

# Algebro-Geometric Solutions of the Camassa–Holm hierarchy

Fritz Gesztesy and Helge Holden

## Abstract

We provide a detailed treatment of the Camassa–Holm (CH) hierarchy with special emphasis on its algebro-geometric solutions. In analogy to other completely integrable hierarchies of soliton equations such as the KdV or AKNS hierarchies, the CH hierarchy is recursively constructed by means of a basic polynomial formalism invoking a spectral parameter. Moreover, we study Dubrovin-type equations for auxiliary divisors and associated trace formulas, consider the corresponding algebro-geometric initial value problem, and derive the theta function representations of algebro-geometric solutions of the CH hierarchy.

## 1. Introduction

Very recently, the Camassa–Holm (CH) equation, also known as the dispersive shallow water equation, as isolated, for instance, in [17] and [18],

$$(1.1) \quad 4u_t - u_{xxt} - 2uu_{xxx} - 4u_x u_{xx} + 24uu_x = 0, \quad (x, t) \in \mathbb{R}^2$$

(choosing a scaling of  $x, t$  that's convenient for our purpose), with  $u$  representing the fluid velocity in  $x$ -direction, received considerable attention. Actually, (1.1) represents the limiting case  $\kappa \rightarrow 0$  of the general Camassa–Holm equation,

$$(1.2) \quad 4v_t - v_{xxt} - 2vv_{xxx} - 4v_x v_{xx} + 24vv_x + 4\kappa v_x = 0, \quad \kappa \in \mathbb{R}, (x, t) \in \mathbb{R}^2.$$

However, in our formalism the general Camassa–Holm equation (1.2) just represents a linear combination of the first two equations in the CH hierarchy

---

*2000 Mathematics Subject Classification:* Primary 35Q53, 58F07; Secondary 35Q51.

*Keywords:* Camassa–Holm hierarchy, algebro-geometric solutions, Dubrovin equations, trace formulas.

and hence we consider without loss of generality (1.1) as the first nontrivial element of the Camassa–Holm hierarchy. Alternatively, one can transform

$$(1.3) \quad v(x, t) \mapsto u(x, t) = v(x - (\kappa/2)t, t) + (\kappa/4)$$

and thereby reduce (1.2) to (1.1).

Various aspects of local existence, global existence, and uniqueness of solutions of (1.1) are treated in [27], [24], [25], [30], [52], [53], [58], [59], [61], wave breaking phenomena are discussed in [23], [26], [28]. Soliton-type solutions (called “peakons”) were extensively studied due to their unusual non-meromorphic (peak-type) behavior, which features a discontinuity in the  $x$ -derivative of  $u$  with existing left and right derivatives of opposite sign at the peak. In this context we refer, for instance, to [3], [5], [7], [8], [10], [12], [13], [14], [17], [18], [51]. Integrability aspects such as infinitely many conservation laws, (bi-)Hamiltonian formalism, Bäcklund transformations, infinite dimensional symmetry groups, etc., are discussed, for instance, in [17], [18], [38], [41] (see also [42]), [57]. The general CH equation (1.2) is shown to give rise to a geodesic flow of a certain right invariant metric on the Bott-Virasoro group in [54]. In the case  $\kappa = 0$ , the CH equation (1.1) corresponds to the geodesic flow on the group of orientation preserving diffeomorphisms of the circle. This follows from the Lie-Poisson structure established in [18] and is also remarked upon in [54]. That the equations define a smooth vector field was first observed by Shkoller in the case of periodic [58] and Dirichlet [59] boundary conditions, which directly leads to the corresponding local existence theory. Scattering data and their evolution under the CH flow are determined in [11] and intimate relations with the classical moment problem and the finite Toda lattice are worked out in [12], [13], and [14]. The case of spatially periodic solutions, the corresponding inverse spectral problem, isospectral classes of solutions, and quasi-periodicity of solutions with respect to time are discussed in [20], [21], [22], and [29]. Moreover, algebro-geometric solutions of (1.1) and their properties are studied in [1], [2], [3], [4], [5], [6], [7], [8] (connections as well as differences between the latter references and our own approach to algebro-geometric solutions will be outlined in the following paragraph). Moreover, even though the following very recent developments are not directly related to the principal topic of this paper, they put the CH equation in a broader context: In [34], a basic integrable shallow water equation, originally introduced in [17], is analyzed in detail. It combines the linear dispersion of the KdV equation with the nonlinear/nonlocal dispersion of the CH equation and contains the KdV and CH equations (as well as an equation studied by Fornberg and Whitham [40]) as special limiting cases. Finally, the three-dimensional viscous Camassa–Holm equations, their connection with the Navier-Stokes equations, estimates for

the Hausdorff and fractal dimension of the associated global attractor, and turbulence theory according to Kolmogorov, Landau, and Lifshitz, are discussed in [39].

Our own approach to algebro-geometric solutions of the CH hierarchy differs from the ones pursued in [1], [2], [3], [4], [5], [6], [7], [8] in several aspects and we will outline some of the differences next. Following previous treatments of the KdV, AKNS, Toda, and Boussinesq hierarchies and the sine-Gordon and massive Thirring models (cf., e.g., [16], [31], [32], [35], [45], [46], [47], [48], [49], [50]), we develop a systematic polynomial recursion formalism for the CH hierarchy and its algebro-geometric solutions. In contrast to the treatments in [3], [7], and [8], we rely on a zero-curvature approach  $U_t - V_x = [V, U]$  (as the compatibility requirement for the system  $\Psi_x = U\Psi$ ,  $\Psi_t = V\Psi$ ) as opposed to their Lax formalism. However, we incorporate important features of the recursion formalism developed in [5] into our zero-curvature approach. Our treatment is comprehensive and self-contained in the sense that it includes Dubrovin-type equations for auxiliary divisors on the associated compact hyperelliptic curve, trace formulas, and theta function representations of solutions, the usual ingredients of such a formalism. Moreover, while [3], [7], [8] focus on solutions of the CH equation itself, we simultaneously derive theta function formulas for solutions of any equation of the CH hierarchy. The key element in our formalism is the solution  $\phi$  of a Riccati-type equation associated with the zero-curvature representation of the CH equation (1.1). Roughly speaking,  $\phi = -z\psi_2/\psi_1$ , where  $\Psi = (\psi_1, \psi_2)^t$  and  $z$  denotes a spectral parameter in  $U$  and  $V$  (cf. (2.42) for more details).  $\phi$  is then used to introduce appropriate auxiliary divisors on the underlying hyperelliptic curve, the Baker-Akhizer vector in the stationary case, etc. Combining  $\phi$  with the polynomial recursion formalism for the CH hierarchy then leads to Dubrovin-type differential equations and trace formulas for  $u$  in terms of auxiliary divisors. Explicit theta function representations for symmetric functions of (projections of) these auxiliary divisors then yield the theta function representations for any algebro-geometric solution  $u$  of the CH hierarchy. Here our strategy differs somewhat from that employed in [3], [7], [8] for the CH equation. While the latter references also employ the trace formula for  $u$  in terms of (projections of) auxiliary divisors, they subsequently rely on generalized theta functions and generalized Jacobians (going back to investigations of Clebsch and Gordan [19]), whereas we stay within the traditional framework familiar from the KdV, AKNS, Toda hierarchies, etc. Finally, we point out a novel feature of our treatment of the CH hierarchy that appears to be without precedent. In Theorems 3.11 and 4.10 we formulate and solve the algebro-geometric initial value problem for the stationary and time-dependent CH hierarchy, in the following

sense. Starting from the initial value problem for auxiliary divisors induced by the Dubrovin-type equations, we define  $u$  in terms of the trace formula involving the (projections of) auxiliary divisors and then prove directly that  $u$  so defined satisfies the corresponding (stationary, resp., time-dependent) equation of the CH hierarchy.

Without going into further details, we note that our constructions extend in a straightforward manner to a closely related hierarchy of completely integrable nonlinear evolution equations, the Dym hierarchy. For different approaches to algebro-geometric solutions of the latter we refer to [3], [6], [9], [33], and [56].

In Section 2 we develop the basic polynomial recursion formalism that defines the CH hierarchy using a zero-curvature approach. Section 3 then treats the stationary CH hierarchy and its algebro-geometric solutions. The corresponding time-dependent results are the subject of Section 4. Appendix Appendix A summarizes the necessary results needed from the theory of compact Riemann surfaces and also serves to establish the notation used throughout this paper. Appendix Appendix B contains a few technical results concerning the polynomial recursion formalism and associated high-energy expansions. Finally, Appendix Appendix C provides a detailed discussion of elementary symmetric functions associated with Dirichlet divisors and their representations in terms of theta functions associated with the underlying hyperelliptic curve. It contains several core results needed in our derivation of algebro-geometric solutions of the CH hierarchy. The results of this appendix apply to a variety of soliton equations and hence are of independent interest.

## 2. The CH hierarchy, recursion relations, and hyperelliptic curves

In this section we provide the basic construction of a completely integrable hierarchy of nonlinear evolution equations in which the Camassa–Holm equation, or dispersive shallow water equation, is the first element in the hierarchy (the higher-order CH equations will turn out to be nonlocal with respect to  $x$ ). We will use a zero-curvature approach and combine it with a polynomial recursion formalism containing a spectral parameter.

Throughout this section we will suppose the following hypothesis.

**Hypothesis 2.1** *In the stationary case we assume that*

$$(2.1) \quad u \in C^\infty(\mathbb{R}), \quad \frac{d^m u}{dx^m} \in L^\infty(\mathbb{R}), \quad m \in \mathbb{N}_0.$$

In the time-dependent case we suppose

$$(2.2) \quad u(\cdot, t) \in C^\infty(\mathbb{R}), \quad \frac{\partial^m u}{\partial x^m}(\cdot, t) \in L^\infty(\mathbb{R}), \quad m \in \mathbb{N}_0, \quad t \in \mathbb{R},$$

$$u(x, \cdot), u_{xx}(x, \cdot) \in C^1(\mathbb{R}), \quad x \in \mathbb{R}.$$

We start by formulating the basic polynomial setup taken essentially from [5]. One defines  $\{f_\ell\}_{\ell \in \mathbb{N}_0}$  recursively by

$$(2.3) \quad f_0 = 1,$$

$$f_{\ell,x} = -2\mathcal{G}(2(4u - u_{xx})f_{\ell-1,x} + (4u_x - u_{xxx})f_{\ell-1}), \quad \ell \in \mathbb{N},$$

where  $\mathcal{G}$  is given by

$$(2.4) \quad \mathcal{G}: L^\infty(\mathbb{R}) \rightarrow L^\infty(\mathbb{R}), \quad (\mathcal{G}v)(x) = \frac{1}{4} \int_{\mathbb{R}} dy e^{-2|x-y|} v(y), \quad x \in \mathbb{R}, \quad v \in L^\infty(\mathbb{R}).$$

One observes that  $\mathcal{G}$  is the resolvent of minus the one-dimensional Laplacian at energy parameter equal to  $-4$ , that is,

$$(2.5) \quad \mathcal{G} = \left( -\frac{d^2}{dx^2} + 4 \right)^{-1}.$$

The first coefficient reads

$$(2.6) \quad f_1 = -2u + c_1,$$

where  $c_1$  is an integration constant. Subsequent coefficients are non local with respect to  $u$ . At each level a new integration constant, denoted by  $c_\ell$ , is introduced. Moreover, we introduce coefficients  $\{g_\ell\}_{\ell \in \mathbb{N}_0}$  and  $\{h_\ell\}_{\ell \in \mathbb{N}_0}$  by

$$(2.7) \quad g_\ell = f_\ell + \frac{1}{2}f_{\ell,x}, \quad \ell \in \mathbb{N}_0,$$

$$(2.8) \quad h_\ell = (4u - u_{xx})f_\ell - g_{\ell+1,x}, \quad \ell \in \mathbb{N}_0.$$

Explicitly, one computes

$$(2.9) \quad f_0 = 1,$$

$$f_1 = -2u + c_1,$$

$$f_2 = 2u^2 + 2\mathcal{G}(u_x^2 + 8u^2) + c_1(-2u) + c_2,$$

$$g_0 = 1,$$

$$g_1 = -2u - u_x + c_1,$$

$$g_2 = 2u^2 + 2uu_x + 2\mathcal{G}(u_x^2 + u_x u_{xx} + 8uu_x + 8u^2)$$

$$+ c_1(-2u - u_x) + c_2,$$

$$\begin{aligned}
h_0 &= 4u + 2u_x, \\
h_1 &= -2u_x^2 - 4uu_x - 8u^2 \\
&\quad - 2\mathcal{G}(u_x u_{xxx} + u_{xx}^2 + 2u_x u_{xx} + 8uu_{xx} + 8u_x^2 + 16uu_x) \\
&\quad + c_1(4u + 2u_x), \text{ etc.}
\end{aligned}$$

For later use it is convenient also to introduce the corresponding homogeneous coefficients  $\hat{f}_\ell$ ,  $\hat{g}_\ell$ , and  $\hat{h}_\ell$  defined by the vanishing of the integration constants  $c_k$ ,  $k = 1, \dots, \ell$ ,

$$(2.10) \quad \hat{f}_0 = f_0 = 1, \quad \hat{f}_\ell = f_\ell|_{c_k=0, k=1, \dots, \ell}, \quad \ell \in \mathbb{N},$$

$$(2.11) \quad \hat{g}_0 = g_0 = 1, \quad \hat{g}_\ell = g_\ell|_{c_k=0, k=1, \dots, \ell}, \quad \ell \in \mathbb{N},$$

$$(2.12) \quad \hat{h}_0 = h_0 = (4u + 2u_x), \quad \hat{h}_\ell = h_\ell|_{c_k=0, k=1, \dots, \ell}, \quad \ell \in \mathbb{N}.$$

Hence,

$$(2.13) \quad f_\ell = \sum_{k=0}^{\ell} c_{\ell-k} \hat{f}_k, \quad g_\ell = \sum_{k=0}^{\ell} c_{\ell-k} \hat{g}_k, \quad h_\ell = \sum_{k=0}^{\ell} c_{\ell-k} \hat{h}_k, \quad \ell \in \mathbb{N}_0,$$

defining

$$(2.14) \quad c_0 = 1.$$

Next, given Hypothesis 2.1, one introduces the  $2 \times 2$  matrix  $U$  by

$$(2.15) \quad U(z, x) = \begin{pmatrix} -1 & 1 \\ z^{-1}(4u(x) - u_{xx}(x)) & 1 \end{pmatrix}, \quad x \in \mathbb{R},$$

and for each  $n \in \mathbb{N}_0$  the following  $2 \times 2$  matrix  $V_n$  by

$$(2.16) \quad V_n(z, x) = \begin{pmatrix} -G_n(z, x) & F_n(z, x) \\ z^{-1}H_n(z, x) & G_n(z, x) \end{pmatrix}, \quad n \in \mathbb{N}_0, z \in \mathbb{C} \setminus \{0\}, x \in \mathbb{R},$$

assuming  $F_n$ ,  $G_n$ , and  $H_n$  to be polynomials of degree  $n$  with respect to  $z$  and  $C^\infty$  in  $x$ . Postulating the zero-curvature condition

$$(2.17) \quad -V_{n,x}(z, x) + [U(z, x), V_n(z, x)] = 0,$$

one finds

$$(2.18) \quad F_{n,x}(z, x) = 2G_n(z, x) - 2F_n(z, x),$$

$$(2.19) \quad z G_{n,x}(z, x) = (4u(x) - u_{xx}(x))F_n(z, x) - H_n(z, x),$$

$$(2.20) \quad H_{n,x}(z, x) = 2H_n(z, x) - 2(4u(x) - u_{xx}(x))G_n(z, x).$$

From (2.18)–(2.20) one infers that

$$(2.21) \quad \frac{d}{dx} \det(V_n(z, x)) = -\frac{1}{z} \frac{d}{dx} \left( zG_n(z, x)^2 + F_n(z, x)H_n(z, x) \right) = 0,$$

and hence

$$(2.22) \quad zG_n(z, x)^2 + F_n(z, x)H_n(z, x) = Q_{2n+1}(z),$$

where the polynomial  $Q_{2n+1}$  of degree  $2n + 1$  is  $x$ -independent. Actually it turns out that it is more convenient to define

$$(2.23) \quad R_{2n+2}(z) = zQ_{2n+1}(z) = \prod_{m=0}^{2n+1} (z - E_m), \quad E_0 = 0, E_1, \dots, E_{2n+1} \in \mathbb{C}$$

so that (2.22) becomes

$$(2.24) \quad z^2G_n(z, x)^2 + zF_n(z, x)H_n(z, x) = R_{2n+2}(z).$$

Next one makes the ansatz that  $F_n$ ,  $H_n$ , and  $G_n$  are polynomials of degree  $n$ , related to the coefficients  $f_\ell$ ,  $h_\ell$ , and  $g_\ell$  by

$$(2.25) \quad F_n(z, x) = \sum_{\ell=0}^n f_{n-\ell}(x)z^\ell,$$

$$(2.26) \quad G_n(z, x) = \sum_{\ell=0}^n g_{n-\ell}(x)z^\ell,$$

$$(2.27) \quad H_n(z, x) = \sum_{\ell=0}^n h_{n-\ell}(x)z^\ell.$$

Insertion of (2.25)–(2.27) into (2.18)–(2.20) then yields the recursion relations (2.3)–(2.4) and (2.7) for  $f_\ell$  and  $g_\ell$  for  $\ell = 0, \dots, n$ . For fixed  $n \in \mathbb{N}$  we obtain the recursion (2.8) for  $h_\ell$  for  $\ell = 0, \dots, n - 1$  and

$$(2.28) \quad h_n = (4u - u_{xx})f_n.$$

(When  $n = 0$  one directly gets  $h_0 = (4u - u_{xx})$ ). Moreover, taking  $z = 0$  in (2.24) yields

$$(2.29) \quad f_n(x)h_n(x) = - \prod_{m=1}^{2n+1} E_m.$$

In addition, one finds

$$(2.30) \quad h_{n,x}(x) - 2h_n(x) + 2(4u(x) - u_{xx}(x))g_n(x) = 0, \quad n \in \mathbb{N}_0.$$

Using the relations (2.7) and (2.28) permits one to write (2.30) as

$$(2.31) \quad \text{s-CH}_n(u) = (u_{xxx} - 4u_x)f_n - 2(4u - u_{xx})f_{n,x} = 0, \quad n \in \mathbb{N}_0.$$

Varying  $n \in \mathbb{N}_0$  in (2.31) then defines the stationary CH hierarchy. We record the first few equations explicitly,

$$(2.32) \quad \begin{aligned} \text{s-CH}_0(u) &= u_{xxx} - 4u_x = 0, \\ \text{s-CH}_1(u) &= -2uu_{xxx} - 4u_x u_{xx} + 24uu_x + c_1(u_{xxx} - 4u_x) = 0, \\ \text{s-CH}_2(u) &= 2u^2 u_{xxx} - 8uu_x u_{xx} - 40u^2 u_x + 2(u_{xxx} - 4u_x)\mathcal{G}(u_x^2 + 8u^2) \\ &\quad - 8(4u - u_{xx})\mathcal{G}(u_x u_{xx} + 8uu_x) \\ &\quad + c_1(-2uu_{xxx} - 4u_x u_{xx} + 24uu_x) + c_2(u_{xxx} - 4u_x) = 0, \text{ etc.} \end{aligned}$$

By definition, the set of solutions of (2.31), with  $n$  ranging in  $\mathbb{N}_0$ , represents the class of algebro-geometric CH solutions. If  $u$  satisfies one of the stationary CH equations in (2.31) for a particular value of  $n$ , then it satisfies infinitely many such equations of order higher than  $n$  for certain choices of integration constants  $c_\ell$ . At times it will be convenient to abbreviate algebro-geometric stationary CH solutions  $u$  simply as CH *potentials*.

For later use we also introduce the corresponding homogeneous polynomials  $\widehat{F}_\ell$ ,  $\widehat{G}_\ell$ , and  $\widehat{H}_\ell$  defined by

$$(2.33) \quad \widehat{F}_\ell(z) = F_\ell(z)|_{c_k=0, k=1, \dots, \ell} = \sum_{k=0}^{\ell} \widehat{f}_{\ell-k} z^k, \quad \ell = 0, \dots, n,$$

$$(2.34) \quad \widehat{G}_\ell(z) = G_\ell(z)|_{c_k=0, k=1, \dots, \ell} = \sum_{k=0}^{\ell} \widehat{g}_{\ell-k} z^k, \quad \ell = 0, \dots, n,$$

$$(2.35) \quad \widehat{H}_\ell(z) = H_\ell(z)|_{c_k=0, k=1, \dots, \ell} = \sum_{k=0}^{\ell} \widehat{h}_{\ell-k} z^k, \quad \ell = 0, \dots, n-1,$$

$$(2.36) \quad \widehat{H}_n(z) = (4u - u_{xx})\widehat{f}_n + \sum_{k=1}^n \widehat{h}_{n-k} z^k.$$



In accordance with our notation introduced in (2.10)–(2.12) and (2.33)–(2.36), the corresponding homogeneous stationary CH equations are then defined by

$$(2.37) \quad \text{s-}\widehat{\text{CH}}_n(u) = \text{s-CH}_n(u) \Big|_{c_\ell=0, \ell=1, \dots, n} = 0, \quad n \in \mathbb{N}_0.$$

Using equations (2.18)–(2.20) one can also derive individual differential equations for  $F_n$  and  $H_n$ . Focusing on  $F_n$  only, one obtains

$$(2.38) \quad \begin{aligned} F_{n,xxx}(z, x) - 4(z^{-1}(4u(x) - u_{xx}(x)) + 1)F_{n,x}(z, x) \\ - 2z^{-1}(4u_x(x) - u_{xxx}(x))F_n(z, x) = 0. \end{aligned}$$

This is of course consistent with (2.25) and (2.3) (applying  $\mathcal{G}^{-1}$  to (2.3)). Multiplying (2.38) with  $F_n$  and integrating the result yields

$$(2.39) \quad F_{n,xx}F_n - 2^{-1}F_{n,x}^2 - 2F_n^2 - 2z^{-1}(4u - u_{xx})F_n^2 = C(z),$$

for some  $C(z)$ , constant with respect to  $x$ . Differentiating (2.18), inserting (2.19) into the resulting equation, and comparing with (2.18) and (2.24) then yields

$$(2.40) \quad C(z) = -2z^{-2}R_{2n+2}(z).$$

Thus,

$$(2.41) \quad \begin{aligned} -(z^2/2)F_{n,xx}(z, x)F_n(z, x) + (z^2/4)F_{n,x}(z, x)^2 \\ + z^2F_n(z, x)^2 + z(4u(x) - u_{xx}(x))F_n(z, x)^2 = R_{2n+2}(z). \end{aligned}$$

Next, we turn to the time-dependent CH hierarchy. Introducing a deformation parameter  $t_n \in \mathbb{R}$  into  $u$  (replacing  $u(x)$  by  $u(x, t_n)$ ), the definitions (2.15), (2.16), and (2.25)–(2.27) of  $U$ ,  $V_n$ , and  $F_n$ ,  $G_n$ , and  $H_n$ , respectively, still apply. The corresponding zero-curvature relation reads

$$(2.42) \quad U_{t_n}(z, x, t_n) - V_{n,x}(z, x, t_n) + [U(z, x, t_n), V_n(z, x, t_n)] = 0, \quad n \in \mathbb{N}_0,$$

which results in the following set of equations

$$(2.43) \quad \begin{aligned} 4u_{t_n}(x, t_n) - u_{xxt_n}(x, t_n) - H_{n,x}(z, x, t_n) + 2H_n(z, x, t_n) \\ - 2(4u(x, t_n) - u_{xx}(x, t_n))G_n(z, x, t_n) = 0, \end{aligned}$$

$$(2.44) \quad F_{n,x}(z, x, t_n) = 2G_n(z, x, t_n) - 2F_n(z, x, t_n),$$

$$(2.45) \quad zG_{n,x}(z, x, t_n) = (4u(x, t_n) - u_{xx}(x, t_n))F_n(z, x, t_n) - H_n(z, x, t_n).$$

Inserting the polynomial expressions for  $F_n$ ,  $H_n$ , and  $G_n$  into (2.44) and (2.45), respectively, first yields recursion relations (2.3) and (2.7) for  $f_\ell$  and  $g_\ell$  for  $\ell = 0, \dots, n$ . For fixed  $n \in \mathbb{N}$  we obtain from (2.43) the recursion (2.8) for  $h_\ell$  for  $\ell = 0, \dots, n-1$  and

$$(2.46) \quad h_n = (4u - u_{xx})f_n.$$

(When  $n = 0$  one directly gets  $h_0 = (4u - u_{xx})$ ). In addition, one finds

$$(2.47) \quad 4u_{t_n}(x, t_n) - u_{xxt_n}(x, t_n) - h_{n,x}(x, t_n) + 2h_n(x, t_n) \\ - 2(4u(x, t_n) - u_{xx}(x, t_n))g_n(x, t_n) = 0, \quad n \in \mathbb{N}_0.$$

Using relations (2.7) and (2.46) permits one to write (2.47) as

$$(2.48) \quad \text{CH}_n(u) = 4u_{t_n} - u_{xxt_n} + (u_{xxx} - 4u_x)f_n - 2(4u - u_{xx})f_{n,x} = 0, \quad n \in \mathbb{N}_0.$$

Varying  $n \in \mathbb{N}_0$  in (2.48) then defines the time-dependent CH hierarchy. We record the first few equations explicitly,

$$(2.49) \quad \begin{aligned} \text{CH}_0(u) &= 4u_{t_0} - u_{xxt_0} + u_{xxx} - 4u_x = 0, \\ \text{CH}_1(u) &= 4u_{t_1} - u_{xxt_1} - 2uu_{xxx} - 4u_xu_{xx} + 24uu_x + c_1(u_{xxx} - 4u_x) = 0, \\ \text{CH}_2(u) &= 4u_{t_2} - u_{xxt_2} + 2u^2u_{xxx} - 8uu_xu_{xx} - 40u^2u_x \\ &\quad + 2(u_{xxx} - 4u_x)\mathcal{G}(u_x^2 + 8u^2) - 8(4u - u_{xx})\mathcal{G}(u_xu_{xx} + 8uu_x) \\ &\quad + c_1(-2uu_{xxx} - 4u_xu_{xx} + 24uu_x) + c_2(u_{xxx} - 4u_x) = 0, \quad \text{etc.} \end{aligned}$$

Similarly, one introduces the corresponding homogeneous CH hierarchy by

$$(2.50) \quad \widehat{\text{CH}}_n(u) = \text{CH}_n(u)|_{c_\ell=0, \ell=1, \dots, n} = 0, \quad n \in \mathbb{N}_0.$$

Up to an inessential scaling of the  $(x, t_1)$  variables,  $\widehat{\text{CH}}_1(u) = 0$  represents the *Camassa–Holm* equation as discussed in [17], [18].

We note that our zero-curvature approach is similar (but not identical) to that sketched in [57]. This is in contrast to almost all other treatments of the CH equation where a Lax equation approach appears to be preferred.

Our recursion formalism was introduced under the assumption of a sufficiently smooth function  $u$  in Hypothesis 2.1. The actual existence of smooth global solutions of the initial value problem associated with the CH hierarchy (2.49) is a nontrivial issue and various aspects of it are discussed, for instance, in [23], [24], [25], [26], [30], [52], [53], [58], [59], [61].

### 3. The stationary CH formalism

This section is devoted to a detailed study of the stationary CH hierarchy and its algebro-geometric solutions. Our principal tool will be a combination of the polynomial recursion formalism introduced in Section 2 and a meromorphic function  $\phi$  (the solution of a Riccati-type equation associated with the zero-curvature representation of (1.1)) on a hyperelliptic curve  $\mathcal{K}_n$  defined in terms of the polynomial  $R_{2n+2}$ .

For major parts of this section we suppose

$$(3.1) \quad u \in C^\infty(\mathbb{R}), \quad \frac{d^m u}{dx^m} \in L^\infty(\mathbb{R}), \quad m \in \mathbb{N}_0,$$

and assume (2.3), (2.4), (2.7), (2.8), (2.15)–(2.17), (2.23), (2.24), (2.25)–(2.27), (2.28)–(2.31), keeping  $n \in \mathbb{N}_0$  fixed.

Returning to (2.24) we infer from (2.26) and (2.9) that  $R_{2n+2}(z) = zQ_{2n+1}(z)$  is a monomial of degree  $2n + 2$  of the form

$$(3.2) \quad R_{2n+2}(z) = \prod_{m=0}^{2n+1} (z - E_m), \quad E_0 = 0, E_1, \dots, E_{2n+1} \in \mathbb{C}.$$

Computing

$$(3.3) \quad \begin{aligned} \det(wI_2 - iV_n(z, x)) &= w^2 - \det(V_n(z, x)) \\ &= w^2 + G_n(z, x)^2 + \frac{1}{z}F_n(z, x)H_n(z, x) \\ &= w^2 + \frac{1}{z^2}R_{2n+2}(z), \end{aligned}$$

that is,

$$(3.4) \quad R_{2n+2}(z) = z^2G_n(z, x)^2 + zF_n(z, x)H_n(z, x)$$

(with  $I_2$  the identity matrix in  $\mathbb{C}^2$ ), we are led to introduce the (possibly singular) hyperelliptic curve  $\mathcal{K}_n$  of arithmetic genus  $n$  defined by

$$(3.5) \quad \mathcal{K}_n: \mathcal{F}_n(z, y) = y^2 - R_{2n+2}(z) = 0.$$

In the following we will occasionally impose further constraints on the zeros  $E_m$  of  $R_{2n+2}$  introduced in (3.2) and assume that

$$(3.6) \quad E_0 = 0, E_1, \dots, E_{2n+1} \in \mathbb{C} \setminus \{0\}.$$

We compactify  $\mathcal{K}_n$  by adding two points at infinity,  $P_{\infty+}, P_{\infty-}$ , with  $P_{\infty+} \neq P_{\infty-}$ , still denoting its projective closure by  $\mathcal{K}_n$ . Hence  $\mathcal{K}_n$  becomes a two-sheeted Riemann surface of arithmetic genus  $n$ . Points  $P$  on  $\mathcal{K}_n \setminus \{P_{\infty\pm}\}$

are denoted by  $P = (z, y)$ , where  $y(\cdot)$  denotes the meromorphic function on  $\mathcal{K}_n$  satisfying  $\mathcal{F}_n(z, y) = 0$ . For additional facts on  $\mathcal{K}_n$  and further notation freely employed throughout this paper, the reader may want to consult Appendix Appendix A.

