INROADS

Real Analysis Exchange
Vol. 41(1), 2015/2016, pp. 233–244

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ABSOLUTE NULL SUBSETS OF THE PLANE WITH BAD ORTHOGONAL PROJECTIONS

Abstract

Under Martin's Axiom, it is proved that there exists an absolute null subset of the Euclidean plane $\mathbf{R}^2$, the orthogonal projections of which on all straight lines in $\mathbf{R}^2$ are absolutely nonmeasurable. A similar but weaker result holds true within the framework of **ZFC** set theory.

Among various set-theoretical operations commonly used in real analysis, the standard projection operation is very important but has a somewhat unpleasant property. Namely, the orthogonal projection of a subset Z of the Euclidean plane $\mathbf{R}^2$ on a straight line lying in $\mathbf{R}^2$ may be of a much more complicated structure than the structure of Z. There are many examples confirming this circumstance. For instance, if Z is a Borel subset of $\mathbf{R}^2$, then the orthogonal projection of Z on the real line $\mathbf{R}$ is, in general, a non-Borel analytic (Suslin) subset of $\mathbf{R}$, and this fact turned out to be a starting point for the emergence and further development of classical descriptive set theory; see, e.g., [7], [11].

Also, simple examples show that the projection of a Lebesgue measurable subset of $\mathbf{R}^2$ may be a Lebesgue nonmeasurable set in $\mathbf{R}$. In the present paper we consider an analogous phenomenon for the so-called absolute null subsets of $\mathbf{R}^2$.

A measure μ defined on some σ-algebra of subsets of $\mathbf{R}$ (respectively, of $\mathbf{R}^2$) is called *continuous* if it vanishes on all singletons of $\mathbf{R}$ (respectively, of $\mathbf{R}^2$).

According to the standard definition, a subset U of $\mathbf{R}$ (respectively, of $\mathbf{R}^2$) is an *absolute null set* or *universal measure zero set* if, for every σ-finite

Mathematical Reviews subject classification: Primary: 28A05, 28D05; Secondary: 03E25
Key words: Absolute null set, Bernstein set, generalized Luzin set, Hamel basis
Received by the editors May 19, 2015
Communicated by: Krzysztof Ciesielski

continuous Borel measure μ on $\mathbf{R}$ (respectively, on $\mathbf{R}^2$), the equality $\mu^*(U) = 0$ holds true, where μ^* denotes the outer measure canonically associated with μ.

Equivalently, U is an absolute null set if and only if there exists no nonzero σ-finite continuous Borel measure on U.

The above definition shows that the absolute null subsets of $\mathbf{R}$ (of $\mathbf{R}^2$) are ultimately small with respect to the class $\mathcal{M}(\mathbf{R})$ (class $\mathcal{M}(\mathbf{R}^2)$) of the completions of all nonzero σ-finite continuous Borel measures on $\mathbf{R}$ (on $\mathbf{R}^2$). In particular, these subsets are *absolutely measurable* with respect to the two above-mentioned classes; i.e., are measurable with respect to any measure belonging to $\mathcal{M}(\mathbf{R})$ ($\mathcal{M}(\mathbf{R}^2)$).

There are several delicate constructions of uncountable absolute null subsets of $\mathbf{R}$ (of $\mathbf{R}^2$). For more details about those constructions, see, e.g., [13] and [16].

A subset X of the real line $\mathbf{R}$ is called *absolutely nonmeasurable* (with respect to the class $\mathcal{M}(\mathbf{R})$) if there exists no measure μ belonging to $\mathcal{M}(\mathbf{R})$ such that $X \in \mathrm{dom}(\mu)$.

This definition shows that absolutely nonmeasurable subsets of $\mathbf{R}$ are extremely bad relative to the class $\mathcal{M}(\mathbf{R})$. It makes sense to note that these subsets of $\mathbf{R}$ can be characterized in purely topological terms, as follows.

Recall that a subset B of $\mathbf{R}$ is a *Bernstein set* if, for each nonempty perfect set $P \subset \mathbf{R}$, the relations $P \cap B \neq \emptyset$ and $P \cap (\mathbf{R} \setminus B) \neq \emptyset$ are satisfied.

Such a set B was first constructed by Bernstein [2] in 1908. In his argument Bernstein essentially relies on an uncountable form of the Axiom of Choice (**AC**) and uses the method of transfinite recursion. Much later, it was recognized that no countable form of **AC** is enough for obtaining B. The importance of Bernstein sets in various topics of general topology, measure theory, and the theory of Boolean algebras is well known; see, e.g., [7], [8], [10], [11], [14], [15].

