

Fredrik Ekström, Centre for Mathematical Sciences, Lund University, Box 118, 22 100 Lund, Sweden. email: fredrike@maths.lth.se

THE FOURIER DIMENSION IS NOT FINITELY STABLE

Abstract

The Fourier dimension is not, in general, stable under finite unions of sets. Moreover, the stability of the Fourier dimension on particular pairs of sets is independent from the stability of the compact Fourier dimension.

1 Introduction

The present note gives an example to show that the Fourier dimension is not stable under finite unions of sets. This improves one of the results in [1], where it was shown that the Fourier dimension is not countably stable. For a more detailed introduction to the Fourier dimension than is given here, see for example [1] and references therein.

The *Fourier transform* of a finite Borel measure μ on \mathbf{R}^d is defined as

$$\widehat{\mu}(\xi) = \int e^{-2\pi i \xi \cdot x} d\mu(x),$$

where $\xi \in \mathbf{R}^d$ and \cdot denotes the Euclidean inner product. The *Fourier dimension* of μ measures how quickly $\widehat{\mu}$ decays at infinity, and is defined by

$$\dim_{\mathbf{F}} \mu = \sup \{s \in [0, d]; \exists C \in \mathbf{R} \text{ such that } |\widehat{\mu}(\xi)| \leq C|\xi|^{-s/2} \text{ for all } \xi\}.$$

If A is a Borel subset of \mathbf{R}^d , then the Fourier dimension of A is defined to be

$$\dim_{\mathbf{F}} A = \sup \{\dim_{\mathbf{F}} \mu; \mu \in \mathcal{P}(A)\},$$

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where $\mathcal{P}(A)$ denotes the set of Borel probability measures on \mathbf{R}^d that give full measure to A . It can be shown that if μ is compactly supported and $0 < \dim_{\mathbf{F}} \mu < d$, then

$$I_s(\mu) := \iint |x - y|^{-s} d\mu(x) d\mu(y)$$

is finite for all $s < \dim_{\mathbf{F}} \mu$; see [2, Lemma 12.12]. From this, it follows that $\dim_{\mathbf{F}} A \leq \dim_{\mathbf{H}} A$ for any Borel set A .

As an alternative way to define the Fourier dimension, one could require that the measure in the definition give full measure to some compact subset of A and not just to A itself. This *compact Fourier dimension* is thus defined as

$$\dim_{\mathbf{FC}} A = \sup \{ \dim_{\mathbf{F}} \mu; \mu \in \mathcal{P}(K), K \subset A \text{ is compact} \}.$$

One of the examples in [1] shows that the compact Fourier dimension is not finitely stable, but that example is not a counterexample to finite stability of the Fourier dimension. Example 2 below produces the opposite situation, namely, sets A' and B' such that

$$\begin{aligned} \dim_{\mathbf{F}}(A' \cup B') &> \max(\dim_{\mathbf{F}} A', \dim_{\mathbf{F}} B'), \\ \dim_{\mathbf{FC}}(A' \cup B') &= \max(\dim_{\mathbf{FC}} A', \dim_{\mathbf{FC}} B'). \end{aligned}$$

2 The example

The following lemma is used in the example. It previously appeared in [1], but is included here for completeness.

Lemma 1. *For any $\varepsilon \in (0, 1]$,*

$$\inf_{\mu} \sup_{j \geq 1} |\widehat{\mu}(j)| \geq \frac{\pi\varepsilon}{8 + 2\pi\varepsilon} \quad \left(\geq \frac{\varepsilon}{5} \right),$$

where the infimum is over all $\mu \in \mathcal{P}([\varepsilon, 1])$ and the supremum is over all positive integers j .

PROOF. Fix $\varepsilon > 0$ and take any $\mu \in \mathcal{P}([\varepsilon, 1])$. If φ is a real-valued continuous function supported on $[0, \varepsilon]$ such that

$$\int \varphi(x) dx = 1 \quad \text{and} \quad \sum_{k=-\infty}^{\infty} |\widehat{\varphi}(k)| < \infty,$$

then

$$0 = \mu(\varphi) = \sum_{k=-\infty}^{\infty} \widehat{\varphi}(k)\overline{\widehat{\mu}(k)} = 1 + 2 \operatorname{Re} \left(\sum_{k=1}^{\infty} \widehat{\varphi}(k)\overline{\widehat{\mu}(k)} \right),$$

and thus

$$\frac{1}{2} \leq \sum_{k=1}^{\infty} |\widehat{\varphi}(k)| |\widehat{\mu}(k)| \leq \left(\sum_{k=1}^{\infty} |\widehat{\varphi}(k)| \right) \left(\sup_{j \geq 1} |\widehat{\mu}(j)| \right).$$