For notational simplicity we will usually tacitly assume that  $n \in \mathbb{N}$ . (The case  $n = 0$  is explicitly treated in Example 3.10).

In the following the roots of the polynomials  $F_n$  and  $H_n$  will play a special role and hence we introduce on  $\mathbb{C} \times \mathbb{R}$

$$(3.7) \quad F_n(z, x) = \prod_{j=1}^n (z - \mu_j(x)), \quad H_n(z, x) = h_0(x) \prod_{j=1}^n (z - \nu_j(x)).$$

Moreover, we introduce

$$(3.8) \quad \hat{\mu}_j(x) = (\mu_j(x), -\mu_j(x)G_n(\mu_j(x), x)) \in \mathcal{K}_n, \quad j = 1, \dots, n, \quad x \in \mathbb{R},$$

$$(3.9) \quad \hat{\nu}_j(x) = (\nu_j(x), \nu_j(x)G_n(\nu_j(x), x)) \in \mathcal{K}_n, \quad j = 1, \dots, n, \quad x \in \mathbb{R},$$

and

$$(3.10) \quad P_0 = (0, 0).$$

The branch of  $y(\cdot)$  near  $P_{\infty\pm}$  is fixed according to

$$(3.11) \quad \lim_{\substack{|z(P)| \rightarrow \infty \\ P \rightarrow P_{\infty\pm}}} \frac{y(P)}{z(P)G_n(z(P), x)} = \mp 1.$$

Due to assumption (3.1),  $u$  is smooth and bounded, and hence  $F_n(z, \cdot)$  and  $H_n(z, \cdot)$  share the same property. Thus, one concludes

$$(3.12) \quad \mu_j, \nu_k \in C(\mathbb{R}), \quad j, k = 1, \dots, n,$$

taking multiplicities (and appropriate reordering) of the zeros of  $F_n$  and  $H_n$  into account.

Next, define the fundamental meromorphic function  $\phi(\cdot, x)$  on  $\mathcal{K}_n$  by

$$(3.13) \quad \phi(P, x) = \frac{y - zG_n(z, x)}{F_n(z, x)}$$

$$(3.14) \quad = \frac{zH_n(z, x)}{y + zG_n(z, x)}, \quad P = (z, y) \in \mathcal{K}_n, \quad x \in \mathbb{R}.$$

Assuming (3.6), the divisor  $(\phi(\cdot, x))$  of  $\phi(\cdot, x)$  is given by

$$(3.15) \quad (\phi(\cdot, x)) = \mathcal{D}_{P_0\hat{\mu}(x)} - \mathcal{D}_{P_{\infty+}\hat{\mu}(x)},$$

taking into account (3.11). Here we abbreviated

$$(3.16) \quad \hat{\underline{\mu}} = \{\hat{\mu}_1, \dots, \hat{\mu}_n\}, \hat{\underline{\nu}} = \{\hat{\nu}_1, \dots, \hat{\nu}_n\} \in \sigma^n \mathcal{K}_n.$$

Given  $\phi(\cdot, x)$ , one defines the associated vector  $\Psi(\cdot, x, x_0)$  on  $\mathcal{K}_n \setminus \{P_{\infty+}, P_{\infty-}, P_0\}$  by

$$(3.17) \quad \Psi(P, x, x_0) = \begin{pmatrix} \psi_1(P, x, x_0) \\ \psi_2(P, x, x_0) \end{pmatrix}, \quad P \in \mathcal{K}_n \setminus \{P_{\infty+}, P_{\infty-}, P_0\}, (x, x_0) \in \mathbb{R}^2,$$

where

$$(3.18) \quad \psi_1(P, x, x_0) = \exp\left(-\frac{1}{z} \int_{x_0}^x dx' \phi(P, x') - (x - x_0)\right),$$

$$(3.19) \quad \psi_2(P, x, x_0) = -\psi_1(P, x, x_0)\phi(P, x)/z.$$

Although  $\Psi$  is formally the analog of the stationary Baker–Akhiezer vector of the stationary CH hierarchy when compared to analogous definitions in the context of the KdV or AKNS hierarchies, its actual properties in a neighborhood of its essential singularity will feature characteristic differences to standard Baker–Akhiezer vectors (cf. Remark 3.5). We summarize the fundamental properties of  $\phi$  and  $\Psi$  in the following result.

**Lemma 3.1** *Suppose (3.1), assume the  $n$ th stationary CH equation (2.31) holds, and let  $P = (z, y) \in \mathcal{K}_n \setminus \{P_{\infty+}, P_{\infty-}, P_0\}$ ,  $(x, x_0) \in \mathbb{R}^2$ . Then  $\phi$  satisfies the Riccati-type equation*

$$(3.20) \quad \phi_x(P, x) - z^{-1}\phi(P, x)^2 - 2\phi(P, x) + 4u(x) - u_{xx}(x) = 0,$$

as well as

$$(3.21) \quad \phi(P, x)\phi(P^*, x) = -\frac{zH_n(z, x)}{F_n(z, x)},$$

$$(3.22) \quad \phi(P, x) + \phi(P^*, x) = -2\frac{zG_n(z, x)}{F_n(z, x)},$$

$$(3.23) \quad \phi(P, x) - \phi(P^*, x) = \frac{2y}{F_n(z, x)},$$

while  $\Psi$  fulfills

$$(3.24) \quad \Psi_x(P, x, x_0) = U(z, x)\Psi(P, x, x_0),$$

$$(3.25) \quad -y\Psi(P, x, x_0) = zV_n(z, x)\Psi(P, x, x_0),$$

$$(3.26) \quad \psi_1(P, x, x_0) = \left(\frac{F_n(z, x)}{F_n(z, x_0)}\right)^{1/2} \exp\left(-\frac{y}{z} \int_{x_0}^x dx' F_n(z, x')^{-1}\right),$$

$$(3.27) \quad \psi_1(P, x, x_0)\psi_1(P^*, x, x_0) = \frac{F_n(z, x)}{F_n(z, x_0)},$$

$$(3.28) \quad \psi_2(P, x, x_0)\psi_2(P^*, x, x_0) = -\frac{H_n(z, x)}{zF_n(z, x_0)},$$

$$(3.29) \quad \psi_1(P, x, x_0)\psi_2(P^*, x, x_0) + \psi_1(P^*, x, x_0)\psi_2(P, x, x_0) = 2\frac{G_n(z, x)}{F_n(z, x_0)},$$

$$(3.30) \quad \psi_1(P, x, x_0)\psi_2(P^*, x, x_0) - \psi_1(P^*, x, x_0)\psi_2(P, x, x_0) = \frac{2y}{zF_n(z, x_0)}.$$

In addition, as long as the zeros of  $F_n(\cdot, x)$  are all simple for  $x \in \Omega$ ,  $\Omega \subseteq \mathbb{R}$  an open interval,  $\Psi(\cdot, x, x_0)$ ,  $x, x_0 \in \Omega$ , is meromorphic on  $\mathcal{K}_n \setminus \{P_0\}$ .

**Proof:** Equation (3.20) follows using the definition (3.13) of  $\phi$  as well as relations (2.18)–(2.20). The other relations, (3.21)–(3.23), are easy consequences of  $y(P^*) = -y(P)$ , (3.13) and (3.14). By (3.17)–(3.19),  $\Psi$  is meromorphic on  $\mathcal{K}_n \setminus \{P_{\infty\pm}\}$  away from the poles  $\hat{\mu}_j(x')$  of  $\phi(\cdot, x')$ . By (2.18), (3.8), and (3.13),

$$(3.31) \quad -\frac{1}{z}\phi(P, x') \underset{P \rightarrow \hat{\mu}_j(x')}{=} \frac{\partial}{\partial x'} \ln(F_n(z, x')) + O(1) \text{ as } z \rightarrow \mu_j(x'),$$

and hence  $\psi_1$  is meromorphic on  $\mathcal{K}_n \setminus \{P_{\infty\pm}\}$  by (3.18) as long as the zeros of  $F_n(\cdot, x)$  are all simple. This follows from (3.18) by restricting  $P$  to a sufficiently small neighborhood  $\mathcal{U}_j$  of  $\{\hat{\mu}_j(x') \in \mathcal{K}_n \mid x' \in \Omega, x' \in [x_0, x]\}$  such that  $\hat{\mu}_k(x') \notin \mathcal{U}_j$  for all  $x' \in [x_0, x]$  and all  $k \in \{1, \dots, n\} \setminus \{j\}$ . Since  $\phi$  is meromorphic on  $\mathcal{K}_n$  by (3.13),  $\psi_2$  is meromorphic on  $\mathcal{K}_n \setminus \{P_{\infty\pm}\}$  by (3.19). The remaining properties of  $\Psi$  can be verified by using the definition (3.17)–(3.19) as well as relations (3.20)–(3.23). In particular, equation (3.26) follows by inserting the definition of  $\phi$ , (3.13), into (3.18), using (2.18). ■

Next, we derive Dubrovin-type equations for  $\mu_j$  and  $\nu_j$ . Since in the remainder of this section we will frequently assume  $\mathcal{K}_n$  to be nonsingular, we list all restrictions on  $\mathcal{K}_n$  in this case,

$$(3.32) \quad E_0 = 0, \quad E_m \in \mathbb{C} \setminus \{0\}, \quad E_m \neq E_{m'} \text{ for } m \neq m', \quad m, m' = 1, \dots, 2n+1.$$

**Lemma 3.2** *Suppose (3.1) and the  $n$ th stationary CH equation (2.31) holds subject to the constraint (3.32) on an open interval  $\tilde{\Omega}_\mu \subseteq \mathbb{R}$ . Moreover, suppose that the zeros  $\mu_j$ ,  $j = 1, \dots, n$ , of  $F_n(\cdot)$  remain distinct and nonzero on  $\tilde{\Omega}_\mu$ . Then  $\{\hat{\mu}_j\}_{j=1, \dots, n}$ , defined by (3.8), satisfies the following first-order system of differential equations*

$$(3.33) \quad \mu_{j,x}(x) = 2\frac{y(\hat{\mu}_j(x))}{\mu_j(x)} \prod_{\substack{\ell=1 \\ \ell \neq j}}^n (\mu_j(x) - \mu_\ell(x))^{-1}, \quad j = 1, \dots, n, \quad x \in \tilde{\Omega}_\mu.$$

Next, assume  $\mathcal{K}_n$  to be nonsingular and introduce the initial condition

$$(3.34) \quad \{\hat{\mu}_j(x_0)\}_{j=1,\dots,n} \subset \mathcal{K}_n$$

for some  $x_0 \in \mathbb{R}$ , where  $\mu_j(x_0) \neq 0$ ,  $j = 1, \dots, n$ , are assumed to be distinct. Then there exists an open interval  $\Omega_\mu \subseteq \mathbb{R}$ , with  $x_0 \in \Omega_\mu$ , such that the initial value problem (3.33), (3.34) has a unique solution  $\{\hat{\mu}_j\}_{j=1,\dots,n} \subset \mathcal{K}_n$  satisfying

$$(3.35) \quad \hat{\mu}_j \in C^\infty(\Omega_\mu, \mathcal{K}_n), \quad j = 1, \dots, n,$$

and  $\mu_j$ ,  $j = 1, \dots, n$ , remain distinct and nonzero on  $\Omega_\mu$ . For the zeros  $\{\nu_j\}_{j=1,\dots,n}$  of  $H_n(\cdot)$  similar statements hold with  $\mu_j$  and  $\Omega_\mu$  replaced by  $\nu_j$  and  $\Omega_\nu$ , etc. In particular,  $\{\hat{\nu}_j\}_{j=1,\dots,n}$ , defined by (3.9), satisfies the system

$$(3.36) \quad \nu_{j,x}(x) = 2 \frac{(4u(x) - u_{xx}(x))y(\hat{\nu}_j(x))}{(4u(x) + 2u_x(x))\nu_j(x)} \prod_{\substack{\ell=1 \\ \ell \neq j}}^n (\nu_j(x) - \nu_\ell(x))^{-1},$$

$$j = 1, \dots, n, \quad x \in \Omega_\nu.$$

**Proof:** We only prove equation (3.33) since the proof of (3.36) follows in an identical manner. Inserting  $z = \mu_j$  into equation (2.18), one concludes from (3.8),

$$(3.37) \quad F_{n,x}(\mu_j) = -\mu_{j,x} \prod_{\substack{\ell=1 \\ \ell \neq j}}^n (\mu_j - \mu_\ell) = 2G_n(\mu_j) = -2y(\hat{\mu}_j)/\mu_j,$$

proving (3.33). The smoothness assertion (3.35) is clear as long as  $\hat{\mu}_j$  stays away from the branch points  $(E_m, 0)$ . In case  $\hat{\mu}_j$  hits such a branch point, one can use the local chart around  $(E_m, 0)$  (with local coordinate  $\zeta = \sigma(z - E_m)^{1/2}$ ,  $\sigma \in \{1, -1\}$ ) to verify (3.35).  $\blacksquare$

Combining the polynomial approach in Section 2 with (3.7) readily yields trace formulas for the CH invariants. For simplicity we just record the simplest case.

**Lemma 3.3** *Suppose (3.1), assume the  $n$ th stationary CH equation (2.31) holds, and let  $x \in \mathbb{R}$ . Then*

$$(3.38) \quad u(x) = \frac{1}{2} \sum_{j=1}^n \mu_j(x) - \frac{1}{4} \sum_{m=0}^{2n+1} E_m.$$

**Proof:** Equation (3.38) follows by considering the coefficient of  $z^{n-1}$  in  $F_n$  in (2.25) which yields

$$(3.39) \quad u = \frac{1}{2} \sum_{j=1}^n \mu_j + \frac{c_1}{2}.$$

The constant  $c_1$  can be determined by considering the coefficient of the term  $z^{2n+1}$  in (2.24), which results in

$$(3.40) \quad c_1 = -\frac{1}{2} \sum_{m=0}^{2n+1} E_m. \quad \blacksquare$$

Next we turn to asymptotic properties of  $\phi$  and  $\psi_j$ ,  $j = 1, 2$ .

**Lemma 3.4** *Suppose (3.1), assume the  $n$ th stationary CH equation (2.31) holds, and let  $P = (z, y) \in \mathcal{K}_n \setminus \{P_{\infty+}, P_{\infty-}, P_0\}$ ,  $x \in \mathbb{R}$ . Then*

$$(3.41) \quad \phi(P, x) \underset{\zeta \rightarrow 0}{=} \begin{cases} -2\zeta^{-1} - 2u(x) + u_x(x) + O(\zeta), & P \rightarrow P_{\infty+}, \\ 2u(x) + u_x(x) + O(\zeta), & P \rightarrow P_{\infty-}, \end{cases} \quad \zeta = z^{-1},$$

$$(3.42) \quad \phi(P, x) \underset{\zeta \rightarrow 0}{=} \frac{\left(\prod_{m=1}^{2n+1} E_m\right)^{1/2}}{f_n(x)} \zeta + O(\zeta^2), \quad P \rightarrow P_0, \quad \zeta = z^{1/2},$$

and

$$(3.43) \quad \psi_1(P, x, x_0) \underset{\zeta \rightarrow 0}{=} \exp(\pm(x - x_0))(1 + O(\zeta)), \quad P \rightarrow P_{\infty\pm}, \quad \zeta = 1/z,$$

$$(3.44)$$

$$\psi_2(P, x, x_0) \underset{\zeta \rightarrow 0}{=} \exp(\pm(x - x_0)) \begin{cases} -2 + O(\zeta), & P \rightarrow P_{\infty+}, \\ (2u(x) + u_x(x))\zeta + O(\zeta^2), & P \rightarrow P_{\infty-}, \end{cases} \quad \zeta = 1/z,$$

$$(3.45)$$

$$\psi_1(P, x, x_0) \underset{\zeta \rightarrow 0}{=} \exp\left(-\frac{1}{\zeta} \int_{x_0}^x dx' \frac{\left(\prod_{m=1}^{2n+1} E_m\right)^{1/2}}{f_n(x')} + O(1)\right), \quad P \rightarrow P_0, \quad \zeta = z^{1/2},$$

$$(3.46)$$

$$\psi_2(P, x, x_0) \underset{\zeta \rightarrow 0}{=} O(\zeta^{-1}) \exp\left(-\frac{1}{\zeta} \int_{x_0}^x dx' \frac{\left(\prod_{m=1}^{2n+1} E_m\right)^{1/2}}{f_n(x')} + O(1)\right), \quad P \rightarrow P_0, \quad \zeta = z^{1/2}.$$



**Proof:** The existence of the asymptotic expansions of  $\phi$  in terms of the appropriate local coordinates  $\zeta = 1/z$  near  $P_{\infty\pm}$  and  $\zeta = z^{1/2}$  near  $P_0$  is clear from the explicit form of  $\phi$  in (3.13). Insertion of the polynomials  $F_n$ ,  $G_n$ , and  $H_n$  into (3.13) then, in principle, yields the explicit expansion coefficients in (3.41) and (3.42). However, a more efficient way to compute these coefficients consists in utilizing the Riccati-type equation (3.20). Indeed, inserting the ansatz

$$(3.47) \quad \phi \underset{z \rightarrow \infty}{=} \phi_1 z + \phi_0 + O(z^{-1})$$

into (3.20) and comparing the leading powers of  $1/z$  immediately yields the first line in (3.41). Similarly, the ansatz

$$(3.48) \quad \phi \underset{z \rightarrow \infty}{=} \phi_0 + \phi_1 z^{-1} + O(z^{-2})$$

inserted into (3.20) then yields the second line in (3.41). Finally, the ansatz

$$(3.49) \quad \phi \underset{z \rightarrow 0}{=} \phi_1 z^{1/2} + \phi_2 z + O(z^{3/2})$$

inserted into (3.20) yields (3.42). Expansions (3.43)–(3.46) then follow from (3.18), (3.19), (3.41), and (3.42).  $\blacksquare$

**Remark 3.5** We note the unusual fact that  $P_0$ , as opposed to  $P_{\infty\pm}$ , is the essential singularity of  $\psi_j$ ,  $j = 1, 2$ . What makes matters worse is the intricate  $x$ -dependence of the leading-order exponential term in  $\psi_j$ ,  $j = 1, 2$ , near  $P_0$ , as displayed in (3.45), (3.46). This is in sharp contrast to standard Baker-Akhiezer functions that feature a linear behavior with respect to  $x$  in connection with their essential singularities of the type  $\exp(c(x - x_0)\zeta^{-1})$  near  $\zeta = 0$ .

Introducing

$$(3.50) \quad \begin{aligned} \underline{\hat{B}}_{Q_0} : \hat{\mathcal{K}}_n \setminus \{P_{\infty+}, P_{\infty-}\} &\rightarrow \mathbb{C}^n, \\ P &\mapsto \underline{\hat{B}}_{Q_0}(P) = (\hat{B}_{Q_0,1}, \dots, \hat{B}_{Q_0,n}) \\ &= \begin{cases} \int_{Q_0}^P \tilde{\omega}_{P_{\infty+}, P_{\infty-}}^{(3)}, & n = 1, \\ \left( \int_{Q_0}^P \eta_2, \dots, \int_{Q_0}^P \eta_n, \int_{Q_0}^P \tilde{\omega}_{P_{\infty+}, P_{\infty-}}^{(3)} \right), & n \geq 2, \end{cases} \end{aligned}$$

where  $\tilde{\omega}_{P_{\infty+}, P_{\infty-}}^{(3)} = z^n dz/y$  (cf. (C.42)) and

$$(3.51) \quad \hat{\beta}_{Q_0} : \sigma^n(\hat{\mathcal{K}}_n \setminus \{P_{\infty+}, P_{\infty-}\}) \rightarrow \mathbb{C}^n,$$

$$\mathcal{D}_{\underline{Q}} \mapsto \hat{\beta}_{Q_0}(\mathcal{D}_{\underline{Q}}) = \sum_{j=1}^n \hat{B}_{Q_0}(Q_j), \underline{Q} = \{Q_1, \dots, Q_n\} \in \sigma^n \hat{\mathcal{K}}_n \setminus \{P_{\infty+}, P_{\infty-}\},$$

choosing identical paths of integration from  $Q_0$  to  $P$  in all integrals in (3.50) and (3.51). Then one obtains the following result, which indicates a characteristic difference between the CH hierarchy and other completely integrable systems such as the KdV and AKNS hierarchies.

**Lemma 3.6** *Assume (3.32) and suppose that  $\{\hat{\mu}_j\}_{j=1,\dots,n}$  satisfies the stationary Dubrovin equations (3.33) on an open interval  $\Omega_\mu \subseteq \mathbb{R}$  such that  $\mu_j$ ,  $j = 1, \dots, n$ , remain distinct and nonzero on  $\Omega_\mu$ . Introducing the associated divisor  $\mathcal{D}_{\hat{\mu}} \in \sigma^n \hat{\mathcal{K}}_n$ ,  $\hat{\mu} = \{\hat{\mu}_1, \dots, \hat{\mu}_n\} \in \sigma^n \hat{\mathcal{K}}_n$ , one computes*

$$(3.52) \quad \frac{d}{dx} \alpha_{Q_0}(\mathcal{D}_{\hat{\mu}(x)}) = -\frac{2}{\Psi_n(\underline{\mu}(x))} \underline{c}(1), \quad x \in \Omega_\mu.$$

*In particular, the Abel map does not linearize the divisor  $\mathcal{D}_{\hat{\mu}(\cdot)}$  on  $\Omega_\mu$ . In addition,*

$$(3.53) \quad \frac{d}{dx} \sum_{j=1}^n \int_{Q_0}^{\hat{\mu}_j(x)} \eta_1 = -\frac{2}{\Psi_n(\underline{\mu}(x))}, \quad x \in \Omega_\mu,$$

$$(3.54) \quad \frac{d}{dx} \hat{\beta}(\mathcal{D}_{\hat{\mu}(x)}) = \begin{cases} 2, & n = 1, \\ 2(0, \dots, 0, 1), & n \geq 2, \end{cases} \quad x \in \Omega_\mu.$$

**Proof:** Let  $x \in \Omega_\mu$ . Then, using

$$(3.55) \quad \frac{1}{\mu_j} = \frac{\prod_{\substack{p=1 \\ p \neq j}}^n \mu_p}{\prod_{m=1}^n \mu_m} = -\frac{\Phi_{n-1}^{(j)}(\underline{\mu})}{\Psi_n(\underline{\mu})}, \quad j = 1, \dots, n,$$

(cf. (C.3), (C.4)) one obtains

$$(3.56) \quad \begin{aligned} \frac{d}{dx} \left( \sum_{j=1}^n \int_{Q_0}^{\hat{\mu}_j} \underline{\omega} \right) &= \sum_{j=1}^n \mu_{j,x} \sum_{k=1}^n \underline{c}(k) \frac{\mu_j^{k-1}}{y(\hat{\mu}_j)} = 2 \sum_{j=1}^n \sum_{k=1}^n \underline{c}(k) \frac{\mu_j^{k-2}}{\prod_{\substack{\ell=1 \\ \ell \neq j}}^n (\mu_j - \mu_\ell)} \\ &= -\frac{2}{\Psi_n(\underline{\mu})} \sum_{j=1}^n \sum_{k=1}^n \underline{c}(k) \frac{\mu_j^{k-1}}{\prod_{\substack{\ell=1 \\ \ell \neq j}}^n (\mu_j - \mu_\ell)} \Phi_{n-1}^{(j)}(\underline{\mu}) \\ &= -\frac{2}{\Psi_n(\underline{\mu})} \sum_{j=1}^n \sum_{k=1}^n \underline{c}(k) (U_n(\underline{\mu}))_{k,j} (U_n(\underline{\mu}))_{j,1}^{-1} \\ &= -\frac{2}{\Psi_n(\underline{\mu})} \sum_{k=1}^n \underline{c}(k) \delta_{k,1} = -\frac{2}{\Psi_n(\underline{\mu})} \underline{c}(1), \end{aligned}$$

using (C.14) and (C.15). (3.53) is just a special case of (3.52) and (3.54) follows as in (3.56) using (C.10).  $\blacksquare$

The analogous results hold for the corresponding divisor  $\mathcal{D}_{\hat{\mu}(x)}$  associated with  $\phi(\cdot, x)$ .

The fact that the Abel map does not provide the proper change of variables to linearize the divisor  $\mathcal{D}_{\hat{\mu}(x)}$  in the CH context is in sharp contrast to standard integrable soliton equations such as the KdV and AKNS hierarchies (cf. also Remark 3.5). The change of variables

$$(3.57) \quad x \mapsto \tilde{x} = \int^x dx' \Psi_n(\underline{\mu}(x'))^{-1}$$

linearizes the Abel map  $\underline{A}_{Q_0}(\mathcal{D}_{\hat{\mu}(\tilde{x})})$ ,  $\tilde{\mu}_j(\tilde{x}) = \mu_j(x)$ ,  $j = 1, \dots, n$ . These facts are well-known and discussed (by different methods) by Constantin and McKean [29], Alber [1], Alber, Camassa, Fedorov, Holm, and Marsden [3], and Alber and Fedorov [7], [8]. The intricate relation between the variables  $x$  and  $\tilde{x}$  is detailed in (3.70). Our approach follows a route similar to Novikov's treatment of the Dym equation [56].

Next we turn to representations of  $\phi$  and  $u$  in terms of the Riemann theta function associated with  $\mathcal{K}_n$ , assuming  $\mathcal{K}_n$  to be nonsingular. In the following, the notation established in Appendices Appendix A–Appendix C will be freely employed. In fact, given the preparatory work collected in Appendices Appendix A–Appendix C, the proof of Theorem 3.7 below will be reduced to a few lines.

