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ON VBG FUNCTIONS AND THE DENJOY-KHINTCHINE INTEGRAL

Abstract

The study of functions of generalized bounded variation (VBG) and generalized absolute continuity (ACG) that appears in Saks's treatise *Theory of the integral* can be thoroughly reworked by using some aspects of the theory of variational measures proposed originally by Ralph Henstock and extended by many others. We present a development of these concepts and use it for a characterization of the Denjoy-Khintchine integral.

1 Introduction

The material in Saks [27] on VBG_* functions, ACG_* functions, and the Denjoy-Perron integral has been thoroughly rewritten in the last few decades. We know now that, associated to any continuous function $F : \mathbb{R} \to \mathbb{R}$, one can define a Borel measure V_F on $\mathbb{R}$ that exactly expresses the variational properties of the function F. In particular, F has bounded variation on an interval if and only if V_F is finite on that interval, and F is VBG_* on a set E if and only if V_F is σ-finite on E. Moreover, F is ACG_* on a closed set E if and only if V_F is absolutely continuous with respect to Lebesgue measure on E. This variational measure has a representation as an integral should the set E be measurable and composed entirely of points of differentiability of F:

$$V_F(E) = \int_E |F'(t)|\, dt.$$

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Finally, this same study establishes that the identity

$$F(x) - F(a) = \int_a^x f(t)\,dt \qquad (a < x \leq b)$$

in the sense of the Denjoy-Perron integral is equivalent to the assertion that $F'(x) = f(x)$ at almost every and V_F-almost every point x of $[a, b]$. This integral is certainly, by now, well-known as the Henstock-Kurzweil integral although the accompanying theory of the variational measures is probably less familiar to non-specialists.

It has also been known, at least as folklore, that the same kind of program can be undertaken for the other related concepts developed in Saks: the classes of functions that are VBG or ACG and for the Denjoy-Khintchine integral. Henstock has always presented his theory of integration as encompassing all or most of the nonabsolutely convergent integrals, certainly including the two most famous, the Denjoy-Perron and the Denjoy-Khintchine integrals.

He had not, however, checked the details. To him it seemed apparent that the characterization of the latter integral by Tolstov [35], [36] as a Perron-type integral would lead easily to a similar characterization using Riemann sums. In [17, pp. 222–223] he put forward the structure that might have worked.

He was mistaken in a key point and mentioned this in [19, Ex. 3, pp. 2–3]. In the latter paper he sketches out, without details, an amended scheme that he proposes will handle the Denjoy-Khintchine integral. The rather cumbersome division space that he sketches there may, perhaps, be adequate for the task, but is not satisfying in any compelling way. A basic problem is that his exposition depends always on his elaborate general theory of integration in division spaces that few readers have been willing to consume.

More recent studies by Ene [13] and Sworowski [31]) have addressed this problem and devised careful schemes that exhibit the Denjoy-Khintchine integral as a Riemann-type integral. For specialists this is probably all we have asked for.

Even so there is a disappointing lack of simplicity in these presentations. The treatment in Saks of these concepts is elegant and satisfying. One would have hoped that a simpler and more obvious construction would join and complement the admirable treatment in Saks.

Our goal in this exposition is to develop the theory for variational measures that capture the concepts of VBG and ACG functions and to use that theory to provide a simpler characterization of the Denjoy-Khintchine integral. The main motivation is indeed simplicity which we hope has been achieved. This can be considered a contribution to study of the variational measures associated with a function, a study that includes, among others, these notable contributions: [2], [3], [4], [10], [11], [12], [13], [18], [19], [20], [25], [26], [28], [29], [30], [31], [33], [34], [37], [39], and [40].

We begin with a review of the material from Saks that we wish to supplement. This review should make the ideas a bit more accessible to the reader. Not everyone interested in this topic will have a working knowledge of that material, nor know on what to focus for this task.

1.1 Weak variation relative to a set

One possible way to capture the essence of Jordan's original 1881 computation [21] of the variation for a function $F : [a, b] \to \mathbb{R}$ can be expressed by this definition.

Definition 1. Let $F : \mathbb{R} \to \mathbb{R}$ and let $\mathcal{I}$ be a collection of closed intervals. Then we write

$$V(F, \mathcal{I}) = \sup \sum_{i=1}^{n} |F(d_i) - F(c_i)|$$

where the supremum is computed using all choices of nonoverlapping intervals $\{[c_i, d_i]\}$ chosen from the collection $\mathcal{I}$.

For the ordinary Jordan variation, certainly

$$V(F, [a, b]) = V(F, \mathcal{I}_0)$$

where $\mathcal{I}_0$ is the collection of all closed subintervals of $[a, b]$.

This definition encourages a generalization by choosing the families $\mathcal{I}$ in the best way for some purpose. For example, let E be an arbitrary set of real numbers. To define the variation of F relative to E one needs to select appropriate families $\mathcal{I}$ that relate in some intimate way to the set E.

One idea that comes to mind immediately (perhaps one should beware of ideas that so readily come to mind) is to take for our class of intervals $\mathcal{I}_E$, the collection of all intervals $[c, d]$ with endpoints c and d in the set E. Then

$$V(F; E) = V(F, \mathcal{I}_E) \tag{1}$$

looks like a reasonable candidate for the variation of F concentrated on E.

Indeed Saks [27, p. 221] uses exactly this notation and refers to $V(F; E)$ as the *weak variation*. "Variation faible" in the first 1933 French version, entitled *Théorie d l'intégrale*, of his treatment of these ideas. The translator's fidelity to the original has led to the standard terminology VB, VB_*, VBG, and VBG_* which preserves the French word order (variation bornée generalisée).

This expression "measures" the variation of a function on a set but, in no way, is this expression a measure: it has no nice analytical properties. Here is Saks's terminology and definitions of the concepts VB and VBG.

Definition 2. Let $F : \mathbb{R} \to \mathbb{R}$ and let E be a set of real numbers. We say that F is VB on E provided that $V(F; E) < \infty$.

Definition 3. Let $F : \mathbb{R} \to \mathbb{R}$ and let E be a set of real numbers. We say that F is VBG on E provided there is a denumerable collection of sets $\{E_n\}$ covering E so that F is VB on E_n for each n.

1.2 E-forms

It is suggested by this definition, and will prove to be quite true, that much of our attention will be on a situation like this, where a set E is expressed as a union of a countable family of subsets.

Let us take on the following terminology from Ene [13] and Sworowski [31]. By an *E-form* we mean simply a countable family $\mathcal{E}$ of sets whose union is all of the set E. Expressed in this language, Definition 3 asserts that F is VBG on a set E if and only if there is an E-form $\mathcal{E}$ for which F is VB on each set $S \in \mathcal{E}$. There is little extra economy here, but the language will pay for itself later on.

If $\mathcal{E}_1$ and $\mathcal{E}_2$ are two E-forms then we write

$$\mathcal{E}_1 \prec \mathcal{E}_2$$

if each set $S \in \mathcal{E}_2$ is a subset of at least one member (possibly more than one member) from $\mathcal{E}_1$. This gives us a partial order, directed in the order of increasing refinement.

If $\mathcal{E}_1$ and $\mathcal{E}_2$ are two E-forms then we write

$$\mathcal{E}_1 \wedge \mathcal{E}_2$$

as the countable collection

$$\{S_1 \cap S_2 : S_1 \in \mathcal{E}_1,\ S_2 \in \mathcal{E}_2\}$$

One can leave out any empty sets. This collection is again a countable collection of sets whose union is E and so is also an E-form. Note that

$$\mathcal{E}_1,\ \mathcal{E}_2 \prec \mathcal{E}_1 \wedge \mathcal{E}_2$$

in the partial order.

1.3 The main features of VB and VBG functions

In working with the Jordan variation on an interval one sees that the notion of bounded variation is intimately related to monotonic functions. Indeed that was precisely the purpose for which Jordan intended the concept.

The same connection is observed here for VB and VBG functions but requires some attention to detail. Let us review these details before introducing the appropriate scheme for defining the weak variational measures. This material is all in Saks [27, pp. 221–222] and a review can also be found in the treatise [8] or the expository account in [15].

Lemma 4. *Let $F : \mathbb{R} \to \mathbb{R}$ and let E be a bounded set of real numbers. Then the following statements are true:*

1. *If F is bounded and monotonic on E then F is VB on E.*
2. *If F is bounded and monotonic on E then there is function $G : \mathbb{R} \to \mathbb{R}$ that is monotonic so that $F(x) = G(x)$ for all $x \in E$.*
3. *If F is unbounded but monotonic on E then F is VBG on E.*

This lemma extends to a similar characterization of functions having finite weak variation on a set.

Lemma 5. *Let $F : \mathbb{R} \to \mathbb{R}$ and let E be a bounded set of real numbers. Then the following are equivalent.*

1. *F is VB on E.*
2. *There is function $G : \mathbb{R} \to \mathbb{R}$ that is of bounded variation so that $F(x) = G(x)$ for all $x \in E$.*

These lemmas explain the connection between ordinary bounded variation and VBG which can be summarized in this way.

Lemma 6. *Let $F : \mathbb{R} \to \mathbb{R}$ and let E be a set of real numbers. Then the following are equivalent.*

1. *F is VBG on E.*
2. *There is an E-form $\mathcal{E}$ consisting of bounded subsets of E so that F is VB on each $S \in \mathcal{E}$.*
3. *There is an E-form $\mathcal{E}$ consisting of bounded subsets of E so that, corresponding to each $S \in \mathcal{E}$ there is a function G_S of bounded variation so that $F(x) = G_S(x)$ for all $x \in S$.*

1.4 Continuous VBG and ACG functions

Since one of our concerns is a characterization of the Denjoy-Khintchine integral, we shall frequently address our attention to continuous functions. The variational ideas are much simplified in this setting. For continuous functions we could simply take the following as a definition for the class of VB and VBG functions. We can add in now the classes of functions AC and ACG from Saks, that are normally described for continuous functions.

Lemma 7. *Let $F : \mathbb{R} \to \mathbb{R}$ be a function that is continuous on a closed bounded set E with endpoints a and b for which $a < b$. Define $G : [a, b] \to \mathbb{R}$ to be that continuous function for which $F(x) = G(x)$ at each point $x \in E$ and that is linear on each interval contiguous to E in $[a, b]$. Then*

1. *F is VB on E if and only if G has bounded variation on $[a, b]$.*
2. *F is AC on E if and only if G is absolutely continuous on $[a, b]$.*

In this special case of continuous functions $F : \mathbb{R} \to \mathbb{R}$ we also have a narrower characterization of VBG and ACG:

1. If F is continuous and VBG on a closed set E then there is an E-form $\mathcal{E}$ consisting of closed, bounded subsets of E so that F is VB on each $S \in \mathcal{E}$.
2. If F is continuous and ACG on a closed set E then there is an E-form $\mathcal{E}$ consisting of closed, bounded subsets of E so that F is AC on each $S \in \mathcal{E}$.

1.5 The category lemma

When E is closed and F is continuous a category argument can be applied to the scheme just described. If E is the union of a sequence $\{E_n\}$ of closed sets, then by the Baire category theorem, there is a portion of E contained in E_n for at least one of these sets. That argument leads to the following characterizations of VBG and ACG functions.

Lemma 8. *Let $F : \mathbb{R} \to \mathbb{R}$ be a function that is continuous on a nonempty closed set E. Then the following are equivalent:*

1. *F is VBG [resp. ACG] on E.*
2. *For every nonempty closed subset S of E there is a nonempty portion $[c, d] \cap S$ on which F is VB [resp. AC].*

3. *F is VBG [resp. ACG] on every subset of E that has Lebesgue measure zero.*

The equivalence of the first two statements is already in Saks [27, pp. 233–234]. The proof requires a category argument in both directions. The third equivalent statement appears to be measure-theoretic in nature but it is not. It too is just a consequence of the Baire category theorem. Every closed set E would contain a dense subset of type $\mathcal{G}_\delta$ that has Lebesgue measure zero. The Baire category theorem can be applied to sets of type $\mathcal{G}_\delta$ as well as to closed sets

For the added third statement, see Ene [10, pp. 11–12] for a simple proof. It seems that this addition to the category lemma is due to Ene, although there were later rediscoveries. There is an identical category lemma available for a variety of similarly defined concepts (including those in Saks with VB and VBG replaced by VB_* and VBG_* and AC and ACG replaced by AC_* and ACG_*).

