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Maximal functions and singular integrals associated to polynomial mappings of $\mathbb{R}^n$

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Abstract

We consider convolution operators on $\mathbb{R}^n$ of the form

$$T_P f(x) = \int_{\mathbb{R}^m} f\big(x - P(y)\big) K(y)\, dy \ ,$$

where P is a polynomial defined on $\mathbb{R}^m$ with values in $\mathbb{R}^n$ and K is a smooth Calderón-Zygmund kernel on $\mathbb{R}^m$. A maximal operator M_P can be constructed in a similar fashion. We discuss weak-type 1–1 estimates for T_P and M_P and the uniformity of such estimates with respect to P. We also obtain L^p-estimates for "supermaximal" operators, defined by taking suprema over P ranging in certain classes of polynomials of bounded degree.

1. Introduction

In this paper we continue the analysis initiated in [3] on certain analogues of singular integral operators and maximal operators, characterized by the fact that the ordinary difference $x - t$ appearing in the convolution is replaced by a rather general polynomial expression $\mathfrak{p}(x, t)$.

In [3] we discussed the one-dimensional case, and introduced the following operators:

(i) the maximal function

$$M_{\mathfrak{p}} f(x) = \sup_{h>0} \frac{1}{2h} \int_{-h}^{h} \big| f\big(\mathfrak{p}(x, t)\big)\big|\, dt \ ,$$

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where $\mathfrak{p}(x,t)$ is a polynomial such that $\mathfrak{p}(x,0)=x$ and the corresponding Hilbert transform

$$H_{\mathfrak{p}}f(x) = p.v.\int_{-\infty}^{+\infty} f\big(\mathfrak{p}(x,t)\big)\,\frac{dt}{t}\ ;$$

(ii) more specifically, the "translation-invariant" operators M_Pf and H_Pf, corresponding to $\mathfrak{p}(x,t)=x-P(t)$, with $P(t)$ a polynomial satisfying $P(0)=0$;

(iii) the "supermaximal function"

$$\mathcal{M}_k f(x) = \sup_{P\in\mathcal{P}_k}\sup_{h>0}\frac{1}{2h}\int_{-h}^{h}\big|f\big(x-P(t)\big)\big|\,dt\ ,$$

where $\mathcal{P}_k$ is the space of polynomials in t of degree at most k and such that $P(0)=0$, and the "superhilbert transform"

$$\mathcal{T}_k f(x) = \sup_{P\in\mathcal{P}_k}\left|\int_{-\infty}^{\infty} f\big(x-P(t)\big)\,\frac{dt}{t}\right|\ .$$

The main results of [3] can be summarized as follows:

(i) the supermaximal function $\mathcal{M}_k$ and the superhilbert transform $\mathcal{T}_k$ are bounded on L^p if and only if[(1)] $p>k$, and they are restricted weak-type k–k ;

(ii) for an arbitrary polynomial $\mathfrak{p}(x,t)$ of degree k in t, one also has boundedness on L^p for $p>k$, and restricted weak-type k–k, for $M_{\mathfrak{p}}$ and $H_{\mathfrak{p}}$; for each k and $p<k$ there exist polynomials of degree k in t for which $M_{\mathfrak{p}}$ and $H_{\mathfrak{p}}$ are unbounded on L^p;

(iii) the translation-invariant operators M_P and H_P satisfy uniform weak-type 1–1 estimates[(2)] for $P\in\mathcal{P}_k$.

In this paper we wish to pose the same kind of questions in higher dimensions and discuss the problems that arise.

In general, the polynomial $\mathfrak{p}(x,t)$ must be replaced by an n-tuple of polynomials

$$\mathfrak{P}(x,y) = \big(\mathfrak{p}_1(x,y),\dots,\mathfrak{p}_n(x,y)\big)$$

(1) We are assuming $p<\infty$. Clearly $\mathcal{M}_k$ is also bounded on L^∞ and $\mathcal{T}_k$ is not.

(2) That they satisfy uniform L^p estimates follows by a transference argument, see [12].

from $\mathbb{R}^n \times \mathbb{R}^m$ to $\mathbb{R}^n$ such that $\mathfrak{P}(x,0) = 0$. However, we shall restrict ourselves to the "translation-invariant" case, where $\mathfrak{P}(x,y) = x - P(y)$ and P is a polynomial from $\mathbb{R}^m$ to $\mathbb{R}^n$.

We must initially fix a family of (possibly non-isotropic) dilations on $\mathbb{R}^m$,

$$y \longmapsto r \cdot y = (r^{\lambda_1} y_1, \dots, r^{\lambda_m} y_m) ,$$

with $\lambda_1, \dots, \lambda_m > 0$. Then the maximal operator M_P on $\mathbb{R}^n$ can be defined as

$$M_P f(x) = \sup_{r>0} \frac{1}{|B_r|} \int_{|y|<r} \big| f\big(x - P(y)\big)\big| \, dy ,$$

where the balls $B_r = r \cdot B_1$ are adapted to the given dilations.

Similarly, if K is a smooth singular kernel adapted to the same dilations, we set

$$T_P f(x) = \int_{\mathbb{R}^m} f\big(x - P(y)\big) K(y) \, dy .$$

It follows from a transference argument that M_P and T_P are bounded on L^p for $p > 1$, and that the bounds are uniform for all P of a given degree [12]. So our main concern will be about weak-type 1–1 estimates, uniformity of these estimates with respect to P, and L^p-estimates for "super"-type operators.

As we start this analysis, we immediately see certain problems and obstructions that are not present in one dimension.

Among the operators included in the above discussion we have the Hilbert transform along the parabola

$$Hf(x_1, x_2) = \int_{-\infty}^{\infty} f(x_1 - t, x_2 - t^2) \, \frac{dt}{t} ,$$

and the companion maximal function. Their endpoint estimates at $p = 1$ are not completely understood; it is known that they map parabolic H^1 into weak L^1 [4, 6] (see also [11]).

We shall stay away from this situation, as well as from those where the image of $\mathbb{R}^m$ under P is a lower dimensional variety. For this reason we shall always assume that $m \geq n$ and that the derivative DP of P has generically(3) rank n. In addition, we will impose that this regularity condition is also satisfied by the "principal parts" of P at 0 and at infinity (as defined in Section 2).

It is convenient to split M_P into two parts: a "local" part, M_P^0, where the supremum is taken only over $r \leq 1$, and a "global" part, M_P^∞, where

(3) Here and in the sequel, "generically" means away from a proper Zariski-closed set, in particular almost everywhere.

the supremum is taken over $r \geq 1$. Similarly, we will separately consider operators T_P^0 whose kernels have compact support (i.e. are singular only at the origin), and operators T_P^∞ with kernels that are locally smooth, and hence singular only at infinity.