We choose a fixed base point  $Q_0$  on  $\mathcal{K}_n \setminus \{P_{\infty+}, P_0\}$ . Let  $\omega_{P_{\infty+}, P_0}^{(3)}$  be a normal differential of the third kind holomorphic on  $\mathcal{K}_n \setminus \{P_{\infty+}, P_0\}$  with simple poles at  $P_{\infty}$  and  $P_0$  and residues 1 and  $-1$ , respectively (cf. (A.22)–(A.27)),

$$(3.58) \quad \omega_{P_{\infty+}, P_0}^{(3)} = \frac{1}{y} \prod_{j=1}^n (z - \lambda_j) dz,$$

$$(3.59) \quad \omega_{P_{\infty+}, P_0}^{(3)} \underset{\zeta \rightarrow 0}{=} (\zeta^{-1} + O(1)) d\zeta \text{ as } P \rightarrow P_{\infty+},$$

$$(3.60) \quad \omega_{P_{\infty+}, P_0}^{(3)} \underset{\zeta \rightarrow 0}{=} (-\zeta^{-1} + O(1)) d\zeta \text{ as } P \rightarrow P_0,$$

where the local coordinates are given by

$$(3.61) \quad \zeta = 1/z \text{ for } P \text{ near } P_{\infty+}, \quad \zeta = \sigma z^{1/2} \text{ for } P \text{ near } P_0, \quad \sigma \in \{1, -1\}.$$

Moreover,

$$(3.62) \quad \int_{a_j} \omega_{P_{\infty+}, P_0}^{(3)} = 0, \quad j = 1, \dots, n,$$

$$(3.63) \quad \int_{Q_0}^P \omega_{P_{\infty+}, P_0}^{(3)} \underset{\zeta \rightarrow 0}{=} \ln(\zeta) + e_0 + O(\zeta) \text{ as } P \rightarrow P_{\infty+},$$

$$(3.64) \quad \int_{Q_0}^P \omega_{P_{\infty+}, P_0}^{(3)} \underset{\zeta \rightarrow 0}{=} -\ln(\zeta) + d_0 + O(\zeta) \text{ as } P \rightarrow P_0$$

for some constants  $e_0, d_0 \in \mathbb{C}$ . We also record

$$(3.65) \quad \underline{A}_{Q_0}(P) - \underline{A}_{Q_0}(P_{\infty\pm}) \underset{\zeta \rightarrow 0}{=} \pm \underline{c}(n)\zeta + O(\zeta^2) \text{ as } P \rightarrow P_{\infty\pm}.$$

In the following it will be convenient to introduce the abbreviations

$$(3.66) \quad \underline{z}(P, \underline{Q}) = \underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{Q}}), \quad P \in \mathcal{K}_n, \\ \underline{Q} = \{Q_1, \dots, Q_n\} \in \sigma^n \mathcal{K}_n,$$

and analogously,

$$(3.67) \quad \hat{z}(P, \hat{Q}) = \hat{\Xi}_{Q_0} - \hat{A}_{Q_0}(P) + \hat{\alpha}_{Q_0}(\mathcal{D}_{\hat{Q}}), \quad P \in \hat{\mathcal{K}}_n, \\ \hat{Q} = \{Q_1, \dots, Q_n\} \in \sigma^n \hat{\mathcal{K}}_n.$$

**Theorem 3.7** *Suppose  $u \in C^\infty(\Omega)$ ,  $u^{(m)} \in L^\infty(\Omega)$ ,  $m \in \mathbb{N}_0$ , and assume the  $n$ th stationary CH equation (2.31) holds on  $\Omega$  subject to the constraint (3.32). Moreover, let  $P \in \mathcal{K}_n \setminus \{P_{\infty+}, P_0\}$  and  $x \in \Omega$ , where  $\Omega \subseteq \mathbb{R}$  is an open interval. In addition, suppose that  $\mathcal{D}_{\hat{\mu}(x)}$ , or equivalently,  $\mathcal{D}_{\hat{\nu}(x)}$ , is nonspecial for  $x \in \Omega$ . Then  $\phi$  and  $u$  admit the representations*

$$(3.68)$$

$$\phi(P, x) = -2 \frac{\theta(\underline{z}(P_{\infty+}, \hat{\mu}(x)))\theta(\underline{z}(P, \hat{\nu}(x)))}{\theta(\underline{z}(P_{\infty+}, \hat{\nu}(x)))\theta(\underline{z}(P, \hat{\mu}(x)))} \exp\left(-\int_{Q_0}^P \omega_{P_{\infty+}, P_0}^{(3)} + e_0\right),$$

$$(3.69)$$

$$u(x) = \frac{1}{2} \sum_{j=1}^n \lambda_j - \frac{1}{4} \sum_{m=0}^{2n+1} E_m + \frac{1}{2} \sum_{j=1}^n U_j \frac{\partial}{\partial w_j} \ln \left( \frac{\theta(\underline{z}(P_{\infty+}, \hat{\mu}(x)) + \underline{w})}{\theta(\underline{z}(P_{\infty-}, \hat{\mu}(x)) + \underline{w})} \right) \Big|_{\underline{w}=0}.$$

Moreover, let  $\tilde{\Omega} \subseteq \Omega$  be such that  $\mu_j$ ,  $j = 1, \dots, n$ , are nonvanishing on  $\tilde{\Omega}$ . Then, the constraint

$$(3.70) \quad 2(x - x_0) = -2 \int_{x_0}^x \frac{dx'}{\prod_{k=1}^n \mu_k(x')} \sum_{j=1}^n \left( \int_{a_j} \tilde{\omega}_{P_{\infty+}, P_{\infty-}}^{(3)} \right) c_j(1) \\ + \ln \left( \frac{\theta(\underline{z}(P_{\infty+}, \hat{\mu}(x)))\theta(\underline{z}(P_{\infty-}, \hat{\mu}(x_0)))}{\theta(\underline{z}(P_{\infty-}, \hat{\mu}(x)))\theta(\underline{z}(P_{\infty+}, \hat{\mu}(x_0)))} \right), \quad x, x_0 \in \tilde{\Omega}$$

holds, with

$$(3.71) \quad \begin{aligned} \hat{\underline{z}}(P_{\infty_{\pm}}, \hat{\underline{\mu}}(x)) &= \hat{\underline{\Xi}}_{Q_0} - \hat{\underline{A}}_{Q_0}(P_{\infty_{\pm}}) + \hat{\underline{\alpha}}_{Q_0}(\mathcal{D}_{\hat{\underline{\mu}}(x)}) \\ &= \hat{\underline{\Xi}}_{Q_0} - \hat{\underline{A}}_{Q_0}(P_{\infty_{\pm}}) + \hat{\underline{\alpha}}_{Q_0}(\mathcal{D}_{\hat{\underline{\mu}}(x_0)}) - 2 \int_{x_0}^x \frac{dx'}{\Psi_n(\underline{\mu}(x'))} \underline{c}(1), \\ &\quad x \in \tilde{\Omega}. \end{aligned}$$

**Proof:** First we temporarily assume that

$$(3.72) \quad \mu_j(x) \neq \mu_{j'}(x), \nu_k(x) \neq \nu_{k'}(x) \text{ for } j \neq j', k \neq k' \text{ and } x \in \tilde{\Omega},$$

for appropriate  $\tilde{\Omega} \subseteq \Omega$ . Since by (3.15),  $\mathcal{D}_{P_0 \hat{\underline{\nu}}(x)} \sim \mathcal{D}_{P_{\infty_+} \hat{\underline{\mu}}(x)}$ , and  $P_{\infty_-} = (P_{\infty_+})^* \notin \{\hat{\underline{\mu}}_1(x), \dots, \hat{\underline{\mu}}_n(x)\}$  by hypothesis, one can apply Theorem A.6 to conclude that  $\mathcal{D}_{\hat{\underline{\nu}}(x)} \in \sigma^n \mathcal{K}_n$  is nonspecial. This argument is of course symmetric with respect to  $\hat{\underline{\mu}}(x)$  and  $\hat{\underline{\nu}}(x)$ . Thus,  $\mathcal{D}_{\hat{\underline{\mu}}(x)}$  is nonspecial if and only if  $\mathcal{D}_{\hat{\underline{\nu}}(x)}$  is. The representation (3.68) for  $\phi$ , subject to (3.72), then follows by combining (3.15), (3.41), (3.42), and Theorem A.5 since  $\mathcal{D}_{\hat{\underline{\mu}}}$  and  $\mathcal{D}_{\hat{\underline{\nu}}}$  are nonspecial. The representation (3.69) for  $u$  on  $\tilde{\Omega}$  follows from the trace formula (3.38) and (C.46) (taking  $k = 1$ ). By continuity, (3.68) and (3.69) extend from  $\tilde{\Omega}$  to  $\Omega$ . Assuming  $\mu_j \neq 0$ ,  $j = 1, \dots, n$ , in addition to (3.72), the constraint (3.70) follows by combining (3.53), (3.54), and (C.45). Equation (3.71) is clear from (3.52). Again the extra assumption (3.72) can be removed by continuity and hence (3.70) and (3.71) extend to  $\tilde{\Omega}$ . ■

**Remark 3.8** While the stationary CH solution  $u$  in (3.69) is of course a meromorphic quasi-periodic function with respect to the new variable  $\tilde{x}$  in (3.57),  $u$  may exhibit a rather intricate behavior with respect to the original variable  $x$ . Generically,  $u$  has an infinite number of branch points of the type

$$(3.73) \quad u(x) \underset{x \rightarrow x_0}{=} O((x - x_0)^{2/3})$$

and

$$(3.74) \quad \tilde{x} - \tilde{x}_0 \underset{x \rightarrow x_0}{=} O((x - x_0)^{1/3}).$$

Moreover, real-valued bounded stationary CH solutions fall into two categories and are either smooth quasi-periodic functions in  $x$ , or else (3.73) and (3.74) hold at infinitely many points (depending on whether or not  $\Psi_n(\underline{\mu})$  is zero-free, cf. (3.57)), as discussed in [3], [7], [8]). We note that (3.70) relates the variables  $x$  and  $\tilde{x}$ .

**Remark 3.9** We emphasized in Remark 3.5 that  $\Psi$  in (3.17)–(3.19) markedly differs from standard Baker-Akhiezer vectors. Hence one cannot expect the usual theta function representation of  $\psi_j$ ,  $j = 1, 2$ , in terms of ratios of theta functions times an exponential term containing a meromorphic differential with a pole at the essential singularity of  $\psi_j$  multiplied by  $(x - x_0)$ . However, combining (C.7) and (C.46), one computes

$$\begin{aligned}
(3.75) \quad F_n(z, x) &= z^n + \sum_{\ell=0}^{n-1} \Psi_{n-\ell}(\underline{\mu}(x)) z^\ell \\
&= z^n + \sum_{k=1}^n \left( \Psi_{n+1-k}(\underline{\lambda}) \right. \\
&\quad \left. - \sum_{j=1}^n c_j(k) \frac{\partial}{\partial w_j} \ln \left( \frac{\theta(\underline{z}(P_{\infty+}, \hat{\underline{\mu}}(x)) + \underline{w})}{\theta(\underline{z}(P_{\infty-}, \hat{\underline{\mu}}(x)) + \underline{w})} \right) \Big|_{\underline{w}=0} z^{k-1} \right) \\
&= \prod_{j=1}^n (z - \lambda_j) - \sum_{j=1}^n \sum_{k=1}^n c_j(k) \frac{\partial}{\partial w_j} \ln \left( \frac{\theta(\underline{z}(P_{\infty+}, \hat{\underline{\mu}}(x)) + \underline{w})}{\theta(\underline{z}(P_{\infty-}, \hat{\underline{\mu}}(x)) + \underline{w})} \right) \Big|_{\underline{w}=0} z^{k-1},
\end{aligned}$$

and hence obtains a theta function representation of  $\psi_1$  upon inserting (3.75) into (3.26). The corresponding theta function representation of  $\psi_2$  is then clear from (3.19) and (3.68).

Next we briefly consider the trivial case  $n = 0$  excluded in Theorem 3.7.

**Example 3.10** Assume  $n = 0$ ,  $P = (z, y) \in \mathcal{K}_0 \setminus \{P_{\infty+}, P_{\infty-}, P_0\}$ , and let  $(x, x_0) \in \mathbb{R}^2$ . Then

$$\begin{aligned}
(3.76) \quad \mathcal{K}_0: \mathcal{F}_0(z, y) &= y^2 - R_2(z) = y^2 - z(z - E_1) = 0, \\
E_0 &= 0, \quad E_1 \in \mathbb{C}, \quad u(x) = -E_1/4, \\
\phi(P, x) &= y - z = -\frac{E_1 z}{y + z}, \\
\psi_1(P, x, x_0) &= \exp(-(y/z)(x - x_0)), \\
\psi_2(P, x, x_0) &= (1 - (y/z)) \exp(-(y/z)(x - x_0)).
\end{aligned}$$

Actually, the general solution of s-CH<sub>0</sub>( $u$ ) =  $u_{xxx} - 4u_x = 0$  is given by

$$(3.77) \quad u(x) = a_1 e^{2x} + a_2 e^{-2x} - (E_1/4), \quad a_j \in \mathbb{C}, \quad j = 1, 2.$$

However, the requirement  $u^{(m)} \in L^\infty(\mathbb{R})$ ,  $m \in \mathbb{N}_0$ , according to (3.1), necessitates the choice  $a_1 = a_2 = 0$  and hence leads to (3.76). The latter corresponds to the trace formula (3.38) in the special case  $n = 0$ .

Finally, we will show that solvability of the Dubrovin equations (3.33) on  $\Omega_\mu \subseteq \mathbb{R}$  in fact implies equation (2.31) on  $\Omega_\mu$ .

**Theorem 3.11** *Fix  $n \in \mathbb{N}$ , assume (3.32), and suppose that  $\{\hat{\mu}_j\}_{j=1,\dots,n}$  satisfies the stationary Dubrovin equations (3.33) on an open interval  $\Omega_\mu \subseteq \mathbb{R}$  such that  $\mu_j$ ,  $j = 1, \dots, n$ , remain distinct and nonzero on  $\Omega_\mu$ . Then  $u \in C^\infty(\Omega_\mu)$  defined by*

$$(3.78) \quad u(x) = \frac{1}{2} \sum_{j=1}^n \mu_j(x) - \frac{1}{4} \sum_{m=0}^{2n+1} E_m$$

satisfies the  $n$ th stationary CH equation (2.31), that is,

$$(3.79) \quad \text{s-CH}_n(u) = 0 \text{ on } \Omega_\mu.$$

**Proof:** Given the solutions  $\hat{\mu}_j = (\mu_j, y(\hat{\mu}_j)) \in C^\infty(\Omega_\mu, \mathcal{K}_n)$ ,  $j = 1, \dots, n$  of (3.33) we introduce

$$(3.80) \quad F_n(z) = \prod_{j=1}^n (z - \mu_j),$$

$$(3.81) \quad G_n(z) = F_n(z) + \frac{1}{2} F_{n,x}(z)$$

on  $\mathbb{C} \times \Omega_\mu$ . The Dubrovin equations imply

$$(3.82) \quad y(\hat{\mu}_j) = \frac{1}{2} \mu_j \mu_{j,x} \prod_{\substack{\ell=1 \\ \ell \neq j}}^n (\mu_j - \mu_\ell) = -\frac{1}{2} \mu_j F_{n,x}(\mu_j) = -\mu_j G_n(\mu_j).$$

Thus

$$(3.83) \quad R_{2n+2}(\mu_j) - \mu_j^2 G_n(\mu_j)^2 = 0, \quad j = 1, \dots, n.$$

Furthermore  $R_{2n+2}(0) = 0$ , and hence there exists a polynomial  $H_n$  such that

$$(3.84) \quad R_{2n+2}(z) - z^2 G_n(z)^2 = z F_n(z) H_n(z).$$

Computing the coefficient of the term  $z^{2n+1}$  in (3.84) one finds

$$(3.85) \quad H_n(z) = (4u + 2u_x)z^n + O(z^{n-1}) \text{ as } |z| \rightarrow \infty.$$

Next, one defines a polynomial  $P_{n-1}$  by

$$(3.86) \quad P_{n-1}(z) = (4u - u_{xx})F_n(z) - H_n(z) - zG_{n,x}(z).$$

Using (3.78), (3.80), (3.81), and (3.85) one infers that indeed  $P_{n-1}$  has degree at most  $n - 1$ . Multiplying (3.86) by  $G_n$ , and replacing the term  $G_n G_{n,x}$  with the result obtained upon differentiating (3.84) with respect to  $x$ , yields

$$(3.87) \quad \begin{aligned} G_n(z)P_{n-1}(z) &= F_n(z)\left((4u - u_{xx})G_n(z) + \frac{1}{2}H_{n,x}(z)\right) \\ &+ \left(\frac{1}{2}F_{n,x}(z) - G_n(z)\right)H_n(z), \end{aligned}$$

and hence

$$(3.88) \quad G_n(\mu_j)P_{n-1}(\mu_j) = 0, \quad j = 1, \dots, n$$

on  $\Omega_\mu$ .

Restricting  $x \in \Omega_\mu$  temporarily to  $x \in \tilde{\Omega}_\mu$ , where

$$(3.89) \quad \begin{aligned} \tilde{\Omega}_\mu &= \{x \in \Omega_\mu \mid F_{n,x}(\mu_j(x), x) = 2iy \frac{\hat{\mu}_j(x)}{\mu_j(x)} \neq 0, j = 1, \dots, n\} \\ &= \{x \in \Omega_\mu \mid \mu_j(x) \notin \{E_0, \dots, E_{2n}\}, j = 1, \dots, n\} \end{aligned}$$

one infers that

$$(3.90) \quad P_{n-1}(\mu_j) = 0, \quad j = 1, \dots, n$$

on  $\mathbb{C} \times \tilde{\Omega}_\mu$ . Since  $P_{n-1}(z)$  has degree at most  $n - 1$ , one concludes

$$(3.91) \quad P_{n-1} = 0 \text{ on } \mathbb{C} \times \tilde{\Omega}_\mu,$$

and hence (2.19), that is,

$$(3.92) \quad zG_{n,x}(z) = (4u - u_{xx})F_n(z) - H_n(z)$$

on  $\mathbb{C} \times \tilde{\Omega}_\mu$ . Differentiating (3.84) with respect to  $x$  and using equations (3.92) and (3.81) one finds

$$(3.93) \quad H_{n,x}(z) = 2F_n(z) - 2(4u - u_{xx})G_n(z)$$

on  $\mathbb{C} \times \tilde{\Omega}_\mu$ . In order to extend these results to  $\Omega_\mu$  we next investigate the case where  $\hat{\mu}_j$  hits a branch point  $(E_m, 0)$ ,  $m \neq 0$ . Hence we suppose

$$(3.94) \quad \mu_{j_0}(x) \rightarrow E_{m_0} \text{ as } x \rightarrow x_0 \in \Omega_\mu,$$

for some  $j_0 \in \{1, \dots, n\}$ ,  $m_0 \in \{1, \dots, 2n + 1\}$ . Introducing

$$(3.95) \quad \zeta_{j_0}(x) = \sigma(\mu_{j_0}(x) - E_{m_0})^{1/2}, \quad \sigma \in \{1, -1\}, \quad \mu_{j_0}(x) = E_{m_0} + \zeta_{j_0}(x)^2,$$



for some  $x$  in an open interval centered around  $x_0$ , the Dubrovin equation (3.33) for  $\mu_{j_0}$  becomes

$$(3.96) \quad \zeta_{j_0, x}(x) \underset{x \rightarrow x_0}{=} \frac{c(\sigma)}{E_{m_0}} \left( \prod_{\substack{m=0 \\ m \neq m_0}}^{2n+1} (E_{m_0} - E_m) \right)^{1/2} \prod_{\substack{k=1 \\ k \neq j_0}}^n (E_{m_0} - \mu_k(x))^{-1} (1 + O(\zeta_{j_0}(x)^2))$$

for some  $|c(\sigma)| = 1$  and hence relations (3.91)–(3.93) extend to  $\Omega_\mu$ . We have now established relations (2.18)–(2.20) on  $\mathbb{C} \times \Omega_\mu$ , and one can now proceed as in Section 2 to obtain (3.79).  $\blacksquare$

## 4. The time-dependent CH formalism

In this section we extend the algebro-geometric formalism of Section 3 to the time-dependent CH hierarchy. For most of this section we will assume the following hypothesis.

**Hypothesis 4.1** *Suppose that  $u: \mathbb{R}^2 \rightarrow \mathbb{C}$  satisfies*

$$(4.1) \quad \begin{aligned} u(\cdot, t) &\in C^\infty(\mathbb{R}), \quad \frac{\partial^m u}{\partial x^m}(\cdot, t) \in L^\infty(\mathbb{R}), \quad m \in \mathbb{N}_0, t \in \mathbb{R}, \\ u(x, \cdot), u_{xx}(x, \cdot) &\in C^1(\mathbb{R}), \quad x \in \mathbb{R}. \end{aligned}$$

The basic problem in the analysis of algebro-geometric solutions of the CH hierarchy consists in solving the time-dependent  $r$ th CH flow with initial data a stationary solution of the  $n$ th equation in the hierarchy. More precisely, given  $n \in \mathbb{N}_0$ , consider a solution  $u^{(0)}$  of the  $n$ th stationary CH equation  $\text{s-CH}_n(u^{(0)}) = 0$  associated with  $\mathcal{K}_n$  and a given set of integration constants  $\{c_\ell\}_{\ell=1, \dots, n} \subset \mathbb{C}$ . Next, let  $r \in \mathbb{N}_0$ ; we intend to construct a solution  $u$  of the  $r$ th CH flow  $\text{CH}_r(u) = 0$  with  $u(t_{0,r}) = u^{(0)}$  for some  $t_{0,r} \in \mathbb{R}$ .

To emphasize that the integration constants in the definitions of the stationary and the time-dependent CH equations are independent of each other, we indicate this by adding a tilde on all the time-dependent quantities. Hence we shall employ the notation  $\tilde{V}_r, \tilde{F}_r, \tilde{G}_r, \tilde{H}_r, \tilde{f}_s, \tilde{g}_s, \tilde{h}_s, \tilde{c}_s$ , etc., in order to distinguish them from  $V_n, F_n, G_n, H_n, f_\ell, g_\ell, h_\ell, c_\ell$ , etc., in the following. In addition, we will follow a more elaborate notation inspired by Hirota's  $\tau$ -function approach and indicate the individual  $r$ th CH flow by a separate time variable  $t_r \in \mathbb{R}$ .

Summing up, we are seeking a solution  $u$  of

$$(4.2) \quad \widetilde{\text{CH}}_r(u) = 4u_{t_r} - u_{xxt_r} + (u_{xxx} - 4u_x)\tilde{f}_r - 2(4u - u_{xx})\tilde{f}_{r,x} = 0,$$

$$u(x, t_{0,r}) = u^{(0)}(x), \quad x \in \mathbb{R},$$

$$(4.3) \quad \text{s-CH}_n(u^{(0)}) = (u_{xxx} - 4u_x)f_n - 2(4u - u_{xx})f_{n,x} = 0,$$

for some  $t_{0,r} \in \mathbb{R}$ ,  $n, r \in \mathbb{N}_0$ , where  $u$  satisfies (4.1). Actually, relying on the isospectral property of the CH flows, we will go a step further and assume (4.3) not only at  $t_r = t_{0,r}$  but for all  $t_r \in \mathbb{R}$ . Hence, we start with

$$(4.4) \quad U_{t_r}(z, x, t_r) - \widetilde{V}_{r,x}(z, x, t_r) + [U(z, x, t_r), \widetilde{V}_r(z, x, t_r)] = 0,$$

$$(4.5) \quad -V_{n,x}(z, x, t_r) + [U(z, x, t_r), V_n(z, x, t_r)] = 0,$$

$$(z, x, t_r) \in \mathbb{C} \times \mathbb{R}^2,$$

where (cf. (2.25)–(2.27))

$$U(z, x, t_r) = \begin{pmatrix} -1 & 1 \\ z^{-1}(4u(x, t_r) - u_{xx}(x, t_r)) & 1 \end{pmatrix},$$

$$(4.6) \quad \widetilde{V}_r(z, x, t_r) = \begin{pmatrix} -\widetilde{G}_r(z, x, t_r) & \widetilde{F}_r(z, x, t_r) \\ z^{-1}\widetilde{H}_r(z, x, t_r) & \widetilde{G}_r(z, x, t_r) \end{pmatrix},$$

$$\widetilde{V}_n(z, x, t_r) = \begin{pmatrix} -G_n(z, x, t_r) & F_n(z, x, t_r) \\ z^{-1}H_n(z, x, t_r) & G_n(z, x, t_r) \end{pmatrix},$$

and

$$(4.7) \quad F_n(z, x, t_r) = \sum_{\ell=0}^n f_{n-\ell}(x, t_r)z^\ell = \prod_{j=1}^n (z - \mu_j(x, t_r)),$$

$$(4.8) \quad G_n(z, x, t_r) = \sum_{\ell=0}^n g_{n-\ell}(x, t_r)z^\ell,$$

$$(4.9) \quad H_n(z, x, t_r) = \sum_{\ell=0}^n h_{n-\ell}(x, t_r)z^\ell = h_0(x, t_r) \prod_{j=1}^n (z - \nu_j(x, t_r)),$$

$$(4.10) \quad h_0(x, t_r) = 4u(x, t_r) + 2u_x(x, t_r),$$

$$(4.11) \quad \widetilde{F}_r(z, x, t_r) = \sum_{s=0}^r \tilde{f}_{r-s}(x, t_r)z^s,$$

$$(4.12) \quad \widetilde{G}_r(z, x, t_r) = \sum_{s=0}^r \tilde{g}_{r-s}(x, t_r)z^s,$$

$$(4.13) \quad \tilde{H}_r(z, x, t_r) = \sum_{s=0}^r \tilde{h}_{r-s}(x, t_r) z^s,$$

$$(4.14) \quad \tilde{h}_0(x, t_r) = 4u(x, t_r) + 2u_x(x, t_r),$$

for fixed  $n, r \in \mathbb{N}_0$ . Here  $f_\ell(x, t_r)$ ,  $\tilde{f}_s(x, t_r)$ ,  $g_\ell(x, t_r)$ ,  $\tilde{g}_s(x, t_r)$ ,  $h_\ell(x, t_r)$ , and  $\tilde{h}_s(x, t_r)$  for  $\ell = 0, \dots, n$ ,  $s = 0, \dots, r$ , are defined as in (2.3), (2.7), and (2.8) with  $u(x)$  replaced by  $u(x, t_r)$ , etc., and with appropriate integration constants. Explicitly, (4.4), (4.5) are equivalent to

$$(4.15) \quad 4u_{t_r}(x, t_r) - u_{xxt_r}(x, t_r) - \tilde{H}_{r,x}(z, x, t_r) + 2\tilde{H}_r(z, x, t_r) \\ - 2(4u(x, t_r) - u_{xx}(x, t_r))\tilde{G}_r(z, x, t_r) = 0,$$

$$(4.16) \quad \tilde{F}_{r,x}(z, x, t_r) = 2\tilde{G}_r(z, x, t_r) - 2\tilde{F}_r(z, x, t_r),$$

$$(4.17) \quad z\tilde{G}_{r,x}(z, x, t_r) = (4u(x, t_r) - u_{xx}(x, t_r))\tilde{F}_r(z, x, t_r) - \tilde{H}_r(z, x, t_r)$$

and

$$(4.18) \quad F_{n,x}(z, x, t_r) = 2G_n(z, x, t_r) - 2F_n(z, x, t_r),$$

$$(4.19) \quad H_{n,x}(z, x, t_r) = 2H_n(z, x, t_r) - 2(4u(x, t_r) - u_{xx}(x, t_r))G_n(z, x, t_r),$$

$$(4.20) \quad zG_{n,x}(z, x, t_r) = (4u(x, t_r) - u_{xx}(x, t_r))F_n(z, x, t_r) - H_n(z, x, t_r),$$

First we will assume the existence of a solution of equations (4.15)–(4.20) and derive an explicit formula for  $u$  in terms of Riemann theta functions. In addition, we will show in Theorem 4.10 that (4.15)–(4.20) and hence the algebro-geometric initial value problem (4.2), (4.3) has a solution at least locally, that is, for  $(x, t_r) \in \Omega$  for some open and connected set  $\Omega \subseteq \mathbb{R}^2$ .

One observes that equations (2.3)–(2.41) apply to  $F_n$ ,  $G_n$ ,  $H_n$ ,  $f_\ell$ ,  $g_\ell$ , and  $h_\ell$  and (2.3)–(2.9), (2.25)–(2.27), with  $n$  replaced by  $r$  and  $c_\ell$  replaced by  $\tilde{c}_\ell$ , apply to  $\tilde{F}_r$ ,  $\tilde{G}_r$ ,  $\tilde{H}_r$ ,  $\tilde{f}_\ell$ ,  $\tilde{g}_\ell$ , and  $\tilde{h}_\ell$ . In particular, the fundamental identity (2.24) holds,

$$(4.21) \quad z^2 G_n(z, x, t_r)^2 + z F_n(z, x, t_r) H_n(z, x, t_r) = R_{2n+2}(z),$$

and the hyperelliptic curve  $\mathcal{K}_n$  is still given by

$$(4.22) \quad \mathcal{K}_n: \mathcal{F}_n(z, y) = y^2 - R_{2n+2}(z) = 0, \quad R_{2n+2}(z) = \prod_{m=0}^{2n+1} (z - E_m),$$

assuming (3.6) for the remainder of this section, that is,

$$(4.23) \quad E_0 = 0, \quad E_1, \dots, E_{2n+1} \in \mathbb{C} \setminus \{0\}.$$

In analogy to equations (3.8), (3.9) we define

$$(4.24) \quad \hat{\mu}_j(x, t_r) = (\mu_j(x, t_r), -\mu_j(x, t_r)G_n(\mu_j(x, t_r), x, t_r)) \in \mathcal{K}_n, \\ j = 1, \dots, n, (x, t_r) \in \mathbb{R}^2,$$

$$(4.25) \quad \hat{\nu}_j(x, t_r) = (\nu_j(x, t_r), \nu_j(x, t_r)G_n(\nu_j(x, t_r), x, t_r)) \in \mathcal{K}_n, \\ j = 1, \dots, n, (x, t_r) \in \mathbb{R}^2.$$

As in Section 3, the regularity assumptions (4.1) on  $u$  imply analogous regularity properties of  $F_n$ ,  $H_n$ ,  $\mu_j$ , and  $\nu_k$ .

Next, one defines the meromorphic function  $\phi(\cdot, x, t_r)$  on  $\mathcal{K}_n$  by

$$(4.26) \quad \phi(P, x, t_r) = \frac{y - zG_n(z, x, t_r)}{F_n(z, x, t_r)}$$

$$(4.27) \quad = \frac{zH_n(z, x, t_r)}{y + zG_n(z, x, t_r)}, \quad P = (z, y) \in \mathcal{K}_n \setminus \{P_{\infty\pm}\}, (x, t_r) \in \mathbb{R}^2.$$

Assuming (4.23), the divisor  $(\phi(\cdot, x, t_r))$  of  $\phi(\cdot, x, t_r)$  reads

$$(4.28) \quad (\phi(\cdot, x, t_r)) = \mathcal{D}_{P_0\hat{\nu}(x, t_r)} - \mathcal{D}_{P_{\infty+}\hat{\mu}(x, t_r)},$$

with

$$(4.29) \quad \hat{\mu} = \{\hat{\mu}_1, \dots, \hat{\mu}_n\}, \hat{\nu} = \{\hat{\nu}_1, \dots, \hat{\nu}_n\} \in \sigma^n \mathcal{K}_n.$$

The corresponding time-dependent vector  $\Psi$ ,

$$(4.30) \quad \Psi(P, x, x_0, t_r, t_{0,r}) = \begin{pmatrix} \psi_1(P, x, x_0, t_r, t_{0,r}) \\ \psi_2(P, x, x_0, t_r, t_{0,r}) \end{pmatrix}, \\ P \in \mathcal{K}_n \setminus \{P_{\infty\pm}\}, (x, x_0, t_r, t_{0,r}) \in \mathbb{R}^4$$

is defined by

$$(4.31) \quad \psi_1(P, x, x_0, t_r, t_{0,r}) = \exp \left( - \int_{t_{0,r}}^{t_r} ds \left( (1/z) \tilde{F}_r(z, x_0, s) \phi(P, x_0, s) + \tilde{G}_r(z, x_0, s) \right) \right. \\ \left. - (1/z) \int_{x_0}^x dx' \phi(P, x', t_r) - (x - x_0) \right),$$

$$(4.32) \quad \psi_2(P, x, x_0, t_r, t_{0,r}) = -\psi_1(P, x, x_0, t_r, t_{0,r}) \phi(P, x, t_r) / z.$$

The properties of  $\phi$  can now be summarized as follows.

