k + (1-\lambda)\,\lambda^k(t-s)$, $k = 0, 1, 2, \ldots$, where $0 < \lambda < 1$ will be specified later. Then for any $m = 1, 2, 3, \ldots$,

$$s_{m+1} - s = \sum_{k=0}^{m}(s_{k+1} - s_k) = (t-s)\,(1-\lambda)\sum_{k=0}^{m}\lambda^k\,.$$

From this we conclude that $s < s_k < t$, and $s_m \longrightarrow t$ as $m \longrightarrow \infty$. By iteration, and the monotonicity of ϱ, we have

$$\begin{aligned}\vartheta(s) &\le \varepsilon^m\,\vartheta(s_m) + \sum_{k=0}^{m-1} \varepsilon^k \varrho(s_{k+1})\,\gamma(s_{k+1}-s_k)\\ &\le \varepsilon^m\vartheta(s_m) + C\varrho(t)\sum_{k=0}^{m-1}\varepsilon^k\gamma((1-\lambda)\,(t-s)\lambda^k)\,.\end{aligned} \tag{3.5}$$

If γ satisfies (3.3), and we choose $\lambda = \tau$, then the above inequality becomes

$$\vartheta(s) \le \varepsilon^m\vartheta(s_m) + C\varrho(t)\,\gamma(1-\tau)\,\gamma(t-s)\sum_{k=0}^{m-1}(\varepsilon\,\gamma(\tau))^k\,.$$

If γ satisfies (3.4), and we choose $\lambda = 1/2$, then inequality (3.5) becomes

$$\vartheta(s) \le \varepsilon^m\vartheta(s_m) + C\varrho(t)\sum_{k=0}^{m-1}\varepsilon^k\Big((k+1)\,\gamma\Big(\frac{1}{2}\Big) + \gamma(t-s)\Big)\,.$$

We now let $m \longrightarrow \infty$. In both cases, we obtain the result as a consequence of the boundedness of ϑ and the sums

$$\sum_{k=0}^{\infty}\delta^k = \frac{1}{1-\delta}\,, \text{ and } \sum_{k=0}^{\infty}(k+1)\,\delta^k = \frac{1}{(1-\delta)^2}\,, \qquad \text{for } 0<\delta<1\,.$$

Given $0 < s < t$, let us make the following convention. Let

$$\begin{aligned}&s(j) := 2^{-j}\,((2^j-1)\,s+t)\,,\\ &\text{and } t(j) := 2^{-j}\,(s+(2^j-1)\,t)\,, \qquad j=1,2,3\,,\end{aligned} \tag{3.6}$$

so that for $j,k = 1,2,3,$ we have $s < s(j+1) < s(j) < t(k) < t(k+1) < t$, and $t(k+1) - t(k) = s(k) - s(k+1) = 2^{k+1}\,(t-s)^{-1}$. We shall also use $\mu(B)$ to denote the following

$$\mu(B) := \Big(\frac{\upsilon(B)}{\omega(B)}\Big)^{1/2}\,.$$

This last notation and the one introduced in (3.6) above will be used for the rest of our discussion without further comment.

Lemma 3.3. *Let h be a twice continuously differentiable function that satisfies* (3.1)*. Let $B \subset\subset B_0$ be a ball, and $u \in H(B_0)$ be a weak solution of $Mu = 0$ on B which has a representative $u = \{u_k\}$ such that $h(u_k)$, $h'(u_k)$, and $h''(u_k)$ are all defined for all k. Suppose that* (1.5) *holds and that $|\mathbf{c}|^2\omega^{-1} \in K$. Given $0 < s < t \leq 1$, with $t/s \lesssim 1$, there are positive constants C, and κ such that*

$$\Big(\fint_{sB} h^2(\widetilde{u})\, v\Big)^{1/2} \leq C\Big(\frac{\mu(B)}{t-s}\Big)^{\kappa} \fint_{tB} |h(\widetilde{u})|\, v\,,$$

provided that the radius of B is sufficiently small.

Proof. For $0 < s < 1$, let

$$I(s) := \Big(\fint_{sB} h^2(\widetilde{u})\, v\Big)^{1/2}.$$

Given $0 < s < t \leq 1$, we can assume without loss of generality, that the v average of $|h(\widetilde{u})|$ over B is 1. Fix $\varphi \in C_c^{\infty}(t(1)B)$ such that $\varphi \equiv 1$ on sB and $\|\nabla\varphi\|_\infty \lesssim ((t-s)\,r)^{-1}$, where r is the radius of B. Let $0 < \vartheta < 1$ such that $(2-\vartheta)/(1-\vartheta) = q$, where q is the exponent in the Sobolev's inequality (2.1). Then, for each k, by the Sobolev's inequality (2.1)

$$\begin{aligned}\Big(\fint_{sB} h^2(u_k) v\Big)^{1/2} &= \Big(\fint_{sB} |h(u_k)|^{2-\vartheta}\, |h(u_k)|^{\vartheta} v\Big)^{1/2}\\ &\leq \Big(\fint_{sB} |h(u_k)|^{(2-\vartheta)/(1-\vartheta)}\, v\Big)^{(1-\vartheta)/2}\\ &\leq \Big(\fint_{t(1)B} |h(u_k)\,\varphi|^{(2-\vartheta)/(1-\vartheta)}\, v\Big)^{(1-\vartheta)/2}\\ &\leq (C\,r\,(s+t))^{2\tau}\Big(\fint_{t(1)B} |\nabla(h(u_k)\,\varphi)|^2\,\omega\Big)^{\tau},\end{aligned}$$

where $\tau := (1-\vartheta)\,q/4$.

In the second inequality we have used, as a result of the assumption $t/s \lesssim 1$, the fact that $v(tB)/v(sB) \lesssim 1$. Consequently, using (1.1) we have

$$\begin{aligned}&\Big(\fint_{sB} h^2(u_k)\, v\Big)^{1/2}\\ &\qquad \leq \Big(\frac{C\,r^2}{\omega(t(1)B)}\Big)^{\tau}\Big(\int \langle A\nabla h(u_k), \nabla h(u_k)\rangle + \int |\nabla\varphi|^2 h^2(u_k)\,\omega\Big)^{\tau},\end{aligned}$$

where the last two integrals are over the ball $t(1)B$. We now pick a subsequence $\{u_k\}$ such that $u_k \longrightarrow \widetilde{u}$ pointwise almost everywhere on B. After using the fact that $\omega \le \upsilon$, we take the limit as $k \longrightarrow \infty$, and argue as in the proof of Lemma 3.1 to obtain the following

$$\Big(⨍_{sB} h^2(\widetilde{u})\, \upsilon \Big)^{1/2} \le \Big(\frac{C\, r^2}{\omega(t(1)B)} \Big)^{\tau} \Big(\int \langle A \nabla h(\widetilde{u}), \nabla h(\widetilde{u}) \rangle + \int |\nabla \varphi|^2 h^2(\widetilde{u})\, \upsilon \Big)^{\tau}.$$

Here again, the last two integrals are over the ball $t(1)B$. By Lemma 3.1, and the fact that $\mu(t(1)B) \lesssim t^{-d}\mu(B)$ for some $d > 0$, we obtain

$$I(s) \le C \Big(\frac{\mu(B)}{t-s} \Big)^{\beta} \Big(⨍_{tB} h^2(\widetilde{u})\, \upsilon \Big)^{\tau},$$

for some positive β. Taking logarithms in the last inequality, and noting that $0 < 2\tau < 1$ we obtain

$$\log I(s) \le 2\tau \log I(t) + \beta \log \Big(\frac{C\mu(B)}{t-s} \Big), \qquad \text{for } \frac{1}{2} \le s < t \le 1.$$

We now apply Lemma 3.2, with $\vartheta(x) = \log I(x)$, $\varrho(x) = 1$, $\gamma(x) = \log(C\mu(B)/x)$, and $\varepsilon = 2\tau$ to obtain

$$I(s) \le C \Big(\frac{\mu(B)}{t-s} \Big)^{\kappa},$$

for some constants C, and κ. On recalling that the υ average of $|h(\widetilde{u})|$ over tB is 1, we get the result.