In Section 2 we discuss weak-type 1–1 estimates for singular integral operators. After a preliminary discussion of operators defined by homogeneous polynomials, we define the "principal part" P_0 of P at the origin and its "principal part" P_∞ at infinity. The construction requires, in each of the two cases, that an appropriate family of dilations be introduced on $\mathbb{R}^n$. We prove that, if P_0 is regular, the operators T_P^0 are weak-type 1–1. Similarly, if P_∞ is regular, the operators T_P^∞ are weak-type 1–1.

In Section 3 we focus on maximal operators. After stating parallel results to those in Section 1 for the maximal operators M_P^0 and M_P^∞, we restrict ourselves to $m = n$, and discuss uniformity of the weak-type 1–1 estimates.

This requires a quantitative formulation of the non-degeneracy condition. We fix at this point two families of dilations on $\mathbb{R}^n$, regarded respectively as the domain and the codomain of P. For fixed $k \in \mathbb{N}$ and $\sigma \in (0,1)$, we take the class $\mathcal{P}_{k,\sigma}^0$ of all polynomials of degree at most k, whose principal part at the origin is regular and homogeneous with repect to the given dilations, and such that the image under P of a ball of radius $r < 1$ (which will be contained in a ball of radius cr) for some $c > 0$, covers at least a portion σ of this containing ball.

We then prove that the weak 1–1 estimate for for M_P^0 are uniform for $P \in \mathcal{P}_{k,\sigma}^0$. Similar classes $\mathcal{P}_{k,\sigma}^\infty$ can be defined in order to obtain a similar result for M_P^∞. Corresponding results —whose details we omit— also hold for singular integrals.

We are not able to say if the estimates blow up as σ tends to zero. This question is clearly related to the open problem of determining whether maximal operators along parabolas or other lower dimensional polynomial manifolds are weak-type 1–1.

In Section 4 we consider supermaximal functions, once again in the setting $m = n$. Here we see an obstruction that forces us to impose, once again, some non-degeneracy condition on the admissible polynomials.

Suppose we take a class $\mathcal{P}$ of polynomials from $\mathbb{R}^n$ to itself containing all linear functions, and define

$$M_{\mathcal{P}} f(x) = \sup_{P \in \mathcal{P}} \sup_{r>0} \frac{1}{|B_r|} \int_{B_r} \big| f\big(x - P(t)\big)\big| \, dt \ .$$

Then this operator dominates the Kakeya maximal operator, which is known to be bounded only on L^∞.

The quantitative non-degeneracy conditions that we need here are slightly weaker that the ones described before. We simply impose that the image of a ball of radius r is contained in a ball of radius r', depending only on P and r, and covers at least a portion σ of this ball.

The corresponding supermaximal operator, that we call $M_{k,\sigma}$, is then bounded on L^p for $p > n(k-1)+1$ and restricted weak-type at this endpoint.

One would expect that this result can be improved if the class of admissible polynomials is further restricted. In Section 5 we consider holomorphic polynomials P of degree at most k from $\mathbb{C} \cong \mathbb{R}^2$ to itself, and show that the corresponding supermaximal operator is bounded on L^p for $p > k$ and restricted weak-type k–k. We plan to return to the matters of supersingular integrals and uniform weak-type 1–1 estimates of singular integrals associated to holomorphic polynomials in a future paper.

Finally, in Section 6 we comment briefly on the case where the difference $x - P(y)$ in the definition of our operators is replaced by the product $x \cdot P(y)^{-1}$ in a nilpotent group law.

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2. Weak-type estimates for singular integrals

Consider a set of dilations on $\mathbb{R}^m$,

$$y \longmapsto r \cdot y = (r^{\lambda_1} y_1, \ldots, r^{\lambda_m} y_m) \ ,$$

with $\lambda_1, \ldots, \lambda_m > 0$. We shall denote by $Q = \lambda_1 + \cdots + \lambda_m$ the associated homogeneous dimension of $\mathbb{R}^m$, and by $|y|$ a homogeneous gauge (to be distinguished from the Euclidean norm $\|y\|$).

Take next a family $\{\varphi_j(y)\}_{j \in \mathbb{Z}}$ of C^1-functions that are supported on the set where $1 < |y| < 4$, have mean value zero and uniformly bounded C^1-norms.

Then the series

$$\sum_{j \in \mathbb{Z}} 2^{-Qj} \varphi_j(2^{-j} \cdot y) \tag{2.1}$$

converges in $\mathcal{S}'(\mathbb{R}^m)$ to a distribution that coincides with a C^1-function away from the origin.

We shall call any distribution that is the sum of a series as in (2.1) a smooth Calderón-Zygmund kernel adapted to the given dilations. It can be shown that this class of kernels includes, for instance, all the homogeneous distributions of degree $-Q+i\gamma$ for some real γ that are C^1 away from the origin (see e.g. [7] for related results in the context of multi-parameter dilations).

Given a smooth Calderón-Zygmund kernel as above and an n-tuple

$$P(y)=\big(p_1(y),\dots,p_n(y)\big)$$

of polynomials on $\mathbb{R}^m$, we construct the operator

$$Tf(x)=\int_{\mathbb{R}^m} f\big(x-P(y)\big)K(y)\,dy\ , \tag{2.2}$$

acting on functions defined on $\mathbb{R}^n$.

We need two technical lemmas. The first concerns scalar-valued polynomials, and it can be found in [8].

Lemma 2.1 *Let $p(x)=\sum_\alpha a_\alpha x^\alpha$ be a (scalar-valued) polynomial of degree k. Then $|p|^{-\delta}$ is locally integrable for $\delta<1/k$ and*

$$\int_{\|x\|<1}|p(x)|^{-\delta}\,dx\le C_\delta\Big(\sum_\alpha |a_\alpha|\Big)^{-\delta}\ .$$

Definition. *We say that a polynomial $P:\mathbb{R}^m\to\mathbb{R}^n$ is* regular *if the Jacobian DP of P has generically rank n.*

This obviously requires that $n\le m$.

If J_ℓ are the minors of order n of DP, we call $J_P(y)=\big(\sum_\ell J_\ell^2(y)\big)^{\frac12}$. Observe that J_P^2 is a polynomial, and if the p_j have degree at most k, then the degree of J_P^2 is at most $M=2n(k-1)$.

The details of the proof of Proposition 2.1 in [9] give us the next lemma.

Lemma 2.2 *Let P be a regular polynomial. If φ is a C^1-function supported on the unit ball in $\mathbb{R}^m$, let μ be the measure on $\mathbb{R}^n$ defined by*

$$\int_{\mathbb{R}^n} f(x)\,d\mu(x)=\int_{\mathbb{R}^m} f\big(P(y)\big)\varphi(y)\,dy\ .$$

Then $d\mu(x)=\psi(x)\,dx$, where ψ is integrable and, for every $\delta<\frac{1}{2n(k-1)}$,

$$\int\big|\psi(x+t)-\psi(x)\big|\,dx\le C\|t\|^\delta\int_{|y|<1}J_P(y)^{-2\delta}\,dy\ , \tag{2.3}$$

where the constant C depends only on δ, on the C^1-norm of φ and on the C^2-norm of P on the unit ball.