1.6 Approximate differentiability of VBG functions

Functions that have bounded variation are differentiable almost everywhere. In the presentation in Saks this is extended to functions that are VBG_*. Saks also treats the differentiability properties of VBG functions. For these the ordinary derivative must be replaced by the approximate derivative. Thus we have the following theorem from Saks [27, p. 222] that he attributes to Denjoy and Khintchine.

Theorem 9 (Denjoy-Khintchine)**.** *Let $F : \mathbb{R} \to \mathbb{R}$ be a measurable function that is VBG on a set E. Then F has a finite approximate derivative $F'_{ap}(x)$ at almost every point x of E.*

In the situation described earlier for VBG functions on a set E, we have presented a sequence of sets $\{E_n\}$ covering E and a sequence of functions $G_n : \mathbb{R} \to \mathbb{R}$ of bounded variation so that $F(x) = G_n(x)$ for each $x \in E \cap E_n$. Then, by the Denjoy-Khintchine theorem, we can also take advantage of the fact that $F'_{ap}(x) = G'_n(x)$ for almost every $x \in E \cap E_n$.

2 Covers

The fundamental notion that we use is that of a cover in certain senses. By a "cover" we mean simply a collection of closed bounded intervals that has some useful property relative to the points of a particular set.

The most famous concepts of this type are the Vitali covers. A family of intervals $\mathcal{I}$ is a *Vitali cover* of a set E if for each $x \in E$ and any $\epsilon > 0$ there is at least one interval $[u, v] \in \mathcal{I}$ that contains x and has length less than ϵ.

Suppose that, for each set $E \subset \mathbb{R}$, there has been defined some collection of covers $\mathfrak{F}[E]$. We can always assume that $\emptyset \in \mathfrak{F}[\emptyset]$; occasionally one might also have $\emptyset \in \mathfrak{F}[E]$ even if E is nonempty. (Frequently $\mathfrak{F}[E]$ will be a filter, but cannot be if it contains the empty set.) The main structural properties that one might need from families of covers in order to develop a theory of variational measures are listed below.

1. [filtering]
   (a) If $\mathcal{I}_1$, $\mathcal{I}_2$ are covers with $\mathcal{I}_1 \in \mathfrak{F}[E]$ and $\mathcal{I}_1 \subset \mathcal{I}_2$, then $\mathcal{I}_2 \in \mathfrak{F}[E]$.
   (b) If $\mathcal{I}_1$, $\mathcal{I}_2$ are covers both of which belong to $\mathfrak{F}[E]$ then the intersection $\mathcal{I}_1 \cap \mathcal{I}_2$, also belongs to $\mathfrak{F}[E]$.
2. [subadditive] If $E \subset \bigcup_{n=1}^{\infty} E_n$ and $\mathcal{I}_n \in \mathfrak{F}[E_n]$, then
$$\bigcup_{n=1}^{\infty} \mathcal{I}_n \in \mathfrak{F}[E].$$
3. [pruning by open sets] If $E \subset G$ where G is open and $\mathcal{I} \in \mathfrak{F}[E]$ then
$$\mathcal{I}(G) = \{I \in \mathcal{I} : I \subset G\} \in \mathfrak{F}[E].$$

It is useful to have a filter, but it is not always needed. The Vitali covers satisfy statements 1(a), 2, and 3 but not statement 1(b) so they are not filtering. In order to construct an outer measure one needs not much more than statement 2. For that measure to be a Borel measure we should add in statement 3. In the next section Lemma 10 is the basis for our construction of measures and can be used in a wide variety of situations.

2.1 Construction of the measures

Let h be a real-valued interval function, i.e., to each interval $[u, v]$ there is assigned a number $h([u, v])$. For any cover $\mathcal{I}$ we write
$$V(h, \mathcal{I}) = \sup \sum_{i=1}^{n} |h([u_i, v_i])|$$
where the supremum is computed using all choices of nonoverlapping intervals $\{[u_i, v_i]\}$ chosen from the collection $\mathcal{I}$. We assume that $V(h, \emptyset) = 0$.

Lemma 10. *Let h be a real-valued interval function and suppose we are given families $\mathfrak{F}[E]$ for each set $E \subset \mathbb{R}$. Define*

$$h_{\mathfrak{F}}(E) = \inf\{V(h, \mathcal{I}) : \mathcal{I} \in \mathfrak{F}[E]\}.$$

If these families satisfy statement 2 [subadditive property] above, then $h_{\mathfrak{F}}$ is an outer measure on the real line. If, in addition, these families satisfy statement 3 [pruning property], then $h_{\mathfrak{F}}$ is a Borel measure on the real line.

Proof. Suppose that $E \subset \bigcup_{n=1}^{\infty} E_n$ and that each $h_{\mathfrak{F}}(E_n) < \infty$. Let $\epsilon > 0$. Choose a cover $\mathcal{I}_n$ from each $\mathfrak{F}[E_n]$ so that

$$h_{\mathfrak{F}}(E_n) \leq V(h, \mathcal{I}_n) \leq h_{\mathfrak{F}}(E_n) + \epsilon 2^{-n}.$$

Then the subadditive property supplies that

$$\mathcal{I} = \bigcup_{n=1}^{\infty} \mathcal{I}_n \in \mathfrak{F}[E].$$

Hence

$$h_{\mathfrak{F}}(E) \leq V(h, \mathcal{I}) \leq \sum_{n=1}^{\infty} V(h, \mathcal{I}_n) \leq \sum_{n=1}^{\infty} h_{\mathfrak{F}}(E_n) + \epsilon.$$

This verifies the inequality $h_{\mathfrak{F}}(E) \leq \sum_{n=1}^{\infty} h_{\mathfrak{F}}(E_n)$ which is what is needed to see that $h_{\mathfrak{F}}$ is an outer measure.

The pruning property is used to check that $h_{\mathfrak{F}}$ is a Borel measure, i.e., that all Borel sets are $h_{\mathfrak{F}}$-measurable. This is equivalent to the requirement that $h_{\mathfrak{F}}$ is a metric outer measure in the language of Carathéodory. (There are numerous textbooks that give an account of Carathéodory's theory; naturally we suggest [7, Chapter 3].)

One needs to show that, if A and B are positively separated, then

$$h_{\mathfrak{F}}(A \cup B) \geq h_{\mathfrak{F}}(A) + h_{\mathfrak{F}}(B).$$

There are disjoint open sets G_1 and G_2 with $A \subset G_1$ and $B \subset G_2$. Take any cover $\mathcal{I}$ from $\mathfrak{F}[A \cup B]$. Write then

$$\mathcal{I}(G_1) = \{I \in \mathcal{I} : I \subset G_1\} \text{ and } \mathcal{I}(G_2) = \{I \in \mathcal{I} : I \subset G_2\}.$$

Observe that $\mathcal{I}(G_1) \in \mathfrak{F}[A]$ and $\mathcal{I}(G_2) \in \mathfrak{F}[B]$. (This makes use of both properties 2 and 3.) Hence

$$h_{\mathfrak{F}}(A) + h_{\mathfrak{F}}(B) \leq V(h, \mathcal{I}(G_1)) + V(h, \mathcal{I}(G_2)) \leq V(h, \mathcal{I}).$$

As this is true for all such choices of $\mathcal{I}$ from $\mathfrak{F}[A \cup B]$ we have the inequality

$$h_{\mathfrak{F}}(A) + h_{\mathfrak{F}}(B) \leq h_{\mathfrak{F}}(A \cup B)$$

that we require. $\square$

2.2 Vitali covering theorem

As a special case of this lemma, we have the following characterization of the Lebesgue measure that is essentially just the Vitali covering theorem expressed in this language.

Theorem 11. *Let λ denote the interval function $\lambda([u,v]) = v - u$ and let $\mathfrak{V}[E]$ be the collection of all Vitali covers of a set E. Then $\lambda_{\mathfrak{V}}$ is Lebesgue outer measure on the real line.*

Proof. Let λ also denote the Lebesgue outer measure on the real line. Take any Vitali cover $\mathcal{I}$ of a set E. Then by the Vitali covering theorem there is a disjointed sequence $\{[u_i, v_i]\} \subset \mathcal{I}$ for which

$$\lambda\left(E \setminus \bigcup_{i=1}^{\infty}[v_i, u_i]\right) = 0 \text{ and } \lambda(E) \leq \sum_{i=1}^{\infty}(v_i - u_i).$$

From this it follows that, for all such $\mathcal{I}$, $\lambda(E) \leq V(\lambda, \mathcal{I})$. Hence $\lambda(E) \leq \lambda_{\mathfrak{V}}(E)$.

In the other direction, if $\lambda(E) < t$ then there is an open set $G \supset E$ with $\lambda(G) < t$. Define $\mathcal{I}$ to be the collection of all intervals $[u,v]$ contained in G. This is a Vitali cover of E and so

$$\lambda(E) \leq \lambda_{\mathfrak{V}}(E) \leq V(\lambda, \mathcal{I}) \leq \lambda(G) < t.$$

The identity $\lambda_{\mathfrak{V}}(E) = \lambda(E)$ now follows since t can be any number larger than $\lambda(E)$. □

2.3 Full, fine, weak, and q-weak covers

As suggested by the measure constructions just reviewed, our task of building a variational measure for a function F that will carry the information about whether F is VBG or ACG on a set can be carried out by a judicious choice of covers. These are the "weak covers" of the next definition. At the same time we review the other, better known, classes of covers that we call here full and fine.

It is rather obvious from the way in which the concepts VB, VBG, AC, and ACG work that a definition of a weak cover that hopes to capture that idea should invoke a sequence of sets $\{E_n\}$ whose union is a set E. In the language we have used in Section 1.2 this is called an E-form.

Definition 12. Let $\mathcal{I}$ be a collection of closed, bounded intervals and E an arbitrary set.

1. $\mathcal{I}$ is said to be a *full cover* of E if for each point $x \in E$ there is a $\delta > 0$ so that all intervals $[x, v]$ with $x < v < x + \delta$ belong to $\mathcal{I}$, and all intervals $[u, x]$ with $x - \delta < u < x$ belong to $\mathcal{I}$.

2. $\mathcal{I}$ is said to be *right-fine* at a point $x \in E$ if for every $\epsilon > 0$ there is at least one interval $[x, v]$ with $x < v < x + \epsilon$ that belongs to $\mathcal{I}$.

3. $\mathcal{I}$ is said to be *left-fine* at a point $x \in E$ if for every $\epsilon > 0$ there is at least one interval $[u, x]$ with $x - \epsilon < u < x$ that belongs to $\mathcal{I}$.

4. $\mathcal{I}$ is said to be a *fine cover* of E if, at every point $x \in E$, $\mathcal{I}$ is either right-fine or left-fine.

5. $\mathcal{I}$ is said to be a *weak cover* of E if there is an E-form $\mathcal{E}$ such that, whenever an interval $[u, v]$ has one endpoint in a set $S \in \mathcal{E}$ and the other endpoint in $\overline{S}$, then necessarily $[u, v]$ must belong to $\mathcal{I}$.

6. $\mathcal{I}$ is said to be a *q-weak* (quite weak cover) of E if there is an E-form $\mathcal{E}$ such that, whenever an interval $[u, v]$ has both endpoints in a set $S \in \mathcal{E}$, then necessarily $[u, v]$ must belong to $\mathcal{I}$.

Note that, according to this definition, every fine cover is also a Vitali cover (see Section 2). In the definition of a fine cover of a set E, however, we require for points $x \in E$ that there be an abundance of small intervals $[u, v]$ in the cover with an endpoint at x. (The Vitali covers, in contrast, require only that these intervals *contain* x).

2.4 Properties of full, fine, weak, and q-weak covers

We know from Section 2.1 what properties we should need in order for a satisfactory construction of measures from these various classes of covers. In the next theorem we summarize all of the properties that play a role at some moment in the theory.

Theorem 13. *The following statements about full, fine, weak, and q-weak covers hold:*

1. *[filtering] The family of all full [weak or q-weak] covers of a set E is filtering.*

   [Note: this is not the case for fine covers or Vitali covers. Also, since a weak cover of a nonempty set might be empty, it is not necessarily a filter.]