Now let χ be the indicator function of $[0, \varepsilon/2]$, and take φ to be the triangle pulse

$$\varphi = \left(\frac{2\chi}{\varepsilon} \right) * \left(\frac{2\chi}{\varepsilon} \right),$$

where $*$ denotes convolution. Then

$$|\widehat{\varphi}(k)| = \left| \frac{2\widehat{\chi}(k)}{\varepsilon} \right|^2 = \operatorname{sinc}^2 \left(\frac{k\pi\varepsilon}{2} \right) \leq \min \left(1, \frac{4}{k^2\pi^2\varepsilon^2} \right),$$

so that

$$\sum_{k=1}^{\infty} |\widehat{\varphi}(k)| \leq \left\lceil \frac{2}{\pi\varepsilon} \right\rceil + \frac{4}{\pi^2\varepsilon^2} \sum_{\lceil \frac{2}{\pi\varepsilon} \rceil + 1}^{\infty} \frac{1}{k^2} \leq \frac{2}{\pi\varepsilon} + 1 + \frac{4}{\pi^2\varepsilon^2} \int_{\frac{2}{\pi\varepsilon}}^{\infty} \frac{1}{x^2} dx = \frac{4 + \pi\varepsilon}{\pi\varepsilon}.$$

It follows that

$$\sup_{j \geq 1} |\widehat{\mu}(j)| \geq \frac{1}{2} \cdot \frac{\pi\varepsilon}{4 + \pi\varepsilon}.$$

□

Example 2. Let $s \in (\sqrt{3} - 1, 1)$ and choose b such that

$$\frac{1-s}{s} < b < \frac{s}{2}$$

(this is possible since $s > \sqrt{3} - 1$). Let $(l_k)_{k=1}^{\infty}$ be a sequence of natural numbers such that

$$\lim_{k \rightarrow \infty} \frac{l_{k+1}}{l_k} = \infty,$$

and set $m_k = \lceil bl_k \rceil$. Whenever $x \in [0, 1]$ is not a dyadic rational, and thus has a unique binary decimal expansion $x = 0.x_1x_2 \dots$, let

$$f(x) = \sup (\{k; x_{l_k+1} \dots x_{l_k+m_k} = 0^{m_k}\} \cup \{0\}).$$

Then for each k ,

$$\lambda\{x; f(x) = \infty\} \leq \lambda\left(\bigcup_{j=k}^{\infty} \{x; x_{l_{j+1}} \dots x_{l_j+m_j} = 0^{m_j}\}\right) \leq \sum_{j=k}^{\infty} 2^{-m_j},$$

where λ denotes Lebesgue measure. The sum converges since (m_k) is eventually strictly increasing, and thus $f(x)$ is finite for λ -a.e. $x \in [0, 1]$. Let

$$A = \{x \in [0, 1]; f(x) \text{ is even}\}, \quad B = \{x \in [0, 1]; f(x) \text{ is odd}\};$$

then $\dim_{\mathbb{F}}(A \cup B) = 1$, since $\lambda(A \cup B) = 1$.

To see that $\dim_{\mathbb{F}} A \leq s$, take any $\mu \in \mathcal{P}(A)$ and define for odd k

$$A_k = \{x \in A; x_{l_{k+1}} \dots x_{l_k+m_k} = 0^{m_k}\}$$

$$A_k^j = \{x \in A_k; f(x) = j\}.$$

If $x \in A_k$, then $f(x) \geq k$, so there must be some even number $j \geq k + 1$ such that $f(x) = j$. Hence for each k ,

$$A_k = \bigcup_{\substack{j \geq k+1 \\ j \text{ even}}} A_k^j.$$

Let

$$\alpha_k = \mu(A_k), \quad \alpha_k^j = \mu(A_k^j),$$

and let P be the set of natural numbers k such that

$$\alpha_k^j \leq \frac{2^{-(m_k+j-k)}}{6} \quad \text{for all even } j \geq k + 1.$$

Suppose first that the set P is infinite. Let $\mu_k = \mu|_{A \setminus A_k}$, and let ν_k be the image of μ_k under the map $x \mapsto 2^{l_k}x \pmod{1}$. The measure ν_k is concentrated on $[2^{-m_k}, 1]$ and has total mass $1 - \alpha_k$. So by Lemma 1, there is some $r_k \geq 1$ such that

$$|\widehat{\mu}_k(2^{l_k}r_k)| = |\widehat{\nu}_k(r_k)| \geq (1 - \alpha_k) \frac{2^{-m_k}}{5}.$$