We go back now to the operator (2.2). We first consider the case where the function P is homogeneous, in the sense that there are dilations on $\mathbb{R}^n$

$$x \longmapsto r \circ x = (r^{\mu_1} x_1, \dots, r^{\mu_n} x_n) \ ,$$

such that $P(r \cdot y) = r \circ P(y)$.

Theorem 2.3 *Assume that P is homogeneous and regular, and that K is a smooth Calderón-Zygmund kernel. Then the operator T in* (2.2) *can be written as*

$$Tf(x) = \int_{\mathbb{R}^n} f(x - x') K'(x') \, dx' \ ,$$

where K' is a "rough" Calderón-Zygmund kernel, in the sense that it satisfies the standard integral condition

$$\int_{|x|>A|t|} \big| K'(x-t) - K'(x) \big| \, dx \leq C \ , \tag{2.4}$$

for some constants $A, C > 0$. Hence T is bounded on L^p for $1 < p < \infty$ and weak-type 1–1.

Proof. Decompose $T = \sum_{j \in \mathbb{Z}} T_j$, where

$$T_j f(x) = \int_{\mathbb{R}^m} f\big(x - P(y)\big) 2^{-Qj} \varphi_j(2^{-j} \cdot y) \, dy = \int_{\mathbb{R}^m} f\big(x - 2^j \circ P(y)\big) \varphi_j(y) \, dy \ .$$

It follows from Lemmas 2.1 and 2.2 that

$$\int_{\mathbb{R}^m} g\big(P(y)\big) \varphi_j(y) \, dy = \int_{\mathbb{R}^n} g(x) \psi_j(x) \, dx \ , \tag{2.5}$$

where the ψ_j are supported on a fixed compact set and satisfy uniform Lipschitz estimates

$$\int \big| \psi_j(x-t) - \psi_j(x) \big| \, dx \leq C \|t\|^\delta \ .$$

Hence,

$$T_j f(x) = \int_{\mathbb{R}^n} f\big(x - 2^j \circ x'\big) \psi_j(x') \, dx' = \int_{\mathbb{R}^n} f(x - x') 2^{-Q'j} \psi_j(2^{-j} \circ x') \, dx' \ ,$$

if Q' is the homogeneous dimension of $\mathbb{R}^n$.

Observe that taking $g \equiv 1$ in (2.5) shows that the ψ_j have mean value zero.

It is now a standard argument to show that the sum $\sum_{j \in \mathbb{Z}} 2^{-Q'j} \psi_j(2^{-j} \circ x)$ satisfies (2.4) and that the T_j are almost orthogonal, so that T is bounded on L^2. Boundedness on L^p and weak-type 1–1 then follows by standard Calderón-Zygmund theory. ■

Remark. As we mentioned in the Introduction, the fact that T is bounded on L^p for $1 < p < \infty$ also follows by a transference argument [12]. This argument also applies to polynomials P that are not regular nor homogeneous, and it gives uniform bounds, for fixed p, over all polynomials of a given degree.

Consider now a polynomial

$$P = (p_1, \dots, p_n) : \mathbb{R}^m \longrightarrow \mathbb{R}^n \ ,$$

and write

$$P(y) = \sum_\alpha v_\alpha y^\alpha \ , \tag{2.6}$$

where the α are multi-indices and the coefficients v_α are in $\mathbb{R}^n$.

If $d(\alpha) = \sum_{j=1}^m \lambda_j \alpha_j$ is the non-isotropic degree of the monomial y^α, let $d_1 < d_2 < \cdots < d_k$ be the different non-isotropic degrees of the various monomials of P.

For $1 \le j \le k$, define

$$V_j = \mathrm{span}\{v_\alpha : d(\alpha) \le d_j\} \ . \tag{2.7}$$

If we also set $V_0 = \{0\}$ and $V_{k+1} = \mathbb{R}^n$, the V_j form a filtration of $\mathbb{R}^n$, i.e.

$$\{0\} \subset V_1 \subseteq V_2 \subseteq \cdots \subseteq V_k \subseteq \mathbb{R}^n \ .$$

For $1 \le j \le k+1$, decompose V_j as $V_{j-1} \oplus W_j$, so that

$$V_j = W_1 \oplus W_2 \oplus \cdots \oplus W_j \ ,$$

and let π_{W_j} denote the projection operator on W_j relative to this decomposition. Define

$$P_0(y) = \sum_{d(\alpha)=d_1} v_\alpha y^\alpha + \sum_{d(\alpha)=d_2} \pi_{W_2}(v_\alpha) y^\alpha + \cdots + \sum_{d(\alpha)=d_k} \pi_{W_k}(v_\alpha) y^\alpha \ . \tag{2.8}$$

Then P_0 is homogeneous, if we introduce the following dilations on $\mathbb{R}^n$:

$$x \longmapsto r \circ x = r^{d_1} \pi_{W_1} x + \cdots + r^{d_{k+1}} \pi_{W_{k+1}} x \tag{2.9}$$

(if $V_k \ne \mathbb{R}^n$, d_{k+1} can be defined arbitrarily).

Observe that

$$P_0(y) = \lim_{r \to 0} r^{-1} \circ P(r \cdot y) \ . \tag{2.10}$$

We call P_0 the *principal part of* P *at* 0.

Definition. *We say that P is non-degenerate at 0 if P_0 is a regular polynomial.*

We must check that this is a good definition, because P_0 depends on the choice of the complementary subspaces W_j (observe that the V_j are intrinsically defined, once P and the dilations on $\mathbb{R}^m$ are assigned, but the W_j are not). Consider therefore a different family $\{W_j'\}$ of complementary subspaces of V_{j-1} in V_j.

Observe that, for $v \in V_j$, $\pi_{W_j'} v = \pi_{W_j'}(\pi_{W_j} v)$, and that $\pi_{W_j'}$ is a bijection from W_j to W_j'. Therefore the map $A : \mathbb{R}^n \to \mathbb{R}^n$ such that $A_{|_{W_j'}} = \pi_{W_j'}$ is invertible.

If

$$P_0'(y) = \sum_{d(\alpha)=d_1} v_\alpha y^\alpha + \sum_{d(\alpha)=d_2} \pi_{W_2'}(v_\alpha) y^\alpha + \cdots + \sum_{d(\alpha)=d_k} \pi_{W_k'}(v_\alpha) y^\alpha \ ,$$

then $P_0'(y) = AP_0(y)$. We conclude that $DP_0'(y)$ has the same rank as $DP_0(y)$.

The following statement provides a more intrinsic understanding of non-degenerate poynomials at 0; it also shows that P_0 is unique modulo semi-trivial transformations.