For the remainder of our discussion, we will assume that the lower order coefficients $\mathbf{b}$ and V of L satisfy both the conditions (1.5) and (2.6).

Perhaps we should remark here that condition (2.6) is not needed in obtaining Harnack's inequality in the uniformly elliptic case. We refer the reader to the paper [8] for a proof.

Using the notation given in (3.6), we state the following.

Lemma 3.4. *Let B be a ball of radius r with $2B \subset B_0$, and let $\widetilde{G}^\rho$ be the approximate Green's function of L on B with pole x_0. Let $1/2 \le s < t < 1$. If $x_0 \in sB$, then for sufficiently small ρ, we have*

$$(1) \qquad \Big(\int_{t(2)B \setminus s(1)B} \langle A \nabla \widetilde{G}^\rho, \nabla \widetilde{G}^\rho \rangle \Big)^{1/2} \le C \frac{r}{\omega(B)} \sqrt{\upsilon(B)} \Big(\frac{\mu(B)}{t-s} \Big)^{\beta},$$

and

$$\left(\int_{t(2)B\smallsetminus s(1)B}(\widetilde{G}^{\rho})^2\upsilon\right)^{1/2} \le C\,\frac{r^2}{\omega(B)}\,\sqrt{\upsilon(B)}\,\Big(\frac{\mu(B)}{t-s}\Big)^{\alpha}, \tag{2}$$

where α, β are constants that depend on σ and the dimension n.

Proof. We follow the idea used in the proof of [7, Theorem 3.8] Let us suppose that B is centered at x_1 and has radius $r > 0$. We cover the annulus $t(2)B \smallsetminus s(1)B$ with k balls B_i that are centered at z_i, and each of radius $(t-s)\,r/4$. We pick the z_i on the sphere $|z-x_1| = (s(1)+t(2))\,r/2$ such that for some $0 < \delta < 1$, $\{\delta B_i\}_{i=1}^k$ is a pairwise disjoint collection with $\delta B_i \subset t(2)B \smallsetminus s(1)B$ for $i = 1, 2, \ldots, k$. Consequently, we note that $k \le C\,(t-s)^{1-n}$ for some constant C. Since sB and the $(9/4)\,B_i$ are disjoint, let us note that G^ρ is a solution of $L^*G^\rho = 0$ on $(9/4)\,B_i$ for $i = 1, 2, \ldots, k$. Furthermore, since $B \subset 8\,(t-s)^{-1}B_i$, and ω is doubling we see that $\mu(2B_i) \lesssim (t-s)^{-\kappa_1}\mu(B)$ for some constant κ_1, and all $i = 1, 2, \ldots, k$. Taking note of this, by Lemma 3.1, and Lemma 3.3 (with $h(\tau) = \tau$) we get the estimations

$$\begin{aligned}
&\int_{t(2)B\smallsetminus s(1)B}\langle A\nabla\widetilde{G}^{\rho}, \nabla\widetilde{G}^{\rho}\rangle \\
&\qquad\le \sum_{i=1}^{k}\int_{B_i}\langle A\nabla\widetilde{G}^{\rho}, \nabla\widetilde{G}^{\rho}\rangle \\
&\qquad\le \frac{C}{r^2\,(t-s)^2}\sum_{i=1}^{k}\int_{(4/3)B_i}(\widetilde{G}^{\rho})^2\upsilon \\
&\qquad\le \frac{C\,\upsilon(B)}{r^2\,(t-s)^2}\sum_{i=1}^{k}\fint_{(4/3)B_i}(\widetilde{G}^{\rho})^2\upsilon \\
&\qquad\le \frac{C\,\upsilon(B)}{r^2\,(t-s)^2}\sum_{i=1}^{k}\Big(\frac{\mu(2B_i)}{t-s}\Big)^{2\kappa_0}\Big(\fint_{2B_i}\widetilde{G}^{\rho}\upsilon\Big)^2 \\
&\qquad\le \frac{C\,\upsilon(B)}{r^2}\Big(\frac{\mu(B)}{t-s}\Big)^{2\kappa_0(\kappa_1+1)}\Big(\fint_{B}\widetilde{G}^{\rho}\upsilon\Big)^2\sum_{i=1}^{k}\frac{\upsilon(B)}{\upsilon(2B_i)}\,.
\end{aligned} \tag{3.7}$$

Since υ is doubling, we also have $\upsilon(B)/\upsilon(2B_i) \lesssim (t-s)^{-\kappa_2}$ for some κ_2. Using this estimation, and applying Theorem A.1 (take a p with $1 < p < \sigma$)

in (3.7) above leads to

$$\begin{aligned}\int_{t(2)B\smallsetminus s(1)B} &\langle A\nabla \widetilde{G}^\rho, \nabla \widetilde{G}^\rho\rangle \\ &\le \frac{C\,k\,\upsilon(B)}{r^2}\Big(\frac{\mu(B)}{t-s}\Big)^{2\kappa_0(\kappa_1+1)+\kappa_2}\Big(\fint_B (\widetilde{G}^\rho)^p \upsilon\Big)^{2/p} \\ &\le C\,k\Big(\frac{\mu(B)}{t-s}\Big)^{d}\upsilon(B)\Big(\frac{r}{\omega(B)}\Big)^2 \\ &\le C\Big(\frac{\mu(B)}{t-s}\Big)^{\kappa}\upsilon(B)\Big(\frac{r}{\omega(B)}\Big)^2.\end{aligned}$$

The proof for the second statement is similar. In fact, it is included in the above proof.

We now state and prove mean-value inequalities for weak solutions. In the theorems that follow all constants will depend only on the parameters occuring in the conditions (1.2), (1.3), and the functions $\eta(|\mathbf{b}|^2\omega^{-1})$, $\eta(\operatorname{div}\mathbf{b})$ and $\eta(V)$.

Theorem 3.1. *Let B be a ball of radius r with $2B \subset B_0$, and $u \in H(B_0)$ be a weak solution of $Lu = 0$ on B. Let $\widetilde{u}$ be the function in L^2_υ associated with u. Then $\widetilde{u}$ is locally bounded on B. If u is non-negative, $0 \le \varepsilon \le 1$, and $1/\sigma^2 \le p \le 2$, then there exist constants C, κ, and r_0, all independent of u, p, and ε such that for $1/2 \le s < t < 1$,*

$$\sup_{sB}\,(\widetilde{u}+\varepsilon)^p \le C\Big(\frac{\mu(B)}{t-s}\Big)^{\kappa}\Big(\fint_{tB}(\widetilde{u}+\varepsilon)^p\,\upsilon\Big),$$

whenever $0 < r \le r_0$.