2. *[full covers are weak] If $\mathcal{I}$ is a full cover of a set E then it is also a weak cover of E.*

3. *[weak covers are q-weak] If $\mathcal{I}$ is a weak cover of a set E then it is also a q-weak cover of E.*

4. *[full covers are fine] If $\mathcal{I}$ is a full cover of a set E then it is also a fine cover and a Vitali cover of E.*

5. *[q-weak covers are nearly fine] If $\mathcal{I}$ is a weak cover of a set E then $\mathcal{I}$ is left-fine and right-fine at nearly every point of E, i.e., with countably many exceptions.*

6. *[weak covers of countable sets] $\mathcal{I} = \emptyset$ is a weak cover of a set E if and only if E is countable.*

7. *[pruning] If $\mathcal{I}$ is a full [fine, q-weak, or weak] cover of a set E and G is an open set containing E then*
$$\mathcal{I}(G) = \{I \in \mathcal{I} : I \subset G\}$$
is also a cover of E in the same sense.

8. *[subadditive] If $\{E_k\}$ is a sequence of sets for which $\mathcal{I}_k$ is a full [fine, weak, q-weak] cover of E_k, then, in the same sense,*
$$\bigcup_{k=1}^{\infty} \mathcal{I}_k \text{ is a cover of every subset of the set } \bigcup_{k=1}^{\infty} E_k.$$

Proof. The most important properties for the purpose of constructing Borel measures are the final two properties, 7 and 8. The other properties are useful in checking relations among the measures.

Statement 1. It is enough for the proof to check that the intersection of two covers (in one of the senses) is again a cover in the same sense. This is available for full, q-weak, and weak covers but fails for fine covers.

The fine covers are just special kind of Vitali covers and we can see easily that the intersection of two fine covers of a set might even be empty. A simple example illustrates this. Let
$$\mathcal{I}_1 = \{[0, 1/n] : n = 1, 2, 3, \dots\}$$
and
$$\mathcal{I}_2 = \{[-1/n, 0] : n = 1, 2, 3, \dots\}.$$

Each is a fine cover of the set containing only the point zero. The intersection $\mathcal{I}_1 \cap \mathcal{I}_2$ is, however, empty.

For full covers the statement is easily proved. Select $\delta_1(x) > 0$ for each $x \in E$ so that $[x, v]$ and $[u, x]$ belong to $\mathcal{I}_1$ for $x < v < x + \delta_1(x)$ and $x - \delta_1(x) < u < x$. Do the same for $\mathcal{I}_2$ with a positive function $\delta_2(x)$. Take $\delta_3(x) = \min\{\delta_1(x), \delta_2(x)\}$. Then $\mathcal{I}_1 \cap \mathcal{I}_2$ must contain every interval $[x, v]$ and $[u, x]$ for $x \in E$ and for which $x < v < x + \delta_3(x)$ and $x - \delta_3(x) < u < x$. This verifies that $\mathcal{I}_1 \cap \mathcal{I}_2$ is a full cover of E.

For weak covers of a set E select two E-form $\mathcal{E}_1$ and $\mathcal{E}_2$ such that whenever an interval $[u, v]$ has one endpoint in a set $S \in \mathcal{E}_i$ $(i = 1, 2)$ and the other endpoint in $\overline{S}$ then $[u, v]$ must belong to $\mathcal{I}_i$ $(i = 1, 2)$. Simply now take

$$\mathcal{E}_3 = \mathcal{E}_1 \wedge \mathcal{E}_2.$$

We check that whenever an interval $[u, v]$ has one endpoint in a set $S \in \mathcal{E}_3$ $(i = 1, 2)$ and the other endpoint in $\overline{S}$ then $[u, v]$ must belong to both $\mathcal{I}_1$ and also to $\mathcal{I}_2$ and hence to their intersection. This verifies that $\mathcal{I}_1 \cap \mathcal{I}_2$ is a weak cover of E. A similar argument will handle the q-weak covers.

Statement 2. Assume that $\mathcal{I}$ is a full cover of a set E. Select, for each $x \in E$, a number $0 < \delta(x) < 1$ so that $[x, v]$ and $[u, x]$ belong to $\mathcal{I}$ for $x < v < x + \delta(x)$ and $x - \delta(x) < u < x$.

We can use δ to decompose E in a familiar way: let

$$A_n = \{x \in E : 1/(n+1) < \delta(x) < 1/n\} \quad (n = 1, 2, 3, \dots)$$

and let

$$A_{nm} = A_n \cap \left[\frac{m-1}{n+1}, \frac{m}{n+1}\right] \quad (m = 0, \pm 1, \pm 2, \pm 3, \dots).$$

Let C_{nm} be the closures of the sets A_{nm}. Consider any interval $[u, v]$ with one endpoint u in a set A_{nm} and the other endpoint v in the closure of that set, i.e., the set C_{nm}. Then $\delta(u) > 1/(n+1)$ and $u < v \leq u + 1/(n+1)$. This means that $[u, v]$ is in $\mathcal{I}$. A similar statement can be made if v is in A_{nm} and the other endpoint in C_{nm}.

Let $\mathcal{E}$ be the E-form obtained from the countable family of all the sets A_{nm}. This E-form evidently verifies that $\mathcal{I}$ is a weak cover of E.

Statement 3. This one is obvious.

Statement 4. It is immediate that every full cover of a set is also a fine cover of that set. In fact there is a duality here between full and fine covers that is worth pointing out. A collection $\mathcal{I}$ is a full cover of a set E if and only if

$\mathcal{I} \cap \mathcal{J}$ is fine at each point of E for all choices of fine covers $\mathcal{J}$ of E. Dually, a collection $\mathcal{I}$ is a fine cover of E if and only if $\mathcal{I} \cap \mathcal{J}$ is fine at each point of E for all choices of full covers $\mathcal{J}$ of E. (We will use this duality concept in Definition 29 to obtain a dual version of the weak covers.)

Statement 5. Assume that $\mathcal{I}$ is a q-weak cover of a set E. Then, by definition there is an E-form $\mathcal{E}$ that witnesses $\mathcal{I}$ to be a q-weak cover. Let $\mathrm{SIS}(\mathcal{E})$ denote the collection of all points x that are semi-isolated (i.e., isolated on one side at least) in at least one set $S \in \mathcal{E}$. This is a countable set which we take to be the exceptional countable set in the statement. If x is a point of E not in this set then there must be a set $S \in \mathcal{E}$ that contains x and x is not isolated on either side in S. This gives an abundance of intervals $[x,s]$ and $[s,x]$ in $\mathcal{I}$ for $s \in S$. In fact $\mathcal{I}$ is both left-fine and right-fine at each point of E except possibly at points belonging to the countable set $\mathrm{SIS}(\mathcal{E})$.

Statement 6. If E is countable, then there is an E-form $\mathcal{E}$ consisting of sets containing exactly one point of E. This is witness to the fact that $\emptyset$ is a weak cover of E. Conversely, if E is not countable and $\mathcal{I}$ is a weak cover of E, then there must be an E-form $\mathcal{E}$ that is witness to the fact that $\mathcal{I}$ is a weak cover of E. At least one set $S \in \mathcal{E}$ must be infinite so that $\mathcal{I}$ must contain infinitely many intervals.

Statement 7. We verify this statement for weak covers. (The full and the fine case are simpler since, in both cases, one needs only verify a local property at each point of the set E.) As usual, there must be an E-form $\mathcal{E}_1$ that is witness to the fact that $\mathcal{I}$ is a weak cover of E.

Let G be an open set containing E and let $\{(a_i, b_i)\}$ be the sequence of open component intervals of G. Consider the countable collection $\mathcal{E}_2$ consisting of all sets

$$\left\{ E \cap \left[a_i + \frac{1}{m}, b_i - \frac{1}{m} \right] \right\}$$

for $m = 1, 2, 3, \dots$ and $i = 1, 2, 3, \dots$. This collection $\mathcal{E}_2$ must be an E-form. Note that if $S \in \mathcal{E}_2$ then $\overline{S} \subset G$.

Now take

$$\mathcal{E}_3 = \mathcal{E}_1 \wedge \mathcal{E}_2$$

and observe that the E-form $\mathcal{E}_3$ is witness to the fact that $\mathcal{I}(G)$ is a weak cover of E. The same argument (even a simpler version) would do the same job for q-weak covers.

Statement 8. This statement is well-known and easy for full and fine covers. Those covers are defined in a pointwise-sense and so the verification can simply verify that a property holds at each point of the set E for which we assume that

$$E \subset \bigcup_{k=1}^{\infty} E_k.$$

We verify this statement only for weak covers. In the case of q-weak covers a similar argument will work. Let $A_1 = E \cap E_1$, let $A_2 = E \cap E_2 \setminus E_1$, $A_3 = E \cap E_3 \setminus (E_1 \cup E_2)$, ... This produces a disjointed sequence of sets $\{A_k\}$ whose union is all of E. Each $\mathcal{I}_k$ is a weak cover of E_k and so also a weak cover of A_k. Thus we can choose, for $k = 1 = 2, 3, \dots$, an A_k-form $\mathcal{E}_k$ that is witness to the fact that $\mathcal{I}_k$ is a weak cover of A_k. Define

$$\mathcal{E} = \bigcup_{k=1}^{\infty} \mathcal{E}_k.$$

This is an E-form. We check to see that it is a witness to the fact that

$$\mathcal{I} = \bigcup_{k=1}^{\infty} \mathcal{I}_k$$

is a weak cover of E. If $S \in \mathcal{E}$ then $S \in \mathcal{E}_k$ for some k. Thus, if $u \in S$ and $v \in \overline{S}$, then necessarily $[u, v] \in \mathcal{I}_k$ so $[u, v] \in \mathcal{I}$. Similarly if $v \in S$ and $u \in \overline{S}$ then $[u, v] \in \mathcal{I}$. □

2.5 Cousin's lemma

Full covers have a well-known partitioning property that makes them particularly useful for handling integrals or proving elementary properties of real functions defined on intervals.

Lemma 14. *[Cousin's Lemma] Suppose that $\mathcal{I}$ is a full cover of an open bounded interval (a, b) and that $\mathcal{I}$ is right-fine at a and left-fine at b. Then, for every closed subinterval $[c, d] \subset [a, b]$, there is a finite subdivision*

$$c = c_0 < c_1 < c_2 < \dots c_{n-1} < c_n = d$$

so that each $[c_{i-1}, c_i] \in \mathcal{I}$ $(i = 1, 2, \dots, n)$, i.e., $[c, d] = \bigcup_{i=1}^{n} [c_{i-1}, c_i]$ expresses $[c, d]$ as the union of finitely many nonoverlapping intervals from $\mathcal{I}$.

This lemma is easily proved by appealing to the Heine-Borel theorem or the nested interval theorem from elementary real analysis. It has been rediscovered many times. Some of us attributed it to Goursat [16] for a while, but the earlier paper of Cousin [9] emerged, and it is now very firmly known as the Cousin lemma. Any earlier 19th century discovery is unlikely now to dislodge this label. There is no shortage of proofs of this, although the exact formulation of the lemma may differ among them.

For the weak covers (with some extra assumptions) there is a similar partitioning property available that requires only a routine Baire category argument. We especially need this lemma for our characterization of the Denjoy-Khintchine integral later on. Note that Lemma 14 follows from Lemma 15 since every full cover is also a weak cover.

Lemma 15. *[Weak version of Cousin's Lemma] Suppose that $\mathcal{I}$ is a weak cover of a closed, bounded interval $[a, b]$ that is right-fine at each point of $[a, b)$ and left-fine at each point of $(a, b]$. Then, for every closed subinterval $[c, d] \subset [a, b]$, there is a finite subdivision*

$$c = c_0 < c_1 < c_2 < \dots c_{n-1} < c_n = d$$

so that each $[c_{i-1}, c_i] \in \mathcal{I}$ $(i = 1, 2, \dots, n)$, i.e.,

$$[c, d] = \bigcup_{i=1}^{n} [c_{i-1}, c_i]$$

expresses $[c, d]$ as the union of finitely many nonoverlapping intervals from $\mathcal{I}$.