**Lemma 4.2** *Assume Hypothesis 4.1 and (4.4), (4.5). Moreover, let  $P = (z, y) \in \mathcal{K}_n \setminus \{P_{\infty+}, P_{\infty-}, P_0\}$  and  $(x, t_r) \in \mathbb{R}^2$ . Then  $\phi$  satisfies*

$$(4.33) \quad \phi_x(P, x, t_r) - z^{-1}\phi(P, x, t_r)^2 - 2\phi(P, x, t_r) + 4u(x, t_r) - u_{xx}(x, t_r) = 0,$$

$$(4.34) \quad \phi_{t_r}(P, x, t_r) = (4u(x, t_r) - u_{xx}(x, t_r))\tilde{F}_r(z, x, t_r) - \tilde{H}_r(z, x, t_r) + 2(\tilde{F}_r(z, x, t_r)\phi(P, x, t_r))_x$$

$$(4.35) \quad = (1/z)\tilde{F}_r(z, x, t_r)\phi(P, x, t_r)^2 + 2\tilde{G}_r(z, x, t_r)\phi(P, x, t_r) - \tilde{H}_r(z, x, t_r),$$

$$(4.36) \quad \phi(P, x, t_r)\phi(P^*, x, t_r) = -\frac{zH_n(z, x, t_r)}{F_n(z, x, t_r)},$$

$$(4.37) \quad \phi(P, x, t_r) + \phi(P^*, x, t_r) = -2\frac{zG_n(z, x, t_r)}{F_n(z, x, t_r)},$$

$$(4.38) \quad \phi(P, x, t_r) - \phi(P^*, x, t_r) = \frac{2y}{F_n(z, x, t_r)}.$$

**Proof:** Equations (4.33) and (4.36)–(4.38) are proved as in Lemma 3.1. To prove (4.35) one first observes that

$$(4.39) \quad (\partial_x - 2((1/z)\phi + 1))(\phi_{t_r} - (1/z)\tilde{F}_r\phi^2 - 2\tilde{G}_r\phi + \tilde{H}_r) = 0$$

using (4.33) and relations (4.15)–(4.17) repeatedly. Thus,

$$(4.40) \quad \phi_{t_r} - \frac{1}{z}\tilde{F}_r\phi^2 - 2\tilde{G}_r\phi + \tilde{H}_r = C \exp\left(2 \int^x dx' ((1/z)\phi + 1)\right),$$

where the left-hand side is meromorphic in a neighborhood of  $P_{\infty-}$ , while the right-hand side is meromorphic near  $P_{\infty-}$  only if  $C = 0$ . This proves (4.35).

Using (4.16) and (4.33) one obtains

$$(4.41) \quad (4u - u_{xx})\tilde{F}_r + 2(\tilde{F}_r\phi)_x = 2\tilde{G}_r\phi + (1/z)\phi^2\tilde{F}_r.$$

Combining this result with (4.35) one concludes that (4.34) holds. ■

Using relations (4.18)–(4.20) and (4.15)–(4.17), we next determine the time evolution of  $F_n$ ,  $G_n$ , and  $H_n$ .

**Lemma 4.3** *Assume Hypothesis 4.1 and (4.4), (4.5). In addition, let  $(z, x, t_r) \in \mathbb{C} \times \mathbb{R}^2$ . Then*

$$(4.42) \quad F_{n,t_r}(z, x, t_r) = 2(G_n(z, x, t_r)\tilde{F}_r(z, x, t_r) - F_n(z, x, t_r)\tilde{G}_r(z, x, t_r)),$$

$$(4.43) \quad zG_{n,t_r}(z, x, t_r) = F_n(z, x, t_r)\tilde{H}_r(z, x, t_r) - H_n(z, x, t_r)\tilde{F}_r(z, x, t_r),$$

$$(4.44) \quad H_{n,t_r}(z, x, t_r) = 2(H_n(z, x, t_r)\tilde{G}_r(z, x, t_r) - G_n(z, x, t_r)\tilde{H}_r(z, x, t_r)).$$

Equations (4.42)–(4.44) are equivalent to

$$(4.45) \quad -V_{n,t_r}(z, x, t_r) + [\tilde{V}_r(z, x, t_r), V_n(z, x, t_r)] = 0.$$

**Proof:** We prove (4.42) by using (4.38) which shows that

$$(4.46) \quad (\phi(P) - \phi(P^*))_{t_r} = -2\frac{yF_{n,t_r}}{F_n^2}.$$

However, the left-hand side of (4.46) also equals

$$(4.47) \quad \phi(P)_{t_r} - \phi(P^*)_{t_r} = \frac{4y}{F_n^2}(\tilde{G}_r F_n - \tilde{F}_r G_n),$$

using (4.35), (4.37), and (4.38). Combining (4.46) and (4.47) proves (4.42). Similarly, to prove (4.43), we use (4.37) to write

$$(4.48) \quad (\phi(P) + \phi(P^*))_{t_r} = -\frac{2z}{F_n^2}(G_{n,t_r} F_n - G_n F_{n,t_r}).$$

Here the left-hand side can be expressed as

$$(4.49) \quad \phi(P)_{t_r} + \phi(P^*)_{t_r} = 2\frac{zG_n}{F_n^2}F_{n,t_r} + \frac{2}{F_n}(\tilde{F}_r H_n - \tilde{H}_r F_n),$$

using (4.35), (4.36), and (4.37). Combining (4.48) and (4.49), using (4.42), proves (4.43). Finally, (4.44) follows by differentiating (2.24), that is,

$$(zG_n)^2 + zF_n H_n = R_{2n+2},$$

with respect to  $t_r$ , and using (4.42) and (4.43). ■

Lemmas 4.2 and 4.3 permit one to characterize  $\Psi$ .

**Lemma 4.4** *Assume Hypothesis 4.1 and (4.4), (4.5). Moreover, let  $P = (z, y) \in \mathcal{K}_n \setminus \{P_{\infty+}, P_{\infty-}, P_0\}$  and  $(x, x_0, t_r, t_{0,r}) \in \mathbb{R}^4$ . Then the Baker–Akhiezer vector  $\Psi$  satisfies*

$$(4.50) \quad \Psi_x(P, x, x_0, t_r, t_{0,r}) = U(z, x, t_r)\Psi(P, x, x_0, t_r, t_{0,r}),$$

$$(4.51) \quad -y\Psi(P, x, x_0, t_r, t_{0,r}) = zV_n(z, x, t_r)\Psi(P, x, x_0, t_r, t_{0,r}),$$

$$(4.52) \quad \Psi_{t_r}(P, x, x_0, t_r, t_{0,r}) = \tilde{V}_r(z, x, t_r)\Psi(P, x, x_0, t_r, t_{0,r}),$$

$$(4.53) \quad \psi_1(P, x, x_0, t_r, t_{0,r}) = \left( \frac{F_n(z, x, t_r)}{F_n(z, x_0, t_{0,r})} \right)^{1/2} \times \\ \times \exp \left( - (y/z) \int_{t_{0,r}}^{t_r} ds \tilde{F}_r(z, x_0, s) F_n(z, x_0, s)^{-1} - (y/z) \int_{x_0}^x dx' F_n(z, x', t_r)^{-1} \right),$$

$$(4.54) \quad \psi_1(P, x, x_0, t_r, t_{0,r})\psi_1(P^*, x, x_0, t_r, t_{0,r}) = \frac{F_n(z, x, t_r)}{F_n(z, x_0, t_{0,r})},$$

$$(4.55) \quad \psi_2(P, x, x_0, t_r, t_{0,r})\psi_2(P^*, x, x_0, t_r, t_{0,r}) = -\frac{H_n(z, x, t_r)}{zF_n(z, x_0, t_{0,r})},$$

$$(4.56) \quad \psi_1(P, x, x_0, t_r, t_{0,r})\psi_2(P^*, x, x_0, t_r, t_{0,r}) \\ + \psi_1(P^*, x, x_0, t_r, t_{0,r})\psi_2(P, x, x_0, t_r, t_{0,r}) = 2\frac{G_n(z, x, t_r)}{F_n(z, x_0, t_{0,r})},$$

$$(4.57) \quad \psi_1(P, x, x_0, t_r, t_{0,r})\psi_2(P^*, x, x_0, t_r, t_{0,r}) \\ - \psi_1(P^*, x, x_0, t_r, t_{0,r})\psi_2(P, x, x_0, t_r, t_{0,r}) = \frac{2y}{zF_n(z, x_0, t_{0,r})}.$$

In addition, as long as the zeros of  $F_n(\cdot, x, t_r)$  are all simple for  $(x, t_r)$ ,  $(x_0, t_{0,r}) \in \Omega$ ,  $\Omega \subseteq \mathbb{R}^2$  open and connected,  $\Psi(\cdot, x, x_0, t_r, t_{0,r})$ ,  $(x, t_r)$ ,  $(x_0, t_{0,r}) \in \Omega$ , is meromorphic on  $\mathcal{K}_n \setminus \{P_0, P_{\infty\pm}\}$ .

**Proof:** By (4.31),  $\psi_1(\cdot, x, x_0, t_r, t_{0,r})$  is meromorphic on  $\mathcal{K}_n \setminus \{P_{\infty\pm}\}$  away from the poles  $\hat{\mu}_j(x_0, s)$  of  $\phi(\cdot, x_0, s)$  and  $\hat{\mu}_k(x', t_r)$  of  $\phi(\cdot, x', t_r)$ . That  $\psi_1(\cdot, x, x_0, t_r, t_{0,r})$  is meromorphic on  $\mathcal{K}_n \setminus \{P_{\infty\pm}\}$  if  $F_n(\cdot, x, t_r)$  has only simple zeros is a consequence of (cf. (3.31))

$$(4.58) \quad -\frac{1}{z}\phi(P, x', t_r) \Big|_{P \rightarrow \hat{\mu}_j(x', t_r)} = \frac{\partial}{\partial x'} \ln(F_n(z, x', t_r)) + O(1)$$

as  $z \rightarrow \mu_j(x', t_r)$ , and

$$(4.59) \quad -\frac{1}{z}\tilde{F}_r(z, x_0, s)\phi(P, x_0, s) \Big|_{P \rightarrow \hat{\mu}_j(x_0, s)} = \frac{\partial}{\partial s} \ln(F_n(z, x_0, s)) + O(1)$$

as  $z \rightarrow \mu_j(x_0, s)$ , using (4.24), (4.26), and (4.42).

This follows from (4.31) by restricting  $P$  to a sufficiently small neighborhood  $\mathcal{U}_j(x_0)$  of  $\{\hat{\mu}_j(x_0, s) \in \mathcal{K}_n \mid (x_0, s) \in \Omega, s \in [t_{0,r}, t_r]\}$  such that  $\hat{\mu}_k(x_0, s) \notin \mathcal{U}_j(x_0)$  for all  $s \in [t_{0,r}, t_r]$  and all  $k \in \{1, \dots, n\} \setminus \{j\}$  and by simultaneously restricting  $P$  to a sufficiently small neighborhood  $\mathcal{U}_j(t_r)$  of  $\{\hat{\mu}_j(x', t_r) \in \mathcal{K}_n \mid (x', t_r) \in \Omega, x' \in [x_0, x]\}$  such that  $\hat{\mu}_k(x', t_r) \notin \mathcal{U}_j(t_r)$  for all  $x' \in [x_0, x]$  and all  $k \in \{1, \dots, n\} \setminus \{j\}$ . By (4.32) and the fact that  $\phi$  is meromorphic on  $\mathcal{K}_n$  one concludes that  $\psi_2$  is meromorphic on  $\mathcal{K}_n \setminus \{P_{\infty\pm}\}$  as well. Relations (4.50) and (4.51) follow as in Lemma 3.1, while the time evolution (4.52) is a consequence of the definition of  $\Psi$  in (4.31), (4.32) as well as (4.35), rewriting

$$(4.60) \quad (1/z)\phi_{t_r} = ((1/z)2\phi\tilde{F}_r + \tilde{G}_r)_x,$$

using (4.17) and (4.34). To prove (4.53) we recall the definition (4.31), that is,

$$(4.61) \quad \begin{aligned} \psi_1(P, x, x_0, t_r, t_{0,r}) &= \exp\left(- (x - x_0) - (1/z) \int_{x_0}^x dx' \phi(P, x', t_r) \right. \\ &\quad \left. - \int_{t_{0,r}}^{t_r} ds \left( (1/z)\tilde{F}_r(z, x_0, s)\phi(P, x_0, s) + \tilde{G}_r(z, x_0, s) \right) \right) \\ &= \left( \frac{F_n(z, x, t_r)}{F_n(z, x_0, t_r)} \right)^{1/2} \exp\left(- (y/z) \int_{x_0}^x dx' F_n(z, x', t_r)^{-1} \right. \\ &\quad \left. - \int_{t_{0,r}}^{t_r} ds \left( (1/z)\tilde{F}_r(z, x_0, s)\phi(P, x_0, s) + \tilde{G}_r(z, x_0, s) \right) \right), \end{aligned}$$

using the calculation leading to (3.26). Equations (4.26) and (4.42) show that

$$(4.62) \quad \begin{aligned} \frac{1}{z} \tilde{F}_r(z, x_0, s)\phi(P, x_0, s) + \tilde{G}_r(z, x_0, s) &= \\ &= \frac{y}{z} \frac{\tilde{F}_r(z, x_0, s)}{F_n(z, x_0, s)} - \frac{1}{2} \frac{F_{n,t_r}(z, x_0, s)}{F_n(z, x_0, s)}, \end{aligned}$$

which inserted into (4.61) yields (4.53). Evaluating (4.53) at the points  $P$  and  $P^*$  and multiplying the resulting expressions yields (4.54). The remaining statements are direct consequences of (4.36)–(4.38) and (4.53).  $\blacksquare$

Next, we turn to the time evolution of the quantities  $\mu_j$  and  $\nu_j$  assuming (3.32), that is,

$$(4.63) \quad E_0 = 0, \quad E_m \in \mathbb{C} \setminus \{0\}, \quad E_m \neq E_{m'} \text{ for } m \neq m', \quad m, m' = 1, \dots, 2n+1.$$



**Lemma 4.5** *Assume Hypothesis 4.1, (4.63), and (4.4), (4.5) on an open and connected set  $\tilde{\Omega}_\mu \subseteq \mathbb{R}^2$ . Moreover, suppose that the zeros  $\mu_j$ ,  $j=1, \dots, n$ , of  $F_n(\cdot)$  remain distinct and nonzero on  $\tilde{\Omega}_\mu$ . Then  $\{\hat{\mu}_j\}_{j=1, \dots, n}$ , defined by (4.24), satisfies the following first-order system of differential equations*

$$(4.64) \quad \mu_{j,x}(x, t_r) = 2 \frac{y(\hat{\mu}_j(x, t_r))}{\mu_j(x, t_r)} \prod_{\substack{\ell=1 \\ \ell \neq j}}^n (\mu_j(x, t_r) - \mu_\ell(x, t_r))^{-1},$$

$$(4.65) \quad \mu_{j,t_r}(x, t_r) = 2\tilde{F}_r(\mu_j(x, t_r), x, t_r) \frac{y(\hat{\mu}_j(x, t_r))}{\mu_j(x, t_r)} \times \\ \times \prod_{\substack{\ell=1 \\ \ell \neq j}}^n (\mu_j(x, t_r) - \mu_\ell(x, t_r))^{-1}, j=1, \dots, n, (x, t_r) \in \tilde{\Omega}_\mu.$$

Next, assume  $\mathcal{K}_n$  to be nonsingular and introduce the initial condition

$$(4.66) \quad \{\hat{\mu}_j(x_0, t_{0,r})\}_{j=1, \dots, n} \subset \mathcal{K}_n$$

for some  $(x_0, t_{0,r}) \in \mathbb{R}^2$ , where  $\mu_j(x_0, t_{0,r}) \neq 0$ ,  $j=1, \dots, n$ , are assumed to be distinct. Then there exists an open and connected set  $\Omega_\mu \subseteq \mathbb{R}^2$ , with  $(x_0, t_{0,r}) \in \Omega_\mu$ , such that the initial value problem (4.64)–(4.66) has a unique solution  $\{\hat{\mu}_j\}_{j=1, \dots, n} \subset \mathcal{K}_n$  satisfying

$$(4.67) \quad \hat{\mu}_j \in C^\infty(\Omega_\mu, \mathcal{K}_n), \quad j=1, \dots, n,$$

and  $\mu_j$ ,  $j=1, \dots, n$ , remain distinct and nonzero on  $\Omega_\mu$ .

For the zeros  $\{\nu_j\}_{j=1, \dots, n}$  of  $H_n(\cdot)$  similar statements hold with  $\mu_j$  and  $\Omega_\mu$  replaced by  $\nu_j$  and  $\Omega_\nu$ , etc. In particular,  $\{\hat{\nu}_j\}_{j=1, \dots, n}$ , defined by (4.25), satisfies the system

$$(4.68) \quad \nu_{j,x}(x, t_r) = \frac{2(4u(x, t_r) - u_{xx}(x, t_r))}{4u(x, t_r) + 2u_x(x, t_r)} \frac{y(\hat{\nu}_j(x, t_r))}{\nu_j(x, t_r)} \prod_{\substack{\ell=1 \\ \ell \neq j}}^n (\nu_j(x, t_r) - \nu_\ell(x, t_r))^{-1},$$

$$(4.69) \quad \nu_{j,t_r}(x, t_r) = \frac{2\tilde{H}_r(\nu_j(x, t_r), x, t_r)}{4u(x, t_r) + 2u_x(x, t_r)} \frac{y(\hat{\nu}_j(x, t_r))}{\nu_j(x, t_r)} \prod_{\substack{\ell=1 \\ \ell \neq j}}^n (\nu_j(x, t_r) - \nu_\ell(x, t_r))^{-1}, \\ j=1, \dots, n, (x, t_r) \in \Omega_\nu.$$

**Proof:** It suffices to prove (4.65) since the argument for (4.69) is analogous and that for (4.64) and (4.68) has been given in the proof of Lemma 3.2. Inserting  $z = \mu_j(x, t_r)$  into (4.42), observing (4.24), yields

$$(4.70) \quad F_{n,t_r}(\mu_j) = -\mu_{j,t_r} \prod_{\substack{\ell=1 \\ \ell \neq j}}^n (\mu_j - \mu_\ell) = 2\tilde{F}_r(\mu_j)G_n(\mu_j) = -2\frac{\tilde{F}_r(\mu_j)}{\mu_j}y(\hat{\mu}_j).$$

The rest is analogous to the proof of Lemma 3.2. ■

Next we note the following trace formula, the  $t_r$ -dependent analog of (3.38).

**Lemma 4.6** *Assume Hypothesis 4.1, (4.4), (4.5), and let  $(x, t_r) \in \mathbb{R}^2$ . Then*

$$(4.71) \quad u(x, t_r) = \frac{1}{2} \sum_{j=1}^n \mu_j(x, t_r) - \frac{1}{4} \sum_{m=1}^{2n+1} E_m.$$

We also record the asymptotic properties of  $\phi$ , the analogs of (3.41) and (3.42).

**Lemma 4.7** *Assume Hypothesis 4.1, (4.4), (4.5), and let  $P = (z, y) \in \mathcal{K}_n \setminus \{P_{\infty+}, P_{\infty-}, P_0\}$ ,  $(x, t_r) \in \mathbb{R}^2$ . Then*

$$(4.72) \quad \phi(P, x, t_r) \underset{\zeta \rightarrow 0}{=} \begin{cases} -2\zeta^{-1} - 2u(x, t_r) + u_x(x, t_r) + O(\zeta), & P \rightarrow P_{\infty+}, \\ 2u(x, t_r) + u_x(x, t_r) + O(\zeta), & P \rightarrow P_{\infty-}, \end{cases} \quad \zeta = z^{-1},$$

$$(4.73) \quad \phi(P, x, t_r) \underset{\zeta \rightarrow 0}{=} \frac{\left( \prod_{m=1}^{2n+1} E_m \right)^{1/2}}{f_n(x, t_r)} \zeta + O(\zeta^2), \quad P \rightarrow P_0, \quad \zeta = z^{1/2}.$$

Since the proofs of Lemmas 4.6 and 4.7 are identical to the corresponding stationary results in Lemmas 3.3 and 3.4 we omit the corresponding details.

Next, recalling the definition of  $\tilde{d}_{r,k}$  and  $\tilde{F}_r(\mu_j)$  introduced in (C.24) and (C.27) and also the definition of  $\hat{B}_{Q_0}$  and  $\hat{\beta}_{Q_0}$  in (3.50) and (3.51), respectively, we now state the analog of Lemma 3.6, thereby underscoring the marked differences between the CH hierarchy and other completely integrable systems such as the KdV and AKNS hierarchies.

**Lemma 4.8** *Assume (4.63) and suppose that  $\{\hat{\mu}_j\}_{j=1,\dots,n}$  satisfies the Dubrovin equations (4.64), (4.65) on an open set  $\Omega_\mu \subseteq \mathbb{R}^2$  such that  $\mu_j$ ,  $j=1, \dots, n$ , remain distinct and nonzero on  $\Omega_\mu$  and that  $\tilde{F}_r(\mu_j) \neq 0$  on  $\Omega_\mu$ ,  $j=1, \dots, n$ . Introducing the associated divisor  $\mathcal{D}_{\underline{\hat{\mu}}} \in \sigma^n \widehat{\mathcal{K}}_n$ ,  $\underline{\hat{\mu}} = \{\hat{\mu}_1, \dots, \hat{\mu}_n\} \in \sigma^n \widehat{\mathcal{K}}_n$ , one computes*

$$(4.74) \quad \frac{\partial}{\partial x} \alpha_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}) = -\frac{2}{\Psi_n(\underline{\mu}(x,t_r))} \mathcal{C}(1), \quad (x,t_r) \in \Omega_\mu,$$

$$(4.75) \quad \frac{\partial}{\partial t_r} \alpha_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}) = -\frac{2}{\Psi_n(\underline{\mu}(x,t_r))} \left( \sum_{k=0}^{r \wedge n} \tilde{d}_{r,k}(\underline{E}) \Psi_k(\underline{\mu}(x,t_r)) \right) \mathcal{C}(1) \\ + 2 \left( \sum_{\ell=1 \vee (n+1-r)}^n \tilde{d}_{r,n+1-\ell}(\underline{E}) \mathcal{C}(\ell) \right), \quad (x,t_r) \in \Omega_\mu.$$

*In particular, the Abel map does not linearize the divisor  $\mathcal{D}_{\underline{\hat{\mu}}(\cdot, \cdot)}$  on  $\Omega_\mu$ . In addition,*

$$(4.76) \quad \frac{\partial}{\partial x} \sum_{j=1}^n \int_{Q_0}^{\hat{\mu}_j(x,t_r)} \eta_1 = -\frac{2}{\Psi_n(\underline{\mu}(x,t_r))}, \quad (x,t_r) \in \Omega_\mu,$$

$$(4.77) \quad \frac{\partial}{\partial x} \widehat{\beta}(\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}) = \begin{cases} 2, & n=1, \\ 2(0, \dots, 0, 1), & n \geq 2, \end{cases} \quad (x,t_r) \in \Omega_\mu,$$

$$(4.78) \quad \frac{\partial}{\partial t_r} \sum_{j=1}^n \int_{Q_0}^{\hat{\mu}_j(x,t_r)} \eta_1 = -\frac{2}{\Psi_n(\underline{\mu}(x,t_r))} \sum_{k=0}^{r \wedge n} \tilde{d}_{r,k}(\underline{E}) \Psi_k(\underline{\mu}(x,t_r)) \\ + 2\tilde{d}_{r,n}(\underline{E}) \delta_{n,r \wedge n}, \quad (x,t_r) \in \Omega_\mu,$$

$$(4.79) \quad \frac{\partial}{\partial t_r} \widehat{\beta}(\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}) = 2 \left( \sum_{s=0}^r \tilde{c}_{r-s} \hat{c}_{s+1-n}(\underline{E}), \dots, \sum_{s=0}^r \tilde{c}_{r-s} \hat{c}_{s+1}(\underline{E}), \right. \\ \left. \sum_{s=0}^r \tilde{c}_{r-s} \hat{c}_s(\underline{E}) \right), \hat{c}_{-\ell}(\underline{E}) = 0, \ell \in \mathbb{N}, \quad (x,t_r) \in \Omega_\mu.$$

**Proof:** Let  $(x,t_r) \in \Omega_\mu$ . Since (4.74), (4.76), and (4.77) are proved as in the stationary context of Lemma 3.6, we focus on the proofs of (4.75), (4.78), and (4.79).

Then, using (4.65), (3.55), (C.11), and (C.9), (C.14), and (C.15) one obtains<sup>1</sup>

$$\begin{aligned}
\frac{\partial}{\partial t_r} \left( \sum_{j=1}^n \int_{Q_0}^{\hat{\mu}_j} \underline{\omega} \right) &= \sum_{j=1}^n \mu_{j,t_r} \sum_{k=1}^n \underline{c}(k) \frac{\mu_j^{k-1}}{y(\hat{\mu}_j)} \\
&= 2 \sum_{j=1}^n \sum_{k=1}^n \underline{c}(k) \frac{\mu_j^{k-1}}{\prod_{\substack{\ell=1 \\ \ell \neq j}}^n (\mu_j - \mu_\ell)} \frac{\tilde{F}_r(\mu_j)}{\mu_j} \\
&= 2 \sum_{j=1}^n \sum_{k=1}^n \underline{c}(k) \frac{\mu_j^{k-1}}{\prod_{\substack{\ell=1 \\ \ell \neq j}}^n (\mu_j - \mu_\ell)} \times \\
&\quad \times \left( - \sum_{m=0}^{r \wedge n} \tilde{d}_{r,m}(\underline{E}) \Psi_m(\underline{\mu}) \frac{\Phi_{n-1}^{(j)}(\underline{\mu})}{\Psi_n(\underline{\mu})} + \sum_{m=1}^{r \wedge n} \tilde{d}_{r,m}(\underline{E}) \Phi_{m-1}^{(j)}(\underline{\mu}) \right) \\
&= -2 \sum_{m=0}^{r \wedge n} \tilde{d}_{r,m}(\underline{E}) \frac{\Psi_m(\underline{\mu})}{\Psi_n(\underline{\mu})} \sum_{k=1}^n \sum_{j=1}^n \underline{c}(k) (U_n(\underline{\mu}))_{k,j} (U_n(\underline{\mu}))_{j,1}^{-1} \\
&\quad + 2 \sum_{m=1}^{r \wedge n} \tilde{d}_{r,m}(\underline{E}) \sum_{k=1}^n \sum_{j=1}^n \underline{c}(k) (U_n(\underline{\mu}))_{k,j} (U_n(\underline{\mu}))_{j,n-m+1}^{-1} \\
&= -\frac{2}{\Psi_n(\underline{\mu})} \sum_{m=0}^{r \wedge n} \tilde{d}_{r,m}(\underline{E}) \Psi_m(\underline{\mu}) \underline{c}(1) + 2 \sum_{m=1}^{r \wedge n} \tilde{d}_{r,m}(\underline{E}) \underline{c}(n-m+1) \\
(4.80) \quad &= -\frac{2}{\Psi_n(\underline{\mu})} \sum_{m=0}^{r \wedge n} \tilde{d}_{r,m}(\underline{E}) \Psi_m(\underline{\mu}) \underline{c}(1) + 2 \sum_{m=1 \vee (n+1-r)}^n \tilde{d}_{r,n+1-m}(\underline{E}) \underline{c}(m).
\end{aligned}$$

Equation (4.78) is just a special case of (4.75) and (4.79) follows as in (4.80) using again (C.9).  $\blacksquare$

The analogous results hold for the corresponding divisor  $\mathcal{D}_{\hat{\mu}(x,t_r)}$  associated with  $\phi(\cdot, x, t_r)$ .

The fact that the Abel map does not effect a linearization of the divisor  $\mathcal{D}_{\hat{\mu}(x,t_r)}$  in the CH context is well-known and discussed (using different approaches) by Constantin and McKean [29], Alber, Camassa, Fedorov, Holm, and Marsden [3], Alber and Fedorov [7], [8]. A change of the variable  $t_1$  in analogy to that in (3.57) in the stationary context, which avoids the use of a meromorphic differential (cf. (3.50), (3.51)) and linearizes the Abel map when considering the  $\text{CH}_1$  flow, is discussed in [1]. That change of variables corresponds to the case  $r = 1$  in (4.83).

---

<sup>1</sup> $m \wedge n = \min(m, n)$ .

Next we turn to one of the principal results of this section, the representations of  $\phi$  and  $u$  in terms of the Riemann theta function associated with  $\mathcal{K}_n$ , assuming  $\mathcal{K}_n$  to be nonsingular. Recalling (3.58)–(3.67), the analog of Theorem 3.7 in the stationary case then reads as follows.