Proposition 2.4 *Let $x \to r \bullet x$ be a family of dilations on $\mathbb{R}^n$ such that*

$$\lim_{r\to 0} r^{-1} \bullet P(r \cdot y) = Q_0(y)$$

exists for every y and defines a regular polynomial. Then there is an invertible linear transformation A of $\mathbb{R}^n$ such that, if P_0 and the V_j are defined by (2.8) *and* (2.7) *respectively, then*

(i) $Q_0(y) = AP_0(y)$;

(ii) *A maps each V_j onto itself;*

(iii) $r \bullet (Av) = A(r \circ v)$;

Proof. Let W_j' be the eigenspaces for the given dilations, so that, for $v \in W_j'$, $r \bullet v = r^{\mu_j} v$ and $\mu_j < \mu_{j+1}$ for every j. Then

$$r^{-1} \bullet P(r \cdot y) = \sum_\alpha \sum_j r^{-\mu_j + d(\alpha)} \pi_{W_j'}(v_\alpha) y^\alpha \ . \tag{2.11}$$

We call $V_j' = \bigoplus_{j' \le j} W_{j'}'$.

By hypothesis, $\mu_j \leq d(\alpha)$ whenever $\pi_{W_j'}(v_\alpha) \neq 0$. Also,

$$Q_0(y) = \sum_j \sum_{\alpha : d(\alpha) = \mu_j} \pi_{W_j'}(v_\alpha) y^\alpha \ .$$

The fact that Q_0 is regular implies that its coefficients span $\mathbb{R}^n$. It follows that all the μ_j are non-isotropic degrees of monomials of P and that $W_j' = \operatorname{span}\{\pi_{W_j'}(v_\alpha) : d(\alpha) = \mu_j\}$.

Since $v_\alpha \in \bigoplus_{\mu_j \leq d(\alpha)} W_j'$, we also have that $V_j' = \operatorname{span}\{v_\alpha : d(\alpha) \leq \mu_j\}$. This means that each V_j' coincides with one of the V_ℓ.

We want to see that every V_ℓ is one of the V_j'. Assume that this is not true. Then, for some k,

$$V_j' = V_k \subset V_{k+1} \subset V_{j+1}' \ .$$

In particular, $\mu_j = d_k < d_{k+1} < \mu_{j+1}$. Let then α be a multi-index of degree d_{k+1} such that $v_\alpha \notin V_k = V_j'$. Then v_α has a non-trivial component in W_{j+1}'. But this implies that $\mu_{j+1} \leq d(\alpha) = d_{k+1}$, which is a contradiction.

A similar argument also shows that if $V_\ell, V_{\ell+1}, \ldots, V_{\ell+p}$ are all the V's that coincide with a given V_j', then $\mu_j = d_\ell$. In fact, if we had $\mu_j = d_{\ell+k}$ with $k \geq 1$, take α with $d(\alpha) = d_\ell$ such that $v_\alpha \notin V_{j-1}'$. Then v_α would have a non-zero component in W_j', so that $\mu_j \leq d_\ell$, which is again a contradiction.

These considerations lead us to the following conclusion: if among the W_ℓ we select the non-trivial ones, $W_{\ell_1}, W_{\ell_2}, \ldots$, with $\ell_1 < \ell_2 < \cdots$, then $\mu_j = d_{\ell_j}$ and $V_j' = V_{\ell_j}$.

Define A by imposing that $A_{|W_{\ell_j}} = \pi_{W_j'}$. Properties (i), (ii) and (iii) are now easy to check. ∎

Remark. Our construction of P_0 is very close to the construction in [10]; it also applies to real-analytic functions on a neighbourhood of 0.

Theorem 2.5 *Let K be a smooth Calderón-Zygmund kernel on $\mathbb{R}^m$ adapted to a given set of dilations and with compact support. Let $P : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ be a polynomial that is non-degenerate at 0. Then the operator T in* (2.2) *can be written as*

$$Tf(x) = \int_{\mathbb{R}^n} f(x - x')K'(x')\, dx' \ ,$$

where K' is a "rough" Calderón-Zygmund kernel with compact support, adapted to the dilations (2.9) *and which satisfies the standard integral condition* (2.4). *Hence T is bounded on L^p for $1 < p < \infty$ and weak-type 1–1.*

Proof. We can write K as

$$\sum_{j\le 0} 2^{-Qj}\varphi_j(2^{-j}\cdot y)+\eta(y)\ ,$$

with the φ_j as in (2.1) and with η a smooth function with compact support (which can be disregarded). As in the proof of Theorem 2.3, we decompose T as the sum $\sum_{j\le 0} T_j$, with

$$T_jf(x)=\int_{\mathbb{R}^m} f\big(x-P(y)\big)2^{-Qj}\varphi_j(2^{-j}\cdot y)\,dy=\int_{\mathbb{R}^m} f\big(x-P(2^j\cdot y)\big)\varphi_j(y)\,dy\ .$$

If we set $P^{(j)}(y)=2^{-j}\circ P(2^j\cdot y)$, then

$$T_jf(x)=\int_{\mathbb{R}^m} f\big(x-2^j\circ P^{(j)}(y)\big)\varphi_j(y)\,dy\ .$$

Observe that $P^{(j)}(y)=P_0(y)+Q_j(y)$, where the coefficients of Q_j are $O(2^{\varepsilon j})$, as $j\to-\infty$, for some $\varepsilon>0$. Since P_0 is regular, it easily follows that, for every j, $DP^{(j)}$ has rank n for generic y.

By Lemma 2.2, there exist integrable functions ψ_j such that

$$\int_{\mathbb{R}^m} g\big(P^{(j)}(y)\big)\varphi_j(y)\,dy=\int_{\mathbb{R}^n} g(x)\psi_j(x)\,dx\ .$$

The support of ψ_j is contained in the image, under $P^{(j)}$, of the ball of radius 4. Since the coefficients of the $P^{(j)}$ are uniformly bounded for $j\le 0$, the ψ_j are all supported on the same ball.