Proof. Let us say that a point $x \in [a, b]$ is regular if there is a neighborhood $(x - \epsilon, x + \epsilon)$ so that $\mathcal{I}$ contains a partition of any subinterval $[c, d]$ of $[a, b]$ that is contained in the open interval $(x - \epsilon, x + \epsilon)$. By the Heine-Borel theorem. the collection of all regular points is an open set $G \subset (a, b)$ for which $\mathcal{I}$ contains a partition of any interval $[c, d]$ contained in G. Moreover, if (a_i, b_i) is a component interval of G we can use the fact that $\mathcal{I}$ is right-fine at a_i and left-fine at b_i to see that $\mathcal{I}$ must contain a partition of any interval $[c, d] \subset [a_i, b_i]$.

If $G \supset [a, b]$ we are done. If not, then $P = [a, b] \setminus G$ is a nonempty, perfect subset of $[a, b]$ with contiguous intervals $\{[a_i, b_i]\}$. We obtain a contradiction from the supposition that P is nonempty. Let $\mathcal{E}$ be an $[a, b]$-form that witnesses $\mathcal{I}$ as a weak cover of $[a, b]$. That means that if u is a member of a set S belonging to $\mathcal{E}$ and if v is in $\overline{S}$, the interval $[u, v]$ is in $\mathcal{I}$ (with a similar version if $v \in S$).

By the Baire category theorem there is a nonempty portion $P \cap [c, d]$ in which one of the sets $S \in \mathcal{E}$ is dense. This means that all intervals $[u, v]$ with one endpoint in $S \cap [c, d]$ and the other endpoint in $P \cap [c, d]$ must belong to $\mathcal{I}$.

From this we shall argue that any interval $[c', d'] \subset [c, d]$ must have a partition that is contained in $\mathcal{I}$. Consider the possible cases: (i) If c', d' in P, then either there is no point s of S between c', d' or there is a point s of S between c', d'. In the former case, for some i, $c' = a_i$ and $d' = b_i$ so $[c', d']$ has a partition from $\mathcal{I}$. In the latter case, both $[c', s]$ and $[s, d']$ are in $\mathcal{I}$. We again have our partition. (ii) If $c' \notin P$ but $d' \in P$, then there is a component interval (a_i, b_i) of G to which c' belongs. Consider the interval

$$[c', d'] = [c', b_i] \cup [b_i, d'].$$

We know that there is a partition of $[c', b_i]$ contained in $\mathcal{I}$ and that $[b_i, d']$ also has a partition contained in $\mathcal{I}$. Consequently, we must have a partition of $[c', d']$ contained in $\mathcal{I}$. (iii) If $c' \in P$ but $d' \notin P$ a similar argument will prevail. Finally, (iv) if $c' \notin P$ and $d' \notin P$ then there is a component interval (a_i, b_i) of G to which c' belongs and there is a different component interval (a_j, b_j) of G to which c' belongs. We know that there is a partition of each of

$$[c', b_i], \ [b_i, a_j], \text{ and } [a_j, d'].$$

There are no other cases to consider. We have shown then that every subinterval of $[c, d]$ has a partition contained in $\mathcal{I}$. This contradicts the definition of P and completes the proof. □

2.6 Full covers, fine covers, and ordinary derivates

The connection between full and fine covers and the process of differentiation is intimate and immediate. That is, in essence, the reason why these covers offer such a simple way of connecting the apparently diverse concepts of measure, derivative, and integral.

For example, consider the two bilateral extreme derivatives $\overline{D}F(x)$ and $\underline{D}F(x)$ of a function F at a point x. Define

$$E^r = \{x : \overline{D}F(x) > r\},$$

$$E_r = \{x : \underline{D}F(x) > r\},$$

and

$$\mathcal{I} = \left\{ [u, v] : \frac{F(v) - F(u)}{v - u} \geq r \right\}.$$

It is easy to check that $\mathcal{I}$ is a full cover of the set E_r and is a fine cover of the (larger) set E^r. This "full/fine covering lemma" can be used in a variety of ways when studying derivatives. Readers familiar with Vitali arguments will have seen this before, but have perhaps not articulated the role of the full covers.

For approximate derivatives one does not see immediately any such weak covering lemma that would unite weak covers and the process of approximate derivation. The intimate relation between approximate derivatives and VBG functions is certainly displayed in Saks. That would suggest some kind of relation involving weak covers. In fact the connection is quite direct and explains why these weak covers play a role in the theory of the Denjoy-Khintchine integral.

We start with the most elementary case where weak covers might arise from a differentiation problem.

2.7 Weak covers and Dini derivatives

There is a weak covering argument that has been used (expressed in different language) for more than a century. One of the earliest uses was by Beppo Levi [24] to show that the set of points where a function has both a right derivative and a left derivative that are unequal would have to be countable. We express it for the upper right Dini derivative.

Lemma 16 (Weak covering lemma (a))**.** *Let $F:\mathbb{R}\to\mathbb{R}$ and suppose that at every point x of a set E the upper right Dini derivative satisfies, for some real number r, the inequality*

$$\overline{D}^{+}F(x) < r.$$

Then the collection

$$\mathcal{I} = \left\{[u,v] : \frac{F(v)-F(u)}{v-u} < r\right\}$$

is a q-weak cover of E.

Proof. This is the simplest of the covering arguments we shall present since it involves no density computations. We see (from the definition of upper right Dini derivative) that, for each u in E, there is a $1 > \delta(u) > 0$ so that

$$\frac{F(v)-F(u)}{v-u} < r$$

provided $u < v < u + \delta(u)$.

We can use δ to decompose E in a familiar way:

$$E_n = \{x \in E : 1/(n+1) < \delta(x) < 1/n\}.$$

Decompose each set E_n into a further sequence of subsets $\{E_{nj}\}$ such that each has diameter less than $1/(n+1)$. Choose any interval $[u,v]$ with endpoints in the same set E_{nj}. In that case $u \in E$ and $v - u < 1/(n+1) < \delta(u)$. Consequently

$$\frac{F(v) - F(u)}{v - u} < r$$

and so $[u, v] \in \mathcal{I}$. We have shown that any interval $[u, v]$ with endpoints in the same set E_{nj} must belong to $\mathcal{I}$. This verifies that $\mathcal{I}$ is a q-weak cover of E. □

Example As an elementary, but perhaps amusing exercise, here is the argument used by Grace Chisholm Young [38] for her theorem on Dini derivatives, but re-expressed in the language of q-weak covers.

> *For an arbitrary function F the set of points x at which the strict inequality $\overline{D}^+ F(x) < \underline{D}^- F(x)$ holds is countable.*

Take any rationals r and s and define

$$E_{rs} = \{x : \overline{D}^+ F(x) < r < s < \underline{D}^- F(x)\}.$$

The collections

$$\mathcal{I}_r = \left\{[u, v] : \frac{F(v) - F(u)}{v - u} < r\right\} \text{ and } \mathcal{I}^s = \left\{[u, v] : \frac{F(v) - F(u)}{v - u} > s\right\}$$

are both q-weak covers of E_{rs}. Thus the intersection $\mathcal{I}_r \cap \mathcal{I}^s = \emptyset$ is also a q-weak cover of E_{rs}. That implies that each such set E_{rs} is countable. Complete the proof by taking the union of these sets over all rationals r and s.

2.8 Weak covers and approximate derivates

For approximate derivatives there is an intimate connection with weak covers.

Lemma 17 (Weak covering lemma (b))**.** *Let $F : \mathbb{R} \to \mathbb{R}$ and suppose that at every point x of a set x the lower bilateral approximate derivative*

$$\underline{AD}F(x) > r.$$

Then the collection

$$\mathcal{I} = \left\{[u, v] : \frac{F(v) - F(u)}{v - u} > r\right\}.$$

is a q-weak cover of E.

Proof. We follow (for easy comparison) the arguments in [27, pp. 238–239] from which the idea for the covering argument is extracted. For the appropriate definitions of approximate derivative, point of density, and point of dispersion we can also refer to [27, §3, p. 220].

Write $F_r(t) = F(t) - rt$ for all t. We see (from the definition of upper approximate limit) that, for each x in E, the set

$$\left\{y : \frac{F(y)-F(x)}{y-x} \leq r\right\} = \left\{y : \frac{F_r(y)-F_r(x)}{y-x} \leq 0\right\}$$

has x as a point of dispersion (on both sides).

Consequently, for each $x \in E$ there is a $\delta(x) > 0$ so that

$$\lambda(\{t : F_r(t) - F_r(x) \leq 0\} \cap [x, x+h]) \leq h/3$$

and

$$\lambda(\{t : F(x) - F_r(t) \leq 0\} \cap [x-h, x]) \leq h/3$$

provided $0 \leq h \leq \delta(x)$. We can use δ to decompose E,

$$E_n = \{x \in E : 1/n < \delta(x) < 1/(n-1)\}.$$

Choose any interval $[u, v]$ with endpoints in the same set E_n and such that $v - u < 1/n$. Set $h = v - u$ and use the above density estimates to see that

$$\lambda(\{t : F_r(t) - F_r(u) \leq 0\} \cap [u, v]) \leq h/3$$

and

$$\lambda(\{t : F_r(v) - F_r(t) \leq 0\} \cap [u, v]) \leq h/3.$$

Thus, the interval (u, v) must contain a point w (many points in fact) for which both

$$F_r(w) - F_r(u) > 0 \text{ and } F_r(v) - F_r(w) > 0.$$

Adding these we have $F_r(v) - F_r(u) > 0$. Thus $F(v) - F(u) > r(v - u)$. By definition $[u, v] \in \mathcal{I}$.

Now it is clear how to verify that $\mathcal{I}$ is a q-weak cover of the set E. Decompose each set E_n into a further sequence of subsets $\{E_{nj}\}$ such that each has diameter less than $1/n$. Then we have shown that any interval $[u, v]$ with endpoints in the same set E_{nj} must belong to $\mathcal{I}$. □

2.9 Weak covers and approximate Dini derivatives

There is one more weak covering argument that we can extract from Saks [27, pp. 238–240]. This uses both the upper and the lower one-sided approximate derivatives, i.e., it uses a pair of the four approximate Dini derivatives. We express it for the upper and lower right approximate Dini derivatives.

Lemma 18 (Weak covering lemma (c))**.** *Let $F : \mathbb{R} \to \mathbb{R}$ and suppose that at every point x of a set E the lower and upper right approximate derivatives satisfy, for some positive number r,*

$$-r < \underline{AD}^{+} F(x) \leq \overline{AD}^{+} F(x) < r.$$

Then the collection

$$\mathcal{I} = \left\{ [u, v] : \left| \frac{F(v) - F(u)}{v - u} \right| < 4r \right\}$$

is a q-weak cover of E.

PROOF. We know (from the definition of approximate limits) that, for each x in E, the set

$$A_{xr} = \left\{ y : \left| \frac{F(y) - F(x)}{y - x} \right| \geq r \right\}$$

has x as a point of dispersion on the right.

Consequently, for each $x \in E$ there is a $\delta(x) > 0$ so that

$$\lambda(A_{xr} \cap [x, x + h]) \leq h/4$$

provided $0 \leq h \leq \delta(x)$. We can use δ to decompose E in the familiar way:

$$E_n = \{x \in E : 1/n < \delta(x) < 1/(n-1)\}.$$

Choose any interval $[u, v]$ with endpoints in the same set E_n and such that $v - u < 1/(2n)$. Set $w = 2v - u$. Then $w - u < 1/n < \delta(u)$ so that

$$\lambda(A_{ur} \cap [u, w]) \leq (w - u)/4.$$

From this we can deduce that

$$\lambda(A_{ur} \cap [v, w]) \leq (w - v)/2$$

since $(w - u)/4 = (w - v)/2$. We also have $w - v < 1/n < \delta(v)$ so that

$$\lambda(A_{vr} \cap [v, w]) \leq (v - w)/4.$$

These inequalities show that there must exist a point y that belongs to the interval $(v, w]$ so that both of the following inequalities must hold:

$$\left|\frac{F(y) - F(u)}{y - u}\right| < r$$

and

$$\left|\frac{F(y) - F(v)}{y - v}\right| < r.$$

(For those familiar with such arguments this corresponds to what we called an "external intersection condition" in [6] in the context of path derivatives.)