Lemma 1. *Let X be a subset of the real line $\mathbf{R}$. The following two assertions are equivalent:*

(1) X is absolutely nonmeasurable with respect to the class $\mathcal{M}(\mathbf{R})$;

(2) X is a Bernstein set in $\mathbf{R}$.

The proof of this lemma is not difficult and can be found, e.g., in [9] or in [10, p. 206].

Some Bernstein sets may possess additional properties of purely algebraic nature.

Example 1. *Consider the real line $\mathbf{R}$ as a vector space V over the field $\mathbf{Q}$ of all rational numbers. Any basis of V is usually called a Hamel basis of $\mathbf{R}$, because such a basis was first constructed by Hamel in [6]. There exists* *a Bernstein set in* $\mathbf{R}$ *which simultaneously is a Hamel basis of* $\mathbf{R}$*; see, for instance, [1], [3], [4, p. 113], [5, p. 11], or [14, p. 221].*

Example 2. *Let* G *be a group of transformations of* $\mathbf{R}$ *with* $\mathrm{card}(G) \leq \mathbf{c}$*, where* $\mathbf{c}$ *denotes the cardinality of the continuum. There exists a Bernstein set* $B \subset \mathbf{R}$ *which is almost invariant under the group* G*. The latter means that for each* $g \in G$*, the inequality*

$$\mathrm{card}(B \triangle g(B)) < \mathbf{c}$$

is valid, where the symbol $\triangle$ *denotes the operation of symmetric difference of two sets; cf. Theorem 21 of Chapter 5 in [14]. In particular, taking the group* Γ *of all homotheties of* $\mathbf{R}$ *with center* 0*, one obtains a Bernstein subset of* $\mathbf{R}$ *which simultaneously is almost invariant under* Γ*.*

Lemma 1 and the previous example allow us to demonstrate the existence of a small subset of $\mathbf{R}^2$ whose projection on every straight line l in $\mathbf{R}^2$ is absolutely nonmeasurable in l. In what follows, the symbol λ_1 stands for the ordinary one-dimensional Lebesgue measure on l and the symbol λ_2 stands for the ordinary two-dimensional Lebesgue measure on $\mathbf{R}^2$.

Theorem 1. *There exists a set* $T \subset \mathbf{R}^2$ *with* $\lambda_2(T) = 0$ *such that the orthogonal projection of* T *on every straight line* l *in* $\mathbf{R}^2$ *is absolutely nonmeasurable in* l*.*

Proof. Take a Bernstein set $X \subset \mathbf{R}$ which is almost invariant under the group Γ (see Example 2), and in $\mathbf{R}^2$ consider the set

$$T = (X \times \{0\}) \cup (\{0\} \times X).$$

This T is contained in the union of the two lines $\mathbf{R} \times \{0\}$ and $\{0\} \times \mathbf{R}$, so $\lambda_2(T) = 0$. Now, let l be a straight line in $\mathbf{R}^2$ and let θ denote the angle between l and $\mathbf{R} \times \{0\}$. We may assume, without loss of generality, that l passes through the origin $(0, 0)$ and that $0 < \theta < \pi/2$. It is not difficult to verify that the orthogonal projection of T on l is congruent to the set

$$T^* = \cos(\theta)X \cup \sin(\theta)X \subset \mathbf{R}.$$

By virtue of the definition of X, we have the inequalities

$$\mathrm{card}((\cos(\theta)X) \triangle X) < \mathbf{c},$$

$$\mathrm{card}((\sin(\theta)X) \triangle X) < \mathbf{c},$$

whence it follows that

$$\mathrm{card}(T^* \triangle X) < \mathbf{c}.$$

Remembering that X is a Bernstein subset of $\mathbf{R}$, we readily conclude that T^* is also a Bernstein set in $\mathbf{R}$, which completes the proof of Theorem 1. $\square$