Proof. Let $B = B(x_1)$ and $1/2 \le s < t \le 1$. Take $x_0 \in sB$, and let $\widetilde{G}^\rho$ be the approximate Green function of L on B with pole at x_0. Pick $\varphi \in C_c^\infty(t(2)B)$ such that $0 \le \varphi \le 1$, $\varphi \equiv 1$ on $s(1)B$ and $\|\nabla\varphi\|_\infty \lesssim ((t-s)\,r)^{-1}$. Let $\varepsilon > 0$, and let $u = \{u_k\}$, $G^\rho = \{G^\rho_k\}$, with $u_k \in \mathrm{Lip}(\overline{B})$, and $G^\rho_k \in \mathrm{Lip}_0(B)$. By Remark 2.3, we can assume that $0 \le G^\rho_k \le C$ for some constant C independent of k. Since $\{\varphi(u_k+\varepsilon)\}$ and $\{\varphi\, G^\rho_k\}$ are bounded in $H_0(B)$ we can select appropriate subsequences that converge weakly in $H_0(B)$. By taking a further subsequence if necessary, we can assume that $G^\rho_k \longrightarrow \widetilde{G}^\rho$ pointwise almost everywhere on B. Taking such weakly convergent subsequences, we have

$$
\begin{aligned}
&\gamma_k + \fint_{B_\rho} (\widetilde{u}+\varepsilon)\,\varphi\,\upsilon \\
&= \int_{t(2)B} \big(\langle A\nabla G_k^\rho, \nabla(\varphi(u_k+\varepsilon))\rangle + \mathbf{b}\cdot\nabla(\varphi(u_k+\varepsilon))\,G_k^\rho + V\varphi(u_k+\varepsilon)\,G_k^\rho\big) \\
&= \delta_k + \int_{t(2)B} \big(\langle A\nabla G_k^\rho, \nabla\varphi\rangle\,(u_k+\varepsilon) + \mathbf{b}\cdot\nabla\varphi(u_k+\varepsilon)\,G_k^\rho + \varepsilon\,V\varphi\,G_k^\rho\big) \\
&\quad - \int_{t(2)B} \langle A\nabla u_k, \nabla\varphi\rangle G_k^\rho\,,
\end{aligned}
$$

where the sequences $\{\gamma_k\}$, and $\{\delta_k\}$ are given by

$$
\gamma_k := D_*(G_k^\rho, \varphi(u_k+\varepsilon)) - \fint_{B_\rho} (\widetilde{u}+\varepsilon)\,\varphi\,\upsilon\,, \qquad \text{and } \delta_k := D(u_k, \varphi\,G_k^\rho)\,.
$$

Therefore, by Cauchy-Schwartz inequality, (1.1), and noting that $\omega \le \upsilon$, we obtain

$$
\begin{aligned}
&\Big|\fint_{B_\rho} (\widetilde{u}+\varepsilon)\,\varphi\,\upsilon\Big| \\
&\le \Big(\int_{t(2)B\smallsetminus s(1)B} \langle A\nabla G_k^\rho, \nabla G_k^\rho\rangle\Big)^{1/2}\Big(\int_{t(2)B} |\nabla\varphi|^2\,(u_k+\varepsilon)^2\,\upsilon\Big)^{1/2} \\
&\quad + \Big(\int_{t(2)B} \langle A\nabla u_k, \nabla u_k\rangle\Big)^{1/2}\Big(\int_{t(2)B\smallsetminus s(1)B} |\nabla\varphi|^2\,(G_k^\rho)^2\,\upsilon\Big)^{1/2} \\
&\quad + \Big(\int_{t(3)B} |\mathbf{b}|^2\omega^{-1}(\psi(u_k+\varepsilon))^2\Big)^{1/2}\Big(\int_{t(2)B\smallsetminus s(1)B} |\nabla\varphi|^2\,(G_k^\rho)^2\,\upsilon\Big)^{1/2} \\
&\quad + |\delta_k| + |\gamma_k| + \varepsilon\int_{t(2)B} |V|\,\varphi\,G_k^\rho\,,
\end{aligned} \tag{3.7}
$$

where $\psi \in C_c^\infty(t(3)B)$ is chosen such that $0 \le \psi \le 1$, $\psi \equiv 1$ on $\operatorname{supp}(|\nabla\varphi|)$ and $\|\nabla\psi\|_\infty \lesssim ((t-s)\,r)^{-1}$. By Lemma A.2, we observe that for sufficiently small r

$$
\begin{aligned}
&\int_{t(3)B} |\mathbf{b}|^2\omega^{-1}(\psi(u_k+\varepsilon))^2 \\
&\qquad\qquad \lesssim \int_{t(3)B} (u_k+\varepsilon)^2\,|\nabla\psi|^2\,\upsilon + \int_{t(3)B} \langle A\nabla u_k, \nabla u_k\rangle\,\psi^2\,.
\end{aligned}
$$

After using this estimate in (3.7), we take the limit as $k \longrightarrow \infty$. To this end, we first observe that $(u_k+\varepsilon)^2\,\zeta \longrightarrow (\widetilde{u}+\varepsilon)^2\,\zeta$ in $L^1_v(B)$ whenever $\zeta \in L^\infty(B)$, and also by Remark 2.2 we note that $\lim_k \gamma_k = 0 = \lim_k \delta_k = 0$. If we now use these observations together with Lemma A.1, and the Lebesgue dominated convergence theorem, we obtain

$$
\begin{aligned}
&\Big| \fint_{B_\rho} (\widetilde{u}+\varepsilon)\,\varphi\, v \Big| \\
&\le \Big(\int_{t(2)B\smallsetminus s(1)B} \langle A\nabla \widetilde{G}^\rho, \nabla \widetilde{G}^\rho\rangle \Big)^{1/2} \Big(\int_{t(2)B} |\nabla\varphi|^2\,(\widetilde{u}+\varepsilon)^2\, v\Big)^{1/2} \\
&\quad + \Big(\int_{t(2)B} \langle A\nabla\widetilde{u}, \nabla\widetilde{u}\rangle \Big)^{1/2} \Big(\int_{t(2)B\smallsetminus s(1)B} |\nabla\varphi|^2\,(\widetilde{G}^\rho)^2\, v\Big)^{1/2} \\
&\quad + C\Big(\int_{t(3)B} (\widetilde{u}+\varepsilon)^2\,|\nabla\psi|^2\, v + \int_{t(3)B} \langle A\nabla\widetilde{u}, \nabla\widetilde{u}\rangle\,\psi^2\Big)^{1/2} \\
&\qquad \cdot \Big(\int_{t(2)B\smallsetminus s(1)B} |\nabla\varphi|^2\,(\widetilde{G}^\rho)^2\, v\Big)^{1/2} + \varepsilon \int_{t(2)B} |V|\,\varphi\,\widetilde{G}^\rho\,.
\end{aligned}
$$