**Theorem 4.9** *Suppose Hypothesis 4.1 and (4.2), (4.3) on  $\Omega$  subject to the constraint (4.63). In addition, let  $P \in \mathcal{K}_n \setminus \{P_{\infty+}, P_0\}$  and  $(x, t_r), (x_0, t_{0,r}) \in \Omega$ , where  $\Omega \subseteq \mathbb{R}^2$  is open and connected. Moreover, suppose that  $\mathcal{D}_{\hat{\mu}(x, t_r)}$ , or equivalently,  $\mathcal{D}_{\hat{\nu}(x, t_r)}$ , is nonspecial for  $(x, t_r) \in \Omega$ . Then  $\phi$  and  $u$  admit the representations*

$$(4.81) \quad \phi(P, x, t_r) = -2 \frac{\theta(\underline{z}(P_{\infty+}, \hat{\mu}(x, t_r)))\theta(\underline{z}(P, \hat{\nu}(x, t_r)))}{\theta(\underline{z}(P_{\infty+}, \hat{\nu}(x, t_r)))\theta(\underline{z}(P, \hat{\mu}(x, t_r)))} \times \\ \times \exp\left(-\int_{Q_0}^P \omega_{P_{\infty+}, P_0}^{(3)} + e_0\right),$$

$$(4.82) \quad u(x, t_r) = \frac{1}{2} \sum_{j=1}^n \lambda_j - \frac{1}{4} \sum_{m=0}^{2n+1} E_m \\ + \frac{1}{2} \sum_{j=1}^n U_j \frac{\partial}{\partial w_j} \ln \left( \frac{\theta(\underline{z}(P_{\infty+}, \hat{\mu}(x, t_r) + \underline{w}))}{\theta(\underline{z}(P_{\infty-}, \hat{\mu}(x, t_r) + \underline{w}))} \right) \Big|_{\underline{w}=0}.$$

Moreover, let  $\tilde{\Omega} \subseteq \Omega$  be such that  $\mu_j, j = 1, \dots, n$ , are nonvanishing on  $\tilde{\Omega}$ . Then, the constraint

$$(4.83) \quad 2(x - x_0) + 2(t_r - t_{0,r}) \sum_{s=0}^r \tilde{c}_{r-s} \hat{c}_s(\underline{E}) = \\ = \left( -2 \int_{x_0}^x \frac{dx'}{\prod_{k=1}^n \mu_k(x', t_r)} \right. \\ \left. - 2 \sum_{k=0}^{r \wedge n} \tilde{d}_{r,k}(\underline{E}) \int_{t_{0,r}}^{t_r} \frac{\Psi_k(\underline{\mu}(x_0, t'))}{\Psi_n(\underline{\mu}(x_0, t'))} dt' \right) \sum_{j=1}^n \left( \int_{a_j} \tilde{\omega}_{P_{\infty+}, P_{\infty-}}^{(3)} \right) c_j(1) \\ + 2(t_r - t_{0,r}) \sum_{\ell=1 \vee (n+1-\ell)}^n \tilde{d}_{r, n+1-\ell}(\underline{E}) \sum_{j=1}^n \left( \int_{a_j} \tilde{\omega}_{P_{\infty+}, P_{\infty-}}^{(3)} \right) c_j(\ell) \\ + \ln \left( \frac{\theta(\underline{z}(P_{\infty+}, \hat{\mu}(x, t_r)))\theta(\underline{z}(P_{\infty-}, \hat{\mu}(x_0, t_{0,r})))}{\theta(\underline{z}(P_{\infty-}, \hat{\mu}(x, t_r)))\theta(\underline{z}(P_{\infty+}, \hat{\mu}(x_0, t_{0,r})))} \right), \\ (x, t_r), (x_0, t_{0,r}) \in \tilde{\Omega}$$

holds, with

$$\begin{aligned}
(4.84) \quad \hat{z}(P_{\infty\pm}, \hat{\mu}(x, t_r)) &= \hat{\Xi}_{Q_0} - \hat{A}_{Q_0}(P_{\infty\pm}) + \hat{\alpha}_{Q_0}(\mathcal{D}_{\hat{\mu}(x, t_r)}) \\
&= \hat{\Xi}_{Q_0} - \hat{A}_{Q_0}(P_{\infty\pm}) + \hat{\alpha}_{Q_0}(\mathcal{D}_{\hat{\mu}(x_0, t_r)}) \\
&\quad - 2 \left( \int_{x_0}^x \frac{dx'}{\Psi_n(\underline{\mu}(x', t_r))} \right) \underline{c}(1) \\
(4.85) \quad &= \hat{\Xi}_{Q_0} - \hat{A}_{Q_0}(P_{\infty\pm}) + \hat{\alpha}_{Q_0}(\mathcal{D}_{\hat{\mu}(x, t_0, r)}) \\
&\quad - 2 \left( \sum_{k=0}^{r \wedge n} \tilde{d}_{r, k}(\underline{E}) \int_{t_0, r}^{t_r} \frac{\Psi_k(\underline{\mu}(x, t'))}{\Psi_n(\underline{\mu}(x, t'))} dt' \right) \underline{c}(1) \\
&\quad + 2(t_r - t_0, r) \left( \sum_{\ell=1 \vee (n+1-r)}^n \tilde{d}_{r, n+1-\ell}(\underline{E}) \underline{c}(\ell) \right), \\
&\hspace{15em} (x, t_r), (x_0, t_0, r) \in \tilde{\Omega}.
\end{aligned}$$

**Proof:** First, let  $\tilde{\tilde{\Omega}} \subseteq \Omega$  be defined by requiring that  $\mu_j$ ,  $j = 1, \dots, n$ , are distinct and nonvanishing on  $\tilde{\tilde{\Omega}}$  and  $\tilde{F}_r(\mu_j) \neq 0$  on  $\tilde{\tilde{\Omega}}$ ,  $j = 1, \dots, n$ . The representation (4.81) for  $\phi$  on  $\tilde{\tilde{\Omega}}$  then follows by combining (4.28), (4.72), (4.73), and Theorem A.5 since  $\mathcal{D}_{\hat{\mu}}$  and  $\mathcal{D}_{\hat{\nu}}$  are simultaneously nonspecial as discussed in the proof of Theorem 3.7. The representation (4.82) for  $u$  on  $\tilde{\tilde{\Omega}}$  follows from the trace formula (4.71) and (C.46) (taking  $k = 1$ ). By continuity, (4.81) and (4.82) extend from  $\tilde{\tilde{\Omega}}$  to  $\tilde{\Omega}$ . The constraint (4.83) then holds on  $\tilde{\tilde{\Omega}}$  by combining (4.76)–(4.79), and (C.45). Equations (4.84) and (4.85) are clear from (4.74) and (4.75). Again by continuity, (4.83)–(4.85) extend from  $\tilde{\tilde{\Omega}}$  to  $\tilde{\Omega}$ .  $\blacksquare$

As discussed by Alber, Camassa, Fedorov, Holm, and Marsden [3], Alber and Fedorov [7], [8], the algebro-geometric CH solution  $u$  in (4.82) is not meromorphic with respect to  $x, t_r$ , in general. In more geometrical terms, the  $\text{CH}_r$  flows evolve on a nonlinear subvariety (corresponding to the constraint (4.83)) of a generalized Jacobian, topologically given by  $J(\mathcal{K}_n) \times \mathbb{C}^*$  ( $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ ). For discussions of generalized Jacobians in this context we refer, for instance, to [37], [43], [44]. Smooth (i.e.,  $C^1$  with respect to  $t_1$  and  $C^3$  and hence  $C^\infty$  with respect to  $x$ ) spatially periodic  $\text{CH}_1$  solutions  $u$  are quasi-periodic in  $t_1$  as shown by Constantin [22].

Without going into details we mention that our approach extends in a straightforward manner to the Dym-type equation,

$$(4.86) \quad v_{xxt} + 2vv_{xxx} + 4v_x v_{xx} - 4\kappa v = 0, \quad \kappa \in \mathbb{R}, (x, t) \in \mathbb{R}^2.$$

The corresponding zero-curvature formalism leads to a trace formula analogous to (4.71) (cf. [3], [7], [8]). One needs to replace the polynomial  $R_{2n+2}(z)$  by  $R_{2n+1}(z) = \prod_{m=0}^{2n} (z - E_m)$ , which results in a branch point  $P_\infty$  at infinity, and replaces the (non-normalized) differential  $\tilde{\omega}_{P_{\infty+}, P_{\infty-}}^{(3)}$  of the third kind by the (non-normalized) differential  $\tilde{\omega}_{P_\infty}^{(2)} = z^n dz/y$  of the second kind, etc. This approach (applied to the Dym equation  $4\rho_t = \rho^3 \rho_{xxx}$ , related to (4.86) by proper variable transformations) was first realized by Novikov [56] and inspired our treatment of the CH hierarchy.

Expressing  $\tilde{F}_r$  in terms of  $\Psi_k(\underline{\mu})$  and hence in terms of the theta function associated with  $\mathcal{K}_n$ , one can use (4.53) to derive a theta function representation of  $\psi_j$ ,  $j = 1, 2$ , in analogy to the stationary case discussed in Remark 3.9. We omit further details.

Up to this point we assumed Hypothesis 4.1 together with the basic equations (4.4) and (4.5). Next, we will show that solvability of the Dubrovin equations (4.64) and (4.65) on  $\Omega_\mu \subseteq \mathbb{R}^2$  in fact implies equations (4.4) and (4.5) on  $\Omega_\mu$  and hence solves the algebro-geometric initial value problem (4.2), (4.3) on  $\Omega_\mu$ . In this context we recall the definition of  $\tilde{F}_r(\mu_j)$  in terms of  $\mu_1, \dots, \mu_n$ , introduced<sup>2</sup> in (C.24), (C.27),

$$(4.87) \quad \tilde{F}_r(\mu_j) = \sum_{k=0}^{r \wedge n} \tilde{d}_{r,k}(\underline{E}) \Phi_k^{(j)}(\underline{\mu}), \quad r \in \mathbb{N}_0, \tilde{c}_0 = 1,$$

$$(4.88) \quad \tilde{d}_{r,k}(\underline{E}) = \sum_{s=0}^{r-k} \tilde{c}_{r-k-s} \hat{c}_s(\underline{E}), \quad k = 0, \dots, r \wedge n,$$

in terms of a given set of integration constants  $\{\tilde{c}_1, \dots, \tilde{c}_r\} \subset \mathbb{C}$ .

**Theorem 4.10** *Fix  $n \in \mathbb{N}$  and assume (4.63). Suppose that  $\{\hat{\mu}_j\}_{j=1, \dots, n}$  satisfies the Dubrovin equations (4.64), (4.65) on an open and connected set  $\Omega_\mu \subseteq \mathbb{R}^2$ , with  $\tilde{F}_r(\mu_j)$  in (4.65) expressed in terms of  $\mu_k$ ,  $k = 1, \dots, n$ , by (4.87), (4.88). Moreover, assume that  $\mu_j$ ,  $j = 1, \dots, n$ , remain distinct and nonzero on  $\Omega_\mu$ . Then  $u \in C^\infty(\Omega_\mu)$  defined by*

$$(4.89) \quad u(x, t_r) = \frac{1}{2} \sum_{j=1}^n \mu_j(x, t_r) - \frac{1}{4} \sum_{m=0}^{2n+1} E_m,$$

satisfies the  $r$ th CH equation (4.2), that is,

$$(4.90) \quad \widetilde{\text{CH}}_r(u) = 0 \quad \text{on } \Omega_\mu,$$

with initial values satisfying the  $n$ th stationary CH equation (4.3).

---

<sup>2</sup> $m \wedge n = \min(m, n)$ .

**Proof:** Given solutions  $\hat{\mu}_j = (\mu_j, y(\hat{\mu}_j)) \in C^\infty(\Omega_\mu, \mathcal{K}_n)$ ,  $j = 1, \dots, n$  of (4.64) and (4.65), we define polynomials  $F_n$ ,  $G_n$ , and  $H_n$  on  $\Omega_\mu$  as in the stationary case, cf. Theorem 3.11, with properties

$$(4.91) \quad F_n(z) = \prod_{j=1}^n (z - \mu_j),$$

$$(4.92) \quad G_n(z) = F_n(z) + (1/2)F_{n,x}(z),$$

$$(4.93) \quad zG_{n,x}(z) = (4u - u_{xx})F_n(z) - H_n(z),$$

$$(4.94) \quad H_{n,x}(z) = 2H_n(z) - 2(4u - u_{xx})G_n(z),$$

$$(4.95) \quad R_{2n+2}(z) = z^2G_n(z)^2 + zF_n(z)H_n(z),$$

treating  $t_r$  as a parameter.

Define polynomials  $\tilde{G}_r$  and  $\tilde{H}_r$  by

$$(4.96) \quad \tilde{G}_r(z) = \tilde{F}_r(z) + (1/2)\tilde{F}_{r,x}(z),$$

$$(4.97) \quad \tilde{H}_r(z) = (4u - u_{xx})\tilde{F}_r(z) - z\tilde{G}_{r,x}(z),$$

respectively. We claim that

$$(4.98) \quad F_{n,t_r}(z) = 2(G_n(z)\tilde{F}_r(z) - F_n(z)\tilde{G}_r(z)).$$

To prove (4.98) one computes from (4.64) and (4.65) that

$$(4.99) \quad F_{n,t_r}(z) = -F_n(z) \sum_{j=1}^n \tilde{F}_r(\mu_j) \mu_{j,x} (z - \mu_j)^{-1},$$

$$(4.100) \quad F_{n,x}(z) = -F_n(z) \sum_{j=1}^n \mu_{j,x} (z - \mu_j)^{-1}.$$

Using (4.92) and (4.96) one sees that (4.98) is equivalent to

$$(4.101) \quad \tilde{F}_{r,x}(z) = \sum_{j=1}^n (\tilde{F}_r(z) - \tilde{F}_r(\mu_j)) \mu_{j,x} (z - \mu_j)^{-1}.$$

Equation (4.101) is proved in Lemma C.5. This in turn proves (4.98). Next, taking the derivative of (4.98) with respect to  $x$  and inserting (4.92) and (4.93), yields

$$(4.102) \quad F_{n,t_r,x}(z) = 2\left((1/z)(4u - u_{xx})F_n(z)\tilde{F}_r(z) - (1/z)H_n(z)\tilde{F}_r(z) + G_n(z)\tilde{F}_{r,x}(z) - 2(G_n(z) - F_n(z))\tilde{G}_r(z) - F_n(z)\tilde{G}_{r,x}(z)\right).$$

On the other hand, by differentiating (4.92) with respect to  $t_r$ , using (4.98) one obtains

$$(4.103) \quad F_{n,t_r,x}(z) = 2(G_{n,t_r}(z) - 2(G_n(z)\tilde{F}_r(z) - F_n(z)\tilde{G}_r(z))).$$



Combining (4.92), (4.96), (4.102), and (4.103) one concludes

$$(4.104) \quad zG_{n,t_r}(z) = F_n(z)\tilde{H}_r(z) - \tilde{F}_r(z)H_n(z).$$

Next, taking the derivative of (4.95) with respect to  $t_r$  and using expressions (4.98) and (4.104) for  $F_{n,t_r}$  and  $G_{n,t_r}$ , respectively, one obtains

$$(4.105) \quad H_{n,t_r}(z) = 2(\tilde{G}_r(z)H_n(z) - G_n(z)\tilde{H}_r(z)).$$

Finally, we compute  $G_{n,x t_r}$  in two different ways. Differentiating (4.104) with respect to  $x$ , using (4.92), (4.96), and (4.94), one finds

$$(4.106) \quad \begin{aligned} zG_{n,x t_r}(z) &= \tilde{H}_{r,x}(z)F_n(z) + 2(G_n(z)\tilde{H}_r(z) - \tilde{G}_r(z)H_n(z)) \\ &+ 2(4u - u_{xx})G_n(z)\tilde{F}_r(z) - 2F_n(z)\tilde{H}_r(z). \end{aligned}$$

Differentiating (4.93) with respect to  $t_r$ , using (4.98) and (4.105), results in

$$(4.107) \quad \begin{aligned} zG_{n,x t_r}(z) &= (u_{t_r} - u_{x x t_r})F_n(z) - 2(\tilde{G}_r(z)H_n(z) - G_n(z)\tilde{H}_r(z)) \\ &+ 2(4u - u_{xx})(G_n(z)\tilde{F}_r(z) - F_n(z)\tilde{G}_r(z)). \end{aligned}$$

Combining (4.106) and (4.107) one concludes

$$(4.108) \quad u_{t_r} - u_{x x t_r} = \tilde{H}_r(z) + 2(4u - u_{xx})\tilde{G}_r(z) - \tilde{H}_r(z)$$

which is equivalent to (4.90). ■

## Appendix A. Hyperelliptic curves and their theta functions

We provide a brief summary of some of the fundamental properties and notations needed from the theory of hyperelliptic curves. More details can be found in some of the standard textbooks [36] and [55], as well as monographs dedicated to integrable systems such as [15, Ch. 2], [46, App. A–C].

Fix  $n \in \mathbb{N}$ . The hyperelliptic curve  $\mathcal{K}_n$  of genus  $n$  used in Sections 3 and 4 is defined by

$$(A.1) \quad \mathcal{K}_n: \mathcal{F}_n(z, y) = y^2 - R_{2n+2}(z) = 0, \quad R_{2n+2}(z) = \prod_{m=0}^{2n+1} (z - E_m),$$

$$(A.2) \quad \{E_m\}_{m=0, \dots, 2n+1} \subset \mathbb{C}, \quad E_m \neq E_{m'} \text{ for } m \neq m', \quad m, m' = 0, \dots, 2n+1.$$

The curve (A.2) is compactified by adding the points  $P_{\infty_+}$  and  $P_{\infty_-}$ ,  $P_{\infty_+} \neq P_{\infty_-}$ , at infinity. One then introduces an appropriate set of  $n + 1$  nonintersecting cuts  $\mathcal{C}_j$  joining  $E_{m(j)}$  and  $E_{m'(j)}$ . We denote

$$(A.3) \quad \mathcal{C} = \bigcup_{j=1}^{n+1} \mathcal{C}_j, \quad \mathcal{C}_j \cap \mathcal{C}_k = \emptyset, \quad j \neq k.$$

Define the cut plane

$$(A.4) \quad \Pi = \mathbb{C} \setminus \mathcal{C},$$

and introduce the holomorphic function

$$(A.5) \quad R_{2n+2}(\cdot)^{1/2}: \Pi \rightarrow \mathbb{C}, \quad z \mapsto \left( \prod_{m=0}^{2n+1} (z - E_m) \right)^{1/2}$$

on  $\Pi$  with an appropriate choice of the square root branch in (A.5). Define

$$(A.6) \quad \mathcal{M}_n = \{(z, \sigma R_{2n+2}(z)^{1/2}) \mid z \in \mathbb{C}, \sigma \in \{1, -1\}\} \cup \{P_{\infty_+}, P_{\infty_-}\}$$

by extending  $R_{2n+2}(\cdot)^{1/2}$  to  $\mathcal{C}$ . The hyperelliptic curve  $\mathcal{K}_n$  is then the set  $\mathcal{M}_n$  with its natural complex structure obtained upon gluing the two sheets of  $\mathcal{M}_n$  crosswise along the cuts. The set of branch points  $\mathcal{B}(\mathcal{K}_n)$  of  $\mathcal{K}_n$  is given by

$$(A.7) \quad \mathcal{B}(\mathcal{K}_n) = \{(E_m, 0)\}_{m=0, \dots, 2n+1}$$

and finite points  $P$  on  $\mathcal{K}_n$  are denoted by  $P = (z, y)$ , where  $y(P)$  denotes the meromorphic function on  $\mathcal{K}_n$  satisfying  $\mathcal{F}_n(z, y) = y^2 - R_{2n+2}(z) = 0$ . Local coordinates near  $P_0 = (z_0, y_0) \in \mathcal{K}_n \setminus \{\mathcal{B}(\mathcal{K}_n) \cup \{P_{\infty_+}, P_{\infty_-}\}\}$  are given by  $\zeta_{P_0} = z - z_0$ , near  $P_{\infty_{\pm}}$  by  $\zeta_{P_{\infty_{\pm}}} = 1/z$ , and near branch points  $(E_{m_0}, 0) \in \mathcal{B}(\mathcal{K}_n)$  by  $\zeta_{(E_{m_0}, 0)} = (z - E_{m_0})^{1/2}$ . The Riemann surface  $\mathcal{K}_n$  defined in this manner has topological genus  $n$ . Moreover, we introduce the holomorphic sheet exchange map (involution)

$$(A.8) \quad *: \mathcal{K}_n \rightarrow \mathcal{K}_n, \quad P = (z, y) \mapsto P^* = (z, -y), \quad P_{\infty_{\pm}} \mapsto P_{\infty_{\pm}}^* = P_{\infty_{\mp}}$$

One verifies that  $dz/y$  is a holomorphic differential on  $\mathcal{K}_n$  with zeros of order  $n - 1$  at  $P_{\infty_{\pm}}$  and hence

$$(A.9) \quad \eta_j = \frac{z^{j-1} dz}{y}, \quad j = 1, \dots, n$$

form a basis for the space of holomorphic differentials on  $\mathcal{K}_n$ . Introducing the invertible matrix  $C$  in  $\mathbb{C}^n$ ,

$$(A.10) \quad C = (C_{j,k})_{j,k=1,\dots,n}, \quad C_{j,k} = \int_{a_k} \eta_j,$$

$$(A.11) \quad \underline{c}(k) = (c_1(k), \dots, c_n(k)), \quad c_j(k) = C_{j,k}^{-1}, \quad j, k = 1, \dots, n,$$

the corresponding basis of normalized holomorphic differentials  $\omega_j$ ,  $j = 1, \dots, n$  on  $\mathcal{K}_n$  is given by

$$(A.12) \quad \omega_j = \sum_{\ell=1}^n c_j(\ell) \eta_\ell, \quad \int_{a_k} \omega_j = \delta_{j,k}, \quad j, k = 1, \dots, n.$$

Here  $\{a_j, b_j\}_{j=1,\dots,n}$  is a homology basis for  $\mathcal{K}_n$  with intersection matrix of the cycles satisfying

$$(A.13) \quad a_j \circ b_k = \delta_{j,k}, \quad j, k = 1, \dots, n.$$

Near  $P_{\infty\pm}$  one infers

$$(A.14) \quad \underline{\omega} = (\omega_1, \dots, \omega_n) = \pm \left( \sum_{j=1}^n \frac{\underline{c}(j) \zeta^{n-j}}{(\prod_{m=0}^{2n+1} (1 - E_m \zeta))^{1/2}} \right) d\zeta$$

$$\underset{\zeta \rightarrow 0}{=} \pm \left( \underline{c}(n) + \left( \frac{1}{2} \underline{c}(n) \sum_{m=0}^{2n+1} E_m + \underline{c}(n-1) \right) \zeta + O(\zeta^2) \right) d\zeta$$

as  $P \rightarrow P_{\infty\pm}$ ,  $\zeta = 1/z$ ,

and

$$(A.15) \quad y(P) \underset{\zeta \rightarrow 0}{=} \mp \left( 1 - \frac{1}{2} \left( \sum_{m=0}^{2n+1} E_m \right) \zeta + O(\zeta^2) \right) \zeta^{-n-1} \text{ as } P \rightarrow P_{\infty\pm}.$$

Similarly, near  $P_0$  one computes

$$(A.16) \quad \underline{\omega} \underset{\zeta \rightarrow 0}{=} -2i (\tilde{Q}^{-1/2} \underline{c}(1) + O(\zeta^2)) d\zeta \text{ as } P \rightarrow P_0,$$

$$\tilde{Q}^{1/2} = \left( \prod_{m=1}^{2n+1} E_m \right)^{1/2}, \quad \zeta = \sigma z^{1/2}, \quad \sigma \in \{1, -1\},$$

using

$$(A.17) \quad y(P) \underset{\zeta \rightarrow 0}{=} i \tilde{Q}^{1/2} \zeta + O(\zeta^3) \text{ as } P \rightarrow P_0, \quad \zeta = \sigma z^{1/2}, \quad \sigma \in \{1, -1\},$$

with the sign of  $\tilde{Q}^{1/2}$  determined by the compatibility of charts.

Associated with the homology basis  $\{a_j, b_j\}_{j=1, \dots, n}$  we also recall the canonical dissection of  $\mathcal{K}_n$  along its cycles yielding the simply connected interior  $\widehat{\mathcal{K}}_n$  of the fundamental polygon  $\partial\widehat{\mathcal{K}}_n$  given by

$$(A.18) \quad \partial\widehat{\mathcal{K}}_n = a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_n^{-1} b_n^{-1}.$$

Let  $\mathcal{M}(\mathcal{K}_n)$  and  $\mathcal{M}^1(\mathcal{K}_n)$  denote the set of meromorphic functions (0-forms) and meromorphic differentials (1-forms) on  $\mathcal{K}_n$ . The residue of a meromorphic differential  $\nu \in \mathcal{M}^1(\mathcal{K}_n)$  at a point  $Q \in \mathcal{K}_n$  is defined by

$$(A.19) \quad \text{res}_Q(\nu) = \frac{1}{2\pi i} \int_{\gamma_Q} \nu,$$

where  $\gamma_Q$  is a counterclockwise oriented smooth simple closed contour encircling  $Q$  but no other pole of  $\nu$ .

Holomorphic differentials are also called Abelian differentials of the first kind (dfk). Abelian differentials of the second kind (dsk)  $\omega^{(2)} \in \mathcal{M}^1(\mathcal{K}_n)$  are characterized by the property that all their residues vanish. They will usually be normalized by demanding that all their  $a$ -periods vanish, that is,

$$(A.20) \quad \int_{a_j} \omega^{(2)} = 0, \quad j = 1, \dots, n.$$

If  $\omega_{P_1, n}^{(2)}$  is a dsk on  $\mathcal{K}_n$  whose only pole is  $P_1 \in \widehat{\mathcal{K}}_n$  with principal part  $\zeta^{-n-2} d\zeta$ ,  $n \in \mathbb{N}_0$  near  $P_1$  and  $\omega_j = (\sum_{m=0}^{\infty} d_{j,m}(P_1) \zeta^m) d\zeta$  near  $P_1$ , then

$$(A.21) \quad \frac{1}{2\pi i} \int_{b_j} \omega_{P_1, n}^{(2)} = \frac{d_{j,m}(P_1)}{m+1}, \quad m = 0, 1, \dots$$

Any meromorphic differential  $\omega^{(3)}$  on  $\mathcal{K}_n$  not of the first or second kind is said to be of the third kind (dtk). A dtk  $\omega^{(3)} \in \mathcal{M}^1(\mathcal{K}_n)$  is usually normalized by the vanishing of its  $a$ -periods, that is,

$$(A.22) \quad \int_{a_j} \omega^{(3)} = 0, \quad j = 1, \dots, n.$$

A normal dtk  $\omega_{P_1, P_2}^{(3)}$  associated with two points  $P_1, P_2 \in \widehat{\mathcal{K}}_n$ ,  $P_1 \neq P_2$  by definition has simple poles at  $P_j$  with residues  $(-1)^{j+1}$ ,  $j = 1, 2$  and vanishing  $a$ -periods. If  $\omega_{P, Q}^{(3)}$  is a normal dtk associated with  $P, Q \in \widehat{\mathcal{K}}_n$ , holomorphic on  $\mathcal{K}_n \setminus \{P, Q\}$ , then

$$(A.23) \quad \frac{1}{2\pi i} \int_{b_j} \omega_{P, Q}^{(3)} = \int_Q^P \omega_j, \quad j = 1, \dots, n,$$

where the path from  $Q$  to  $P$  lies in  $\widehat{\mathcal{K}}_n$  (i.e., does not touch any of the cycles  $a_j, b_j$ ).