Example 3. *Let H be a Hamel basis of $\mathbf{R}$ which simultaneously is a Bernstein set in $\mathbf{R}$; see Example 1. We may assume, without loss of generality, that $H = \{h_\xi : \xi < \alpha\}$, where α denotes the least ordinal number of cardinality $\mathbf{c}$. According to the definition of H, any nonzero element $x \in \mathbf{R}$ admits a unique representation*

$$x = q_1 h_{\xi_1} + q_2 h_{\xi_2} + ... + q_n h_{\xi_n},$$

where $n > 0$, $q_1 = q_1(x)$, $q_2 = q_2(x)$,..., $q_n = q_n(x)$ are some rational numbers distinct from zero, and $(\xi_1, \xi_2, ..., \xi_n)$ is a strictly increasing sequence of ordinals, all of which are strictly less than α. Further, let us put

$$K' = \{x \in \mathbf{R} : q_n(x) > 0\}.$$

Obviously, we may write

$$K' \cup (-K') = \mathbf{R} \setminus \{0\}, \quad K' \cap (-K') = \emptyset.$$

Moreover, since $H \subset K'$ and $-H \subset \mathbf{R} \setminus K'$, we conclude that both K' and $-K'$ are Bernstein sets in $\mathbf{R}$. Now, denoting

$$K = (K' \times \{0\}) \cup (\{0\} \times (-K')),$$

we infer that $\lambda_2(K) = 0$. At the same time, considering in $\mathbf{R}^2$ the straight line

$$l = \{(x, y) : x - y = 0\},$$

one can easily deduce that the orthogonal projection of K on l coincides with the set $l \setminus \{(0,0)\}$, so is λ_1-measurable in l. This fact explains why in the proof of Theorem 1 we appealed to the aid of an almost Γ-invariant Bernstein subset of $\mathbf{R}$.

The natural question arises whether it is possible to strengthen Theorem 1 and to establish the existence of an absolute null subset of $\mathbf{R}^2$ (with respect to the class $\mathcal{M}(\mathbf{R}^2)$), the orthogonal projections of which on all straight lines in $\mathbf{R}^2$ are absolutely nonmeasurable in those lines. In this context, let us immediately note that such a generalization of Theorem 1 is not realizable within **ZFC** set theory. Indeed, a model of **ZFC** was constructed in which the Continuum Hypothesis (**CH**) fails to be true and in which all uncountable absolute null subsets of $\mathbf{R}^2$ have cardinalities ω_1, where ω_1 denotes the least uncountable cardinal number; for more details, see [12], [13]. Since the cardinality of any Bernstein set is equal to $\mathbf{c}$, in the above-mentioned model of **ZFC** there exists no absolute null subset of $\mathbf{R}^2$ whose orthogonal projection on $\mathbf{R} \times \{0\}$ is absolutely nonmeasurable with respect to the class $\mathcal{M}(\mathbf{R})$.

Nevertheless, by using Martin's Axiom (**MA**), it becomes possible to substantially strengthen Theorem 1 in terms of absolute null subsets of $\mathbf{R}^2$. For this purpose, we need the notion of a $\mathbf{c}$-*Luzin set* in $\mathbf{R}$.

A set $X \subset \mathbf{R}$ is called a $\mathbf{c}$-Luzin subset of $\mathbf{R}$ if $\mathrm{card}(X) = \mathbf{c}$ and, for every first category set $F \subset \mathbf{R}$, the inequality $\mathrm{card}(F \cap X) < \mathbf{c}$ is satisfied.

It is well known that, under Martin's Axiom, there exist $\mathbf{c}$-Luzin subsets of $\mathbf{R}$; see, e.g., [13]. In our further consideration, two $\mathbf{c}$-Luzin sets in $\mathbf{R}$ with certain specific properties will play a key role.

Lemma 2. *Assuming Martin's Axiom, every* $\mathbf{c}$*-Luzin subset of* $\mathbf{R}$ *is an absolute null set in* $\mathbf{R}$*.*

Lemma 3. *The product of two absolute null subsets of* $\mathbf{R}$ *is an absolute null subset of* $\mathbf{R}^2$*.*

Lemmas 2 and 3 are well known, so we omit their detailed proofs here. Actually, Lemma 3 is Theorem 8.1 from [13], and Lemma 2 is readily implied by the following two assertions:

(i) Assuming Martin's Axiom, any set $X \subset \mathbf{R}$ with $\mathrm{card}(X) < \mathbf{c}$ is an absolute null subset of $\mathbf{R}$;

(ii) Every σ-finite Borel measure on $\mathbf{R}$ is concentrated on a first category subset of $\mathbf{R}$.

In connection with (i), see again [13].

In connection with (ii), see e.g. Chapter 16 in [15] where a more general result than (ii) is discussed for σ-finite Borel measures on metric spaces.