Now put these inequalities to work to obtain the estimate

$$\left|\frac{F(v) - F(u)}{v - u}\right| < 4r$$

for u, v in E_n and $v - u < 1/(2n)$. The details are simply these:

$$|F(v) - F(u)| \leq |F(y) - F(u)| + |F(y) - F(v)| < r(y - u) + r(y - v)$$

$$\leq r(w - u) + r(w - v) = 2r(v - u) + 2r(v - u) = 4r(v - u).$$

By definition then, $[u, v] \in \mathcal{I}$.

To see that $\mathcal{I}$ is a q-weak cover of E let $\{E_{nk}\}$ be a decomposition of each set E_n into subsets of diameter smaller than $1/(2n)$. Consequently any interval $[u, v]$ with endpoints in the same set $\{E_{nk}\}$ must belong to $\mathcal{I}$ as we have just seen. □

2.10 Approximately full covers

We have just seen an intimate connection between q-weak covers and approximate derivatives. If one were embarking, however, on a study of approximate derivatives the following definition would be the most likely choice.

Definition 19. Let $\mathcal{I}$ be a collection of closed intervals and $E \subset \mathbb{R}$. We say that $\mathcal{I}$ is an *approximately full cover* of E if, for each $x \in E$ there is a measurable set A_x that has x as a point of density so that every interval $[u, x]$ with $u \in A_x$ belongs to $\mathcal{I}$ and every interval $[x, v]$ with $u \in A_x$ belongs to $\mathcal{I}$.

This concept has been used to study the approximately continuous integral of Burkill. Henstock alludes to this occasionally in his writing, but one should see instead Bullen [5] who uses approximately full covers as part of an extensive study of this integral. The survey article of Skvortsov et al. [30] is an essential reference if one wishes to clarify some of the claims made in the literature as well as for an account of the variational measures that would be associated with approximately full covers.

These covers have most of the useful properties of full, weak, and q-weak covers. Like the others, they are also filtering and have the subadditive and pruning properties of Theorem 13 . Clearly every full cover is also an approximately full cover.

There are two properties that are worth pointing out, even though our study here will not make further use of this concept. The first is well-known and is discussed and exploited in [5], [30], and [32] among others. The second property would be useful for relating any proposed variational measure based on approximately full covers to the weak variational measures we study in this article.

Lemma 20 (Approximate Cousin lemma)**.** *Suppose that $\mathcal{I}$ is an approximately full cover of a bounded, open interval (a,b) that is right fine at a and left fine at b. Then, for every closed subinterval $[c,d]\subset[a,b]$, there is a finite subdivision*

$$c=c_0<c_1<c_2<\dots c_{n-1}<c_n=d$$

so that each $[c_{i-1},c_i]\in\mathcal{I}$ $(i=1,2,\dots,n)$.

Lemma 21 (Relation to q-weak covers)**.** *Suppose that $\mathcal{I}$ is an approximately full cover of a set E. Let $\mathcal{I}_1$ denote the collection that includes $\mathcal{I}$ but also any interval $[u,w]$ for which $[u,v]$ and $[v,w]$ belong to $\mathcal{I}$ for some $u<w<v$. Then $\mathcal{I}_1$ is a q-weak cover of E.*

Proof. This is already part of the proof of Lemma 16 although it was not there expressed in the language of approximately full covers. □

3 Variational measures

We are now in a position to define the variational measures associated with a function that are our main object of study. Each of the classes of covers (full, fine, weak and q-weak) gives rise to a measure.

Definition 22. Let $F:\mathbb{R}\to\mathbb{R}$ and let E be a set of real numbers. Then we define

1. $V_F(E)=\inf\{V(F,\mathcal{I}):\ \mathcal{I}$ is a full cover of $E\}$.
2. $v_F(E)=\inf\{V(F,\mathcal{I}):\ \mathcal{I}$ is a fine cover of $E\}$.
3. $W_F(E)=\inf\{V(F,\mathcal{I}):\ \mathcal{I}$ is a weak cover of $E\}$.

4. $W_F^q(E) = \inf\{V(F,\mathcal{I}) :\ \mathcal{I}$ is a q-weak cover of $E\}$.

The immediate properties that we can obtain are that these are Borel measures with obvious relations among them.

Theorem 23. *Let $F : \mathbb{R} \to \mathbb{R}$ be an arbitrary function. Then the set functions V_F, v_F, W_F^q, and W_F are Borel measures on $\mathbb{R}$ and these inequalities hold:*

$$v_F \leq V_F \text{ and } W_F^q \leq W_f \leq V_F.$$

Proof. That these are Borel measures follows from Lemma 10. The inequalities derive immediately from the facts that all full covers are fine covers, all weak covers are q-weak covers, and all full covers are weak covers. □

The full variational measures V_F play a well-known role in studying and characterizing the notions VBG$_*$ and ACG$_*$. We propose the closely related (and smaller) variational measures W_F and W_F^q to play a similar role in studying and characterizing the notions VBG and ACG. As we shall see the two measures W_F and W_F^q agree for continuous functions. It is not hard, however, to see that they are different in general. Let $F(x) = 0$ for x rational and $F(x) = 1$ for x irrational; such a function is VBG. The smaller measure W_F^q vanishes on all sets, but $W_F([0,1]) = \infty$.

3.1 Weak covering arguments for VBG and ACG functions

We now summarize what we can extract from the theory of VBG and ACG functions reviewed in Section 1.4 in the language of covers and variational measures.

First lemma To begin, we note how and in what way the concept of VBG is completely captured by the measures W_F and W_F^q.

Lemma 24. *Let $F : \mathbb{R} \to \mathbb{R}$ and $E \subset \mathbb{R}$. Then*

1. *$W_F^q(E) \leq V(F;E)$.*
2. *If E is closed, then $W_F(E) \leq V(F;E)$.*
3. *F is VBG on E if and only if W_F^q is σ-finite on E.*
4. *If F is continuous on the set $\overline{E}$, then $W_F(E) = W_F^q(E)$.*
5. *If F is continuous on $\overline{E}$, then F is VBG on E if and only if W_F is σ-finite on E.*

PROOF. *Statement 1.* Let $\mathcal{I}_E$ denote the collection of all intervals $[u,v]$ with both endpoints in the set E. Then $\mathcal{I}_E$ is a q-weak cover of E and, by definition, $V(F;E) = V(F,\mathcal{I}_E)$. Thus

$$W_F^q(E) \leq V(F,\mathcal{I}_E) = V(F;E).$$

Statement 2. If E is closed, then $\mathcal{I}_E$ (as above) is now a weak cover of E and so

$$W_F(E) \leq V(F,\mathcal{I}_E) = V(F;E).$$

Statement 3. From statement 1 it follows that if F is VBG on E then W_F^q must be σ-finite on E. Suppose then that $W_F^q(E) < \infty$. There is a q-weak cover $\mathcal{C}$ with $V(F,\mathcal{C}) < \infty$. Select an E-form $\mathcal{E}$ that witnesses $\mathcal{C}$ as a q-weak cover. Check that

$$V(F;S) \leq V(F,\mathcal{C}) < \infty$$

for each $S \in \mathcal{E}$. Consequently, F is VBG on E.

Statement 4. If $W_F^q(E) = \infty$, then certainly $W_F(E) = W_F^q(E)$. If $W_F^q(E) < \infty$ then F is VBG and we can use the classical theory of continuous VBG functions.

We know from Lemma 7 that we must have

a. A sequence of closed bounded sets $\{E_n\}$ whose union includes all of $\overline{E}$.

b. For each n, a continuous function $G_n : \mathbb{R} \to \mathbb{R}$ of bounded variation function so that $F(x) = G_n(x)$ for each $x \in E_n$.

c. For each n, $F'_{ap}(x) = G'_n(x)$ for almost every $x \in E_n$.

We can insist too that G_n is linear and continuous in the contiguous intervals to E_n if that helps. For our lemma we can exploit property (b) by using a version of the Vitali theorem for the continuous functions of bounded variation G_n:

$$V_{G_n} = v_{G_n}.$$

(See [34, Theorem 6.29].)

Using the agreement of F and G_n on the closed set E_n, we see that, for any set $S \subset E_n$,

$$v_{G_n}(S) \leq W_{G_n}^q(S) = W_F^q(S) \leq W_F(S) = W_{G_n}(S) \leq V_{G_n}(S).$$

Thus $W_F^q(S) = W_F(S)$ for all subsets of the closed set E_n. As these are Borel measures the same must be true for all subsets of E which are contained in the union of the E_n.

Statement 5. This now follows from Statements 3 and 4. □

Second lemma Let us examine the proof of the first lemma to see if there is more that we can extract. In the case that F is ACG (not merely VBG) the functions G_n can be chosen to be absolutely continuous. About such functions we know that

$$V_{G_n}(S) = \int_S |G_n'(x)|\, dx \tag{2}$$

for all measurable sets S and, consequently V_{G_n} vanishes on all sets of Lebesgue measure zero.

Applying this to any measure zero subset Z of E, we have that

$$W_F(Z) \leq \sum_{n=1}^{\infty} W_F(Z \cap E_n) = \sum_{n=1}^{\infty} W_{G_n}(Z \cap E_n) = \sum_{n=1}^{\infty} V_{G_n}(Z \cap E_n) = 0.$$

Consequently W_F vanishes on all measure zero subsets of E. This proves one direction in the following lemma.

Lemma 25. *Let $F : \mathbb{R} \to \mathbb{R}$, $E \subset \mathbb{R}$ and suppose that F is continuous on $\overline{E}$. Then F is ACG on E if and only if F is VBG on E and the measure W_F is zero on all subsets of E that have Lebesgue measure zero.*

Proof. The other direction that needs proof starts by assuming that W_F is absolutely continuous with respect to Lebesgue measure and that F is VBG. Thus, we shall use our scheme to deduce that the functions G_n as above are absolutely continuous (we know as yet that they are continuous and have bounded variation).

But if $N \subset E_n$ is a set of measure zero and $W_F(N) = 0$, then $W_{G_n}(N) = 0$. If N is a set of measure zero that is inside any complementary interval to E_n, then, since we insisted that G_n is linear and continuous in the contiguous intervals to E_n, then again $W_{G_n}(N) = 0$. So for any set N of measure zero $W_{G_n}(N) = 0$. That implies that G_n is absolutely continuous for each n and exhibits F as an ACG function on E. □

As a corollary we can invoke the category lemma (Lemma 8) and improve this, for closed sets E, by dropping the need to assume in advance that F is VBG.

Corollary 26. *Let $F : \mathbb{R} \to \mathbb{R}$, let $E \subset \mathbb{R}$ be a closed set and suppose that F is continuous on E. Then F is ACG on E if and only if the measure W_F is zero on all subsets of E that have Lebesgue measure zero.*

Third lemma Can we extract yet more from our proof? The identity (2) provides

$$W_F(S) = W_{G_n}(S) = V_{G_n}(S) = \int_S |G_n'(x)|\, dx = \int_S |F_{ap}'(x)|\, dx$$

for all measurable sets $S \subset E_n$. But the sequence $\{E_n\}$ consists of closed sets whose union is all of E. As W_F is a measure this identity must hold for all measurable sets $S \subset E$. This establishes the next of our lemmas.

Lemma 27. *Let $F : \mathbb{R} \to \mathbb{R}$ and $E \subset \mathbb{R}$. If F is continuous on $\overline{E}$ and also F is ACG on E then, for all measurable subsets S of E,*

$$W_F^q(S) = W_F(S) = \int_S |F_{ap}'(x)|\, dx. \tag{3}$$

Fourth lemma Yet again the same covering argument gives us one more useful observation to be extracted. We shall need it for the characterization of the Denjoy-Khintchine integral.

The theory of the Henstock-Kurzweil integral can be applied here to each of the absolutely continuous functions G_n. Let g_n be any function that is almost everywhere equal to G_n'. Then, for any $\epsilon_n > 0$, we can find a gauge $\delta_n : \mathbb{R} \to \mathbb{R}$ so that

$$\sum_{i=1}^{n} |G_n(v_i) - G_n(u_i) - g_n(w_i)(v_i - u_i)| < \epsilon_n \tag{4}$$

whenever $\{([u_i, v_i], w_i) : i = 1, 2, 3, \dots, n\}$ is a packing finer than δ_n. To be a packing requires only that the collection of intervals $\{[u_i, v_i] : i = 1, 2, 3, \dots, n\}$ are pairwise nonoverlapping. To be finer than δ_n requires $v_i - u_i < \delta_n(w_i)$ where w_i is either u_i or v_i.