An application of Lemma 3.1 (where we take $h(\tau)=\tau$), and Lemma 3.4 leads to the estimation

$$
\begin{aligned}
\Big| \fint_{B_\rho} (\widetilde{u}+\varepsilon)\varphi v\Big| &\le C\Big(\frac{\mu(B)}{t-s}\Big)^{\kappa} \Big(\Big(\fint_{tB} (\widetilde{u}+\varepsilon)^2\, v\Big)^{1/2} + \Big(\fint_{tB} \widetilde{u}^2\, v\Big)^{1/2}\Big) \\
&\quad + \varepsilon \int_{t(2)B} |V|\,\varphi\,\widetilde{G}^\rho\,,
\end{aligned}
\tag{3.8}
$$

for some positive constants κ, and C that might change from time to time. If $\varepsilon = 0$, we take the limit as $\rho \longrightarrow 0$. Recalling that $\varphi \equiv 1$ on sB and that $x_0 \in sB$ is arbitrary, we obtain

$$
\sup_{sB} |\widetilde{u}| \le C\Big(\frac{\mu(B)}{t-s}\Big)^{\kappa} \Big(\fint_{tB} \widetilde{u}^{\,2}\, v\Big)^{1/2},
$$

showing that $\widetilde{u}$ is locally bounded on B. Suppose now $0 < \varepsilon \le 1$, and $u \ge 0$. Then $\varepsilon \le \widetilde{u}+\varepsilon$, and $0 \le \widetilde{u} \le \widetilde{u}+\varepsilon$ almost everywhere on B. Using these facts and Lemma A.3, the inequality (3.8) becomes

$$
\begin{aligned}
&\fint_{B_\rho} (\widetilde{u}+\varepsilon)\,\varphi\, v \\
&\le C\Big(\frac{\mu(B)}{t-s}\Big)^{\kappa} \Big(\fint_{tB} (\widetilde{u}+\varepsilon)^2 v\Big)^{1/2} + C\,\eta(|\mathbf{b}|^2\omega^{-1}+V)(2\,r)\sup_{tB}\,((\widetilde{u}+\varepsilon)\,\varphi)\,.
\end{aligned}
$$

We take the limit as $\rho \longrightarrow 0$ to conclude that

$$\begin{aligned}&\sup_{sB}\,(\varphi(\widetilde{u}+\varepsilon))\\&\le C\Big(\frac{\mu(B)}{t-s}\Big)^{\kappa}\Big(\fint_{tB}(\widetilde{u}+\varepsilon)^2\,\upsilon\Big)^{1/2}+C\,\eta(|\mathbf{b}|^2\omega^{-1}+V)(2\,r)\sup_{tB}\,((\widetilde{u}+\varepsilon)\,\varphi)\,.\end{aligned}$$

We now wish to apply Lemma 3.2 with the functions

$$\vartheta(s)=\sup_{sB}\,(\widetilde{u}+\varepsilon)\,\varphi\,,\quad \varrho(t)=C\Big(\fint_{tB}(\widetilde{u}+\varepsilon)^2\,\upsilon\Big)^{1/2}\,,\quad \text{and } \gamma(s)=\Big(\frac{\mu(B)}{s}\Big)^{\kappa}\,.$$

Since υ is doubling, we see that ϱ is almost increasing on $(1/2,1)$. We now choose r_0 such that $C\mu(B)^{\kappa}\,\eta(|\mathbf{b}|^2\omega^{-1}+V)(2\,r_0)<1$, where C is the constant appearing on the second term of the right-hand side of the last inequality above. Then with the choice of $\varepsilon := C\,\eta(|\mathbf{b}|^2\omega^{-1}+V)(2\,r_0)$, for $0<r\le r_0$, Lemma 3.2 is applicable and we obtain

$$\sup_{sB}\,(\widetilde{u}+\varepsilon)\le C\Big(\frac{\mu(B)}{t-s}\Big)^{\kappa}\Big(\fint_{tB}(\widetilde{u}+\varepsilon)^2\upsilon\Big)^{1/2}\,,\qquad \frac{1}{2}\le s<t<1\,. \tag{3.9}$$

That $\varphi\equiv 1$ on sB has been used in the above inequality.

Now suppose that $0<p<2$, and let

$$I(s):=C\Big(\fint_{sB}(\widetilde{u}+\varepsilon)^2\,\upsilon\Big)^{1/2}\,.$$

Without loss of generality, assume that the υ-average of $(\widetilde{u}+\varepsilon)^p$ over the ball B is 1. Using the doubling condition of υ, and (3.9), it is easy to see that

$$I(s)\le\big(\sup_{sB}\,(\widetilde{u}+\varepsilon)\big)^{\theta}\le\Big(\frac{\mu(B)}{t-s}\Big)^{\kappa}\theta\,I(t)^{\theta}\,,$$

where $\theta:=(2-p)/2$. From this we obtain, noting that $0<\theta<1$,

$$\log I(s)\le\kappa\log\Big(\frac{C\mu(B)}{(t-s)^{\tau}}\Big)+\theta\log I(t)\,,\qquad \frac{1}{2}\le s<t<1\,.$$

We now let $\vartheta(s)=\log I(s)$, $\gamma(s)=\log\,(C\mu(B)\,s^{-\tau})$, $\varrho(s)=\kappa$, and apply Lemma 3.2 again to get

$$I(s)\le C\Big(\frac{\mu(B)}{t-s}\Big)^{\kappa/p}\,,\qquad \text{for } \frac{1}{\sigma^2}\le p<2\,,$$

and for some positive constants C, and κ, independent of p. Therefore, recalling that the υ-average of $(\widetilde{u}+\varepsilon)^p$ over B is 1, we obtain the result for all $1/\sigma^2 \le p \le 2$.

Remark 3.1. By Hölder inequality, Theorem 3.1 also holds for $p > 2$ with C, and κ replaced by C^p, and $p\,\kappa$, respectively.

Theorem 3.2. *Let B be a ball of radius r with $2B \subset B_0$, and $u \in H(B_0)$ be a non-negative weak solution of $Lu = 0$ on B. Let $\widetilde{u}$ be the function in L^2_υ associated with u, and $0 < \varepsilon \le 1$. If $-1 \le p \le 1/\sigma^2$, then there exist constants C, κ, and r_0, all independent of u, p, and ε such that for $1/2 \le s < t < 1$*

$$\sup_{sB} (\widetilde{u}+\varepsilon)^p \le C\Big(\frac{\mu(B)}{t-s}\Big)^{\kappa}\Big(\fint_{tB} (\widetilde{u}+\varepsilon)^p\, \upsilon\Big)\,,$$

whenever $0 < r \le r_0$.

Proof. Let us first consider the case $0 < p \le 1/\sigma^2$. Let $u := \{u_k\}$ be a non-negative solution so that $u_k \ge 0$ for $k = 1, 2, 3, \ldots$ For each k, and some $\varepsilon > 0$, let $z_k := u_k + \varepsilon$. Now for $-1 < \beta \le 1/\sigma - 1$, let us define $\psi_k := \varphi^2\, z_k^{\beta}$, where φ is as in the proof Theorem 3.1 above. Then $\|\psi_k\|_0$ is bounded in k, and hence we can pick a subsequence, still denoted by $\{\psi_k\}$ such that ψ_k converges weakly in $H_0(B)$.