Lemma 4. *Under Martin's Axiom, there exists an absolute null subset Z of* $\mathbf{R}^2$ *such that, for every straight line l in* $\mathbf{R}^2$*, the equality* $\mathrm{card}(l \cap Z) = \mathbf{c}$ *holds true.*

Proof. Denote by α the least ordinal number of cardinality $\mathbf{c}$ and define:

$\{l_\xi : \xi < \alpha\}$ = the family of all straight lines in $\mathbf{R}^2$ not parallel to the coordinate axes $\mathbf{R} \times \{0\}$ and $\{0\} \times \mathbf{R}$;

$\{B_\xi : \xi < \alpha\}$ = the family of all those Borel subsets of $\mathbf{R}$ which are of first category in $\mathbf{R}$.

According to the definition of $\{l_\xi : \xi < \alpha\}$, every straight line l in $\mathbf{R}^2$ given by the equation

$$y = ax + b \qquad (a \in \mathbf{R},\ b \in \mathbf{R},\ a \neq 0)$$

belongs to $\{l_\xi : \xi < \alpha\}$, and we may additionally suppose that l occurs in $\{l_\xi : \xi < \alpha\}$ continuum many times.

Now, by using the method of transfinite recursion, construct two injective families

$$\{x_\xi : \xi < \alpha\} \subset \mathbf{R}, \quad \{y_\xi : \xi < \alpha\} \subset \mathbf{R}.$$

Assume that, for an ordinal number $\xi < \alpha$, the partial families $\{x_\zeta : \zeta < \xi\}$ and $\{y_\zeta : \zeta < \xi\}$ of points in $\mathbf{R}$ have already been constructed. Consider the line l_ξ. The canonical equation corresponding to this line is of the form

$$y = a_\xi x + b_\xi \qquad (a_\xi \in \mathbf{R},\ b_\xi \in \mathbf{R},\ a_\xi \neq 0).$$

Using Martin's Axiom and keeping in mind the relation $a_\xi \neq 0$, it is not difficult to check that there exists a point $x' \in \mathbf{R}$ satisfying the following two conditions:

$$x' \notin (\cup\{B_\zeta : \zeta < \xi\}) \cup \{x_\zeta : \zeta < \xi\} \cup \{y_\zeta : \zeta < \xi\};$$

$$a_\xi x' + b_\xi \notin (\cup\{B_\zeta : \zeta < \xi\}) \cup \{x_\zeta : \zeta < \xi\} \cup \{y_\zeta : \zeta < \xi\}.$$

We then put $x_\xi = x'$ and $y_\xi = a_\xi x' + b_\xi$.

Proceeding in this manner, we obtain the required two injective α-sequences $\{x_\xi : \xi < \alpha\}$ and $\{y_\xi : \xi < \alpha\}$ of points of $\mathbf{R}$. Further, we put

$$X = \{x_\xi : \xi < \alpha\}, \quad Y = \{y_\xi : \xi < \alpha\}.$$

It immediately follows from our construction that both X and Y are $\mathbf{c}$-Luzin subsets of $\mathbf{R}$.

By virtue of Lemmas 2 and 3, the product set $Z' = X \times Y$ is an absolute null subset of $\mathbf{R}^2$.

Also, it can easily be seen that every line l_ξ $(\xi < \alpha)$ meets Z' in continuum many points.

Finally, let g be a rotation of $\mathbf{R}^2$ about the origin $(0,0)$, whose corresponding angle is θ, where $0 < \theta < \pi/2$, and let

$$Z = Z' \cup g(Z').$$

Then Z is an absolute null subset of $\mathbf{R}^2$, too, and has continuum many common points with every straight line lying in $\mathbf{R}^2$. This completes the proof of the lemma. □

As a straightforward consequence of Lemma 4, we obtain that the orthogonal projection of the absolute null set Z on any line l in $\mathbf{R}^2$ coincides with l.

In this context, it should be mentioned that, under Martin's Axiom, the existence of an absolute null subset of $\mathbf{R}^2$, the orthogonal projection of which on every line $l \subset \mathbf{R}^2$ coincides with l, was also shown by Zindulka; see Corollary 3.7 in [17].

Theorem 2. *Assuming Martin's Axiom, there exists an absolute null subset T of $\mathbf{R}^2$, the orthogonal projection of which on every straight line $l \subset \mathbf{R}^2$ is an absolutely nonmeasurable subset of l.*

Proof. Let Z be as in Lemma 4. We shall construct a set $T \subset Z$ with the desired properties.