Explicitly, one obtains

$$(A.24) \quad \omega_{P_{\infty+}, P_{\infty-}}^{(3)} = \frac{z^n dz}{y} + \sum_{j=1}^n \gamma_j \omega_j = \frac{\prod_{j=1}^n (z - \lambda_j) dz}{y},$$

$$(A.25) \quad \omega_{P_1, P_{\infty+}}^{(3)} = \frac{1}{2} \frac{y + y_1}{z - z_1} \frac{dz}{y} - \frac{\prod_{j=1}^n (z - \tilde{\lambda}_j) dz}{2y},$$

$$(A.26) \quad \omega_{P_1, P_{\infty-}}^{(3)} = \frac{1}{2} \frac{y + y_1}{z - z_1} \frac{dz}{y} + \frac{\prod_{j=1}^n (z - \lambda'_j) dz}{2y},$$

$$(A.27) \quad \omega_{P_1, P_2}^{(3)} = \left( \frac{y + y_1}{z - z_1} - \frac{y + y_2}{z - z_2} \right) \frac{dz}{2y} + \lambda''_n \frac{\prod_{j=1}^{n-1} (z - \lambda''_j) dz}{y},$$

$P_1, P_2 \in \mathcal{K}_n \setminus \{P_{\infty+}, P_{\infty-}\},$

where  $\gamma_j, \lambda_j, \tilde{\lambda}_j, \lambda'_j, \lambda''_j \in \mathbb{C}$ ,  $j = 1, \dots, n$ , are uniquely determined by the requirement of vanishing  $a$ -periods and we abbreviated  $P_j = (z_j, y_j)$ ,  $j = 1, 2$ . (If  $n = 0$ , we use the standard convention that the product over an empty index set is replaced by 1.) We shall always assume (without loss of generality) that all poles of  $\text{dsk}$ 's and  $\text{dtk}$ 's on  $\mathcal{K}_n$  lie on  $\widehat{\mathcal{K}}_n$  (i.e., not on  $\partial\widehat{\mathcal{K}}_n$ ). Define the matrix  $\tau = (\tau_{j,\ell})_{j,\ell=1,\dots,n}$  by

$$(A.28) \quad \tau_{j,\ell} = \int_{b_j} \omega_\ell, \quad j, \ell = 1, \dots, n.$$

Then

$$(A.29) \quad \text{Im}(\tau) > 0, \quad \text{and} \quad \tau_{j,\ell} = \tau_{\ell,j}, \quad j, \ell = 1, \dots, n.$$

Associated with  $\tau$  one introduces the period lattice

$$(A.30) \quad L_n = \{z \in \mathbb{C}^n \mid z = \underline{m} + \tau \underline{n}, \underline{m}, \underline{n} \in \mathbb{Z}^n\}$$

and the Riemann theta function associated with  $\mathcal{K}_n$  and the given homology basis  $\{a_j, b_j\}_{j=1,\dots,n}$ ,

$$(A.31) \quad \theta(\underline{z}) = \sum_{\underline{n} \in \mathbb{Z}^n} \exp(2\pi i(\underline{n}, \underline{z}) + \pi i(\underline{n}, \tau \underline{n})), \quad \underline{z} \in \mathbb{C}^n,$$

where  $(\underline{u}, \underline{v}) = \sum_{j=1}^n \bar{u}_j v_j$  denotes the scalar product in  $\mathbb{C}^n$ . It has the fundamental properties

$$(A.32) \quad \theta(z_1, \dots, z_{j-1}, -z_j, z_{j+1}, \dots, z_n) = \theta(\underline{z}),$$

$$(A.33) \quad \theta(\underline{z} + \underline{m} + \tau \underline{n}) = \exp(-2\pi i(\underline{n}, \underline{z}) - \pi i(\underline{n}, \tau \underline{n})) \theta(\underline{z}), \quad \underline{m}, \underline{n} \in \mathbb{Z}^n.$$

Next, fix a base point  $Q_0 \in \mathcal{K}_n \setminus \{P_{0\pm}, P_{\infty\pm}\}$ , denote by  $J(\mathcal{K}_n) = \mathbb{C}^n/L_n$  the Jacobi variety of  $\mathcal{K}_n$ , and define the Abel map  $\underline{A}_{Q_0}$  by

$$(A.34) \quad \underline{A}_{Q_0}: \mathcal{K}_n \rightarrow J(\mathcal{K}_n), \quad \underline{A}_{Q_0}(P) = \left( \int_{Q_0}^P \omega_1, \dots, \int_{Q_0}^P \omega_n \right) \pmod{L_n}, \quad P \in \mathcal{K}_n.$$

Similarly, we introduce

$$(A.35) \quad \underline{\alpha}_{Q_0}: \text{Div}(\mathcal{K}_n) \rightarrow J(\mathcal{K}_n), \quad \mathcal{D} \mapsto \underline{\alpha}_{Q_0}(\mathcal{D}) = \sum_{P \in \mathcal{K}_n} \mathcal{D}(P) \underline{A}_{Q_0}(P),$$

where  $\text{Div}(\mathcal{K}_n)$  denotes the set of divisors on  $\mathcal{K}_n$ . Here  $\mathcal{D}: \mathcal{K}_n \rightarrow \mathbb{Z}$  is called a divisor on  $\mathcal{K}_n$  if  $\mathcal{D}(P) \neq 0$  for only finitely many  $P \in \mathcal{K}_n$ . (In the main body of this paper we will choose  $Q_0$  to be one of the branch points, i.e.,  $Q_0 \in \mathcal{B}(\mathcal{K}_n)$ , and for simplicity we will always choose the same path of integration from  $Q_0$  to  $P$  in all Abelian integrals.) For subsequent use in Remark A.7 we also introduce

$$(A.36) \quad \widehat{\underline{A}}_{Q_0}: \widehat{\mathcal{K}}_n \rightarrow \mathbb{C}^n, \\ P \mapsto \widehat{\underline{A}}_{Q_0}(P) = (\widehat{A}_{Q_0,1}(P), \dots, \widehat{A}_{Q_0,n}(P)) = \left( \int_{Q_0}^P \omega_1, \dots, \int_{Q_0}^P \omega_n \right)$$

and

$$(A.37) \quad \widehat{\underline{\alpha}}_{Q_0}: \text{Div}(\widehat{\mathcal{K}}_n) \rightarrow \mathbb{C}^n, \quad \mathcal{D} \mapsto \widehat{\underline{\alpha}}_{Q_0}(\mathcal{D}) = \sum_{P \in \widehat{\mathcal{K}}_n} \mathcal{D}(P) \widehat{\underline{A}}_{Q_0}(P).$$

In connection with divisors on  $\mathcal{K}_n$  we shall employ the following (additive) notation,

$$(A.38) \quad \mathcal{D}_{Q_0\underline{Q}} = \mathcal{D}_{Q_0} + \mathcal{D}_{\underline{Q}}, \quad \mathcal{D}_{\underline{Q}} = \mathcal{D}_{Q_1} + \dots + \mathcal{D}_{Q_m}, \\ \underline{Q} = \{Q_1, \dots, Q_m\} \in \sigma^m \mathcal{K}_n, \quad Q_0 \in \mathcal{K}_n, \quad m \in \mathbb{N},$$

where for any  $Q \in \mathcal{K}_n$ ,

$$(A.39) \quad \mathcal{D}_Q: \mathcal{K}_n \rightarrow \mathbb{N}_0, \quad P \mapsto \mathcal{D}_Q(P) = \begin{cases} 1 & \text{for } P = Q, \\ 0 & \text{for } P \in \mathcal{K}_n \setminus \{Q\}, \end{cases}$$

and  $\sigma^m \mathcal{K}_n$  denotes the  $m$ th symmetric product of  $\mathcal{K}_n$ . In particular,  $\sigma^m \mathcal{K}_n$  can be identified with the set of nonnegative divisors  $0 \leq \mathcal{D} \in \text{Div}(\mathcal{K}_n)$  of degree  $m \in \mathbb{N}$ .

For  $f \in \mathcal{M}(\mathcal{K}_n) \setminus \{0\}$ ,  $\omega \in \mathcal{M}^1(\mathcal{K}_n) \setminus \{0\}$  the divisors of  $f$  and  $\omega$  are denoted by  $(f)$  and  $(\omega)$ , respectively. Two divisors  $\mathcal{D}, \mathcal{E} \in \text{Div}(\mathcal{K}_n)$  are called equivalent, denoted by  $\mathcal{D} \sim \mathcal{E}$ , if and only if  $\mathcal{D} - \mathcal{E} = (f)$  for some  $f \in \mathcal{M}(\mathcal{K}_n) \setminus \{0\}$ . The divisor class  $[\mathcal{D}]$  of  $\mathcal{D}$  is then given by  $[\mathcal{D}] = \{\mathcal{E} \in \text{Div}(\mathcal{K}_n) \mid \mathcal{E} \sim \mathcal{D}\}$ . We recall that

$$(A.40) \quad \begin{aligned} \deg((f)) &= 0, \quad \deg((\omega)) = 2(n-1), \\ f &\in \mathcal{M}(\mathcal{K}_n) \setminus \{0\}, \quad \omega \in \mathcal{M}^1(\mathcal{K}_n) \setminus \{0\}, \end{aligned}$$

where the degree  $\deg(\mathcal{D})$  of  $\mathcal{D}$  is given by  $\deg(\mathcal{D}) = \sum_{P \in \mathcal{K}_n} \mathcal{D}(P)$ . It is customary to call  $(f)$  (respectively,  $(\omega)$ ) a principal (respectively, canonical) divisor.

Introducing the complex linear spaces

$$(A.41) \quad \mathcal{L}(\mathcal{D}) = \{f \in \mathcal{M}(\mathcal{K}_n) \mid f = 0 \text{ or } (f) \geq \mathcal{D}\}, \quad r(\mathcal{D}) = \dim_{\mathbb{C}} \mathcal{L}(\mathcal{D}),$$

$$(A.42) \quad \mathcal{L}^1(\mathcal{D}) = \{\omega \in \mathcal{M}^1(\mathcal{K}_n) \mid \omega = 0 \text{ or } (\omega) \geq \mathcal{D}\}, \quad i(\mathcal{D}) = \dim_{\mathbb{C}} \mathcal{L}^1(\mathcal{D}),$$

( $i(\mathcal{D})$  the index of specialty of  $\mathcal{D}$ ) one infers that  $\deg(\mathcal{D})$ ,  $r(\mathcal{D})$ , and  $i(\mathcal{D})$  only depend on the divisor class  $[\mathcal{D}]$  of  $\mathcal{D}$ . Moreover, we recall the following fundamental facts.

**Theorem A.1** *Let  $\mathcal{D} \in \text{Div}(\mathcal{K}_n)$ ,  $\omega \in \mathcal{M}^1(\mathcal{K}_n) \setminus \{0\}$ . Then*

$$(A.43) \quad i(\mathcal{D}) = r(\mathcal{D} - (\omega)), \quad n \in \mathbb{N}_0.$$

*The Riemann-Roch theorem reads*

$$(A.44) \quad r(-\mathcal{D}) = \deg(\mathcal{D}) + i(\mathcal{D}) - n + 1, \quad n \in \mathbb{N}_0.$$

*By Abel's theorem,  $\mathcal{D} \in \text{Div}(\mathcal{K}_n)$ ,  $n \in \mathbb{N}$ , is principal if and only if*

$$(A.45) \quad \deg(\mathcal{D}) = 0 \text{ and } \underline{\alpha}_{Q_0}(\mathcal{D}) = \underline{0}.$$

*Finally, assume  $n \in \mathbb{N}$ . Then  $\underline{\alpha}_{Q_0} : \text{Div}(\mathcal{K}_n) \rightarrow J(\mathcal{K}_n)$  is surjective (Jacobi's inversion theorem).*

Next we introduce

$$(A.46) \quad \underline{W}_0 = \{0\} \subset J(\mathcal{K}_n), \quad \underline{W}_m = \underline{\alpha}_{Q_0}(\sigma^m \mathcal{K}_n), \quad m \in \mathbb{N}$$

and note that while  $\sigma^m \mathcal{K}_n \not\subset \sigma^n \mathcal{K}_n$  for  $m < n$ , one has  $\underline{W}_m \subseteq \underline{W}_n$  for  $m < n$ . Thus  $\underline{W}_m = J(\mathcal{K}_n)$  for  $m \geq n$  by Jacobi's inversion theorem.

Denote by  $\underline{\Xi}_{Q_0} = (\Xi_{Q_0,1}, \dots, \Xi_{Q_0,n})$  the vector of Riemann constants,

$$(A.47) \quad \Xi_{Q_0,j} = \frac{1}{2}(1 + \tau_{j,j}) - \sum_{\substack{\ell=1 \\ \ell \neq j}}^n \int_{a_\ell} \omega_\ell(P) \int_{Q_0}^P \omega_j, \quad j = 1, \dots, n.$$

**Theorem A.2** *The set  $\underline{W}_{n-1} + \underline{\Xi}_{Q_0} \subset J(\mathcal{K}_n)$  is the complete set of zeros of  $\theta$  on  $J(\mathcal{K}_n)$ , that is,*

$$(A.48) \quad \theta(X) = 0 \text{ if and only if } X \in \underline{W}_{n-1} + \underline{\Xi}_{Q_0}$$

(i.e., if and only if  $X = (\underline{\alpha}_{Q_0}(\mathcal{D}) + \underline{\Xi}_{Q_0}) \pmod{L_n}$  for some  $\mathcal{D} \in \sigma^{n-1}\mathcal{K}_n$ ). The set  $\underline{W}_{n-1} + \underline{\Xi}_{Q_0}$  has complex dimension  $n - 1$ .

**Theorem A.3** *Let  $\underline{\mathcal{D}}_Q \in \sigma^n \mathcal{K}_n$ ,  $\underline{Q} = \{Q_1, \dots, Q_n\}$ . Then*

$$(A.49) \quad 1 \leq i(\underline{\mathcal{D}}_Q) = s$$

if and only if there are  $s$  pairs of the type  $\{P, P^*\} \subseteq \{Q_1, \dots, Q_n\}$  (this includes, of course, branch points for which  $P = P^*$ ). Obviously, one has  $s \leq n/2$ .

**Remark A.4** While  $\theta(\underline{z})$  is well-defined (in fact, entire) for  $\underline{z} \in \mathbb{C}^n$ , it is not well-defined on  $J(\mathcal{K}_n) = \mathbb{C}^n/L_n$  because of (A.33). Nevertheless,  $\theta$  is a “multiplicative function” on  $J(\mathcal{K}_n)$  since the multipliers in (A.33) cannot vanish. In particular, if  $\underline{z}_1 = \underline{z}_2 \pmod{L_n}$ , then  $\theta(\underline{z}_1) = 0$  if and only if  $\theta(\underline{z}_2) = 0$ . Hence it is meaningful to state that  $\theta$  vanishes at points of  $J(\mathcal{K}_n)$ . Since the Abel map  $\underline{A}_{Q_0}$  maps  $\mathcal{K}_n$  into  $J(\mathcal{K}_n)$ , the function  $\theta(\underline{A}_{Q_0}(P) - \underline{\xi})$  for  $\underline{\xi} \in \mathbb{C}^n$ , becomes a multiplicative function on  $\mathcal{K}_n$ . Again it makes sense to say that  $\theta(\underline{A}_{Q_0}(\cdot) - \underline{\xi})$  vanishes at points of  $\mathcal{K}_n$ .

**Theorem A.5** *Let  $\underline{Q} = \{Q_1, \dots, Q_n\} \in \sigma^n \mathcal{K}_n$  and assume  $\underline{\mathcal{D}}_Q$  to be non-special, that is,  $i(\underline{\mathcal{D}}_Q) = 0$ . Then*

$$(A.50) \quad \theta(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P) + \alpha_{Q_0}(\underline{\mathcal{D}}_Q)) = 0 \text{ if and only if } P \in \{Q_1, \dots, Q_n\}.$$

**Theorem A.6** *Suppose  $\underline{\mathcal{D}}_{\hat{\mu}} \in \sigma^n \mathcal{K}_n$  is nonspecial,  $\hat{\mu} = \{\hat{\mu}_1, \dots, \hat{\mu}_n\}$ , and  $\hat{\mu}_{n+1} \in \mathcal{K}_n$  with  $\hat{\mu}_{n+1}^* \notin \{\hat{\mu}_1, \dots, \hat{\mu}_n\}$ . Let  $\{\hat{\lambda}_1, \dots, \hat{\lambda}_{n+1}\} \subset \mathcal{K}_n$  with  $\mathcal{D}_{\hat{\lambda}_{n+1}} \sim \mathcal{D}_{\hat{\mu}_{n+1}}$  (i.e.,  $\mathcal{D}_{\hat{\lambda}_{n+1}} \in [\mathcal{D}_{\hat{\mu}_{n+1}}]$ ). Then any  $n$  points  $\hat{\nu}_j \in \{\hat{\lambda}_1, \dots, \hat{\lambda}_{n+1}\}$ ,  $j = 1, \dots, n$  define a nonspecial divisor  $\underline{\mathcal{D}}_{\hat{\nu}} \in \sigma^n \mathcal{K}_n$ ,  $\hat{\nu} = \{\hat{\nu}_1, \dots, \hat{\nu}_n\}$ .*

**Proof:** Since  $i(\mathcal{D}_P) = 0$  for all  $P \in \mathcal{K}_1$ , there is nothing to prove in the special case  $n = 1$ . Hence we assume  $n \geq 2$ . Let  $Q_0 \in \mathcal{B}(\mathcal{K}_n)$  be a fixed branch point of  $\mathcal{K}_n$  and suppose that  $\underline{\mathcal{D}}_{\hat{\nu}}$  is special. Then by Theorem A.3 there is a pair  $\{\hat{\nu}, \hat{\nu}^*\} \subset \{\hat{\nu}_1, \dots, \hat{\nu}_n\}$  such that

$$(A.51) \quad \underline{\alpha}_{Q_0}(\underline{\mathcal{D}}_{\hat{\nu}}) = \underline{\alpha}_{Q_0}(\underline{\mathcal{D}}_{\hat{\nu}^*}),$$



where  $\underline{\hat{\nu}} = \{\hat{\nu}_1, \dots, \hat{\nu}_n\} \setminus \{\hat{\nu}, \hat{\nu}^*\} \in \sigma^{n-2}\mathcal{K}_n$ . Let  $\hat{\nu}_{n+1} \in \{\hat{\lambda}_1, \dots, \hat{\lambda}_{n+1}\} \setminus \{\hat{\nu}_1, \dots, \hat{\nu}_n\}$  so that  $\{\hat{\nu}_1, \dots, \hat{\nu}_{n+1}\} = \{\hat{\lambda}_1, \dots, \hat{\lambda}_{n+1}\} \subset \sigma^{n+1}\mathcal{K}_n$ . Then

$$(A.52) \quad \begin{aligned} \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}_{n+1}}) &= \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}_{n+1}}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\lambda}}_{n+1}}) \\ &= \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}_{n+1}}) = -\underline{A}_{Q_0}(\hat{\mu}_{n+1}^*) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}}), \end{aligned}$$

and hence by Theorem A.2 and (A.52),

$$(A.53) \quad 0 = \theta(\underline{\Xi}_{Q_0} + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}_{n+1}})) = \theta(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(\hat{\mu}_{n+1}^*) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}}).$$

Since by hypothesis  $\mathcal{D}_{\underline{\hat{\mu}}}$  is nonspecial and  $\hat{\mu}_{n+1}^* \notin \{\hat{\mu}_1, \dots, \hat{\mu}_n\}$ , (A.53) contradicts Theorem A.5. Thus,  $\mathcal{D}_{\underline{\hat{\nu}}}$  is nonspecial.  $\blacksquare$

**Remark A.7** In Sections 3 and 4 we frequently deal with theta function expressions of the type

$$(A.54) \quad \phi(P) = \frac{\theta(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P) + \underline{\alpha}_{Q_0}(\mathcal{D}_1))}{\theta(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P) + \underline{\alpha}_{Q_0}(\mathcal{D}_2))} \exp\left(\int_{Q_0}^P \omega_{Q_1, Q_2}^{(3)}\right), \quad P \in \mathcal{K}_n$$

and

$$(A.55) \quad \psi(P) = \frac{\theta(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P) + \underline{\alpha}_{Q_0}(\mathcal{D}_1))}{\theta(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P) + \underline{\alpha}_{Q_0}(\mathcal{D}_2))} \exp\left(-c \int_{Q_0}^P \Omega^{(2)}\right), \quad P \in \mathcal{K}_n,$$

where  $\mathcal{D}_j \in \sigma^n \mathcal{K}_n$ ,  $j = 1, 2$ , are nonspecial positive divisors of degree  $n$ ,  $c \in \mathbb{C}$  is a constant,  $Q_j \in \mathcal{K}_n \setminus \{P_{\infty_1}, \dots, P_{\infty_N}\}$ , where  $\{P_{\infty_1}, \dots, P_{\infty_N}\}$ ,  $N \in \mathbb{N}$ , denotes the set of points of  $\mathcal{K}_n$  at infinity,  $\omega_{Q_1, Q_2}^{(3)}$  is a normal differential of the third kind, and  $\Omega^{(2)}$  a normalized differential of the second kind with singularities contained in  $\{P_{\infty_1}, \dots, P_{\infty_N}\}$ . In particular, one has

$$(A.56) \quad \int_{a_j} \omega_{Q_1, Q_2}^{(3)} = \int_{a_j} \Omega^{(2)} = 0, \quad j = 1, \dots, n.$$

Even though we agree to always choose identical paths of integration from  $P_0$  to  $P$  in all Abelian integrals (A.54) and (A.55), this is not sufficient to render  $\phi$  and  $\psi$  single-valued on  $\mathcal{K}_n$ . To achieve single-valuedness, one needs to replace  $\mathcal{K}_n$  by its simply connected canonical dissection  $\widehat{\mathcal{K}}_n$  and then replace  $\underline{A}_{Q_0}$ ,  $\underline{\alpha}_{Q_0}$  in (A.54) and (A.55), with  $\widehat{\underline{A}}_{Q_0}$ ,  $\widehat{\underline{\alpha}}_{Q_0}$  as introduced in (A.36) and (A.37). In particular, one regards  $a_j, b_j$ ,  $j = 1, \dots, n$ , as curves (being a part of  $\partial\widehat{\mathcal{K}}_n$ , cf. (A.18)) and not as homology classes. Moreover, to render  $\phi$  single-valued on  $\widehat{\mathcal{K}}_n$  one needs to assume in addition that

$$(A.57) \quad \widehat{\underline{\alpha}}_{Q_0}(\mathcal{D}_1) - \widehat{\underline{\alpha}}_{Q_0}(\mathcal{D}_2) = 0$$

(as opposed to merely  $\underline{\alpha}_{Q_0}(\mathcal{D}_1) - \underline{\alpha}_{Q_0}(\mathcal{D}_2) = 0 \pmod{L_n}$ ). Similarly, in connection with  $\psi$ , one introduces the vector of  $b$ -periods  $\underline{U}^{(2)}$  of  $\Omega^{(2)}$  by

$$(A.58) \quad \underline{U}^{(2)} = (U_1^{(2)}, \dots, U_g^{(2)}), \quad U_j^{(2)} = \frac{1}{2\pi i} \int_{b_j} \Omega^{(2)}, \quad j = 1, \dots, n,$$

and then renders  $\psi$  single-valued on  $\widehat{\mathcal{K}}_n$  by requiring

$$(A.59) \quad \widehat{\underline{\alpha}}_{Q_0}(\mathcal{D}_1) - \widehat{\underline{\alpha}}_{Q_0}(\mathcal{D}_2) = c \underline{U}^{(2)}$$

(as opposed to merely  $\underline{\alpha}_{Q_0}(\mathcal{D}_1) - \underline{\alpha}_{Q_0}(\mathcal{D}_2) = c \underline{U}^{(2)} \pmod{L_n}$ ). These statements easily follow from (A.23) and (A.33) in the case of  $\phi$  and simply from (A.33) in the case of  $\psi$ . In fact, by (A.33),

$$(A.60) \quad \widehat{\underline{\alpha}}_{Q_0}(\mathcal{D}_1 + \mathcal{D}_{Q_1}) - \widehat{\underline{\alpha}}_{Q_0}(\mathcal{D}_2 + \mathcal{D}_{Q_2}) \in \mathbb{Z}^n,$$

respectively,

$$(A.61) \quad \widehat{\underline{\alpha}}_{Q_0}(\mathcal{D}_1) - \widehat{\underline{\alpha}}_{Q_0}(\mathcal{D}_2) - c \underline{U}^{(2)} \in \mathbb{Z}^n,$$

suffice to guarantee single-valuedness of  $\phi$ , respectively,  $\psi$  on  $\widehat{\mathcal{K}}_n$ . Without the replacement of  $\underline{A}_{Q_0}$  and  $\underline{\alpha}_{Q_0}$  by  $\widehat{\underline{A}}_{Q_0}$  and  $\widehat{\underline{\alpha}}_{Q_0}$  in (A.54) and (A.55) and without the assumptions (A.57) and (A.59) (or (A.60) and (A.61)),  $\phi$  and  $\psi$  are multiplicative (multi-valued) functions on  $\mathcal{K}_n$ , and then most effectively discussed by introducing the notion of characters on  $\mathcal{K}_n$  (cf. [36, Sect. III.9]). For simplicity, we decided to avoid the latter possibility and throughout this paper will tacitly always assume (A.57) and (A.59) without particularly emphasizing this convention each time it is used.

## Appendix B. High-Energy Expansions

In this appendix we study the relationship between the homogeneous coefficients  $\hat{f}_\ell$  and nonhomogeneous coefficients  $f_\ell$  of the polynomial  $F_n$ , discuss the high-energy expansion of  $F_n/y$ , and use it to derive a nonlinear recursion relation for  $\hat{f}_\ell$ ,  $\ell \in \mathbb{N}_0$ .

Let

$$(B.1) \quad \{E_m\}_{m=0, \dots, 2n+1} \text{ for some } n \in \mathbb{N}_0$$

$$(B.2) \quad \text{and } \eta \in \mathbb{C} \text{ such that } |\eta| < \min\{|E_0|^{-1}, \dots, |E_{2n+1}|^{-1}\}.$$

Then

$$(B.3) \quad \left( \prod_{m=0}^{2n+1} (1 - E_m \eta) \right)^{-1/2} = \sum_{k=0}^{\infty} \hat{c}_k(\underline{E}) \eta^k,$$

where

$$(B.4) \quad \hat{c}_0(\underline{E}) = 1,$$

$$\hat{c}_k(\underline{E}) = \sum_{\substack{j_0, \dots, j_{2n+1}=0 \\ j_0 + \dots + j_{2n+1} = k}}^k \frac{(2j_0 - 1)!! \cdots (2j_{2n+1} - 1)!!}{2^k j_0! \cdots j_{2n+1}!} E_0^{j_0} \cdots E_{2n+1}^{j_{2n+1}}, \quad k \in \mathbb{N}.$$

The first few coefficients explicitly read

$$(B.5) \quad \hat{c}_0(\underline{E}) = 1, \quad \hat{c}_1(\underline{E}) = \frac{1}{2} \sum_{m=0}^{2n+1} E_m,$$

$$\hat{c}_2(\underline{E}) = \frac{1}{4} \sum_{\substack{m_1, m_2=0 \\ m_1 < m_2}}^{2n+1} E_{m_1} E_{m_2} + \frac{3}{8} \sum_{m=0}^{2n+1} E_m^2, \quad \text{etc.}$$

Similarly,

$$(B.6) \quad \left( \prod_{m=0}^{2n+1} (1 - E_m \eta) \right)^{1/2} = \sum_{k=0}^{\infty} c_k(\underline{E}) \eta^k,$$

where

$$(B.7) \quad c_0(\underline{E}) = 1,$$

$$c_k(\underline{E}) = \sum_{\substack{j_0, \dots, j_{2n+1}=0 \\ j_0 + \dots + j_{2n+1} = k}}^k \frac{(2j_0 - 3)!! \cdots (2j_{2n+1} - 3)!!}{2^k j_0! \cdots j_{2n+1}!} E_0^{j_0} \cdots E_{2n+1}^{j_{2n+1}}, \quad k \in \mathbb{N}.$$

The first few coefficients explicitly are given by

$$(B.8) \quad c_0(\underline{E}) = 1, \quad c_1(\underline{E}) = -\frac{1}{2} \sum_{m=0}^{2n+1} E_m,$$

$$c_2(\underline{E}) = \frac{1}{4} \sum_{\substack{m_1, m_2=0 \\ m_1 < m_2}}^{2n+1} E_{m_1} E_{m_2} - \frac{1}{8} \sum_{m=0}^{2n+1} E_m^2, \quad \text{etc.}$$

Here we used the abbreviations

$$(B.9) \quad (2q - 1)!! = 1 \cdot 3 \cdots (2q - 1), \quad q \in \mathbb{N}, \quad (-3)!! = -1, \quad (-1)!! = 1.$$

**Theorem B.1** *Assume*

$$u \in C^\infty(\mathbb{R}), d^m u/dx^m \in L^\infty(\mathbb{R}), m \in \mathbb{N}_0, \text{s-CH}_n(u) = 0,$$

and suppose  $P = (z, y) \in \mathcal{K}_n \setminus \{P_{\infty+}, P_{\infty-}\}$ . Then  $F_n/y$  has the following convergent expansion as  $P \rightarrow P_{\infty\pm}$ ,

$$(B.10) \quad \frac{F_n(z)}{y} \underset{\zeta \rightarrow 0}{=} \mp \sum_{\ell=0}^{\infty} \hat{f}_\ell \zeta^{\ell+1},$$

with  $\zeta = 1/z$  the local coordinate near  $P_{\infty\pm}$  described in Appendix Appendix A and  $\hat{f}_\ell$  the homogeneous coefficients  $f_\ell$  in (2.10). In particular,  $\hat{f}_\ell$  can be computed from the nonlinear recursion relation

$$(B.11) \quad \hat{f}_0 = 1, \quad \hat{f}_1 = -2u,$$

$$\begin{aligned} \hat{f}_{\ell+1} = \mathcal{G} \left( \sum_{k=1}^{\ell} (\hat{f}_{\ell+1-k,xx} \hat{f}_k - \frac{1}{2} \hat{f}_{\ell+1-k,x} \hat{f}_{k,x} - 2 \hat{f}_{\ell+1-k} \hat{f}_k) \right. \\ \left. + 2(u_{xx} - 4u) \sum_{k=0}^{\ell} \hat{f}_{\ell-k} \hat{f}_k \right), \quad \ell \in \mathbb{N}, \end{aligned}$$

assuming

$$(B.12) \quad \hat{f}_\ell \in L^\infty(\mathbb{R}), \quad \ell \in \mathbb{N}.$$

Moreover, one infers for the  $E_m$ -dependence of the integration constants  $c_\ell$ ,  $\ell = 0, \dots, n$ , in  $F_n$ ,

$$(B.13) \quad c_\ell = c_\ell(\underline{E}), \quad \ell = 0, \dots, n$$

and

$$(B.14) \quad f_\ell = \sum_{k=0}^{\ell} c_{\ell-k}(\underline{E}) \hat{f}_k, \quad \ell = 0, \dots, n,$$

$$(B.15) \quad \hat{f}_\ell = \sum_{k=0}^{\ell} \hat{c}_{\ell-k}(\underline{E}) f_k, \quad \ell = 0, \dots, n.$$

**Proof:** Dividing  $F_n$  by  $R_{2n+2}^{1/2}$  (temporarily fixing the branch of  $R_{2n+2}(z)^{1/2}$  as  $z^{n+1}$  near infinity), one obtains

$$(B.16) \quad \frac{F_n(z)}{R_{2n+2}(z)^{1/2}} \underset{|z| \rightarrow \infty}{=} \left( \sum_{k=0}^{\infty} \hat{c}_k(\underline{E}) z^{-k} \right) \left( \sum_{\ell=0}^n f_\ell z^{-\ell-1} \right) = \sum_{\ell=0}^{\infty} \check{f}_\ell z^{-\ell-1}$$

for some coefficients  $\check{f}_\ell$  to be determined next. Dividing (2.41) by  $R_{2n+2}$ , and inserting the expansion (B.16) into the resulting equation then yields the recursion relation (B.11) (with  $\hat{f}_\ell$  replaced by  $\check{f}_\ell$ ). More precisely, for  $\check{f}_1$  one originally obtains the relation

$$(B.17) \quad -\check{f}_{1,xx} + 4\check{f}_1 = 2(u_{xx} - 4u), \quad \text{that is,} \quad \left( -\frac{d^2}{dx^2} + 4 \right) (\check{f}_1 + 2u) = 0.$$

Thus,

$$(B.18) \quad \check{f}_1(x) = -2u(x) + a_1 e^{2x} + b_1 e^{-2x}$$

for some  $a_1, b_1 \in \mathbb{C}$ , and hence the requirement  $\check{f}_1 \in L^\infty(\mathbb{R})$  then yields  $a_1 = b_1 = 0$ . The open sign of  $\check{f}_0$  has been chosen such that  $\check{f}_0 = \hat{f}_0 = 1$ .