As the C^2-norms of the $P^{(j)}$ are also uniformly bounded for $j\le 0$, the bounds for the L^1-Lipschitz estimates on the ψ_j will only depend on $\int_{|y|<4} J_{P^{(j)}}(y)^{-2\delta}\,dy$, by Lemma 2.2. Since $P_0\ne 0$, Lemma 2.1 implies that these quantities are also uniformly bounded for $j\le 0$. The conclusion follows as in the proof of Theorem 2.3. ∎

The principal part at infinity of the polynomial in (2.6) is defined in a similar way. For $1\le j\le k$, let

$$\widetilde{V}_j=\operatorname{span}\{v_\alpha : d(\alpha)\ge d_j\}\ .$$

Then the $\widetilde{V}_j$ form a descending filtration of $\mathbb{R}^n$, i.e.

$$\{0\}\subset\widetilde{V}_k\subseteq\widetilde{V}_{k-1}\subseteq\cdots\subseteq\widetilde{V}_1\subseteq\mathbb{R}^n\ .$$

For $0 \le j \le k$, decompose $\widetilde{V}_j$ as $\widetilde{V}_{j+1} \oplus \widetilde{W}_j$, so that

$$\widetilde{V}_j = \widetilde{W}_j \oplus \cdots \oplus \widetilde{W}_k \ .$$

Define

$$P_\infty(y) = \sum_{d(\alpha)=d_k} v_\alpha y^\alpha + \sum_{d(\alpha)=d_{k-1}} \pi_{\widetilde{W}_{k-1}}(v_\alpha) y^\alpha + \cdots + \sum_{d(\alpha)=d_1} \pi_{\widetilde{W}_1}(v_\alpha) y^\alpha \ .$$

Then P_∞ is homogeneous with respect to the dilations on $\mathbb{R}^n$:

$$x \longmapsto r \bullet x = r^{d_0} \pi_{\widetilde{W}_0} x + \cdots + r^{d_k} \pi_{\widetilde{W}_k} x \ , \tag{2.12}$$

and

$$P_\infty(y) = \lim_{r \to \infty} r^{-1} \bullet P(r \cdot y) \ .$$

Definition. *We say that P is non-degenerate at infinity if P_∞ is a regular polynomial.*

Theorem 2.6 *Let K be a smooth Calderón-Zygmund kernel on $\mathbb{R}^m$ adapted to a given set of dilations and supported away from the origin. Let $P : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ be a polynomial that is non-degenerate at infinity. Then the operator T in* (2.2) *can be written as*

$$Tf(x) = \int_{\mathbb{R}^n} f(x - x') K'(x') \, dx' \ ,$$

where K' is a "rough" Calderón-Zygmund kernel, adapted to the dilations (2.12)*, locally integrable, and which satisfies the standard integral condition* (2.4)*. Hence T is bounded on L^p for $1 < p < \infty$ and weak-type 1–1.*

The proof is essentially the same as that of Theorem 2.5.

3. Maximal functions

Theorem 2.5 and 2.6 have their companion maximal theorems.

Theorem 3.1 *Let $P : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ be non-degenerate at 0. Then the maximal operator*

$$M_P^0 f(x) = \sup_{0<r\le 1} \frac{1}{|B_r|} \int_{|y|<r} \big| f\big(x - P(y)\big)\big| \, dy \tag{3.1}$$

is weak type 1–1.

Proof. Restricting the supremum to $r = 2^j$, $j \le 0$, and introducing a positive smooth cutoff function in the integrals, then

$$M_P^0 f(x) \le C \sup_{j\le 0} \int \big|f\big(x-2^j\circ P^{(j)}(y)\big)\big|\varphi(y)\,dy = \sup_{j\le 0}\int |f(x-2^j\circ x')|\psi_j(x')\,dx',$$

where the ψ_j have uniformly bounded supports and integral Lipschitz norms. The conclusion follows from weak-type vector-valued inequalities (see [12] page 80). ■

In the same way one shows that if P is non-degenerate at ∞, the maximal operator M_P^∞, defined by taking the supremum in (3.1) over $r \ge 1$, is also weak type 1–1.

Turning now to matters of uniformity, we may observe that, because of (2.3), the weak 1–1 estimates on M_P^0 are uniform in P, as long as there are $\delta, c, c' > 0$ such that for each j

$$P^{(j)}(B_1) \subset B_c\ , \qquad \int_{B_1} J_{P^{(j)}}(y)^{-\delta}\,dy \le c'\ . \tag{3.2}$$

When $m = n$ it turns out that we can recast this condition in a more readily verifiable form, for which we need some preliminary lemmas. Both of these lemmas also play a rôle in Section 4. The first lemma involves rather refined considerations related to Bézout's theorem (see [5], p.223).

Lemma 3.2 *Let $P = (p_1, p_2, \dots, p_n) : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a polynomial, such that $J_P = \det DP$ is not identically zero. Let $\deg p_j = m_j$ and $M = m_1 m_2 \cdots m_n$. Then there are M open subsets $A_1, A_2, \dots, A_M$ of $\mathbb{R}^n$ such that*

(i) the A_j are pairwise disjoint and they cover $\mathbb{R}^n$ except for a set of measure zero;

(ii) the restriction of P to A_j is a diffeomorphism with its image.

Proof. Call $E = \{x : J_P(x) = 0\} \subset \mathbb{R}^n$ and $\tilde{E} = \{(x, P(x)) : x \in E\} \subset \mathbb{R}^{2n}$. Then $\tilde{E}$ is an algebraic scheme (i.e. a finite union of algebraic varieties) of dimension strictly smaller than n. It follows from the Tarski-Seidenberg theorem that $P(E) = \pi_2(\tilde{E})$ is a semi-algebraic subset of $\mathbb{R}^n$ of dimension strictly smaller than n, and such is also its Zariski closure E' in $\mathbb{R}^n$ (see [1] pp. 45, 46). It follows that E' is closed also in the Euclidean topology and that $|E'| = 0$.

Let $y = (y_1, y_2, \dots, y_n) \notin E'$; we claim that the set $P^{-1}(y)$ contains at most M points. In order to see this, let $\widetilde{P}$ the natural extension of P to a function from $\mathbb{C}^n$ to $\mathbb{C}^n$, and call

$$q_j(z_0, z_1, \dots, z_n) = z_0^{m_j}\tilde{p}_j\left(\frac{z_1}{z_0}, \dots, \frac{z_n}{z_0}\right) - y_j z_0^{m_j}\ .$$

Let V_j be the subscheme of the complex projective space $\mathbb{P}^n_{\mathbb{C}}$ determined by the equation $q_j = 0$. Since $\mathbb{C}$ is algebraically closed, each irreducible component of V_j has dimension $n-1$ at least. If, on the other hand, one component had dimension n, then p_j would be constant, contradicting the assumption that J_P is not identically zero. It follows that V_j is a pure dimensional subscheme of dimension $n-1$ and degree m_j.

Let φ denote the immersion of $\mathbb{C}^n$ into $\mathbb{P}^n_{\mathbb{C}}$ given by

$$\varphi(z_1, \dots, z_n) = \mathbb{C}(1, z_1, \dots, z_n) \ .$$

If $x \in P^{-1}(y) \subset \mathbb{R}^n$, then $J_P(x) \neq 0$, so that x is isolated in $\widetilde{P}^{-1}(y) \subset \mathbb{C}^n$. It follows that $\varphi(x)$ is isolated in $\bigcap_{j=1}^n V_j$. Hence each element in $\varphi\big(P^{-1}(y)\big)$ is an irreducible component of $\bigcap_{j=1}^n V_j$ of degree greater than or equal to 1. By Bézout's theorem,

$$\#\big(P^{-1}(y)\big) \leq \sum_{x \in P^{-1}(y)} \deg(\{x\}) \leq M \ .$$

Let now $E'' = P^{-1}(E')$. Then E'' is closed (in the Zariski and hence in the Euclidean topology) and is a proper subset of $\mathbb{R}^n$ (otherwise, we would have $P(\mathbb{R}^n) = E'$, which contradicts the assumption that J_P is not identically zero). Hence $|E''| = 0$.