Let us first notice that

$$\nabla\psi_k = \beta\,\varphi^2\, z_k^{\beta-1}\,\nabla u_k + 2\,\varphi\, z_k^{\beta}\,\nabla\varphi\,, \quad \nabla(z_k^{(\beta+1)/2}) = \frac{\beta+1}{2}\, z_k^{(\beta-1)/2}\,\nabla u_k\ .$$

Therefore, using these we can write

$$\begin{aligned}\langle A\nabla u_k, \nabla\psi_k\rangle &= \frac{4\,\beta}{(\beta+1)^2}\,\langle A\nabla(z_k^{(\beta+1)/2}), \nabla(z_k^{(\beta+1)/2})\rangle\,\varphi^2 \\ &\qquad + \frac{4}{\beta+1}\,\langle A\nabla(z_k^{(\beta+1)/2}), \nabla\varphi\rangle\, z_k^{(\beta+1)/2}\,\varphi\ .\end{aligned}$$

From this and noting that $0 < \beta+1 \le (\sigma-1)^{-1}\,|\beta|$, one readily obtains

$$\begin{aligned}&(\sigma-1)\int_B \langle A\nabla(z_k^{(\beta+1)/2}), \nabla(z_k^{(\beta+1)/2})\rangle\,\varphi^2 \\ &\quad \le \int_B |\mathbf{b}|\,|\nabla(z_k^{(\beta+1)/2})|\, z_k^{(\beta+1)/2}\,\varphi^2 + \int_B |V|\,(z_k^{(\beta+1)/2}\varphi)^2 \\ &\quad + \int_B |\langle A\nabla(z_k^{(\beta+1)/2}), \nabla\varphi\rangle|\, z_k^{(\beta+1)/2}\,\varphi + |\delta_k|\,,\end{aligned}$$

where $\delta_k := D(u_k, \psi_k)$.

Let r_0 be chosen such that $\eta(|\mathbf{b}|^2\omega^{-1} + V)(2\,r_0) \le (\sigma-1)^2/32$. After use of the Cauchy-Schwartz inequality and Lemma A.2, and then collecting terms we obtain, for $0 < r \le r_0$,

$$\int_B \langle A\nabla(z_k^{(\beta+1)/2}), \nabla(z_k^{(\beta+1)/2})\rangle \varphi^2 \le C\Big(\int_B \langle A\nabla\varphi, \nabla\varphi\rangle\, z_k^{\beta+1} + |\delta_k|\Big),$$

where C is a positive constant independent of β. If we recall that $\varphi \equiv 1$ on sB and that $\|\nabla\varphi\|_\infty \le C\,((t-s)\,r)^{-1}$, we see that by (1.1)

$$\int_{sB} |\nabla(z_k^{(\beta+1)/2})|^2\,\omega \le C\Big(\frac{1}{r^2\,(t-s)^2}\int_{tB} z_k^{\beta+1}\,v + |\delta_k|\Big).$$

We now apply the Poincarè Inequality (2.2), and arguing as in [4] (note that $s\,(t-s)^{-1} \ge 1$, and $\mu(B) \ge 1$) we obtain

$$\Big(\fint_{sB} z_k^{((\beta+1)/2)q}\,v\Big)^{1/q} \le C\Big(\frac{s}{t-s}\,\mu(B)\Big)\Big(\fint_{tB} z_k^{\beta+1}\,v\Big)^{1/2} + C\,s\,r\Big(\frac{|\delta_k|}{\omega(sB)}\Big)^{1/2}.$$

Now let us take the limit as $k \longrightarrow \infty$ in the above inequality. Let us first observe, by Remark 2.2 that $\delta_k \longrightarrow 0$, and that $z_k \longrightarrow \widetilde{u} + \varepsilon$ in $L_v^{\beta+1}$ (as $z_k \longrightarrow \widetilde{u} + \varepsilon$ in L_v^2). Thus letting $k \longrightarrow \infty$, and $m = \beta + 1$, the above inequality reduces to

$$\Big(\fint_{sB} (\widetilde{u}+\varepsilon)^{\sigma m}\,v\Big)^{1/(\sigma m)} \le C^{2/m}\Big(\frac{s}{t-s}\,\mu(B)\Big)^{2/m}\Big(\fint_{tB}(\widetilde{u}+\varepsilon)^m\,v\Big)^{1/m}, \tag{3.10}$$

for $0 < m \le 1/\sigma$ and $1/2 \le s < t < 1$. Here we have used that $q = 2\,\sigma$. We now wish to iterate this inequality by taking the starting value of m as any fixed p with $0 < p \le 1/\sigma^2$. Let j be the positive integer such that $1/\sigma^2 \le \sigma^j\,p < 1/\sigma$. Since $\sigma^k p < 1/\sigma$ for $k = 0, 1, 2\ldots, j$, we iterate the inequality (3.10) j times for successive entries of t and s in the sequence $s_k = s + (t-s)/(k+1)$, $k = 0, 1, 2\ldots$, and successive entries of m and $\sigma\,m$ in the sequence $\{\sigma^k p\}_{k=0}^j$. We obtain

$$\Big(\fint_{s_jB}(\widetilde{u}+\varepsilon)^{\sigma^j p}\,v\Big)^{1/\sigma^j p} \le \prod_{k=0}^{j-1}(C\,a_k\,\mu(B))^{1/\sigma^k p}\Big(\fint_B(\widetilde{u}+\varepsilon)^p\,v\Big)^{1/p},$$

where $a_k := s_{k+1}/(s_k - s_{k+1})$. Noting that $1/\sigma^2 \le \sigma^j p < 1/\sigma$, we can apply Theorem 3.1 to obtain

$$\begin{aligned}\sup_{s_{j+1}B}(\widetilde{u}+\varepsilon) &\le C^{1/\sigma^j p}\Big(\frac{\mu(B)\,a_j}{s_{j+1}}\Big)^{\kappa/\sigma^j p}\Big(\fint_{s_j B}(\widetilde{u}+\varepsilon)^{\sigma^j p}\,\upsilon\Big)^{1/\sigma^j p}\\ &\le \prod_{k=0}^{j}(Ca_k\mu(B))^{\theta/\sigma^k p}\Big(\fint_B(\widetilde{u}+\varepsilon)^p\,\upsilon\Big)^{1/p},\end{aligned}$$

for some $\theta > 0$. We have used $1/2 \le s_{j+1}$ in obtaining the last inequality. If we now observe that $s < s_{j+1}$, and

$$\prod_{k=0}^{\infty}(C\,a_k\,\mu(B))^{\theta/\sigma^k p} \le \Big(\frac{C\mu(B)}{t-s}\Big)^{\kappa/p},$$

for some constants C, and κ that depend on σ, we obtain

$$\sup_{sB}(\widetilde{u}+\varepsilon) \le C^{1/p}\Big(\frac{C\mu(B)}{t-s}\Big)^{\kappa/p}\Big(\fint_B(\widetilde{u}+\varepsilon)^p\,\upsilon\Big)^{1/p},$$

which is the desired result when $0 < p \le 1/\sigma^2$.