This can be translated into a useful statement about the function F on the set E_n since F and G_n agree on E_n and $G_n'(x)$ and $F_{ap}'(x)$ agree almost everywhere on E_n.

Lemma 28. *Suppose that F is continuous on a closed set E and AC on each member of a sequence of closed sets $\{E_n\}$ whose union is E. Let $f(x)$ be any function that agrees almost everywhere with $F_{ap}'(x)$ on E_n.*

Then, for any $\epsilon_n > 0$, we can find a gauge $\delta_n : E_n \to \mathbb{R}^+$ so that

$$\sum_{i=1}^{n} |F(v_i) - F(u_i) - f(w_i)(v_i - u_i)| < \epsilon_n \tag{5}$$

whenever $\{([u_i, v_i], w_i) : i = 1, 2, 3, \dots, n\}$ is a packing with u_i, v_i, $w_i = u_i$ or $w_i = v_i$, u_i, $v_i \in E_n$ and $v_i - u_i < \delta_n(w_i)$.

3.2 The Vitali covering theorem for VBG functions

The Vitali covering theorem on the real line is usually asserted for Lebesgue measure and always in a form that arrives at some conclusion similar to

$$\lambda\left(E \setminus \bigcup_{n=1}^{\infty}[u_n, v_n]\right) = 0.$$

Our version of this theorem in Section 2.2 had the equivalent but unusual conclusion

$$v_\lambda = V_\lambda = \lambda$$

where λ is Lebesgue outer measure. What we claim also as a Vitali covering theorem is the identity

$$v_F = V_F$$

which holds for continuous functions F if and only if F is VBG_*. A proof is given in [34, Chapter 6]. This material is known (although I would not say well-known). It is an essential part of the study of VBG_* functions.

A proposed "weak" version is our concern now. In order to develop a Vitali theorem for the weak measures we need a dual measure w_F to accompany the weak measure W_F in the same way that v_F and V_F are dual measures. The following definition captures this.

Definition 29. A collection $\mathcal{I}$ of closed, bounded intervals is said to be a *weak fine cover* of a set E provided for any weak cover $\mathcal{J}$ of E the collection $\mathcal{I} \cap \mathcal{J}$ is fine at nearly every point in E.

By "nearly" we mean fine at all but countably many points. Thus a weak fine cover is also a fine cover (and hence also a Vitali cover) if countable sets are ignored. Note that there is essentially the same duality between weak covers and weak fine covers as there is between full covers and fine covers.

Theorem 30. *Suppose that $F : \mathbb{R} \to \mathbb{R}$. For any set E we define*

$$w_F(E) = \inf\{V(F, \mathcal{I}) : \mathcal{I} \text{ a weak fine cover of } E)\}.$$

Then w_F is a Borel measure on $\mathbb{R}$ and $w_F \le W_F \le V_F$. If F is continuous then

$$v_F \le w_F \le W_F \le V_F.$$

Proof. By Lemma 10 we can establish that w_F is a Borel measure by showing that the class of weak fine covers of any set E has both the required subadditive property and the pruning property.

We start with the pruning property. Suppose that $\mathcal{I}$ is a weak fine cover of E and that G is an open set that contains E. Take any weak cover $\mathcal{J}$ of E then

$$\mathcal{I}(G) \cap \mathcal{J} = \mathcal{I} \cap \mathcal{J}(G).$$

We know that $\mathcal{J}(G)$ is a weak cover of E and thus we know that the intersection $\mathcal{I} \cap \mathcal{J}(G)$ is fine at nearly every point of E. But that means that $\mathcal{I}(G) \cap \mathcal{J}$ is fine at nearly every point of E for any choice of $\mathcal{J}$. By definition then, $\mathcal{I}(G)$ is a weak fine cover of E.

Let us check the subadditive property. Suppose that $E \subset \bigcup_{n=1}^{\infty} E_n$ and we have, for each n, a collection $\mathcal{I}_n$ that is a weak fine cover of E_n. Define

$$\mathcal{I} = \bigcup_{n=1}^{\infty} \mathcal{I}_n.$$

Take any weak cover $\mathcal{J}$ of E. Then $\mathcal{I} \cap \mathcal{J}$ must be fine at nearly every point in E_n. As this is true for all n, there are at most countably many points in E at which $\mathcal{I} \cap \mathcal{J}$ fails to be fine. By definition, $\mathcal{I}$ is a fine weak cover of E.

Because of these two properties, we can conclude that w_F is a Borel measure. The inequality $w_F \leq W_F$ follows from the simple fact that every weak cover of a set is also a weak fine cover. The inequality $v_F \leq w_F$ for continuous functions follows from the fact that every weak fine cover of a set is a fine cover of all but a countable subset and the fact that v_F vanishes on all countable sets for continuous functions. $\square$

Theorem 31 (Weak Vitali covering theorem)**.** *Suppose that F is VBG on a set E and continuous on $\overline{E}$. Then,*

$$w_F(S) = W_F(S) \textit{ for all } S \subset E.$$

PROOF. The weak Vitali covering theorem is obtained from the Vitali theorem $V_G = v_G$ for continuous functions of bounded variation G. As usual, find a sequence of bounded closed sets $\{E_n\}$ covering E and a sequence of continuous functions $\{G_n\}$ of bounded variation with $F(x) = G_n(x)$ on $E \cap E_n$. We know then that $v_{G_n} = V_{G_n}$. We also know, for continuous functions, that

$$v_{G_n} \leq w_{G_n} \leq W_{G_n} \leq V_{G_n}.$$

But $W_F = W_{G_n}$ and $w_F = w_{G_n}$ on subsets of $E \cap E_n$ because the two functions F and G_n have identical values on the closed set E_n. Putting these together and using the fact that these are measures we obtain that $w_F = W_F$ on all subsets of E. $\square$

3.3 Some variational inequalities

Lemma 32. *Let $F : \mathbb{R} \to \mathbb{R}$ be a function such that $|F'_{ap}(x)| > r$ at every point x of a set E. Then*

$$\mathcal{I}_r = \left\{ [u,v] : \left| \frac{F(v) - F(u)}{v - u} \right| > r \right\}$$

is a q-weak cover of E. Consequently $W_F^q(E) \geq r\lambda(E)$.

Proof. To establish that $\mathcal{I}_r$ is a q-weak cover of E is identical to the proof of Lemma 17 (where we assumed the weaker hypothesis that $\underline{AD}F(x) > r$ instead). Let $\mathcal{I}$ be an arbitrary weak cover of E. Then $\mathcal{I}_r \cap \mathcal{I}$ is also a weak cover of E. It is moreover nearly a Vitali cover of E. Thus, taking advantage of the Vitali covering theorem (i.e., Theorem 11) we must have

$$r\lambda(E) \leq V(r\lambda, \mathcal{I} \cap \mathcal{I}_r) \leq V(F, \mathcal{I} \cap \mathcal{I}_r) \leq V(F, \mathcal{I}).$$

As this holds for all choices of $\mathcal{I}$, it follows that $W_F^q(E) \geq r\lambda(E)$. □

A similar lemma is proved in a nearly identical fashion.

Lemma 33. *Let $F : \mathbb{R} \to \mathbb{R}$ be a function such that $|F'_{ap}(x)| < s$ at every point x of a set E. Then*

$$\mathcal{I}^s = \left\{ [u,v] : \left| \frac{F(v) - F(u)}{v - u} \right| < s \right\}$$

is a q-weak cover of E. Consequently $W_F^q(E) \leq s\lambda(E)$.

Proof. We can establish that $\mathcal{I}^s$ is a q-weak cover of E by the same methods as the preceding lemma. Take any open set G containing E. Then $\mathcal{I}^s(G)$ is also a weak cover of E. We must have

$$W_F^q(E) \leq V(F, \mathcal{I}^s(G)) \leq V(s\lambda, \mathcal{I}^s(G)) \leq s\lambda(G).$$

As this holds for all choices of G it follows that $W_F^q(E) \leq s\lambda(E)$. □

3.4 Zero approximate derivatives and finite approximate derivatives

As a result of Lemmas 32 and 33 there is a close connection between weak variation zero and approximate derivative zero. There is a similar connection between finite weak variational measure and finite approximate derivatives. We express this connection in the next two theorems.

Theorem 34. *Let $F : \mathbb{R} \to \mathbb{R}$. The following assertions hold:*

1. *If, at every point x of a set E, the approximate derivative $F'_{ap}(x)$ is zero, then $W_F^q(E) = 0$.*
2. *If $W_F^q(E) = 0$, then at almost every point x of E the approximate derivative $F'_{ap}(x)$ is zero.*

Proof. Let $F : \mathbb{R} \to \mathbb{R}$ be a function such that $s > |F'_{ap}(x)| = 0$ at every point x of a bounded set E. Then by Lemmas 33, we know that $W_F^q(E) \leq s\lambda(E)$. If this is true for all $s > 0$, then $W_F^q(E) = 0$. If E is unbounded, then this shows that every bounded portion of E has W_F^q-measure zero. Thus, in that case too, $W_F^q(E) = 0$.

The opposite direction is nearly identical but with the exclusion of some set of measure zero. The assumption that $W_F^q(E) = 0$ requires F to be VBG on E, and hence, F has a finite approximate derivative at almost every point of E. Let N_1 be the set of points $x \in E$ at which $F'_{ap}(x)$ fails to exist; we know now that $\lambda(N_1) = 0$. Let N_{m+1} be the set of points $x \in E$ at which $|F'_{ap}(x)| > 1/m$ for $m = 1, 2, 3, \ldots$. Then by Lemma 32, we have the inequality

$$0 = mW_F^q(E) \geq mW_F^q(N_{m+1}) \geq \lambda(N_{m+1}).$$

Thus, N_{m+1} has Lebesgue measure zero. The set $N = \bigcup_{m=1}^{\infty} N_m$ has Lebesgue measure zero too since it is a union of countably many sets of measure zero. Since N contains every point at which either $F'_{ap}(x)$ fails to exist or at which $F'_{ap}(x) \neq 0$, we have proved that, at almost every point x of E, the approximate derivative $F'_{ap}(x)$ is zero. □

A similar version for finite approximate derivatives can be stated without proof. Note that we already know that VBG functions have finite approximate derivatives almost everywhere.

Theorem 35. *Let $F : \mathbb{R} \to \mathbb{R}$. The following assertions hold:*

1. *If, at every point x of a set E of Lebesgue measure zero, the approximate derivative $F'_{ap}(x)$ exists and is finite, then $W_F^q(E) = 0$.*
2. *If $W_F^q(E) < \infty$, then at almost every point x of E the approximate derivative $F'_{ap}(x)$ exists and is finite.*

3.5 Representation of the q-weak variational measures

If F is a function that is differentiable in the ordinary sense at every point of a measurable set E, then the identity

$$V_F(E) = \int_E |F'(x)|\, dx$$

is known (see [34, Theorem 6.8]). There is an identical version for the approximate derivative replacing the full variational measure with the q-weak one. We have already seen such a representation in Lemma 27, but there was the added assumption that F was continuous.

Theorem 36. *Let $F : \mathbb{R} \to \mathbb{R}$ be a measurable function that has a finite approximate derivative $F'_{ap}(x)$ at every point x of a measurable set E. Then*

$$W^q_F(E) = \int_E |F'_{ap}(x)|\, dx. \tag{6}$$

Proof. We can assume that E is a measurable set of finite measure. Write $f(x) = |F'_{ap}(x)|$ for convenience. This is a measurable function defined on E.

