INROADS

Real Analysis Exchange
Vol. 41(2), 2016, pp. 347–350

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A NON SELF-SIMILAR SET

Abstract

For each $s \in (0,1]$, we give an example of a nowhere dense perfect set E contained in the unit interval with $\dim_{\mathcal{H}}(E) = s$, which is not an attractor for any iterated function system composed of weak contractions. This answers a problem posed by Zoltán Buczolich at the Summer Symposium in Real Analysis XXXIX (June 8-13, 2015, St. Olaf College, Northfield, MN).

1 Introduction

Let X be a complete metric space with $\mathcal{S} = \{S_1, \ldots, S_N\}$ a finite set of contraction maps from X to itself. Call a non-empty compact subset E of X *an attractor* for the iterated function system (IFS) $\mathcal{S}$ if $E = \bigcup_{i=1}^{N} S_i(E) = \mathcal{S}(E)$ ([5], [4]). Since $\mathcal{S}$ is a contraction on the compact metric space $(\mathcal{H}, \mathcal{K}(X))$ comprised of the non-empty compact subsets of X endowed with the Hausdorff metric, there exists a unique compact set $E \subseteq X$ such that $E = \mathcal{S}(E)$. Take $\mathcal{T} = \{E \in \mathcal{K}(X) : E = \mathcal{S}(E); \mathcal{S}$ a finite collection of contraction maps$\}$ to be the set of attractors for contractive systems defined on X. From [2] and [3] one sees that $\mathcal{T}$ is always an F_σ subset of $\mathcal{K}(X)$ and, in the case that $X = [0,1]^n$, the set $\mathcal{T}$ is of the first category. In particular, there is a set $\mathcal{K}^{\star\star}$ comprised of

Mathematical Reviews subject classification: Primary: 26A18, 28A78; Secondary: 28A80
Key words: iterated function system, attractor, self-similar set, Cantor set
Received by the editors July 23, 2015
Communicated by: Zoltán Buczolich
*This research has been partially supported by the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

certain nowhere dense perfect sets contained in the irrationals which is residual in $\mathcal{K}([0,1])$ and has the property that $\mathcal{K}^{\star\star} \cap \mathcal{T} = \emptyset$.

In this brief note, for each $s \in (0,1]$ we give a simple construction of a nowhere dense perfect set E contained in the unit interval with $\dim_{\mathcal{H}}(E) = s$, which is not invariant with respect to any iterated function system comprised of weak contractions. That is, $E \neq \mathcal{S}(E)$ whenever $\mathcal{S} = \{S_1, \ldots, S_N\}$, and $d(S_i(x), S_i(y)) < d(x,y)$ for $i = 1, 2, \ldots, N$.

This example answers a problem posed by Zoltán Buczolich at the Summer Symposium in Real Analysis XXXIX (June 8-13, 2015, St. Olaf College, Northfield, MN). Its construction exploits ideas found in [6] and [1].

2 Notations

Let (X,d) be a metric space. A map $f : X \to X$ is a *contraction* if there exists a constant $M \in (0,1)$ such that, for each x, y in X,

$$d(f(x), f(y)) \leq M d(x,y).$$

A map $f : X \to X$ is a *weak contraction* if, for each x, y in X, $x \neq y$,

$$d(f(x), f(y)) < d(x,y).$$

Let A and B be subsets of X. Set $d(A,B) = \inf\{d(x,y) : x \in A, y \in B\}$. By $|A|$ we denote the diameter of A.

3 Main result

Theorem 1. *For each $s \in (0,1]$, there exists a subset E of $[0,1]$, nowhere dense and perfect, with $\dim_{\mathcal{H}} E = s$, that is not the attractor for any iterated function system composed of weak contractions.*

Proof.

1. Construction of E.

   Fix $s \in (0,1]$. The set E is defined as

   $$E = \{0\} \cup \Big\{ \bigcup_{n=1}^{\infty} E_n \Big\},$$

   where the sets E_n are taken so that, for each n:

   (a) $|E_n| = \frac{1}{5}\frac{1}{2^n}$,

(b) $s_n = \dim_{\mathcal{H}} E_n = s - \frac{s}{n+1}$; hence, $\{s_n\}$ is an increasing sequence with $\lim_n s_n = \sup_n s_n = s$,

(c) $\min E_n = \frac{1}{2^n}$, and

(d) $0 < \mathcal{H}^{s_n}(E_n) < 1$.

One notes immediately that

2. $d(E_n, E \setminus E_n) > |E_n|$,

3. $\dim_{\mathcal{H}} E = s$ and $\mathcal{H}^s(E) = 0$, and

4. if $m > n$, then $\mathcal{H}^{s_m}(f(E_n) \cap E_m) = 0$, for any Lipschitz map f.

5. We will be interested in two types of behavior inherent to weak contractions defined on E.

   Let $f : E \to E$ be a weak contraction.

   (a) Suppose $f : E \to E$, and $f(0) = 0$. We first show that, for any n, $f(E_n) \subset E \setminus \bigcup_{i=1}^{n} E_i$. In fact, there exists $x \in E_n$ such that $d(0, x) = d(0, E_n)$. Therefore, $d(0, f(x)) < d(0, x) = d(0, E_n)$. Hence $f(x) \in E \setminus \bigcup_{i=1}^{n} E_i$. The conclusion follows from the observation that

   $$|f(E_n)| < |E_n| < d(E_n, E \setminus E_n).$$

   (b) Suppose $f : E \to E$, and $f(0) \neq 0$. We show that there exists $M \in \mathbb{N}$ such that $\mathcal{H}^{s_m}(f(E) \cap E_m) = 0$ whenever $m > M$. Then $f(0) = x \in E_n$, for some n. Now, f continuous implies the existence of some $N \in \mathbb{N}$ such that $f(E_m) \subseteq E_n$ whenever $m > N$. Consider $L = \bigcup_{i=1}^{N} E_i$. Then $\mathcal{H}^{s_m}(f(L) \cap E_m) = 0$ for any $m > N$, and $\mathcal{H}^{s_m}(f(E) \cap E_m) = 0$ for any $m > M = \max\{N, n\}$.

6. Consider $\mathcal{S} = \{S_1, \ldots, S_t\}$ where each S_i is a weak contraction from $[0, 1]$ to itself. Consider two subsets of $\mathcal{S}$, $\mathcal{A}$ and $\mathcal{B}$, so defined. Let $S_i \in \mathcal{A}$ if $S_i(0) = 0$ and $S_i \in \mathcal{B}$ if $S_i(0) \neq 0$. If $S_i \in \mathcal{B}$, then there exists N_i such that $\mathcal{H}^{s_m}(S_i(E) \cap E_m) = 0$ for any $m > N_i$. Let $N^\star = \max\{N_i : S_i \in \mathcal{B}\}$. Fix E_k, $k > N^\star$. If $S_i \in \mathcal{B}$, then $\mathcal{H}^{s_k}(S_i(E) \cap E_k) = 0$. If $S_i \in \mathcal{A}$, then ${S_i}^{-1}(E_k) \subseteq \bigcup_{j=1}^{k-1} E_j$. Thus, $\mathcal{H}^{s_k}(S_i(E) \cap E_k) = 0$. We conclude that $\sum_{i=1}^{t} \mathcal{H}^{s_k}(S_i(E) \cap E_k) = 0$ and consequently $E_k \not\subseteq \mathcal{S}(E)$.

□

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