We now take up the remaining case $-1 \le p < 0$.

For $-1 \le \beta < 0$, let $\widetilde{G}^\rho$ be the approximate Green function of $L_0 + \mathbf{b}\cdot\nabla + \beta V$, and D^β be the bilinear form associated with this operator. As a consequence of Remark 2.3, we choose a representative $G^\rho = \{G_k^\rho\}$ such that $0 \le G_k^\rho \le C$ for some positive constant C, independent of k. Therefore, $\{\varphi\, z_k^\beta\}$, and $\{\varphi\, z_k^{\beta-1} G_k^\rho\}$, are bounded in $H_0(B)$, and we take subsequences that are weakly convergent $H_0(B)$. By considering further subsequences if necessary, we assume that $u_k \longrightarrow \widetilde{u}$, and $G_k^\rho \longrightarrow \widetilde{G}^\rho$ pointwise almost everywhere on B. For such sequences, let us now notice that

$$\begin{aligned}&\langle A\nabla G_k^\rho, \nabla(\varphi\, z_k^\beta)\rangle + \mathbf{b}\cdot\nabla(\varphi\, z_k^\beta)\, G_k^\rho + \beta\, V\,\varphi\, z_k^\beta\, G_k^\rho\\ &\quad= \langle A\nabla G_k^\rho, \nabla\varphi\rangle\, z_k^\beta\\ &\qquad+ \beta\,\langle A\nabla G_k^\rho, \nabla u_k\rangle\,\varphi\, z_k^{\beta-1} + \mathbf{b}\cdot\nabla u_k(\beta\,\varphi\, z_k^{\beta-1}\, G_k^\rho) + \mathbf{b}\cdot\nabla\varphi(z_k^\beta\, G_k^\rho)\\ &\qquad+ V u_k(\beta\,\varphi\, z_k^{\beta-1}\, G_k^\rho) + \varepsilon V(\beta\,\varphi\, z^{\beta-1} G_k^\rho)\\ &\quad= \langle A\nabla G_k^\rho, \nabla\varphi\rangle\, z_k^\beta + \langle A\nabla u_k, \nabla(\beta\,\varphi\, z_k^{\beta-1}\, G_k^\rho)\rangle\\ &\qquad- \langle A\nabla u_k, \nabla(\beta\,\varphi\, z_k^{\beta-1})\rangle\, G_k^\rho + \mathbf{b}\cdot\nabla u_k(\beta\,\varphi\, u^{\beta-1}\, G_k^\rho)\\ &\qquad+ \mathbf{b}\cdot\nabla\varphi(z_k^\beta\, G_k^\rho) + V u_k(\beta\,\varphi\, z_k^{\beta-1}\, G_k^\rho) + \varepsilon\, V(\beta\,\varphi\, z_k^{\beta-1}\, G_k^\rho).\end{aligned}$$

Observing that

$$\begin{aligned}&\langle A\nabla u_k, \nabla(\beta\,\varphi\, z_k^{\beta-1})\rangle\, G_k^\rho \\ &\qquad = \beta\,(\beta-1)\,\langle A\nabla u_k, \nabla u_k\rangle\, \varphi\, z_k^{\beta-2}\, G_k^\rho + \langle A\nabla u_k, \nabla\varphi\rangle\,(\beta\, z_k^{\beta-1}\, G_k^\rho)\,,\end{aligned}$$

and that the first term on the right-hand side of the equation is positive, we conclude

$$\begin{aligned}\gamma_k + &\fint_{B_\rho} \varphi\, z_k^\beta\, \upsilon \\ &= \int (\langle A\nabla G_k^\rho, \nabla(\varphi\, z_k^\beta)\rangle + \mathbf{b}\cdot\nabla(\varphi\, z_k^\beta)\, G_k^\rho + \beta\, V\,\varphi\, z_k^\beta\, G_k^\rho) \\ &\le \int (\langle A\nabla G_k^\rho, \nabla\varphi\rangle\, z_k^\beta + \mathbf{b}\cdot\nabla\varphi (z_k^\beta\, G_k^\rho) - \beta\,\langle A\nabla u_k, \nabla\varphi\rangle\,(z_k^{\beta-1}\, G_k^\rho)) \\ &\quad + \delta_k + \int \varepsilon\, V(\beta\,\varphi\, z_k^{\beta-1}\, G_k^\rho)\,,\end{aligned}$$

where the integrals on the right are carried over the ball tB, and

$$\delta_k := D^\beta(u_k, \beta\,\varphi\, z_k^{\beta-1}\, G_k^\rho)\,, \qquad \text{and} \qquad \gamma_k := D_*^\beta(G_k^\rho, \varphi\, z_k^\beta) - \fint_{B_\rho} \varphi\, z_k^\beta\, \upsilon\,.$$

By Remark 2.2, we notice that $\delta_k \longrightarrow 0$, and $\gamma_k \longrightarrow 0$, as $k \longrightarrow \infty$. So taking the limit in k, we invoke Lemma A.1, and the Lebesgue dominated convergence theorem to obtain the inequality

$$\begin{aligned}\fint_{B_\rho} (\widetilde{u}+\varepsilon)^\beta\, \varphi\, \upsilon \le \int_{tB} &\langle A\nabla \widetilde{G}^\rho, \nabla\varphi\rangle\, (\widetilde{u}+\varepsilon)^\beta + \mathbf{b}\cdot\nabla\varphi((\widetilde{u}+\varepsilon)^\beta\, \widetilde{G}^\rho) \\ &- \int_{tB} \beta\,\langle A\nabla\widetilde{u}, \nabla\varphi\rangle\,(\widetilde{u}^{\beta-1}\, \widetilde{G}^\rho) + \int_{tB} |V|\,(\varepsilon\,\varphi\,(\widetilde{u}+\varepsilon)^\beta)\, \widetilde{G}^\rho\,.\end{aligned}$$

Now, if we recall that $-1 \le \beta < 0$, and $0 < \varepsilon \le 1$, then by Lemma A.3, the last integral is not bigger than $\eta(|\mathbf{b}|^2\omega^{-1} + V)(2\,r)\sup_{tB}((\widetilde{u}+\varepsilon)^\beta\,\varphi)$. This last observation together with an application of the Cauchy-Schwartz inequality, and (1.1) leads to (recall that $\omega \le \upsilon$)

$$\begin{aligned}
&\fint_{B_\rho} (\widetilde{u}+\varepsilon)^\beta \,\varphi\,\upsilon \\
&\le \Big(\int_{t(2)B\smallsetminus s(1)B} \langle A\nabla \widetilde{G}^\rho, \nabla \widetilde{G}^\rho\rangle\Big)^{1/2} \Big(\int_{t(2)B} |\nabla\varphi|^2\,(\widetilde{u}+\varepsilon)^{2\beta}\,\upsilon\Big)^{1/2} \\
&\quad + \Big(\int_B |\mathbf{b}|^2\omega^{-1}(\psi(\widetilde{u}+\varepsilon)^\beta)^2\Big)^{1/2} \Big(\int_{t(2)B\smallsetminus s(1)B} |\nabla\varphi|^2\,(\widetilde{G}^\rho)^2\,\upsilon\Big)^{1/2} \\
&\quad + \Big(\int_{t(2)B} \langle A\nabla(\widetilde{u}+\varepsilon)^\beta, \nabla(\widetilde{u}+\varepsilon)^\beta\rangle\Big)^{1/2} \Big(\int_{t(2)B\smallsetminus s(1)B} |\nabla\varphi|^2\,(\widetilde{G}^\rho)^2\,\upsilon\Big)^{1/2} \\
&\quad + \eta(|\mathbf{b}|^2\omega^{-1}+V)(2\,r)\sup_{tB}\,((\widetilde{u}+\varepsilon)^\beta\,\varphi)\,.
\end{aligned}$$