In what follows the symbol $l(z, z')$ will denote the straight line passing through two distinct points z and z' in $\mathbf{R}^2$.

Also, for any point $t \in \mathbf{R}^2$ and for any straight line $l \subset \mathbf{R}^2$, we will denote by the symbol $\mathrm{pr}_l(t)$ the orthogonal projection of t on l.

As earlier, let α be the least ordinal number of cardinality $\mathbf{c}$.

Let $\{(l_\xi, P_\xi) : \xi < \alpha\}$ be an injective enumeration of all pairs (l, P), where l is a straight line in $\mathbf{R}^2$ and P is a nonempty perfect subset of l.

Starting with this α-sequence $\{(l_\xi, P_\xi) : \xi < \alpha\}$, we define by transfinite recursion two disjoint injective families

$$\{t_\xi : \xi < \alpha\} \subset Z, \quad \{t'_\xi : \xi < \alpha\} \subset Z.$$

Suppose that, for an ordinal $\xi < \alpha$, the two partial families

$$\{t_\zeta : \zeta < \xi\} \subset Z, \quad \{t'_\zeta : \zeta < \xi\} \subset Z$$

have already been defined. Take the pair (l_ξ, P_ξ) and introduce the following notation:

$T_\xi = \{t_\zeta : \zeta < \xi\}$;

$T'_\xi = \{t'_\zeta : \zeta < \xi\}$;

$\mathcal{L}_\xi =$ the family of all those straight lines in $\mathbf{R}^2$ which pass through one of the points from $T_\xi \cup T'_\xi$ and, simultaneously, are perpendicular to one of the straight lines from $\{l_\zeta : \zeta \leq \xi\}$;

$S_\xi =$ the set of all points $z \in \mathbf{R}^2$ such that $\mathrm{pr}_{l_\xi}(z) \in P_\xi$.

Keeping in mind the relations

$$\mathrm{card}(\xi) < \mathrm{card}(\alpha) = \mathbf{c}, \quad \mathrm{card}(T_\xi \cup T'_\xi) < \mathbf{c},$$

we immediately get the inequality $\mathrm{card}(\mathcal{L}_\xi) < \mathbf{c}$. In addition, remembering the property of Z described in the formulation of Lemma 4, we obtain that every straight line in $\mathbf{R}^2$ intersecting P_ξ and perpendicular to l_ξ is entirely contained in the set S_ξ and has continuum many common points with Z.

These circumstances imply the existence of two points

$$t \in S_\xi \cap Z, \quad t' \in S_\xi \cap Z$$

satisfying the following two conditions:

(a) $\mathrm{pr}_{l_\xi}(t) \neq \mathrm{pr}_{l_\xi}(t')$ and the straight line $l(t,t')$ is not perpendicular to any straight line from the family $\{l_\zeta : \zeta < \xi\}$;

(b) $t \notin \cup\mathcal{L}_\xi$ and $t' \notin \cup\mathcal{L}_\xi$.

We then put $t_\xi = t$ and $t'_\xi = t'$.

Proceeding in this manner, we come to the two disjoint injective α-sequences

$$\{t_\xi : \xi < \alpha\} \subset Z, \quad \{t'_\xi : \xi < \alpha\} \subset Z.$$

Finally, we define

$$T = \{t_\xi : \xi < \alpha\}, \qquad T' = \{t'_\xi : \xi < \alpha\},$$

and claim that T is as required.

Indeed, first of all, T is an absolute null set in $\mathbf{R}^2$, because T is a subset of the absolute null set Z.

Let l be an arbitrary straight line in $\mathbf{R}^2$. There exists an ordinal $\xi < \alpha$ such that $l = l_\xi$. From the transfinite construction described above it follows that:

(c) the orthogonal projection $\mathrm{pr}_l(T)$ of T on l has common points with every nonempty perfect subset of l and the orthogonal projection $\mathrm{pr}_l(T')$ of T' on l also has common points with every nonempty perfect subset of l;

(d) $\mathrm{card}(\mathrm{pr}_l(T) \cap \mathrm{pr}_l(T')) \leq \mathrm{card}(\xi) + 1$.

Indeed, to show the validity of (c), it suffices to note that for any nonempty perfect subset P of l, we have $(l, P) = (l_\beta, P_\beta)$, where $\beta < \alpha$, and

$$\mathrm{pr}_l(t_\beta) \in P_\beta, \quad \mathrm{pr}_l(t'_\beta) \in P_\beta$$

by virtue of our transfinite construction.