For  $\ell \geq 2$  one obtains similarly

$$(B.19) \quad -\check{f}_{\ell+1,xx} + 4\check{f}_{\ell+1} = \left( \sum_{k=1}^{\ell} (\check{f}_{\ell+1-k,xx} \check{f}_k - \frac{1}{2} \check{f}_{\ell+1-k,x} \check{f}_{k,x} - 2\check{f}_{\ell+1-k} \check{f}_k) \right. \\ \left. + 2(u_{xx} - 4u) \sum_{k=0}^{\ell} \check{f}_{\ell-k} \check{f}_k \right), \quad \ell \geq 1,$$

and hence,

$$(B.20) \quad \check{f}_{\ell+1} = \mathcal{G} \left( \sum_{k=1}^{\ell} (\check{f}_{\ell+1-k,xx} \check{f}_k - \frac{1}{2} \check{f}_{\ell+1-k,x} \check{f}_{k,x} - 2\check{f}_{\ell+1-k} \check{f}_k) \right. \\ \left. + 2(u_{xx} - 4u) \sum_{k=0}^{\ell} \check{f}_{\ell-k} \check{f}_k \right) + a_{\ell+1} e^{2x} + b_{\ell+1} e^{-2x}, \quad \ell \geq 1$$

for some  $a_{\ell+1}, b_{\ell+1} \in \mathbb{C}$ . Again the requirement  $\check{f}_{\ell+1} \in L^\infty(\mathbb{R})$  then yields  $a_{\ell+1} = b_{\ell+1} = 0$ ,  $\ell \geq 1$ . Introducing  $\hat{f}_\ell$  by (2.10) with  $c_k = 0$ ,  $k \in \mathbb{N}$ , and  $\check{f}_\ell$  by (B.11), a straightforward computation shows that

$$(B.21) \quad \check{f}_{\ell,x} = \mathcal{G} \left( \sum_{k=1}^{\ell-1} (f_{\ell-k,xxx} - 4f_{\ell-k,x}) f_k - \sum_{k=0}^{\ell-1} 2(-2(u_{xx} - 4u) f_{\ell-k-1,x} \right. \\ \left. + (4u_x - u_{xxx}) f_{\ell-k-1}) \right) f_k \\ = \mathcal{G} \left( -\sum_{k=1}^{\ell-1} \mathcal{G}^{-1} f_{\ell-k,x} f_k + \sum_{k=0}^{\ell-1} (\mathcal{G}^{-1} f_{\ell-k,x}) f_k \right) \\ = \hat{f}_{\ell,x}, \quad \ell \in \mathbb{N}.$$

Hence,

$$(B.22) \quad \check{f}_\ell = \hat{f}_\ell + d_\ell, \quad \ell \in \mathbb{N}$$

for some constants  $d_\ell \in \mathbb{C}$ ,  $\ell \in \mathbb{N}$ . Since  $d_0 = d_1 = 0$  by inspection, we next proceed by induction on  $\ell$  and suppose that

$$(B.23) \quad d_k = 0 \text{ and hence } \check{f}_k = \hat{f}_k \text{ for } k = 0, \dots, \ell.$$

Thus, (B.11) and (B.22) imply

$$(B.24) \quad \check{f}_{\ell+1} = \mathcal{G}\{\dots\} = \hat{f}_{\ell+1} + d_{\ell+1},$$

where  $\{\dots\}$  denotes the expression on the right-hand side of (B.11) in terms of  $\check{f}_k = \hat{f}_k$ ,  $k = 0, \dots, \ell$ . Hence,

$$(B.25) \quad \{\dots\} - \hat{f}_{\ell+1} + \alpha_{\ell+1}e^{2x} + \beta_{\ell+1}e^{-2x} = \mathcal{G}^{-1}d_{\ell+1} = 4d_{\ell+1}$$

for some constants  $\alpha_{\ell+1}, \beta_{\ell+1} \in \mathbb{C}$ . Since  $\{\dots\} - \hat{f}_{\ell+1} \in L^\infty(\mathbb{R})$ , one concludes once more that  $\alpha_{\ell+1} = \beta_{\ell+1} = 0$ . Moreover, since  $\{\dots\} - \hat{f}_{\ell+1}$  contains no constants by construction, one concludes  $d_{\ell+1} = 0$  and hence

$$(B.26) \quad \check{f}_\ell = \hat{f}_\ell \text{ for all } \ell \in \mathbb{N}_0.$$

Thus, we proved

$$(B.27) \quad \frac{F_n(z)}{R_{2n+2}(z)^{1/2}} \Big|_{|z| \rightarrow \infty} = \left( \sum_{k=0}^{\infty} \hat{c}_k(\underline{E}) z^{-k} \right) \left( \sum_{\ell=0}^n f_\ell z^{-\ell-1} \right) = \sum_{\ell=0}^{\infty} \hat{f}_\ell z^{-\ell-1}$$

and hence (B.10). A comparison of coefficients in (B.27) then proves (B.15). Next, multiplying (B.3) and (B.6), a comparison of coefficients of  $z^{-k}$  yields

$$(B.28) \quad \sum_{\ell=0}^k \hat{c}_{k-\ell}(\underline{E}) c_\ell(\underline{E}) = \delta_{k,0}, \quad k \in \mathbb{N}_0.$$

Thus, one computes

$$(B.29) \quad \begin{aligned} \sum_{m=0}^{\ell} c_{\ell-m}(\underline{E}) \hat{f}_m &= \sum_{m=0}^{\ell} \sum_{k=0}^m c_{\ell-m}(\underline{E}) \hat{c}_{m-k}(\underline{E}) f_k \\ &= \sum_{k=0}^{\ell} \sum_{p=k}^{\ell} c_{\ell-p}(\underline{E}) \hat{c}_{p-k}(\underline{E}) f_k \\ &= \sum_{k=0}^{\ell} \left( \sum_{m=0}^{\ell-k} c_{\ell-k-m}(\underline{E}) \hat{c}_m(\underline{E}) \right) f_k \\ &= f_\ell, \quad \ell = 0, \dots, n, \end{aligned}$$

applying (B.28). Hence one obtains (B.14) and thus (B.13). ■

## Appendix C. Symmetric Functions and their Theta Function Representations

In this appendix we consider Dubrovin-type equations for auxiliary divisors  $\mathcal{D}_{\underline{\mu}}$  of degree  $n$  on  $\mathcal{K}_n$  and study in detail elementary symmetric functions associated with the projections  $\mu_j$  of  $\hat{\mu}_j$ ,  $j = 1, \dots, n$ . In addition to various applications of Lagrange interpolation formulas we derive explicit theta function representations of elementary symmetric functions of  $\mu_j$ ,  $j = 1, \dots, n$ . While some of the material of this appendix is classical, some parts are taken from [45] (cf. also [46, App. F and G]), and [56]. Proofs are only presented for results that do not appear to belong to the standard arsenal of the literature on hierarchies of soliton equations. Our principal results on theta function representations derived in Sections 3 and 4 are based on Theorem C.6. The results of this appendix apply to a variety of soliton equations and hence are of independent interest.

Assuming  $n \in \mathbb{N}$  to be fixed and introducing

$$(C.1) \quad \mathcal{S}_k = \{\underline{\ell} = (\ell_1, \dots, \ell_k) \in \mathbb{N}^k \mid \ell_1 < \dots < \ell_k \leq n\}, \quad 1 \leq k \leq n,$$

$$(C.2) \quad \mathcal{I}_k^{(j)} = \{\underline{\ell} = (\ell_1, \dots, \ell_k) \in \mathcal{S}_k \mid \ell_m \neq j\}, \quad 1 \leq k \leq n-1, \quad 1 \leq j \leq n,$$

one defines

$$(C.3) \quad \Psi_0(\underline{\mu}) = 1, \quad \Psi_k(\underline{\mu}) = (-1)^k \sum_{\underline{\ell} \in \mathcal{S}_k} \mu_{\ell_1} \cdots \mu_{\ell_k}, \quad 1 \leq k \leq n,$$

$$(C.4) \quad \Phi_0^{(j)}(\underline{\mu}) = 1, \quad \Phi_k^{(j)}(\underline{\mu}) = (-1)^k \sum_{\underline{\ell} \in \mathcal{I}_k^{(j)}} \mu_{\ell_1} \cdots \mu_{\ell_k}, \quad 1 \leq k \leq n-1,$$

$$\Phi_n^{(j)}(\underline{\mu}) = 0, \quad 1 \leq j \leq n,$$

where  $\underline{\mu} = (\mu_1, \dots, \mu_n) \in \mathbb{C}^n$ . Explicitly, one verifies

$$(C.5) \quad \Psi_1(\underline{\mu}) = -\sum_{\ell=1}^n \mu_\ell, \quad \Psi_2(\underline{\mu}) = \sum_{\substack{\ell_1, \ell_2=1 \\ \ell_1 < \ell_2}}^n \mu_{\ell_1} \mu_{\ell_2}, \quad \text{etc.},$$

$$(C.6) \quad \Phi_1^{(j)}(\underline{\mu}) = -\sum_{\substack{\ell=1 \\ \ell \neq j}}^n \mu_\ell, \quad \Phi_2^{(j)}(\underline{\mu}) = \sum_{\substack{\ell_1, \ell_2=1 \\ \ell_1, \ell_2 \neq j \\ \ell_1 < \ell_2}}^n \mu_{\ell_1} \mu_{\ell_2}, \quad \text{etc.}$$

Introducing

$$(C.7) \quad F_n(z) = \prod_{j=1}^n (z - \mu_j) = \sum_{\ell=0}^n f_{n-\ell} z^\ell = \sum_{\ell=0}^n \Psi_{n-\ell}(\underline{\mu}) z^\ell,$$

one infers  $(F'_n(z) = \partial F_n(z)/\partial z)$

$$(C.8) \quad F'_n(\mu_k) = \prod_{\substack{j=1 \\ j \neq k}}^n (\mu_k - \mu_j).$$

The general form of Lagrange's interpolation theorem (cf., e.g., [46, App. F], [60, App. E]) then reads as follows.

**Theorem C.1** *Assume that  $\mu_1, \dots, \mu_n$  are  $n$  distinct complex numbers. Then*

$$(C.9) \quad \sum_{j=1}^n \frac{\mu_j^{m-1}}{F'_n(\mu_j)} \Phi_k^{(j)}(\underline{\mu}) = \delta_{m,n-k} - \Psi_{k+1}(\underline{\mu}) \delta_{m,n+1},$$

$$m = 1, \dots, n+1, \quad k = 0, \dots, n-1.$$

The simplest Lagrange interpolation formula reads in the case  $k = 0$ ,

$$(C.10) \quad \sum_{j=1}^n \frac{\mu_j^{m-1}}{F'_n(\mu_j)} = \delta_{m,n}, \quad m = 1, \dots, n.$$

For use in the main text we also recall the following results.

**Lemma C.2** ([45], [46, App. F]) *Assume that  $\mu_1, \dots, \mu_n$  are  $n$  distinct complex numbers. Then*

$$(C.11) \quad (i) \quad \Psi_{k+1}(\underline{\mu}) + \mu_j \Phi_k^{(j)}(\underline{\mu}) = \Phi_{k+1}^{(j)}(\underline{\mu}), \quad k = 0, \dots, n-1,$$

$$j = 1, \dots, n.$$

$$(C.12) \quad (ii) \quad \sum_{\ell=0}^k \Psi_{k-\ell}(\underline{\mu}) \mu_j^\ell = \Phi_k^{(j)}(\underline{\mu}), \quad k = 0, \dots, n, \quad j = 1, \dots, n.$$

$$(C.13) \quad (iii) \quad \sum_{\ell=0}^{k-1} \Phi_{k-1-\ell}^{(j)}(\underline{\mu}) z^\ell = \frac{1}{z - \mu_j} \left( \sum_{\ell=0}^k \Psi_{k-\ell}(\underline{\mu}) z^\ell - \Phi_k^{(j)}(\underline{\mu}) \right),$$

$$k = 0, \dots, n, \quad j = 1, \dots, n.$$

Next, assuming  $\mu_j \neq \mu_{j'}$  for  $j \neq j'$ , introduce the  $n \times n$  matrix  $U_n(\underline{\mu})$  by

$$(C.14) \quad U_1(\underline{\mu}) = 1, \quad U_n(\underline{\mu}) = \left( \frac{\mu_k^{j-1}}{\prod_{\substack{m=1 \\ m \neq k}}^n (\mu_k - \mu_m)} \right)_{j,k=1}^n,$$

where  $\underline{\mu} = (\mu_1, \dots, \mu_n) \in \mathbb{C}^n$ .



**Lemma C.3** ([45], [46, App. G]) *Suppose  $\mu_j \in \mathbb{C}$ ,  $j = 1, \dots, n$ , are  $n$  distinct complex numbers. Then*

$$(C.15) \quad U_n(\underline{\mu})^{-1} = \left( \Phi_{n-k}^{(j)}(\underline{\mu}) \right)_{j,k=1}^n.$$

Next, we express  $f_\ell$ ,  $F_n(\mu_j)$ , and  $\widetilde{F}_r(\mu_j)$  in terms of elementary symmetric functions of  $\mu_1, \dots, \mu_n$ . We start with the homogeneous expressions denoted by  $\hat{f}_\ell$  and  $\widehat{F}_r(\mu_j)$ , where  $c_k = 0$ ,  $k = 0, \dots, \ell$  and  $\tilde{c}_s = 0$ ,  $s = 1, \dots, r$ . Let  $\hat{c}_k(\underline{E})$ ,  $k \in \mathbb{N}_0$ , be defined as in (B.4) and suppose  $r \in \mathbb{N}_0$ . Then, combining (B.15) and (C.7) one infers

$$(C.16) \quad \hat{f}_\ell = \sum_{k=0}^{\ell} \hat{c}_{\ell-k}(\underline{E}) \Psi_k(\underline{\mu}), \quad \ell = 0, \dots, n.$$

Next, we turn to  $\widehat{F}_r(\mu_j)$ .

**Lemma C.4** *Let  $r \in \mathbb{N}_0$ . Then<sup>3</sup>,*

$$(C.17) \quad \widehat{F}_r(\mu_j) = \sum_{s=(r-n) \vee 0}^r \hat{c}_s(\underline{E}) \Phi_{r-s}^{(j)}(\underline{\mu}).$$

**Proof:** By definition

$$(C.18) \quad \widehat{F}_r(z) = \sum_{\ell=0}^r \hat{f}_{r-\ell} z^\ell = \sum_{\ell=0}^r z^\ell \sum_{m=0}^{(r-\ell) \wedge n} \Psi_m(\underline{\mu}) \hat{c}_{r-\ell-m}(\underline{E}).$$

Consider first the case  $r \leq n$ . Then

$$(C.19) \quad \widehat{F}_r(z) = \sum_{s=0}^r \hat{c}_s(\underline{E}) \sum_{\ell=0}^{r-s} \Psi_{r-\ell-s}(\underline{\mu}) z^\ell$$

and hence

$$(C.20) \quad \widehat{F}_r(\mu_j) = \sum_{s=0}^r \hat{c}_s(\underline{E}) \Phi_{r-s}^{(j)}(\underline{\mu}),$$

using (C.12).

---

<sup>3</sup> $m \vee n = \max\{m, n\}$ .

In the case where  $r \geq n + 1$  we find (applying (C.7))

$$\begin{aligned}
(C.21) \quad \widehat{F}_r(z) &= \sum_{m=0}^n \Psi_m(\underline{\mu}) \sum_{s=0}^{r-m} z^{r-m-s} \widehat{c}_s(\underline{E}) \\
&= \sum_{s=0}^{r-n} \widehat{c}_s(\underline{E}) \left( \sum_{\ell=0}^n \Psi_\ell(\underline{\mu}) z^{n-\ell} \right) z^{r-n-s} \\
&\quad + \sum_{s=r-n+1}^r \widehat{c}_s(\underline{E}) \sum_{\ell=0}^{r-s} \Psi_\ell(\underline{\mu}) z^{r-s-\ell} \\
&= F_n(z) \sum_{s=0}^{r-n} \widehat{c}_s(\underline{E}) z^{r-n-s} + \sum_{s=r-n+1}^r \widehat{c}_s(\underline{E}) \sum_{\ell=0}^{r-s} \Psi_\ell(\underline{\mu}) z^{r-s-\ell} \\
&= F_n(z) \sum_{s=0}^{r-n} \widehat{c}_s(\underline{E}) z^{r-n-s} + \sum_{s=r-n+1}^r \widehat{c}_s(\underline{E}) \sum_{\ell=0}^{r-s} \Psi_{r-s-\ell}(\underline{\mu}) z^\ell.
\end{aligned}$$

Hence

$$(C.22) \quad \widehat{F}_r(\mu_j) = \sum_{s=r-n+1}^r \widehat{c}_s(\underline{E}) \Phi_{r-s}^{(j)}(\underline{\mu}),$$

using (C.12) again. ■

Introducing

$$(C.23) \quad d_{\ell,k}(\underline{E}) = \sum_{m=0}^{\ell-k} c_{\ell-k-m}(\underline{E}) \widehat{c}_m(\underline{E}), \quad k = 0, \dots, \ell, \quad \ell = 0, \dots, n,$$

$$(C.24) \quad \widetilde{d}_{r,k}(\underline{E}) = \sum_{s=0}^{r-k} \widetilde{c}_{r-k-s} \widehat{c}_s(\underline{E}), \quad k = 0, \dots, r \wedge n,$$

for a given set of constants  $\{\widetilde{c}_s\}_{s=1, \dots, r} \subset \mathbb{C}$ , the corresponding nonhomogeneous quantities  $f_\ell$ ,  $F_n(\mu_j)$ , and  $\widetilde{F}_r(\mu_j)$  are then given by<sup>4</sup>

$$(C.25) \quad f_\ell = \sum_{k=0}^{\ell} c_{\ell-k}(\underline{E}) \widehat{f}_k = \sum_{k=0}^{\ell} d_{\ell,k}(\underline{E}) \Psi_k(\underline{\mu}), \quad \ell = 0, \dots, n,$$

$$(C.26) \quad F_n(\mu_j) = \sum_{\ell=0}^n c_{n-\ell}(\underline{E}) \widehat{F}_\ell(\mu_j) = \sum_{\ell=0}^n d_{n,\ell}(\underline{E}) \Phi_\ell^{(j)}(\underline{\mu}), \quad c_0 = 1,$$

$$(C.27) \quad \widetilde{F}_r(\mu_j) = \sum_{s=0}^r \widetilde{c}_{r-s} \widehat{F}_s(\mu_j) = \sum_{k=0}^{r \wedge n} \widetilde{d}_{r,k}(\underline{E}) \Phi_k^{(j)}(\underline{\mu}), \quad r \in \mathbb{N}_0, \quad \widetilde{c}_0 = 1,$$

using (B.13) and (B.14). Here  $c_k(\underline{E})$ ,  $k \in \mathbb{N}_0$ , is defined by (B.7).

---

<sup>4</sup> $m \wedge n = \min\{m, n\}$ .

Next, we prove a result needed in the proof of Theorem 4.10.

**Lemma C.5** *Suppose  $r \in \mathbb{N}_0$ ,  $(x, t_r) \in \Omega_\mu$ , where  $\Omega_\mu \subseteq \mathbb{R}^2$  is open and connected, and assume  $\mu_j \neq \mu_{j'}$  on  $\Omega_\mu$  for  $j \neq j'$ ,  $j, j' = 1, \dots, n$ . Then,*

$$(C.28) \quad \tilde{F}_{r,x}(z, x, t_r) = \sum_{j=1}^n \left( \tilde{F}_r(\mu_j(x, t_r), x, t_r) - \tilde{F}_r(z, x, t_r) \right) \frac{\mu_{j,x}(x, t_r)}{(z - \mu_j(x, t_r))}.$$

**Proof:** It suffices to prove (C.28) for the homogeneous case where  $\tilde{F}_r$  is replaced by  $\hat{F}_r$ . Using

$$(C.29) \quad \Psi_{k,x}(\underline{\mu}) = - \sum_{j=1}^n \mu_{j,x} \Phi_{k-1}^{(j)}(\underline{\mu}), \quad k = 0, \dots, n,$$

with the convention

$$(C.30) \quad \Phi_{-1}^{(j)}(\underline{\mu}) = 0, \quad j = 1, \dots, n,$$

one computes for  $r \leq n$ ,

$$(C.31) \quad \begin{aligned} \hat{F}_{r,x}(z) &= \sum_{s=0}^r \hat{c}_s(\underline{E}) \sum_{\ell=0}^{r-s} \Psi_{r-s-\ell,x}(\underline{\mu}) z^\ell \\ &= - \sum_{j=1}^n \mu_{j,x} \sum_{s=0}^r \hat{c}_s(\underline{E}) \sum_{\ell=0}^{r-s} \Phi_{r-s-\ell-1}^{(j)}(\underline{\mu}) z^\ell \\ &= \sum_{j=1}^n \mu_{j,x} (z - \mu_j)^{-1} \sum_{s=0}^r \hat{c}_s(\underline{E}) \left( \Phi_{r-s}^{(j)}(\underline{\mu}) - \sum_{\ell=0}^{r-s} \Psi_{r-s-\ell}(\underline{\mu}) z^\ell \right) \\ &= \sum_{j=1}^n \left( \hat{F}_r(\mu_j) - \hat{F}_r(z) \right) \mu_{j,x} (z - \mu_j)^{-1}, \end{aligned}$$

applying (C.13), (C.19), and (C.20). For  $r \geq n+1$  one obtains from (C.13), (C.21), and (C.22),

$$(C.32) \quad \begin{aligned} \hat{F}_{r,x}(z) &= F_{n,x}(z) \sum_{s=0}^{r-n} \hat{c}_s(\underline{E}) z^{r-n-s} + \sum_{s=r-n+1}^r \hat{c}_s(\underline{E}) \sum_{\ell=0}^{r-s} \Psi_{r-s-\ell,x}(\underline{\mu}) z^\ell \\ &= -F_n(z) \sum_{j=1}^n \mu_{j,x} (z - \mu_j)^{-1} \sum_{s=0}^{r-n} \hat{c}_s(\underline{E}) z^{r-n-s} \\ &\quad - \sum_{j=1}^n \mu_{j,x} \sum_{s=r-n+1}^r \hat{c}_s(\underline{E}) \sum_{\ell=0}^{r-s} \Phi_{r-s-\ell-1}^{(j)}(\underline{\mu}) z^\ell \end{aligned}$$

$$\begin{aligned}
&= -F_n(z) \sum_{j=1}^n \mu_{j,x}(z - \mu_j)^{-1} \sum_{s=0}^{r-n} \hat{c}_s(\underline{E}) z^{r-n-s} \\
&\quad + \sum_{j=1}^n \mu_{j,x}(z - \mu_j)^{-1} \sum_{s=r-n+1}^r \hat{c}_s(\underline{E}) \left( \Phi_{r-s}^{(j)}(\underline{\mu}) - \sum_{\ell=0}^{r-s} \Psi_{r-s-\ell}(\underline{\mu}) z^\ell \right) \\
&= \sum_{j=1}^n \left( \hat{F}_r(\mu_j) - \hat{F}_r(\underline{\mu}) \right) \mu_{j,x}(z - \mu_j)^{-1}.
\end{aligned}$$

■

Next we turn to a detailed discussion of elementary symmetric functions of  $\{\mu_1, \dots, \mu_n\}$ . Given the nonsingular hyperelliptic curve  $\mathcal{K}_n$  in (A.1), (A.2), we introduce the first-order Dubrovin-type system

$$\begin{aligned}
\text{(C.33)} \quad \frac{\partial \mu_j(\underline{v})}{\partial v_k} &= \Phi_{n-k}^{(j)}(\underline{\mu}(\underline{v})) \frac{y(\hat{\mu}_j(\underline{v}))}{\prod_{\substack{m=1 \\ m \neq j}}^n (\mu_j(\underline{v}) - \mu_m(\underline{v}))}, \\
j, k &= 1, \dots, n, \quad \underline{v} = (v_1, \dots, v_n) \in \mathcal{V},
\end{aligned}$$

with initial conditions

$$\text{(C.34)} \quad \{\hat{\mu}_j(\underline{v}_0)\}_{j=1, \dots, n} \subset \mathcal{K}_n$$

for some  $\underline{v}_0 \in \mathcal{V}$ , where  $\mathcal{V} \subseteq \mathbb{C}^n$  is an open connected set such that  $\mu_j$  remain distinct on  $\mathcal{V}$ ,  $\mu_j \neq \mu_{j'}$  for  $j \neq j'$ ,  $j, j' = 1, \dots, n$ . One then obtains, using (C.33) and (C.9),

$$\begin{aligned}
\text{(C.35)} \quad \frac{\partial}{\partial v_k} \sum_{j=1}^n \int_{Q_0}^{\hat{\mu}_j(\underline{v})} \frac{z^{k-1} dz}{y} &= \sum_{j=1}^n \frac{\mu_j(\underline{v})^{k-1}}{y(\hat{\mu}_j(\underline{v}))} \frac{\partial \mu_j(\underline{v})}{\partial v_k} \\
&= \sum_{j=1}^n \frac{\mu_j(\underline{v})^{k-1}}{y(\hat{\mu}_j(\underline{v}))} \Phi_{n-k}^{(j)}(\underline{\mu}(\underline{v})) \frac{y(\hat{\mu}_j(\underline{v}))}{\prod_{\substack{m=1 \\ m \neq j}}^n (\mu_j(\underline{v}) - \mu_m(\underline{v}))} \\
&= \sum_{j=1}^n \Phi_{n-k}^{(j)}(\underline{\mu}(\underline{v})) \frac{\mu_j(\underline{v})^{k-1}}{\prod_{\substack{m=1 \\ m \neq j}}^n (\mu_j(\underline{v}) - \mu_m(\underline{v}))} = 1,
\end{aligned}$$

implying

$$\begin{aligned}
\text{(C.36)} \quad \sum_{j=1}^n \int_{Q_0}^{\hat{\mu}_j(\underline{v})} \frac{z^{k-1} dz}{y} - \sum_{j=1}^n \int_{Q_0}^{\hat{\mu}_j(\underline{v}_0)} \frac{z^{k-1} dz}{y} &= (\underline{v})_k - (\underline{v}_0)_k, \\
k &= 1, \dots, n, \quad \underline{v}, \underline{v}_0 \in \mathcal{V}.
\end{aligned}$$

Moreover, introducing

$$(C.37) \quad v_{n+1}(\underline{v}) = \sum_{j=1}^n \int_{Q_0}^{\hat{\mu}_j(\underline{v})} \frac{z^n dz}{y},$$

one then computes as in (C.35)

$$(C.38) \quad \frac{\partial v_{n+1}(\underline{v})}{\partial v_k} = -\Psi_{n+1-k}(\underline{\mu}(\underline{v})), \quad k = 1, \dots, n,$$

using

$$(C.39) \quad \sum_{\ell=1}^n \Phi_{n-p}^{(\ell)}(\underline{\mu}) \frac{\mu_\ell^n}{\prod_{\substack{q=1 \\ q \neq \ell}}^n (\mu_\ell - \mu_q)} = -\Psi_{n+1-p}(\underline{\mu}), \quad p = 1, \dots, n$$

(cf. (C.9)). Thus, one concludes

$$(C.40) \quad \prod_{j=1}^n (z - \mu_j(\underline{v})) = \sum_{\ell=0}^n \Psi_{n-\ell}(\underline{\mu}(\underline{v})) z^\ell = z^n - \sum_{k=1}^n \frac{\partial v_{n+1}(\underline{v})}{\partial v_k} z^{k-1}, \quad \underline{v} \in \mathcal{V},$$

whenever  $\underline{\mu}$  satisfies (C.33).