Here $\psi \in C_c^\infty(t(3)B)$ is chosen such that $0 \le \psi \le 1$, $\psi \equiv 1$ on $\operatorname{supp}(|\nabla\varphi|)$, and $\|\nabla\psi\|_\infty \le C\,((t-s)\,r)^{-1}$. We now proceed as in the proof of Theorem 3.1, to estimate the integrals on the right. We point out that in applying the Caccioppoli estimate, we use the function $h(\tau) := (\tau+\varepsilon)^\beta$ in Lemma 3.1. Therefore,

$$\begin{aligned}
\sup_{sB}\,((\widetilde{u}+\varepsilon)^\beta\,\varphi) &\le C\Big(\frac{\mu(B)}{t-s}\Big)^\kappa \Big(\fint_{tB} (\widetilde{u}+\varepsilon)^{2\beta}\,\upsilon\Big)^{1/2} \\
&\qquad + \eta(|\mathbf{b}|^2\omega^{-1}+V)(2\,r)\sup_{tB}((\widetilde{u}+\varepsilon)^\beta\,\varphi)\,,
\end{aligned}$$

for $-1 \le \beta < 0$.

We now appeal to Lemma 3.2 to conclude that, for sufficiently small r,

$$\sup_{sB}\,((\widetilde{u}+\varepsilon)^\beta\,\varphi) \le C\Big(\frac{\mu(B)}{t-s}\Big)^\kappa \Big(\fint_{tB} (\widetilde{u}+\varepsilon)^{2\beta}\,\upsilon\Big)^{1/2},$$

for some positive constants C, and κ. Finally, we invoke Lemma 3.3 (with $h(\tau) := (\tau+\varepsilon)^\beta$ again) to obtain

$$\sup_{sB}\,((\widetilde{u}+\varepsilon)^\beta\,\varphi) \le C\Big(\frac{\mu(B)}{t-s}\Big)^\kappa \Big(\fint_{tB} (\widetilde{u}+\varepsilon)^{\beta}\,\upsilon\Big),$$

for some constants C, and κ.

Noting that $\varphi \equiv 1$ on sB, we obtain the claimed inequality for $-1 \le p < 0$.

Remark 3.2. By Hölder inequality, Theorem 3.2 continues to hold for $p < -1$ with C, and κ replaced by C^{-p}, and $-p\,\kappa$, respectively.

4. Harnack's inequality.

Theorem 3.2 does not provide anything new when $p = 0$. As a replacement, we have an estimate provided by the following Lemma.

Lemma 4.1. *Let $B \subset\subset B_0$ be a ball of radius r, and let $u \in H(B_0)$ be a non-negative weak solution of $Lu = 0$ in B. For $\varepsilon > 0$, and $1/2 \le s < 1$ define $N := N(\varepsilon, s, \widetilde{u})$ by*

$$\log N := \fint_{sB} \log\left(\widetilde{u} + \varepsilon\right) \upsilon \,.$$

Then for $\lambda > 0$, and $1/2 \le s < 1$, we have

$$\upsilon\Big(\Big\{x \in sB : \ \Big|\log\Big(\frac{\widetilde{u}+\varepsilon}{N}\Big)\Big| > \lambda\Big\}\Big) \le \frac{C\mu(B)}{\lambda\,(1-s)}\,\upsilon(sB)\,,$$

for some constant C, whenever r is sufficiently small.

Proof. Let $\varphi \in C_c^\infty(tB)$ satisfy $\varphi \equiv 1$ on sB, $0 \le \varphi \le 1$, and $\|\nabla\varphi\|_\infty \lesssim ((1-s)\,r)^{-1}$. Let $u = \{u_k\}$, $u_k \in \mathrm{Lip}(\overline{B}_0)$, $u_k \ge 0$, and for each k, define $\psi_k := \varphi^2\,(u_k + \varepsilon)^{-1}$. Then

$$\nabla\psi_k = -\varphi^2\,(u_k+\varepsilon)^{-2}\,\nabla u_k + 2\,\varphi\,(u_k+\varepsilon)^{-1}\,\nabla\varphi\,,$$

and $\|\psi_k\|_0$ is bounded in $H_0(B)$. Therefore, we pick a weakly convergent subsequence still denoted by $\{\psi_k\}$. With this sequence, we have

$$\begin{aligned}\langle A\nabla u_k, \nabla u_k\rangle\,\varphi^2(u_k+\varepsilon)^{-2} &\\ = -T_k + \mathbf{b}\cdot\nabla u_k(u_k+\varepsilon)^{-1}\,\varphi^2 + V u_k\,(u_k+\varepsilon)^{-1}\,\varphi^2 &\\ + 2\,\langle A\nabla u_k, \nabla\varphi\rangle\,\varphi(u_k+\varepsilon)^{-1}\,, &\end{aligned} \tag{4.1}$$

where

$$T_k = \langle A\nabla u_k, \nabla\psi_k\rangle + \mathbf{b}\cdot\nabla u_k\,\psi_k + V u_k\,\psi_k\ .$$

Integrating (4.1) over B, and using Hölder inequality followed by Cauchy-Schwartz inequality (and also using the fact that $u_k\,(u_k+\varepsilon)^{-1} \le 1$), we obtain

$$\begin{aligned}\int_B \langle A\nabla \log(u_k+\varepsilon), \nabla \log(u_k+\varepsilon)\rangle \varphi^2 \\ &= \delta_k + \int_B (\mathbf{b}\cdot\nabla u_k\,(u_k+\varepsilon)^{-1}\varphi^2 + V u_k\,(u_k+\varepsilon)^{-1}\varphi^2) \\ &\quad + \int_B 2\,\langle A\nabla u_k, \nabla\varphi\rangle\,\varphi(u_k+\varepsilon)^{-1} \\ &\le |\delta_k| + \int_B (|\mathbf{b}|^2\omega^{-1} + |V|)\,\varphi^2 + 4\int_B \langle A\nabla\varphi, \nabla\varphi\rangle \\ &\quad + \frac{1}{2}\int_B \langle A\nabla u_k, \nabla u_k\rangle\,(u_k+\varepsilon)^{-2}\,\varphi^2\,,\end{aligned}$$

where $\delta_k = -D(u_k\,\psi_k)$. Therefore, by Lemma 3.1, and (1.1) we have

$$\begin{aligned}\int_B \langle A\nabla \log(u_k+\varepsilon), \nabla \log(u_k+\varepsilon)\rangle \varphi^2 \\ &\le (2\,\eta(|\mathbf{b}|^2\omega^{-1}+|V|)(3\,r)+8)\int_B |\nabla\varphi|^2\,\upsilon + 2\,|\delta_k|\,.\end{aligned}$$