To show the validity of (d), it suffices to observe that if two ordinal numbers $\zeta < \alpha$ and $\eta < \alpha$ are such that $\max(\zeta, \eta) > \xi$, then the line $l(t_\zeta, t'_\eta)$ cannot be perpendicular to $l = l_\xi$; see (a) and (b). Consequently, if a point x belongs to $\mathrm{pr}_l(T) \cap \mathrm{pr}_l(T')$, then

$$x = \mathrm{pr}_l(t_\zeta) = \mathrm{pr}_l(t'_\eta),$$

where

$$t_\zeta \in T, \quad t'_\eta \in T', \quad \zeta \leq \xi, \quad \eta \leq \xi,$$

whence it follows that the cardinality of the set $\mathrm{pr}_l(T) \cap \mathrm{pr}_l(T')$ does not exceed $\mathrm{card}(\xi) + 1$.

The relations (c) and (d) directly imply that both $\mathrm{pr}_l(T)$ and $\mathrm{pr}_l(T')$ are Bernstein subsets of l, so $\mathrm{pr}_l(T)$ is absolutely nonmeasurable with respect to the class $\mathcal{M}(l)$.

Theorem 2 has thus been proved. □

Remark 1. *In the literature, the notion of a strong measure zero set was introduced by Borel many years ago and yields an interesting representative of the so-called small sets; cf. [12], [13], [16]. Recall that a subset X of $\mathbf{R}$ has strong measure zero if, for every sequence $\{\varepsilon_n : n = 0,1,2,...\}$ of strictly positive real numbers, there exists a sequence $\{\Delta_n : n = 0,1,2,...\}$ of open intervals in $\mathbf{R}$ which collectively cover X and*

$$\lambda_1(\Delta_n) < \varepsilon_n \qquad (n = 0,1,2,...).$$

The analogous notion makes sense for the plane $\mathbf{R}^2$ (in the corresponding definition, open intervals should be replaced by open squares and λ_1 should be replaced by λ_2). It is not difficult to show that every strong measure zero set is an absolute null set; see [13]. However, in contrast to absolute null sets in $\mathbf{R}$ (in $\mathbf{R}^2$), the existence of uncountable strong measure zero subsets of $\mathbf{R}$ (of $\mathbf{R}^2$) cannot be established within the framework of **ZFC** *set theory; see [12], [13]. At the same time, under Martin's Axiom, any* $\mathbf{c}$*-Luzin set in $\mathbf{R}$ (in $\mathbf{R}^2$) has strong measure zero; see [13]. For strong measure zero subsets of $\mathbf{R}^2$ no analogue of Theorem 2 is true. Indeed, if Z is a strong measure zero subset of $\mathbf{R}^2$, then the orthogonal projection of Z on the coordinate axis $\mathbf{R} \times \{0\}$ is a strong measure zero subset of $\mathbf{R} \times \{0\}$, so is measurable.*

Remark 2. *A set $Z \subset \mathbf{R}^2$ is called absolutely nonmeasurable with respect to the class $\mathcal{M}(\mathbf{R}^2)$ if there exists no measure μ belonging to this class such that $Z \in \mathrm{dom}(\mu)$. Actually, the absolutely nonmeasurable sets with respect to $\mathcal{M}(\mathbf{R}^2)$ are identical with the Bernstein subsets of $\mathbf{R}^2$ (this fact is a direct analogue of Lemma 1 and its proof does not differ from the proof of Lemma 1; the same argument works for any uncountable Polish topological space). If Z is an arbitrary subset of $\mathbf{R}^2$ absolutely nonmeasurable with respect to the class $\mathcal{M}(\mathbf{R}^2)$ and l is an arbitrary straight line in $\mathbf{R}^2$, then the orthogonal projection of Z on l coincides with the whole of l. Indeed, take any point $t \in l$ and consider the straight line l' perpendicular to l and passing through t. Since l' is a nonempty perfect subset of $\mathbf{R}^2$ and Z is a Bernstein set in $\mathbf{R}^2$, we obviously have $Z \cap l' \neq \emptyset$. Consequently, $t \in \mathrm{pr}_l(Z)$ and so $l = \mathrm{pr}_l(Z)$. In particular, we see that the orthogonal projection of an absolutely nonmeasurable subset of $\mathbf{R}^2$ on any straight line l in $\mathbf{R}^2$ turns out to be absolutely measurable with respect to the class of all measures defined on various σ-algebras of subsets of l.*

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