We use the variational estimates of Lemmas 32 and 33 to obtain measure inequalities for sets of the form

$$D_{rs} = \{x \in E : r < |f(x)| \le s\}.$$

Write

$$D_{rsn} = \{x \in E : r < |f(x)| < s + 1/n\} \quad (n = 1, 2, 3, \dots).$$

Then we have

$$r\lambda(D_{rs}) \le W^q_F(D_{rs}) \text{ and } W^q_F(D_{rs}) \le W^q_F(D_{rsn}) \le (s + 1/n)\lambda(D_{rsn}).$$

Since

$$D_{rs} = \bigcap_{n=1}^{\infty} D_{rsn}$$

we conclude, from standard measure-theoretic properties, that

$$r\lambda(D_{rs}) \le W^q_F(D_{rs}) \le s\lambda(D_{rs}). \tag{7}$$

Define the set

$$Z = \{x \in E : f(x) = 0\}$$

and note that

$$W^q_F(Z) = \int_Z f(x)\, dx = 0.$$

Let $\theta > 1$ and use the sequence

$$0 < \dots < \theta^{-3} < \theta^{-2} < \theta^{-1} < 1 < \theta^1 < \theta^2 < \theta^3 < \dots$$

to define the sets

$$E_n = \{x \in E : \theta^n < f(x) \le \theta^{n+1}\} \quad (n = 0, \pm 1, \pm 2, \dots).$$

One has to verify that E is the disjoint union of all of these sets Z, E_0, E_1, E_{-1}, … and that these are all measurable sets. As W_F^q vanishes on subsets of E that have Lebesgue measure zero, these sets are also W_F^q-measurable.

We make use of the variational estimates (7) just obtained to deduce that

$$\theta^n \lambda(E_n) \le W_F^q(E_n) \le \theta^{n+1}\lambda(E_n).$$

We compute that

$$\theta^{-1} W_F^q(E_n) \le \theta^n \lambda(E_n) \le \int_{E_n} f(x)\,dx \le \theta^{n+1}\lambda(E_n) \le \theta W_F^q(E_n).$$

We also have

$$W_F^q(Z) = \int_Z f(x)\,dx = 0.$$

Now, summing all of these inequalities, we obtain

$$\theta^{-1} W_F^q(E) \le \int_E f(x)\,dx \le \theta W_F^q(E).$$

This is valid for all choices of $\theta > 1$ and so the identity (6) in the theorem must follow for W_F^q. □

4 Generalized bounded variation

One of our goals was to obtain a way of rewriting and extending the theory from Saks about functions of generalized bounded variation (VBG) and generalized absolute continuity (ACG). We can review some of the highlights here.

4.1 Generalized bounded variation and variational measures

We can now summarize how the measures (full, fine, and weak) relate to the classical notions of generalized bounded variation explored in Saks. Saks credits Denjoy and Khintchine with the first versions of the theory of these generalized variations. Denjoy, in particular, developed his ideas for continuous functions and focused only on the variational ideas on closed sets. The exposition in Saks has long been considered definitive (at least until the emergence of the theory of these variational measures many years later).

Let us state some of the facts as a theorem. If we restrict ourselves to continuous functions the details are simplified.

Theorem 37. *Let $F : \mathbb{R} \to \mathbb{R}$ be a continuous function, $[a, b]$ an interval and $E \subset \mathbb{R}$. Then*

1. *$V_F([a, b]) = V(F, [a, b])$, i.e., the total variation of F (which may be infinite).*
2. *If F is locally of bounded variation, then*
$$v_F = w_F = W_F^q = W_F = V_F.$$
3. *F is VBG_* on E if and only if V_F is σ-finite on E.*
4. *F is ACG_* on E if and only if V_F is σ-finite on E and absolutely continuous with respect to Lebesgue measure on E.*
5. *F is VBG on E if and only if W_F is σ-finite on E.*
6. *F is ACG on E if and only if W_F is σ-finite on E and absolutely continuous with respect to Lebesgue measure on E.*
7. *If F is VBG_* on E then $v_F = V_F$ on all subsets of E.*
8. *If F is VBG on E then $w_F = W_F$ on all subsets of E.*
9. *Suppose that E is closed. Then F is VBG_* on E if and only if V_F is σ-finite on every subset of E that has Lebesgue measure zero.*
10. *Suppose that E is closed. Then F is ACG_* on E if and only if V_F vanishes on every subset of E that has Lebesgue measure zero.*
11. *Suppose that E is closed. Then F is VBG on E if and only if W_F is σ-finite on every subset of E that has Lebesgue measure zero.*
12. *Suppose that E is closed. Then F is ACG on E if and only if W_F vanishes on every subset of E that has Lebesgue measure zero.*

For discontinuous functions There are some differences for discontinuous functions. In order for V_F to be σ-finite on a set E it would be necessary and sufficient for F to be VBG$_*$ on E and also locally bounded at each point of E. In order for F to be VBG on E, it would be necessary and sufficient for the q-weak measure W_F^q to be σ-finite on E. The measures W_F^q and W_F do not necessarily agree for discontinuous functions. The two "Vitali" theorems $v_F = V_F$ and $w_F = W_F$ generally require continuity.

4.2 Criteria for the classes VBG and ACG

Saks [27, pp. 237–240] gives a number of criteria that can be used to deduce that a function is VBG or ACG from information about its derivatives, either its Dini derivatives, its approximate derivates, or its approximate Dini derivatives. These criteria correspond precisely to the three covering lemmas we presented in Sections 2.7, 2.8, and 2.9.

Theorem 38. *The following statements are sufficient conditions for a function $F : \mathbb{R} \to \mathbb{R}$ to be VBG on a set E.*

1. *At every point x of E one at least of the four Dini derivatives*

$$\overline{D}^+F(x),\ \underline{D}^+F(x),\ \overline{D}^-F(x),\quad \textit{or}\ \ \underline{D}^-F(x)$$

is finite.

2. *At every point x of E one at least of the two approximate extreme derivatives*

$$\overline{AD}F(x) \quad \textit{or} \quad \underline{AD}F(x)$$

is finite.

3. *At every point x of E the pairs of the approximate Dini derivatives satisfy either*

$$-\infty < \underline{AD}^+F(x) \leq \overline{AD}^+F(x) < \infty$$

or

$$-\infty < \underline{AD}^-F(x) \leq \overline{AD}^-F(x) < \infty.$$

Proof. See the conditions in Lemmas 16, 17, and 18. The proofs are simply applications of those lemmas. One can consult Saks for the rather straightforward details. The only real work that needs to be done is the covering argument itself. □

Corollary 39. *Suppose that $F : \mathbb{R} \to \mathbb{R}$ is continuous, and that, at every point x of a set E, the pairs of the approximate Dini derivatives satisfy either*

$$-\infty < \underline{AD}^+F(x) \leq \overline{AD}^+F(x) < \infty$$

or

$$-\infty < \underline{AD}^-F(x) \leq \overline{AD}^-F(x) < \infty.$$

Then F is ACG on E.

4.3 Lusin's condition (N)

A function F is said to satisfy Lusin's condition (N) on a set E provided the Lebesgue measure of the image set $F[Z]$ is zero for every measure zero subset Z of E. There is a close connection with the concepts we are studying here. In particular, Saks proves that

> [27, Theorem 6.7] *In order that a function F which is continuous and VB on a bounded closed set E be AC on E, it is necessary and sufficient that F fulfill Lusin's condition (N) on this set.*

This connects continuous ACG functions with functions that are both VBG and satisfy Lusin's condition (N).

There is another perspective that we can have on this feature. The lemma shows that absolute continuity of the measure W_F necessarily requires that F satisfy Lusin's condition (N).

Lemma 40. *Suppose that F is continuous. Then, for every set $E \subset \mathbb{R}$,*

$$\lambda(F[E]) \leq W_F(E). \tag{8}$$

Proof. We shall use in the proof the similar inequality for the full variational measures:

$$\lambda(G[E]) \leq V_G(E)$$

where λ is the Lebesgue outer measure and $G[E]$ is the image under the function G of the set E. For a proof we refer the reader to [34, Theorem 6.7]. If G is continuous and has bounded variation, then we would know more:

$$\lambda(G[E]) \leq W_G(E) = V_G(E).$$

Let us address the inequality (8). If $W_F(E) = \infty$ there is nothing here to prove. Suppose that $W_F(E) < \infty$. Then F is VBG on E and we can construct the usual sequence of closed sets $\{E_n\}$ and usual sequence of continuous functions of bounded variation $\{G_n\}$ with $F = G_n$ on E_n. Let $\{E'_n\}$ be a pairwise disjoint sequence constructed from $\{E_n\}$ so as to have the same union, so that $E'_n \subset E_n$, and the sets $\{E'_n\}$ are Borel sets.

This means that

$$\lambda(F[E]) \leq \sum_{n=1}^{\infty} \lambda(F[E \cap E'_n]) = \sum_{n=1}^{\infty} \lambda(G_n[E \cap E'_n])$$

$$\leq \sum_{n=1}^{\infty} W_{G_n}(E \cap E'_n) = \sum_{n=1}^{\infty} W_F(E \cap E'_n) = W_F(E).$$

□

5 The Denjoy-Khintchine integral

So far, the q-weak covers and the q-weak variational measures W_F^q offer a different and more useful exposition for VBG functions. For continuous functions, the weak covers and the weak variational measures W_F offer a way to study both the VBG and ACG concepts. We now to turn to our other project—to provide a characterization of the Denjoy-Khintchine integral that arises from the same considerations. This should be considered a presentation and simplification of some of the material that can be found in Ene [13] and Sworowski [31] for that integral.

5.1 Packings and partitions

We have defined a cover to simply mean a collection $\mathcal{I}$ of closed intervals. For an integration theory, we prefer covering relations. By a *covering relation* we shall mean a collection of pairs $([u, v], w)$ where $[u, v]$ is a closed interval and w is either the endpoint u or the endpoint v.

Definition 41. If a covering relation

$$\pi = \{([u_i, v_i], w_i) : i = 1, 2, 3, \dots, n\}$$

is finite and the intervals $\{[u_i, v_i]\}$ in this collection do not overlap, then π is said to be a *packing*. If, moreover,

$$\bigcup_{i=1}^{n} [u_i, v_i] = [a, b],$$

then π is said to be a *partition* of the interval $[a, b]$.

We can extend our definitions of full covers, fine covers, and weak covers to similar concepts for coverings. Thus, full coverings, for example, now can be used in any instance in which a full cover would have appeared. The simplest version of the theory for our purposes, however, is simply to announce which packings or partitions are to be employed.

Definition 42. Let $\delta : E \to \mathbb{R}^+$ be a gauge on a set E. We say that a packing

$$\pi = \{([u_i, v_i], w_i) : i = 1, 2, 3, \dots, n\}$$

is *finer* than δ provided that, for each i, w_i is a point of E, and either $w_i = u_i$ or $w_i = v_i$, and the corresponding interval $[u_i, v_i]$ has length smaller than $\delta(w_i)$.

Definition 43. Let $\mathcal{E}$ be an E-form. We say that a packing

$$\pi = \{([u_i, v_i], w_i) : i = 1, 2, 3, \ldots, n\}$$

is *compatible* with $\mathcal{E}$ provided that $w_i = u_i$ or $w_i = v_i$ is a point of some S from $\mathcal{E}$ and the corresponding interval $[u_i, v_i]$ has endpoints in the set $\overline{S}$.

There is a relation between these two concepts that is closely connected to a fact that we have proved earlier: every full cover of a set is also a weak cover. The proof is identical with that for Statement 2 of Theorem 13.

Lemma 44. *Let $\delta : E \to \mathbb{R}^+$ be a gauge on a set E. Then there must exist an E-form $\mathcal{E}$ so that any packing π that is compatible with $\mathcal{E}$ is necessarily finer than δ.*

5.2 Characterization of the Denjoy-Perron integral

In order to motivate our characterization of the Denjoy-Khintchine integral, let us summarize some of the facts already well-known for the narrower Denjoy-Perron integral.

Theorem 45. *Let f, F be real functions defined on a closed bounded interval $[a, b]$. The following five statements are equivalent:*

1. *f is Denjoy integrable in the restricted sense on $[a, b]$ and F is an indefinite integral for f on that interval.*
2. *F is ACG_* on $[a, b]$ and $F'(x) = f(x)$ at almost every point $x \in (a, b)$.*
3. *$F'(x) = f(x)$ at almost every and V_F-almost every point $x \in [a, b]$.*
4. *For each $\epsilon > 0$ there is a gauge δ on $[a, b]$ such that*

$$\sum_{i=1}^{n} |F(v_i) - F(u_i) - f(w_i)(v_i - u_i)| < \epsilon \tag{9}$$

   for every packing $\{([u_i, v_i], w_i)\}$ that is finer than δ.