Since $\varphi\equiv 1$ on sB, on taking note of (1.1) again we have, for small r

$$\int_{sB} |\nabla\log(u_k+\varepsilon)|^2\,\omega \le \frac{C\upsilon(B)}{r^2\,(t-s)^2} + 2\,|\delta_k|\,.$$

By Poincaré Lemma (2.2), we have

$$⨍_{sB} \Big|\log(u_k+\varepsilon) - (\log(u_k+\varepsilon))_{sB,\upsilon}\Big|^2\,\upsilon \le \frac{C\,\upsilon(B)}{(t-s)^2} + \frac{C\,r^2}{\omega(B)}\,|\delta_k|\,.$$

We now take the limit as $k\longrightarrow\infty$. Note that $\log(u_k+\varepsilon)\longrightarrow\log(\widetilde{u}+\varepsilon)$ in $L^2_\upsilon(B)$, and hence also

$$(\log(u_k+\varepsilon))_{sB,\upsilon} = ⨍_{sB}\log(u_k+\varepsilon)\,\upsilon \longrightarrow ⨍_{sB}\log(\widetilde{u}+\varepsilon)\,\upsilon := \log N\,.$$

After taking the limit as $k\longrightarrow\infty$, we use Chebyshev's inequality followed by Hölder inequality to obtain

$$\begin{aligned}\upsilon\Big(\Big\{x\in\alpha\,B:\ \Big|\log\Big(\frac{\widetilde{u}+\varepsilon}{N}\Big)\Big| > \lambda\Big\}\Big) &\le \frac{1}{\lambda}\int_{sB} |\log(\widetilde{u}+\varepsilon) - \log N|\,\upsilon \\ &\le \frac{C\mu(B)}{\lambda\,(t-s)}\,\upsilon(sB)\,,\end{aligned}$$

as desired.

The following real-variable fact is the final ingredient needed to obtain the Harnack inequality. It is proved in the same way as [11, Lemma 3] (see the comment following [3, Lemma (3.14)]).

Bombieri's Lemma. *Let $\mu > 0$, and υ be a doubling measure. Let f be a non-negative bounded function on a ball B. Suppose there are positive constants C, d such that*

$$(1)\quad \sup_{sB}(f^p) \le \frac{C}{(t-s)^d} \fint_{tB} f^p\,\upsilon\,, \qquad 0 < p < \frac{1}{\mu}\,, \qquad \frac{1}{2} \le s < t \le 1\,,$$

$$(2)\qquad \upsilon(\{x \in B:\ \log f(x) > \lambda\}) \le \frac{C\mu}{\lambda}\,\upsilon(B)\,, \qquad \lambda > 0\,.$$

Then there are constants γ and δ so that for all $0 < \alpha < 1$, we have

$$\sup_{\alpha B} f \le \exp\left(\frac{\gamma\mu}{(1-\alpha)^\delta}\right).$$

We can now state the Harnack's inequality

Theorem 4.1 (Harnack's inequality)**.** *Let B be a ball of radius r with $4B \subset B_0$. Suppose $u \in H(B_0)$ is a non-negative weak solution of $Lu = 0$ on B, and let $\widetilde{u}$ be the function in $L^2_\upsilon(B_0)$ associated with u. There are constants C and r_0 depending only on the parameters in* (1.2), (1.3), (1.5) *and* (2.6) *such that*

$$\sup_B \widetilde{u} \le \exp\left(C\,\frac{\upsilon(B)}{\omega(B)}\right) \inf_B \widetilde{u}\,,$$

whenever $0 < r \le r_0$.

Proof. The proof relies on Theorems 3.1, 3.2, and Lemma 4.1 together with Bombieri's Lemma. Since u is non-negative, we recall that $\widetilde{u} \ge 0$. Given $0 < \varepsilon \le 1$, we apply Bombieri's Lemma to the functions $(\widetilde{u}+\varepsilon)/N$, and $N/(\widetilde{u}+\varepsilon)$. with N defined by

$$N = \fint_{(3/2)B} \log(\widetilde{u}+\varepsilon)\,\upsilon\,.$$

The parameters α, B, and μ are taken to be $2/3$, $(3/2)\,B$, and $\mu(2B)$, respectively. Following the arguments detailed in [3], one obtains the inequality

$$\sup_B\,(\widetilde{u}+\varepsilon) \leq \exp\left(C\,\frac{v(B)}{\omega(B)}\right)\inf_B(\widetilde{u}+\varepsilon)\,,$$

We now let $\varepsilon \longrightarrow 0$ to get the desired Harnack's inequality stated in Theorem 4.1.

Acknowledgments. I would like to record my indebtedness to professor Cristian E. Gutierrez for his constant encouragement and his invaluable advice throughout the preparation of this paper. I would also like to thank the referee for the useful comments to improve the exposition.

References.

[1] Aizenman, M., Simon, B., Brownian motion and Harnack's inequality for Schrödinger operators. *Comm. Pure Appl. Math.* **35** (1982), 209-273.

[2] Chanillo, S., Wheeden, R. L., Weighted Poincaré and Sobolev inequalities and estimates for the Peano maximal function. *Amer. J. Math.* **107** (1985), 1191-1226.

[3] Chanillo, S., Wheeden, R. L., Harnack's inequality and mean-value inequalities for solutions of degenerate elliptic equations. *Comm. Partial Diff. Eq.* **11** (1986), 1111-1134.

[4] Chanillo, S., Wheeden, R. L., Existence and estimates of Green's function for degenerate elliptic equations. *Scuola Norm. Sup. Pisa* **XV** (1989), 309-340.

[5] Chiarenza, F., Fabes, E., Garofalo, N., Harnack's inequality for Schrödinger operators and continuity of solutions. *Proc. Amer. Math. Soc.* **98** (1986), 415-425.

[6] Fabes, E., Kenig, C. E., Serapioni, P., The local properties of solutions of degenerate elliptic equations. *Comm. Partial Diff. Eq.* **7** (1982), 77-116.

[7] Gutierrez, C. E., Harnack's inequality for degenerate Schrödinger operators. *Trans. Amer. Math. Soc.* **312** (1989), 403-419.

[8] Kurata, K., Continuity and Harnack's inequality for solutions of elliptic partial differential equations of second order. *Indiana Univ. Math. J.* **43** (1994), 411-440.

[9] Lu, G., Harnack's inequality for a class of degenerate Schrödinger operators formed by vector fields. *Trans. Amer. Math. Soc.* **312** (1989), 403-419.

[10] Mohammed, A., Uniform integrability of approximate Green functions of some degenerate elliptic operators. *Pacific J. Math.* **199** (2001), 467-492.

[11] Moser, J., On a pointwise estimate for parabolic differential equations. *Comm. Pure Appl. Math.* **24** (1971), 727-740.

[12] Strömberg, J. O., Torchinsky, A., *Weighted Hardy spaces.* Lecture Notes in Math. **1381** (1989), 83-128. Springer-Verlag.

Recibido: 3 de marzo de 2000

Ahmed Mohammed
Department of Mathematical Sciences
Ball State University
Muncie, IN 47306-0490, U.S.A.
amohammed@bsu.edu