In order to derive theta function representations of the elementary symmetric functions  $\Psi_k(\underline{\mu})$  of  $\mu_1(\underline{v}), \dots, \mu_n(\underline{v})$ ,  $k = 1, \dots, n$  we recall that  $\mathcal{K}_n$  corresponds to the curve  $y^2 = \prod_{m=0}^{2n+2} (z - E_m)$  with pairwise distinct  $E_m \in \mathbb{C}$ ,  $m = 0, \dots, 2n+2$  (cf. (A.1) and (A.2)). Using the notation established in Appendix Appendix A,  $v_{n+1}(\underline{v})$  can be written as

$$(C.41) \quad v_{n+1}(\underline{v}) = \sum_{j=1}^n \int_{Q_0}^{\hat{\mu}_j(\underline{v})} \frac{z^n dz}{y} = \sum_{j=1}^n \int_{Q_0}^{\hat{\mu}_j(\underline{v})} \tilde{\omega}_{P_{\infty+}, P_{\infty-}}^{(3)},$$

where

$$(C.42) \quad \tilde{\omega}_{P_{\infty+}, P_{\infty-}}^{(3)} = z^n dz / y$$

represents a differential of the third kind with simple poles at  $P_{\infty+}$  and  $P_{\infty-}$  and corresponding residues  $+1$  and  $-1$ , respectively. This differential is not normalized, that is, the  $a$ -periods of  $\tilde{\omega}_{P_{\infty+}, P_{\infty-}}^{(3)}$  are not all vanishing. We also introduce the notation

$$(C.43) \quad \begin{aligned} \underline{z}(P, \underline{Q}) &= \underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{Q}}), \\ P &\in \mathcal{K}_n, \quad \underline{Q} = \{Q_1, \dots, Q_n\} \in \sigma^n \mathcal{K}_n, \end{aligned}$$

$$(C.44) \quad \hat{\underline{z}}(P, \underline{Q}) = \hat{\underline{\Xi}}_{Q_0} - \hat{\underline{A}}_{Q_0}(P) + \hat{\underline{a}}_{Q_0}(\mathcal{D}_{\underline{Q}}),$$

$$P \in \hat{\mathcal{K}}_n, \quad \underline{Q} = \{Q_1, \dots, Q_n\} \in \sigma^n \hat{\mathcal{K}}_n$$

in connection with  $\mathcal{K}_n$  and  $\hat{\mathcal{K}}_n$ , respectively. Moreover, we conveniently choose  $Q_0 \in \partial \hat{\mathcal{K}}_n$  (e.g., the initial point of the curve  $a_1 \subset \partial \hat{\mathcal{K}}_n$ ).

**Theorem C.6** *Suppose  $\mathcal{D}_{\hat{\underline{\mu}}} \in \sigma^n \hat{\mathcal{K}}_n$  is nonspecial,  $\hat{\underline{\mu}} = \{\hat{\mu}_1, \dots, \hat{\mu}_n\} \in \sigma^n \hat{\mathcal{K}}_n$ . Then,*

$$(C.45) \quad \sum_{j=1}^n \int_{Q_0}^{\hat{\mu}_j} \tilde{\omega}_{P_{\infty+}, P_{\infty-}}^{(3)} = \sum_{j=1}^n \left( \int_{a_j} \tilde{\omega}_{P_{\infty+}, P_{\infty-}}^{(3)} \right) \left( \sum_{k=1}^n \int_{Q_0}^{\hat{\mu}_k} \omega_j - \sum_{k=1}^n \int_{a_k} (\hat{\underline{A}}_{Q_0})_j \omega_k \right)$$

$$+ \ln \left( \frac{\theta(\hat{\underline{z}}(P_{\infty+}, \hat{\underline{\mu}}))}{\theta(\hat{\underline{z}}(P_{\infty-}, \hat{\underline{\mu}}))} \right)$$

and

$$(C.46) \quad \Psi_{n+1-k}(\underline{\mu}) = \Psi_{n+1-k}(\underline{\lambda}) - \sum_{j=1}^n c_j(k) \frac{\partial}{\partial w_j} \ln \left( \frac{\theta(\underline{z}(P_{\infty+}, \hat{\underline{\mu}}) + \underline{w})}{\theta(\underline{z}(P_{\infty-}, \hat{\underline{\mu}}) + \underline{w})} \right) \Big|_{\underline{w}=0},$$

$k = 1, \dots, n$ , with  $\underline{\lambda} = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$  introduced in (A.24).

**Proof:** Let  $\mathcal{D}_{\hat{\underline{\mu}}} \in \sigma^n \hat{\mathcal{K}}_n$  be a nonspecial divisor on  $\hat{\mathcal{K}}_n$ ,  $\hat{\underline{\mu}} = \{\hat{\mu}_1, \dots, \hat{\mu}_n\} \in \sigma^n \hat{\mathcal{K}}_n$ . Introducing

$$(C.47) \quad \tilde{\Omega}^{(3)}(P) = \int_{Q_0}^P \tilde{\omega}_{P_{\infty+}, P_{\infty-}}^{(3)}, \quad P \in \mathcal{K}_n \setminus \{P_{\infty+}, P_{\infty-}\},$$

we can render  $\tilde{\Omega}^{(3)}(\cdot)$  single-valued on

$$(C.48) \quad \hat{\mathcal{K}}_n = \hat{\mathcal{K}}_n \setminus \Sigma,$$

where  $\Sigma$  denotes the union of cuts

$$(C.49) \quad \Sigma = \Sigma(P_{\infty+}) \cup \Sigma(P_{\infty-}), \quad \Sigma(P_{\infty+}) \cap \Sigma(P_{\infty-}) = \{Q_0\},$$

with  $\Sigma(P_{\infty+})$  (resp.,  $\Sigma(P_{\infty-})$ ) a cut connecting  $Q_0$  and  $P_{\infty+}$  (resp.,  $P_{\infty-}$ ) through the open interior  $\hat{\mathcal{K}}_n$  (i.e., avoiding all curves  $a_j, b_j, a_j^{-1}, b_j^{-1}$ ,  $j = 1, \dots, n$ , with the exception of the point  $Q_0 \in \partial \hat{\mathcal{K}}_n$ ), avoiding the points  $\hat{\mu}_j$ ,  $j = 1, \dots, n$ . The left and right side of the cut  $\Sigma(P_{\infty\pm})$  is denoted by

$\Sigma(P_{\infty_{\pm}})_{\ell}$  and  $\Sigma(P_{\infty_{\pm}})_r$ . The oriented boundary  $\partial\widehat{\mathcal{K}}_n$  of  $\widehat{\mathcal{K}}_n$ , in obvious notation, is then given by

$$(C.50) \quad \partial\widehat{\mathcal{K}}_n = \Sigma(P_{\infty_+})_{\ell} \cup \Sigma(P_{\infty_+})_r \cup \Sigma(P_{\infty_-})_{\ell} \cup \Sigma(P_{\infty_-})_r \cup \partial\widehat{\mathcal{K}}_n,$$

that is, it consists of  $\partial\widehat{\mathcal{K}}_n$  together with the piece from  $Q_0$  to  $P_{\infty_+}$  along the left side of the cut  $\Sigma(P_{\infty_+})$  and then back to  $Q_0$  along the right side of  $\Sigma(P_{\infty_+})$ , plus the corresponding pieces from  $Q_0$  to  $P_{\infty_-}$  and back to  $Q_0$  along the cut  $\Sigma(P_{\infty_-})$ , preserving orientation. Introducing the meromorphic differential,

$$(C.51) \quad \nu = d\ln(\theta(\underline{z}(\cdot, \underline{\hat{\mu}}))),$$

the residue theorem applied to  $\widetilde{\Omega}^{(3)}\nu$  yields

$$(C.52) \quad \begin{aligned} \int_{\partial\widehat{\mathcal{K}}_n} \widetilde{\Omega}^{(3)}\nu &= \sum_{j=1}^n \left( \left( \int_{a_j} \widetilde{\omega}_{P_{\infty_+}, P_{\infty_-}}^{(3)} \right) \left( \int_{b_j} \nu \right) - \left( \int_{b_j} \widetilde{\omega}_{P_{\infty_+}, P_{\infty_-}}^{(3)} \right) \left( \int_{a_j} \nu \right) \right) \\ &+ \int_{\Sigma} \widetilde{\Omega}^{(3)}\nu = 2\pi i \sum_{P \in \widehat{\mathcal{K}}_n} \text{res}_P(\widetilde{\Omega}^{(3)}\nu). \end{aligned}$$

Investigating separately the items occurring in (C.52) then yields the following facts:

$$(C.53) \quad \sum_{P \in \widehat{\mathcal{K}}_n} \text{res}_P(\widetilde{\Omega}^{(3)}\nu) = \sum_{j=1}^n \widetilde{\Omega}^{(3)}(\hat{\mu}_j) = \sum_{j=1}^n \int_{Q_0}^{\hat{\mu}_j} \widetilde{\omega}_{P_{\infty_+}, P_{\infty_-}}^{(3)},$$

$$(C.54) \quad \int_{a_j} \nu = 0, \quad j = 1, \dots, n,$$

$$(C.55) \quad \int_{b_j} \nu = 2\pi i \left( (\widehat{\Xi}_{Q_0})_j - (\widehat{A}_{Q_0}(R(a_j)))_j + (\widehat{\alpha}_{Q_0}(\mathcal{D}_{\hat{\mu}}))_j \right) - i\pi\tau_{j,j},$$

$$j = 1, \dots, n,$$

applying (A.33) in (C.54) and (C.55). Here  $R(a_j)$  denotes the end point of  $a_j \subset \partial\widehat{\mathcal{K}}_n$ ,  $j = 1, \dots, n$ . In addition, the cut  $\Sigma$  produces the contribution

$$(C.56) \quad \int_{\Sigma} \widetilde{\Omega}^{(3)}\nu = 2\pi i \left( \int_{Q_0}^{P_{\infty_+}} \nu - \int_{Q_0}^{P_{\infty_-}} \nu \right) = 2\pi i \int_{P_{\infty_-}}^{P_{\infty_+}} \nu = 2\pi i \ln \left( \frac{\theta(\hat{z}(P_{\infty_+}, \underline{\hat{\mu}}))}{\theta(\hat{z}(P_{\infty_-}, \underline{\hat{\mu}}))} \right),$$

since (by an application of the residue theorem)

$$(C.57) \quad \tilde{\Omega}^{(3)}(\hat{\mu}_\ell) - \tilde{\Omega}^{(3)}(\hat{\mu}_r) = \pm 2\pi i, \quad \hat{\mu}_\ell \in \Sigma(P_{\infty\pm})_\ell, \hat{\mu}_r \in \Sigma(P_{\infty\pm})_r,$$

where  $\hat{\mu}_\ell \in \Sigma(P_{\infty\pm})_\ell$  and  $\hat{\mu}_r \in \Sigma(P_{\infty\pm})_r$  are on opposite sides of the cut  $\Sigma(P_{\infty\pm})$ .

Recalling the well-known results,

$$(C.58) \quad (\hat{A}_{Q_0}(R(a_j)))_j = \frac{1}{2} + \int_{a_j} (\hat{A}_{Q_0})_j \omega_j, \quad j = 1, \dots, n,$$

$$(C.59) \quad (\hat{\Xi}_{Q_0})_j = \frac{1}{2}(1 + \tau_{j,j}) - \sum_{\substack{k=1 \\ k \neq j}}^n \int_{a_k} (\hat{A}_{Q_0})_j \omega_k, \quad j = 1, \dots, n,$$

equations (C.52)–(C.59) imply

$$(C.60) \quad \sum_{j=1}^n \int_{Q_0}^{\hat{\mu}_j} \tilde{\omega}_{P_{\infty+}, P_{\infty-}}^{(3)} = \sum_{j=1}^n \left( \int_{a_j} \tilde{\omega}_{P_{\infty+}, P_{\infty-}}^{(3)} \right) \times \\ \left( \sum_{k=1}^n \int_{Q_0}^{\hat{\mu}_k} \omega_j - \sum_{k=1}^n \int_{a_k} (\hat{A}_{Q_0})_j \omega_k \right) \\ + \ln \left( \frac{\theta(\hat{\underline{z}}(P_{\infty+}, \hat{\underline{\mu}}))}{\theta(\hat{\underline{z}}(P_{\infty-}, \hat{\underline{\mu}}))} \right).$$

This proves (C.45).

In the following we will apply (C.60) to  $\hat{\mu}_j$ ,  $j = 1, \dots, n$  satisfying the first-order system (C.33), (C.34) on some open connected set  $\mathcal{V}$  such that  $\mu_j$ ,  $j = 1, \dots, n$ , remain distinct on  $\mathcal{V}$  and  $\Phi_{n-k}^{(j)}(\underline{\mu}) \neq 0$  on  $\mathcal{V}$ ,  $j, k = 1, \dots, n$ . Using (A.12), (C.35), and (C.9) one computes

$$(C.61) \quad \frac{\partial}{\partial v_k} (\hat{\alpha}_{Q_0}(\mathcal{D}_{\hat{\underline{\mu}}(\underline{v})}))_j = \frac{\partial}{\partial v_k} \sum_{\ell=1}^n \int_{Q_0}^{\hat{\mu}_\ell(\underline{v})} \omega_j = \sum_{\ell, m=1}^n \frac{\partial}{\partial v_k} \int_{Q_0}^{\hat{\mu}_\ell(\underline{v})} c_j(m) \eta_m \\ = \sum_{\ell, m=1}^n c_j(m) \frac{\mu_\ell(\underline{v})^{m-1}}{y(\hat{\mu}_\ell(\underline{v}))} \frac{\partial}{\partial v_k} \mu_\ell(\underline{v}) \\ = \sum_{\ell, m=1}^n c_j(m) \Phi_{n-k}^{(\ell)} \frac{\mu_\ell(\underline{v})^{m-1}}{\prod_{\substack{\ell'=1 \\ \ell' \neq \ell}}^n (\mu_\ell(\underline{v}) - \mu_{\ell'}(\underline{v}))} \\ = c_j(k), \quad \underline{v} \in \mathcal{V}.$$



Thus, (C.40) and (C.61) imply

$$(C.62) \quad \Psi_{n+1-k}(\underline{\mu}(\underline{v})) = -\frac{\partial v_{n+1}(\underline{v})}{\partial v_k} = -\sum_{j=1}^n c_j(k) \left( \int_{a_j} \tilde{\omega}_{P_{\infty+}, P_{\infty-}}^{(3)} \right) \\ - \sum_{j=1}^n c_j(k) \frac{\partial}{\partial w_j} \ln \left( \frac{\theta(\underline{z}(P_{\infty+}, \hat{\underline{\mu}}(\underline{v})) + \underline{w})}{\theta(\underline{z}(P_{\infty-}, \hat{\underline{\mu}}(\underline{v})) + \underline{w})} \right) \Big|_{\underline{w}=0}, \\ \underline{v} \in \mathcal{V}, \quad k = 1, \dots, n.$$

We replaced  $\hat{\underline{z}}$  by  $\underline{z}$  to arrive at (C.62) using properties (A.33) of  $\theta$ . If  $\hat{\mu}_j$ ,  $j = 1, \dots, n$ , are distinct and  $\Phi_{n-k}^{(j)}(\underline{\mu}) \neq 0$ ,  $j, k = 1, \dots, n$ , we can choose  $\hat{\mu}_j(\underline{v}_0) = \hat{\mu}_j$ ,  $j = 1, \dots, n$ , and obtain (C.46). The general case where  $\mathcal{D}_{\hat{\underline{\mu}}}$  is nonspecial, then follows from (C.62) by continuity, choosing  $\mathcal{V}$  such that there exists a sequence  $\underline{v}_p \in \mathcal{V}$  with  $\hat{\underline{\mu}}(\underline{v}_p) \rightarrow \hat{\underline{\mu}}$  as  $p \rightarrow \infty$ . Finally, invoking the normal differential of the third kind in (A.24),  $\omega_{P_{\infty+}, P_{\infty-}}^{(3)} = \prod_{j=1}^n (z - \lambda_j) dz/y$ , corresponding to  $\tilde{\omega}_{P_{\infty+}, P_{\infty-}}^{(3)} = z^n dz/y$ , a simple computation, combining (A.9), (A.10), (A.11), (A.24), and the normalization  $\int_{a_j} \omega_{P_{\infty+}, P_{\infty-}}^{(3)} = 0$ ,  $j = 1, \dots, n$ , yields

$$(C.63) \quad \sum_{j=1}^n c_j(k) \left( \int_{a_j} \tilde{\omega}_{P_{\infty+}, P_{\infty-}}^{(3)} \right) = \Psi_{n+1-k}(\underline{\Delta}), \quad k = 1, \dots, n.$$

Equations (C.62) and (C.63) complete the proof of (C.46). ■

Formulas (C.40), (C.45), and (C.46) (without explicit proofs and without the explicit form of the constant terms on the right-hand sides of (C.45) and (C.46)) have been used in [56] in the course of deriving algebro-geometric solutions of the Dym equation. Our approach based on the Dubrovin-type system (C.33) appears to be new. It can easily be adapted to the case of KdV-type hyperelliptic curves branched at infinity (cf. [46, App. F]). Since solutions of 1+1-dimensional soliton equations typically can be expressed in terms of trace formulas involving elementary symmetric functions of (projections of) auxiliary divisors, results of the type of (C.46) are of general interest in this context.

**Acknowledgments.** We are indebted to Mark Alber, Darryl Holm, and Jerry Marsden for helpful comments and many hints regarding the literature.

## References

- [1] ALBER, M. S.:  $N$ -component integrable systems and geometric asymptotics. In *Integrability: The Seiberg-Witten and Whitham equations* (H. W. Braden and I. M. Krichever, editors). Gordon and Breach Science Publishers, Singapore, 2000, 213–228.
- [2] ALBER, M. S., CAMASSA, R., FEDOROV, YU. N., HOLM, D. D. AND MARS DEN, J. E.: On billiard solutions of nonlinear PDE's. *Phys. Lett. A* **264** (1999), 171–178.
- [3] ALBER, M. S., CAMASSA, R., FEDOROV, YU. N., HOLM, D. D. AND MARS DEN, J. E.: The complex geometry of weak piecewise smooth solutions of integrable nonlinear PDE's of shallow water and Dym type. *Comm. Math. Phys.* **221** (2001), 197–227.
- [4] ALBER, M. S., CAMASSA, R. AND GEKHTMAN, M.: Billiard weak solutions of nonlinear PDE's and Toda flows. In: *SIDE III—Symmetries and Integrability of Difference Equations* (D. Levi and O. Ragnisco, editors). CRM Proceedings and Lecture Notes **25**, 1–10. Amer. Math. Soc., Providence, RI, 2000.
- [5] ALBER, M. S., CAMASSA, R., HOLM, D. D. AND MARS DEN, J. E.: The geometry of peaked solitons and billiard solutions of a class of integrable PDE's. *Lett. Math. Phys.* **32** (1994), 137–151.
- [6] ALBER, M. S., CAMASSA, R., HOLM, D. D. AND MARS DEN, J. E.: On the link between umbilic geodesics and soliton solutions of nonlinear PDE's. *Proc. Roy. Soc. London Ser. A* **450** (1995), 677–692.
- [7] ALBER, M. S. AND FEDOROV, YU. N.: Wave solutions of evolution equations and Hamiltonian flows on nonlinear subvarieties of generalized Jacobians. *J. Phys. A* **33** (2000), 8409–8425.
- [8] ALBER, M. S. AND FEDOROV, YU. N.: Algebraic geometrical solutions for certain evolution equations and Hamiltonian flows on nonlinear subvarieties of generalized Jacobians. *Inverse Problems* **17** (2001), 1017–1042.
- [9] ALBER, M. S., LUTHER, G. G. AND MILLER, C. A.: On soliton-type solutions of equations associated with  $N$ -component systems. *J. Math. Phys.* **41** (2000), 284–316.
- [10] ALBER, M. S. AND MILLER, C.: Peakon solitons of the shallow water equation. *Appl. Math. Lett* **14** (2001), 93–98.
- [11] BEALS, R., SATTINGER, D. H. AND SZMIGIELSKI, J.: Acoustic scattering and the extended Korteweg–de Vries hierarchy. *Adv. Math.* **140** (1998), 190–206.
- [12] BEALS, R., SATTINGER, D. H. AND SZMIGIELSKI, J.: Multi-peakons and a theorem of Stieltjes. *Inverse Problems* **15** (1999), no. 1, L1–L4.

- [13] BEALS, R., SATTINGER, D. H. AND SZMIGIELSKI, J.: Multipeakons and the classical moment problem. *Adv. in Math.* **154** (2000), 229–257.
- [14] BEALS, R., SATTINGER, D. H. AND SZMIGIELSKI, J.: Peakons, strings, and the finite Toda lattice. *Comm. Pure Appl. Math.* **54** (2001), 91–106.
- [15] BELOKOLOS, E. D., BOBENKO, A. I., ENOL'SKII, V. Z., ITS, A. R. AND MATVEEV, V. B.: *Algebro-Geometric Approach to Nonlinear Integrable Equations*. Springer, Berlin, 1994.
- [16] BULLA, W., GESZTESY, F., HOLDEN, H. AND TESCHL, G.: Algebro-geometric quasi-periodic finite-gap solutions of the Toda and Kac-van Moerbeke hierarchy. *Mem. Amer. Math. Soc.* **135** (1998), no. 641, 1–79.
- [17] CAMASSA, R. AND HOLM, D. D.: An integrable shallow water equation with peaked solitons. *Phys. Rev. Lett.* **71** (1993), 1661–1664.
- [18] CAMASSA, R., HOLM, D. D. AND HYMAN, J. M.: A new integrable shallow water equation. *Adv. Appl. Mech.* **31** (1994), 1–33.
- [19] CLEBSCH, A. AND GORDAN, P.: *Theorie der Abelschen Funktionen*, Teubner, Leipzig, 1866.
- [20] CONSTANTIN, A.: On the Cauchy problem for the periodic Camassa–Holm equation. *J. Differential Equations* **141** (1997), 218–235.
- [21] CONSTANTIN, A.: On the inverse spectral problem for the Camassa–Holm equation. *J. Funct. Anal.* **155** (1998), 352–363.
- [22] CONSTANTIN, A.: Quasi-periodicity with respect to time of spatially periodic finite-gap solutions of the Camassa–Holm equation. *Bull. Sci. Math.* **122** (1998), 487–494.
- [23] CONSTANTIN, A.: Existence of permanent and breaking waves for a shallow water equation: a geometric approach. *Ann. Inst. Fourier (Grenoble)* **50** (2000), 321–362.
- [24] CONSTANTIN, A. AND ESCHER, J.: Global existence and blow-up for a shallow water equation. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.(4)* **26** (1998), 303–328.
- [25] CONSTANTIN, A. AND ESCHER, J.: Global weak solutions for a shallow water equation. *Indiana Univ. Math. J.* **47** (1998), 1527–1545.
- [26] CONSTANTIN, A. AND ESCHER, J.: Wave breaking for nonlinear nonlocal shallow water equations. *Acta Math.* **181** (1998), 229–243.
- [27] CONSTANTIN, A. AND ESCHER, J.: Well-posedness, global existence, and blow-up phenomena for a periodic quasi-linear hyperbolic equation. *Comm. Pure Appl. Math.* **51** (1998), 475–504.
- [28] CONSTANTIN, A. AND ESCHER, J.: On the blow-up rate and the blow-up set of breaking waves for a shallow water equation. *Math. Z.* **233** (2000), 75–91.

- [29] CONSTANTIN, A. AND MCKEAN, H. P.: A shallow water equation on the circle. *Comm. Pure Appl. Math.* **52** (1999), 949–982.
- [30] CONSTANTIN, A. AND MOLINET, L.: Global weak solutions for a shallow water equation. *Comm. Math. Phys.* **211** (2000), 45–61.
- [31] DICKSON, R., GESZTESY, F. AND UNTERKOFLE, K.: Algebro-geometric solutions of the Boussinesq hierarchy. *Rev. Math. Phys.* **11** (1999), 823–879.
- [32] DICKSON, R., GESZTESY, F. AND UNTERKOFLE, K.: A new approach to the Boussinesq hierarchy. *Math. Nachr.* **198** (1999), 51–108.
- [33] DMITRIEVA, L. A.: Finite-gap solutions of the Harry Dym equation. *Phys. Lett. A* **182** (1993), 65–70.
- [34] DULLIN, H. R., GOTTWALD, G. AND HOLM, D. D.: An integrable shallow water equation with linear and nonlinear dispersion. *Phys. Rev. Lett.* **87** (2001), 4501–4504.
- [35] ENOLSKII, V. Z., GESZTESY, F. AND HOLDEN, H.: The classical massive Thirring system revisited. In: *Stochastic Processes, Physics and Geometry: New Interplays. I: A Volume in Honor of Sergio Albeverio* (F. Gesztesy, H. Holden, J. Jost, S. Paycha, M. Röckner, and S. Scarlatti, editors). CMS Conference Proceedings **28**, Amer. Math. Soc., Providence, RI, 2000, 163–200.
- [36] FARKAS, H. M. AND KRA, I.: *Riemann Surfaces*, second edition, Springer, New York, 1992.
- [37] FEDOROV, YU.: Classical integrable systems and billiards related to generalized Jacobians. *Acta Appl. Math.* **55** (1999), 251–301.
- [38] FISHER, M. AND SCHIFF, J.: The Camassa Holm equation: conserved quantities and the initial value problem. *Phys. Lett. A* **259** (1999), 371–376.
- [39] FOIAS, C., HOLM, D. D. AND TITI, E. S.: The three dimensional viscous Camassa–Holm equations, and their relation to the Navier–Stokes equations and turbulence theory. *J. Dynam. Diff. Eq.* **14** (2002), 1–35.
- [40] FORNBERG, B. AND WITHAM, G. B.: A numerical and theoretical study of certain nonlinear wave phenomena. *Philos. Trans. Roy. Soc. London Ser. A* **289** (1978), 373–404.
- [41] FUCHSSTEINER, B.: Some tricks from the symmetry-toolbox for nonlinear equations: generalizations of the Camassa–Holm equation. *Phys. D* **95** (1996), 229–243.
- [42] FUCHSSTEINER, B. AND FOKAS, A. S.: Symplectic structures, their Bäcklund transformations and hereditary symmetries. *Phys. D* **4** (1981), 47–66.
- [43] GAGNON, L., HARNAD, J., WINTERNITZ, P. AND HURTUBISE, J.: Abelian integrals and the reduction method for an integrable Hamiltonian system. *J. Math. Phys.* **26** (1985), 1605–1612.

- [44] GAVRILOV, L.: Generalized Jacobians of spectral curves and completely integrable systems. *Math. Z.* **230** (1999), 487–508.
- [45] GESZTESY, F. AND HOLDEN, H.: Dubrovin equations and integrable systems on hyperelliptic curves. *Math. Scand.* **91** (2002), 91–126.
- [46] GESZTESY, F. AND HOLDEN, H.: *Soliton Equations and Their Algebro-Geometric Solutions. Vol. I: (1 + 1)-Dimensional Continuous Models*. Cambridge Studies in Advanced Mathematics, Cambridge Univ. Press, Cambridge, 2003.
- [47] GESZTESY, F. AND HOLDEN, H.: A combined sine-Gordon and modified Korteweg–de Vries hierarchy and its algebro-geometric solutions. In: *Differential Equations and Mathematical Physics* (R. Weikard and G. Weinstein, editors), 133–173. Studies in Advanced Mathematics **16**, Amer. Math. Soc. and International Press, Providence and Boston, 2000.
- [48] GESZTESY, F. AND HOLDEN, H.: Darboux-type transformations and hyperelliptic curves. *J. Reine Angew. Math.* **527** (2000), 151–183.
- [49] GESZTESY, F. AND RATNASEELAN, R.: An alternative approach to algebro-geometric solutions of the AKNS hierarchy. *Rev. Math. Phys.* **10** (1998), 345–391.
- [50] GESZTESY, F., RATNASEELAN, R. AND TESCHL, G.: The KdV hierarchy and associated trace formulas. In: *Recent Developments in Operator Theory and Its Applications* (I. Gohberg, P. Lancaster, and P. N. Shivakumar, editors). Operator Theory: Advances and Applications **87**, Birkhäuser, Basel, (1996), 125–163.
- [51] JOHNSON, R. S.: Camassa–Holm, Korteweg–de Vries and related models for water waves. *J. Fluid Mech.* **455** (2002), 63–82.
- [52] MARSDEN, J. E., RATIU, T. S. AND SHKOLLER, S.: The geometry and analysis of the averaged Euler equations and a new diffeomorphism group. *Geom. Funct. Anal.* **10** (2000), 582–599.
- [53] MARSDEN, J. E. AND SHKOLLER, S.: Global well-posedness for the Lagrangian averaged Navier-Stokes (LANS- $\alpha$ ) equations on bounded domains. *R. Soc. Lond. Philos. Trans. Ser. A Math. Phys. Eng. Sci.* **359** (2001), 1449–1468.
- [54] MISIOLEK, G.: A shallow water equation as a geodesic flow on the Bott–Virasoro group. *J. Geom. Phys.* **24** (1998), 203–208.
- [55] MUMFORD, D.: *Tata Lectures on Theta II*. Progress in Mathematics **43**, Birkhäuser, Boston, 1984.
- [56] NOVIKOV, D. P.: Algebraic-geometric solutions of the Harry Dym equation. *Siberian Math. J.* **40** (1999), 136–140.
- [57] SCHIFF, J.: Zero curvature formulations of dual hierarchies. *J. Math. Phys.* **37** (1996), 1928–1938.

- [58] SHKOLLER, S.: Geometry and curvature of diffeomorphism groups with  $H^1$  metric and mean hydrodynamics. *J. Funct. Anal.* **160** (1998), 337–365.
- [59] SHKOLLER, S.: On incompressible averaged Lagrangian hydrodynamics. Preprint, arXiv:math.AP/9908109.
- [60] TODA, M.: *Theory of Nonlinear Lattices*, second edition. Solid-State Sciences **20**, Springer, Berlin, 1989.
- [61] XIN, Z. AND ZHANG, P.: On the weak solutions to a shallow water equation. *Comm. Pure Appl. Math.* **53** (2000), 1411–1433.

*Recibido:* 20 de junio de 2001

Fritz Gesztesy  
Department of Mathematics  
University of Missouri  
Columbia, MO 65211, USA  
fritz@math.missouri.edu

<http://www.math.missouri.edu/people/fgesztesy.html>

Helge Holden  
Department of Mathematical Sciences  
Norwegian University of Science and Technology  
NO-7491 Trondheim, Norway  
holden@math.ntnu.no

<http://www.math.ntnu.no/~holden/>