The map $P : \mathbb{R}^n \setminus E'' \longrightarrow \mathbb{R}^n \setminus E'$ is a local diffeomorphism at each point of its domain and at most M-to-one. Any point $y \in \mathbb{R}^n \setminus E'$, has a neighbourhood B_y such that $P^{-1}(U_y)$ is the disjoint union of neighbourhoods $U_1, \dots, U_q$ of the elements $x_1, \dots, x_q \in P^{-1}(y)$, $(q \leq M)$, and $P : U_j \to B_y$ is a diffeomorphism.

If the B_y have been chosen from a countable basis of balls, a simple inductive construction shows that we can reduce ourselves to the following situation: $\mathbb{R}^n \setminus E'$ is covered, up to a set of measure zero, by open subsets B'_j that are pairwise disjoint and, in addition, $P^{-1}(B'_j)$ is the disjoint union of at most M open sets $U'_{j,1}, \dots, U'_{j,q_j}$ such that $P : U'_{j,i} \to B'_j$ is a diffeomorphism for every i, j.

It is also easy to see that the $U'_{j,i}$ cover $\mathbb{R}^n \setminus E''$ up to a set F of measure zero. If it were not so, let x_0 be a point of positive density in F. If W is a neighbourhood of x_0, then we would have

$$\big|P(W \cap F)\big| = \int_{W \cap F} |J_P(x)| \, dx > 0 \, ,$$

contradicting the fact that $P(F)$ is disjoint from each B'_j.

We now set $A_i = \bigcup_j U'_{j,i}$ for $i = 1, 2, \dots, M$ and this concludes the proof. ∎

The second lemma sharpens Lemma 2.1 and is a direct consequence of Theorem 7.1 in [2].

Lemma 3.3 *Let $p(x)$ be a (scalar-valued) polynomial on $\mathbb{R}^n$ with $\deg p \le d$. There is a constant C_d, depending only on d, such that*

$$\frac{1}{|B_r|}\big|\{x \in B_r : |p(x)| < \alpha\}\big| \le C_d \frac{\alpha^{1/d}}{\left(\frac{1}{|B_r|}\int_{B_r}|p(x)|\,dx\right)^{1/d}} \ . \tag{3.3}$$

Proof. By scaling, we may assume that $r = 1$ and that $\int_{B_1}|p(x)|\,dx = 1$. By equivalence of norms on a finite dimensional space, there are constants c_1 and c_2, depending only on d, such that $|p(x)| \le c_1$ for $x \in B_1$ and $\inf_{B_1}|\partial^\beta p| \ge c_2$ for some multi-index β with $0 \le |\beta| \le d$.

Then the estimate is trivial for $\alpha \ge c_1$. For $\alpha < c_1$, Theorem 7.1 in [2] implies that

$$\big|\{x \in B_1 : |p(x)| < \alpha\}\big| \le C_d\ \alpha^{1/|\beta|}\ ,$$

which easily gives the required estimate. ■

Condition (3.3) is the end-point (non-isotropic) version of Corollary 3 in [8], stating that polynomials of degree at most d are uniformly in A_p for $p > d+1$.

This uniform A_p-condition tells us that condition (3.2) is implied, for δ small enough, by the following:

$$P^{(j)}(B_1) \subset B_c\ , \qquad \int_{B_1} J_{P^{(j)}}(y)\,dy \ge c'\ . \tag{3.2'}$$

At this point we concentrate our attention on the case $m = n$, where we can give a neater form to (3.2'): in fact, as a consequence of Lemma 3.2,

$$\int_{B_1} J_{P^{(j)}}(y)\,dy \sim \big|P^{(j)}(B_1)\big|\,.$$

Suppose that two families of dilations on $\mathbb{R}^n$ have been fixed, together with two corresponding homogeneous gauges, and call B_r and B'_r the respective non-isotropic balls.

Given an integer k and a number σ, $0 < \sigma \le 1$, let $\mathcal{P}^0_{k,\sigma}$ be the class of polynomials $P : \mathbb{R}^n \to \mathbb{R}^n$ such that $P(0) = 0$, $\deg P \le k$, and moreover there is $c = c(P) > 0$ such that

$$P(B_r) \subset B'_{cr}\ , \qquad \big|P(B_r)\big| \ge \sigma|B'_{cr}|\ , \quad \text{for } 0 < r \le 1. \tag{3.4}$$

Theorem 3.4 *Let $P \in \mathcal{P}^0_{k,\sigma}$, for some fixed pair of families of dilations. Then P is non-degenerate at 0, and the weak type 1–1 bound for M^0_P depends only on k and σ.*

Proof. Let $\cdot, \bullet$ denote the dilations in the domain and the codomain of P respectively. To begin with, observe that replacing P by $c^{-1} \bullet P$ does not alter the norm of M^0_P. Hence we can assume that $c = 1$ in (3.4).

Let $P_r(y) = r^{-1} \bullet P(r \cdot y)$. The first inequality in (3.4) implies that the P_r are uniformly bounded for $r \leq 1$. Therefore there is a sequence $r_\nu \to 0$ such that P_{r_ν} tends to some homogeneous polynomial Q. The second inequality in (3.4) tells us that Q is regular. By Proposition 2.4, P is non-degenerate at 0.

Now, as we have already observed, (3.4) and Lemma 3.2 imply (3.2'), which, combined with the uniform A_p condition, gives (3.2). ■

We remark that every P that is non-degenerate at 0 belongs to some class $\mathcal{P}^0_{k,\sigma}$ for appropriate dilations.

Defining classes $\mathcal{P}^\infty_{k,\sigma}$ in a similar fashion, we obtain uniform bounds for the operators M^∞_P. The details are left to the reader, as well as the formulation of similar results for singular integrals.

4. Supermaximal functions

We switch now our attention to supermaximal functions. As in [3], these are defined by taking a supremum not only in r for a fixed P, but also over P ranging in a given class of polynomials. We keep the assumption that $m = n$.

It is convenient at this point to introduce new classes $\mathcal{Q}_{k,\sigma}$, consisting of polynomials from $\mathbb{R}^n$ to itself, of degree at most k, with $P(0) = 0$, and such that for every $r > 0$ there is $r' = r'(P, r) > 0$ for which

$$P(B_r) \subset B'_{r'}\ , \qquad \big|P(B_r)\big| \geq \sigma |B'_{r'}|\ . \tag{4.1}$$

Clearly, $\mathcal{P}^0_{k,\sigma} \subset \mathcal{Q}_{k,\sigma}$.