For $k \in P$,

$$\alpha_k = \sum_{\substack{j \geq k+1 \\ j \text{ even}}} \alpha_k^j \leq \frac{2^{-m_k}}{6},$$

and hence

$$\begin{aligned} (2^{l_k} r_k)^{s/2} |\widehat{\mu}(2^{l_k} r_k)| &\geq 2^{sl_k/2} (|\widehat{\mu}_k(2^{l_k} r_k)| - \alpha_k) \\ &\geq 2^{sl_k/2 - m_k} \left(\frac{1 - \alpha_k}{5} - \frac{1}{6} \right), \end{aligned}$$

where the exponent is positive for large k since $b < s/2$. Thus if $(k_i)_{i=1}^\infty$ is an enumeration of P , then

$$\limsup_{|\xi| \rightarrow \infty} |\xi|^{s/2} |\widehat{\mu}(\xi)| \geq \lim_{i \rightarrow \infty} (2^{l_{k_i}} r_{k_i})^{s/2} |\widehat{\mu}(2^{l_{k_i}} r_{k_i})| = \infty,$$

so $\dim_{\mathbb{F}} \mu \leq s$.

Suppose on the other hand that P is finite. If k is odd and $j \geq k + 1$ is even, then

$$A_k^j \subset \{x \in [0, 1]; x_{l_{k+1}} \dots x_{l_k+m_k} = 0^{m_k} \text{ and } x_{l_{j+1}} \dots x_{l_j+m_j} = 0^{m_j}\} \subset \bigcup_p I_p,$$

where $\{I_p\}$ are $2^{l_j - m_k}$ intervals, each of length $2^{-(l_j+m_j)}$. Thus,

$$\begin{aligned} I_s(\mu) &\geq \sum_{p=1}^{2^{l_j - m_k}} \iint_{I_p \times I_p} |x - y|^{-s} d\mu(x) d\mu(y) \\ &\geq 2^{s(l_j+m_j)} \sum_{p=1}^{2^{l_j - m_k}} \mu(I_p)^2 \geq 2^{s(l_j+m_j)} \frac{(\alpha_k^j)^2}{2^{l_j - m_k}}, \end{aligned}$$

where the inequality $\|\cdot\|_2^2 \geq d^{-1} \|\cdot\|_1^2$ for norms in \mathbf{R}^d is used in the last step. If $k \notin P$, then $j = j(k)$ can be chosen such that the last expression is greater than or equal to

$$\frac{1}{36} \cdot 2^{\wedge((s(1+b) - 1)l_j - 2j - m_k)}.$$

Because $b > (1 - s)/s$ and l_j grows exponentially with j , there is some $\varepsilon > 0$ such that the exponent is at least

$$\varepsilon l_j - m_k \geq \varepsilon l_{k+1} - \lceil bl_k \rceil$$

whenever k (and hence j) is large enough. Since $\mathbf{N} \setminus P$ contains arbitrarily large odd k , it follows that $I_s(\mu) = \infty$, and thus $\dim_{\mathbb{F}} \mu \leq s$ in this case too.

This shows that $\dim_{\mathbb{F}} A \leq s$, and similar reasoning shows that $\dim_{\mathbb{F}} B \leq s$ as well. Thus the Fourier dimension is not finitely stable.

Consider now the F_σ -sets

$$A' = \bigcup_{k \text{ even}} \overline{f^{-1}(k)}, \quad B' = \bigcup_{k \text{ odd}} \overline{f^{-1}(k)},$$

which are slightly larger than A and B . From the inclusion

$$\overline{f^{-1}(k)} \subset \overline{\{x; x_{l_k+1} \dots x_{l_k+m_k} = 0^{m_k}\}} \cap \bigcap_{j=k+1}^{\infty} \overline{\{x; x_{l_j+1} \dots x_{l_j+m_j} \neq 0^{m_j}\}},$$

it follows that the difference sets $A' \setminus A$ and $B' \setminus B$ only contain dyadic rationals, and in particular, they are countable. Thus A' and B' have the same Fourier dimensions as A and B , respectively, using that any measure that gives positive mass to a countable set has Fourier dimension 0. In particular,

$$\dim_{\mathbb{F}}(A' \cup B') > \max(\dim_{\mathbb{F}} A', \dim_{\mathbb{F}} B').$$

On the other hand, Proposition 5 in [1] implies that the compact Fourier dimension is finitely stable on F_σ -sets, and thus

$$\dim_{\text{FC}}(A' \cup B') = \max(\dim_{\text{FC}} A', \dim_{\text{FC}} B').$$

References

- [1] F. Ekström, T. Persson and J. Schmeling, *On the Fourier dimension and a modification*, J. Fract. Geom., to appear. Preprint: arXiv:1406.1480v3.
- [2] P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces: Fractals and Rectifiability*, Cambridge Univ. Press, Cambridge, 1995.