Theorem 4.1 *Define*

$$M_{k,\sigma} f(x) = \sup_{r>0\,,\,P\in\mathcal{Q}_{k,\sigma}} \frac{1}{|B_r|} \int_{B_r} \big| f\big(x - P(x')\big)\big|\, dx'\ .$$

Then $M_{k,\sigma}$ is bounded on L^p for $p > \bar{p} = n(k-1) + 1$ and is restricted weak type $\bar{p}$ – $\bar{p}$.

Remark. In dimension 1, let $\mathcal{P}_k$ consist of all polynomials of degree at most k which vanish at the origin. Then $\mathcal{Q}_{k,1/2} = \mathcal{P}_k$, because, if $(-r', r')$ is the smallest interval containing $P(-r, r)$, then this set contains at least one of the two halves of the interval. This shows that Theorem 2.1 of [3] is the special case $n = 1$ of Theorem 4.1.

The proof of Theorem 4.1 follows the same lines as that of Theorem 2.1 in [3]. We emphasize those points where modifications are needed(4).

Proof. It is sufficient to prove that $M_{k,\sigma}$ is restricted weak type $\bar{p}$–$\bar{p}$.

Take $f = \chi_S$, the characteristic function of a set S, and $P \in \mathcal{Q}_{k,\sigma}$. Then J_P is not identically zero. We can then apply Lemma 3.2. Let M and A_j be as above and observe that $M \le k^n$. We have

$$
\begin{aligned}
\frac{1}{|B_r|}\int_{B_r} f\big(x - P(x')\big)\,dx' &= \frac{1}{|B_r|}\sum_{j=1}^{M}\int_{P(A_j\cap B_r)} f(x-y)\frac{1}{\big|J_P\big(P_{(j)}^{-1}(y)\big)\big|}\,dy \\
&= \int_{P(B_r)} f(x-y)g(y)\,dy\ , \qquad (4.2)
\end{aligned}
$$

where $P_{(j)}^{-1}$ denotes the inverse function of $P_{|A_j}$ and

$$
g(y) = \frac{1}{|B_r|}\sum_{j=1}^{M}\frac{1}{\big|J_P\big(P_{(j)}^{-1}(y)\big)\big|}\chi_{P(A_j\cap B_r)}(y)\ .
$$

As in [3], we obtain that

$$
\begin{aligned}
\int_{P(B_r)} & f(x-y)g(y)\,dy \le \\
&\le \Big(\frac{1}{\big|P(B_r)\big|}\int_{P(B_r)} f(x-y)^{\bar{p}}\,dy\Big)^{1/\bar{p}}\big|P(B_r)\big|^{1/\bar{p}}\|g\|_{L^{\bar{p}',\infty}(P(B_r))}\ . \qquad (4.3)
\end{aligned}
$$

By (4.1),

$$
\begin{aligned}
\frac{1}{\big|P(B_r)\big|}\int_{P(B_r)} f(x-y)^{\bar{p}}\,dy &\le \frac{1}{\sigma|B'_{r'}|}\int_{B'_{r'}} f(x-y)^{\bar{p}}\,dy \\
&\le \frac{1}{\sigma}\mathcal{M}(f^{\bar{p}})(x)\ , \qquad (4.4)
\end{aligned}
$$

where $\mathcal{M}(f^{\bar{p}})$ is the Hardy-Littlewood maximal function of $f^{\bar{p}}$.

(4)We take this opportunity to point out two minor errors in [3], at the bottom of page 123 and at the top of page 124 respectively. The proof that follows provides the appropriate corrections.

Consider the set

$$\{y \in P(B_r) : |g(u)| > \lambda\} \subseteq \bigcup_{j=1}^{M} \Big\{ y \in P(A_j \cap B_r) : \big|J_P\big(P_{(j)}^{-1}(y)\big)\big|^{-1} > \frac{\lambda|B_r|}{M} \Big\} .$$

Then

$$\begin{aligned} \big|\{y \in P(B_r) : |g(u)| > \lambda\}\big| &\leq \sum_{j=1}^{M} \int_{\Big\{y \in P(A_j\cap B_r) : \big|J_P\big(P_{(j)}^{-1}(y)\big)\big| < \frac{M}{\lambda|B_r|}\Big\}} dy \\ &= \sum_{j=1}^{M} \int_{\big\{x \in A_j\cap B_r : |J_P(x)| < \frac{M}{\lambda|B_r|}\big\}} |J_P(x)|\, dx \\ &\leq \frac{M}{\lambda|B_r|} \Big|\Big\{ x \in B_r : \big|J_P(x)\big| < \frac{M}{\lambda|B_r|}\Big\}\Big| . \end{aligned}$$

By Lemma 3.3, observing that $\deg J_P \leq n(k-1) = \bar{p} - 1$,

$$\Big|\Big\{ x \in B_r : \big|J_P(x)\big| < \frac{M}{\lambda|B_r|}\Big\}\Big| \leq \frac{C_k|B_r|}{\lambda^{\bar{p}'-1}\big(\int_{B_r}|J_P(x)|\,dx\big)^{1/(\bar{p}-1)}} . \tag{4.5}$$

Since $\int_{B_r}|J_P(x)|\,dx \geq \big|P(B_r)\big|$, we have

$$\|g\|_{L^{\bar{p}',\infty}\big(P(B_r)\big)} \leq C_k \big|P(B_r)\big|^{-1/\bar{p}} . \tag{4.6}$$

Finally, putting together (4.2), (4.4) and (4.6), we have

$$\frac{1}{|B_r|}\int_{B_r} f\big(x - P(x')\big)\,dx' \leq C_{k,\sigma}(\mathcal{M}f^{\bar{p}})^{1/\bar{p}}(x) ,$$

whence

$$M_{k,\sigma}f(x) \leq C_{k,\sigma}(\mathcal{M}f^{\bar{p}})^{1/\bar{p}}(x) .$$

The conclusion follows from the weak-type 1–1 estimate for the Hardy-Littlewood maximal function. ∎

5. Holomorphic polynomials in $\mathbb{C}$

We do not know if the exponent $\bar{p} = n(k-1)+1$ in Theorem 4.1 is sharp. However, it is quite possible to have positive results for $p < \bar{p}$ if the class $\mathcal{Q}_{k,\sigma}$ of admissible polynomials is replaced by a proper subclass.

We discuss here the situation where $n = 2$, $\mathbb{R}^2$ is identified with $\mathbb{C}$ (with isotropic dilations both in the domain and in the codomain), and the polynomials are assumed to be holomorphic.

We then call $\mathcal{H}_k$ the class of holomorphic polynomials P on $\mathbb{C}$ of degree at most k and such that $P(0) = 0$.

Lemma 5.1 *For every k there is $\sigma_k > 0$ such that $\mathcal{H}_k \subset \mathcal{Q}_{k,\sigma_k}$.*

Proof. If $P(z) = \sum_{j=1}^k a_j z^j$, then $P(B_r) \subseteq B_{r'}$ with $r' = \sum_{j=1}^k |a_j| r^j$. On the other hand,

$$\big|P(B_r)\big| \geq \frac{1}{k} \int_{|z|<r} |P'(z)|^2 \, dz = \frac{\pi}{k} \sum_{j=1}^{k} j |a_j|^2 r^{2j} \geq c_k {r'}^2 ,$$

by the orthogonality of monomials in $L^2(B_r)$. ∎

Theorem 5.2 *The supermaximal operator*

$$\tilde{M}_k f(z) = \sup_{r>0\,,\, P \in \mathcal{H}_k} \frac{1}{|B_r|} \int_{B_r} \big|f\big(z - P(z')\big)\big| \, dz'$$

is bounded on L^p for $p > k$ and restricted weak-type k–k.

The proof is based on the following improvement of Lemma 3.3 in the context of holomorphic polynomials.

Lemma 5.3 *Let q be a holomorphic polynomial of degree less than or equal to d. Then*

$$\big|\big\{z \in B_r : |q(z)| < \alpha\big\}\big| \leq C_d \frac{|B_r|^{1+1/d} \alpha^{2/d}}{\Big(\int_{B_r} |q(z)|^2 \, dz\Big)^{1/d}} .$$

Proof. We can assume that q has degree d and, modulo a change of scale, that $r = 1$ and $\int_{B_1} |q(z)|^2 \, dz = 1$. Arguing as in [8], there is $z_0 \in B_1$ such that $|q(z_0)| \geq c$, where c only depends on d. Let $z_1, \dots, z_d$ be the (possibly repeated) roots of q. Then

$$|q(z)| \geq c \frac{|q(z)|}{|q(z_0)|} = c \prod_{j=1}^{d} \frac{|z - z_j|}{|z_0 - z_j|} .$$

Hence

$$\big\{z \in B_1 : |q(z)| < \alpha\big\} \subseteq \bigcup_{j=1}^{d} \big\{z \in B_1 : |z - z_j| < c^{-1/d} \alpha^{1/d} |z_0 - z_j|\big\} .$$

We can consider only small values of α, so that if $|z_j| \geq 10$, the corresponding set on the right-hand side is empty. Therefore,

$$\begin{aligned} \big|\big\{z \in B_1 : |q(z)| < \alpha\big\}\big| \;&\leq\; \sum_{j : |z_j| < 10} \big|\big\{z \in B_1 : |z - z_j| < c^{-1/d} \alpha^{1/d} |z_0 - z_j|\big\}\big| \\ &\leq\; C_d \, \alpha^{2/d} . \end{aligned}$$

∎

The proof of Theorem 5.2 now follows the same lines as that of Theorem 4.1.

It is natural at this point to look for an analogue of Theorem 3 in [3], viz. the uniform weak-type 1–1 estimates for the maximal functions associated to polynomials in $\mathcal{H}_k$. Such a result is not contained in Theorem 3.4, due to the fact that $\mathcal{H}_k$ is not a finite union of classes $\mathcal{P}^0_{k,\sigma}$ or $\mathcal{P}^\infty_{k,\sigma}$.

On the other hand, the proof of Theorem 3 in [3] relies on the comparison between the size of a polynomial and that of its derivative away from the zeroes of the polynomial itself. It is not surprising that such estimates are valid not only on the real axis, but on the whole complex plane. We report the relevant part of the statement of Lemma 2.5 in [3], leaving the minor modifications required in the proof to the reader.

Lemma 5.4 *Let $P \in \mathcal{H}_k$ with leading coefficient 1, and with roots $z_1 = 0, z_2, \dots, z_k$ ordered so that*

$$0 = |z_1| \le |z_2| \le \cdots \le |z_k| \ .$$

There are constants $c(k) \ge 1$ and $\varepsilon(k) > 0$, depending only on k, such that if $A > c(k)$ and j, ℓ are such that $\ell - j \ge 3$ and

$$|z_\nu| < A^j < A^\ell < |z_{\nu+1}|$$

for some ν, then, for $A^{j+1} < |z| < A^{\ell-1}$,

$$\left(1 - \frac{1}{A}\right)^{k-1} |z_{\nu+1}| \cdots |z_k| \, |z|^\nu \le |P(z)| \le \left(1 + \frac{1}{A}\right)^{k-1} |z_{\nu+1}| \cdots |z_k| \, |z|^\nu \ ; \tag{5.1}$$

$$\left| \frac{zP'(z)}{P(z)} \right| \ge \varepsilon(k) \ . \tag{5.2}$$

Theorem 5.5 *Let $P \in \mathcal{H}_k$. Then*

$$M_P f(z) = \sup_{r>0} \frac{1}{|B_r|} \int_{B_r} \left| f\big(z - P(z')\big) \right| dz'$$

is weak-type 1–1 with bounds that depend only on k.

For the proof, see Lemma 3.8 and Theorem 3.9 in [3]. The only modification consists in the observation that the annulus where $A^{j+1} < |z| < A^{\ell-1}$ can be decomposed as the union of at most k regions where P is one-to-one.

6. Comments on extensions to nilpotent Lie groups

Assume now that $\mathbb{R}^n$ (i.e. the codomain of P) is endowed with a nilpotent Lie group structure, and that the product of two elements x and x' is expressed by polynomials in x and x' (e.g. in canonical coordinates of some kind). We can then modify the above definitions of maximal and singular integral operators by substituting the difference $x-P(y)$ with the product $x \cdot P(y)^{-1}$.

Then all the arguments given in the previous sections apply, *as long as the dilations on* $\mathbb{R}^n$ *that are involved are group automorphisms.* This is a rather severe restriction, and it is natural to ask if it can be removed.

In order to see that this is not a trivial question, consider the following simple example: P is the identity map from $\mathbb{R}^{2n+1}$ to the Heisenberg group H_n, and we have isotropic dilations on the domain space. We can then ask if the "isotropic" Hardy-Littlewood maximal function on H_n is weak-type 1–1.

Isotropic dilations are not group automorphisms, but the "parabolic" dilations $(z,t) \mapsto (rz, r^2 t)$ are (here $z \in \mathbb{R}^{2n}$ and $t \in \mathbb{R}$). So, the isotropic Hardy-Littlewood maximal function can be rescaled by means of the parabolic dilations, and it preserves all its boundedness properties. In the limit, we obtain the maximal function along the z-plane:

$$Mf(z,t) = \sup_{r>0} r^{-2n} \int_{|w|<r} \left| f\big((z,t)\cdot(w,0)^{-1}\big) \right| dw \ .$$

If we then knew that the isotropic Hardy-Littlewood maximal function is weak-type 1–1 on H_n, then we would also know that M is weak-type 1–1. But this problem is of the same nature as the weak-type 1–1 boundedness problem for the maximal function along the parabola in the plane, because the z-plane in H_n is a "curved" surface (e.g. it has non-zero rotational curvature, [12]).

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