5. *For each $\epsilon > 0$ there is a gauge δ on $[a, b]$ such that, for any subinterval $[c, d]$ of $[a, b]$,*

$$\left| F(d) - F(c) - \sum_{i=1}^{n} f(w_i)(v_i - u_i) \right| < \epsilon \tag{10}$$

   for every partition $\{([u_i, v_i], w_i)\}$ of $[c, d]$ that is finer than δ.

The first two statements are classical (i.e., about a century old now). Denjoy's integral is defined by a countable, transfinite sequence of extensions of the Lebesgue integral. The second statement is commonly called the descriptive version and is due to Lusin. A further characterization was given by Perron at the same time; it has some technical uses for developing properties of the integral but was severely (and intemperately) criticized by Denjoy as a nonconstructive fantasy. Zygmund in Math Reviews [MR0031096 (11,99d)] recounts this in a characteristically gentle way: "One also finds a criticism of what is usually called Perron's definition of integral. This is merely one aspect of the author's [Denjoy] distrust of a certain type of mathematical reasoning. One may not share the author's views here in their entirety, and still be in agreement with him about the importance of constructive definitions in the theory of integrals."

A full account of the Denjoy integral, the Lusin characterization, and the Perron version appears in Saks [27, Chapter VIII].

The fourth and fifth assertions are now half a century old and are due to Henstock. At first these were noticed only by specialists, but eventually the fact that a formally simple exposition of this integral using Riemann sums was possible attracted significant attention. The late Bob Bartle was awarded the Paul R. Halmos–Lester R. Ford Award in 1997 for expository excellence as a result of his account of this integral [1] published in the American Mathematical Monthly. That the integral had been known for nearly forty years prior is a good indication of how little mainstream attention it had attracted as well as how the material in Saks had fallen from favor. There are numerous sources for these ideas (many in the bibliography below, including [34]).

5.3 A variational characterization of the Denjoy-Khintchine integral

We present now our first version of Theorem 45 for the Denjoy-Khintchine integral. Note that the statement in the theorem assuming that F is already given to be continuous makes this less satisfying than Theorem 45 since, in that theorem, the continuity of F can be deduced from the variational statements. Another "defect" is that there is no statement about partitions possible in this version since they may not exist. Both of these difficulties are patched up in Section 5.5 below.

Theorem 46. *Let f, F be real functions defined on a closed bounded interval $[a,b]$. Suppose that F is continuous. The following four statements are equivalent:*

1. *f is Denjoy integrable in the wide sense on $[a,b]$ and F is an indefinite integral for f on that interval.*

2. *F is ACG on $[a,b]$ and $F'_{ap}(x) = f(x)$ at almost every point $x \in (a,b)$.*

3. *$F'_{ap}(x) = f(x)$ at almost every and W_F-almost every point $x \in (a,b)$.*

4. *For each $\epsilon > 0$ there is an $[a,b]$-form $\mathcal{E}$ such that*

$$\sum_{i=1}^{n} |F(v_i) - F(u_i) - f(w_i)(v_i - u_i)| < \epsilon \tag{11}$$

for every packing $\{([u_i, v_i], w_i)\}$ that is compatible with $\mathcal{E}$.

Proof. The equivalence of the first two statements is classical and can be found, in detail, in Saks [27, Chapter VIII]. The descriptive characterization (i.e., statement 2) is due to Lusin as was the similar characterization in Theorem 45. The third statement is equivalent to the second since that condition implies that F is ACG on $[a,b]$ and, conversely, all ACG functions have that property.

We show that the fourth statement in the theorem is implied by the second. Thus, we assume that F is ACG. Let $\epsilon > 0$. Using Lemma 28, we know that there is an $[a,b]$-form $\mathcal{E}_1$ consisting of a sequence of closed sets $\{E_n\}$ and, for each $E_n \in \mathcal{E}$, a gauge $\delta_n : E_n \to \mathbb{R}^+$ so that

$$\sum_{i=1}^{n} \left|F(v_i) - F(u_i) - F'_{ap}(w_i)(v_i - u_i)\right| < \epsilon\, 2^{-n} \tag{12}$$

whenever $\{([u_i, v_i], w_i) : i = 1, 2, 3, \dots, n\}$ is a packing with $w_i = u_i$ or $w_i = v_i$, u_i, $v_i \in E_n$ and $v_i - u_i < \delta_n(w_i)$.

Construct another $[a,b]$-form $\mathcal{E}_2$ (no longer consisting of closed sets) so that the sets in $\mathcal{E}_2$ are pairwise disjoint and

$$\mathcal{E}_1 \prec \mathcal{E}_2.$$

Just write $A_1 = E_1$, $A_2 = E_2 \setminus E_1$, $A_3 = E_3 \setminus (E_1 \cup E_2)$, ... Then $\mathcal{E}_2$ will be defined as the collection of sets A_1, A_2, A_3, ... Define the gauge $\delta : [a,b] \to \mathbb{R}^+$ by setting $\delta(x) = \delta_n(x)$ if $x \in A_n$.

By Lemma 44, there is an $[a,b]$-form $\mathcal{E}_3$ so that any packing compatible with $\mathcal{E}_3$ is also finer than δ. Take then

$$\mathcal{E} = \mathcal{E}_2 \wedge \mathcal{E}_3.$$

Suppose that $\{([u_i, v_i], w_i) : i = 1, 2, 3, \ldots, n\}$ is a packing that is compatible with the $[a, b]$-form $\mathcal{E}$. Then, by the construction, the packing is finer than δ and, at the same time, compatible with $\mathcal{E}_1$. Take any element $([u_i, v_i], w_i)$. By definition, $w_i \in A_n \subset E_n$ for some n and $v_i - u_i < \delta(w_i)$ and u_i, v_i belong to $\overline{A_n} \subset E_n$.

Thus, using (12), we can estimate that the total of the elements of the sum (11) for which w_i belongs to A_n for some fixed n is smaller than $\epsilon\, 2^{-n}$. It follows that the entire sum is smaller than ϵ. This completes the proof in the one direction.

We complete the proof by showing that the fourth statement in the theorem implies that F is ACG and that f is almost everywhere the approximate derivative of F. We assume that $\epsilon > 0$ and that we have been given an $[a, b]$-form $\mathcal{E}$ for which the inequality (11) holds for all such packings.

We first verify that F is ACG. Let N be any subset of $[a, b]$ of Lebesgue measure zero and proceed with the goal of proving that $W_F(N) = 0$. Write $N_0 = \{x \in N : f(x) = 0\}$ and

$$N_k = \{x \in N : k - 1 < |f(x)| \leq k\} \quad (k = 1, 2, 3, \ldots).$$

These sets exhaust all of the set N. It is enough, then, to check that $W_F(N_k) = 0$ for each $k = 0, 1, 2, \ldots$.

Fix k. Choose an open set G_k containing N_k of measure less than ϵ. Define a gauge δ_k on N_k by requiring for each $x \in N_k$ that $(x - \delta_k(x), x + \delta_k(x) \subset G_k$. By Lemma 44, there is an N_k-form $\mathcal{E}_1$ so that packings compatible with that form are finer than δ_k.

Define

$$\mathcal{E}_2 = \{S \cap N_k : S \in \mathcal{E}\}.$$

This is also an N_k-form. Consequently.

$$\mathcal{E}_3 = \mathcal{E}_1 \wedge \mathcal{E}_2$$

is yet again an N_k-form.

If $\{([u_i, v_i], w_i) : i = 1, 2, 3, \ldots, n\}$ is any packing that is compatible with the N_k-form $\mathcal{E}_3$. then we know that

$$\sum_{i=1}^{n} |F(v_i) - F(u_i) - f(w_i)(v_i - u_i)| < \epsilon.$$

But we also know that each $[u_i, v_i] \subset G_k$ and each $w_i \in N_k$ so that

$$\sum_{i=1}^{n} |f(w_i)(v_i - u_i)| \leq k\lambda(G_k) < k\epsilon.$$

This means that
$$\sum_{i=1}^{n} |F(v_i) - F(u_i)| < \epsilon(1+k).$$
We can use the N_k-form $\mathcal{E}_3$ to construct a weak cover $\mathcal{I}$ of N_k for which we would have
$$W_F(N_k) \leq V(F, \mathcal{I})) < \epsilon(1+k).$$
This verifies that $W_F(N_k) = 0$ for each k and hence $W_F(N) = 0$ as required, so W_F is absolutely continuous with respect to Lebesgue measure on $[a,b]$. By Theorem 37, it follows that F is ACG on $[a,b]$.

Our second and final task in this direction is to show that $F'_{ap}(x) = f(x)$ almost everywhere in $[a,b]$. Since F is ACG, we know that the approximate derivative $F'_{ap}(x)$ exists at almost every point of the interval. Take any function g for which $F'_{ap}(x) = g(x)$ almost everywhere in $[a,b]$. We need to show that $f = g$ almost everywhere.

We know from the first half of the proof that there is an $[a,b]$-form $\mathcal{E}_1$ for which the inequality
$$\sum_{i=1}^{n} |F(v_i) - F(u_i) - g(w_i)(v_i - u_i)| < \epsilon \tag{13}$$
holds for all packings compatible with $\mathcal{E}_1$. But we were given an $[a,b]$-form $\mathcal{E}$ for which the inequality
$$\sum_{i=1}^{n} |F(v_i) - F(u_i) - f(w_i)(v_i - u_i)| < \epsilon \tag{14}$$
holds for all packings compatible with $\mathcal{E}$. Together the inequalities (13) and (14) evidently show that
$$\sum_{i=1}^{n} |g(w_i) - f(w_i)|\,(v_i - u_i) < 2\epsilon$$
holds for all packings compatible with $\mathcal{E} \wedge \mathcal{E}_1$.

Define
$$B_k = \{x \in [a,b] : |f(x) - g(x)| > 1/k\} \quad (k = 1, 2, 3, \dots).$$
The union of these sets B_k contains every point at which f and g differ. Fix k and define
$$\mathcal{E}_4 = \{S \cap B_k : S \in \mathcal{E} \wedge \mathcal{E}_1\}.$$

This is a B_k-form. For this we see that

$$\sum_{i=1}^{n} k^{-1}(v_i - u_i) \leq \sum_{i=1}^{n} |g(w_i) - f(w_i)|\,(v_i - u_i) < 2\epsilon$$

holds for all packings compatible with $\mathcal{E}_4$.

We can use the B_k-form $\mathcal{E}_4$ to construct a weak cover $\mathcal{I}$ of B_k for which we would have

$$V(\lambda, \mathcal{I})) < 2k\epsilon.$$

Any weak cover is also a Vitali cover of all of B_k except possibly for a countable subset (which we can ignore). By the Vitali covering theorem (i.e., the version given as Theorem 11) this means that the Lebesgue measure of B_k must be zero. Consequently $f = g$ almost everywhere since the set of points where the identity does not hold is exactly the union of the sequence $\{B_k\}$. □

5.4 Composite pairs

It is essential for the purposes of a characterization of integrals in this kind of theory to have some version of the Cousin lemma available. Certainly we know from Lemma 14 that if we are given an arbitrary gauge on an interval $[a, b]$, then there must exist a partition π of that interval so that π is finer than δ.

We might hope for a similar statement. Is it true that for any $[a, b]$-form $\mathcal{E}$ there must exist a partition π of that interval so that π is compatible with $\mathcal{E}$? The answer is no. For the weak version of the Cousin lemma (Lemma 15) we needed covers that were also fine at each point. Consequently, we need to allow more elements to a packing to be sure of the existence of a partition.

We use the device and the language from Ene [13] and Sworowski [31]. Recall that SIS($\mathcal{E}$) denotes the countable set of points that are semi-isolated in at least one member S of the form $\mathcal{E}$. Let us say that $(\mathcal{E}, \delta)$ is a composite pair for a set E if $\mathcal{E}$ is an E-form and δ is a gauge defined at each point of the countable set SIS($\mathcal{E}$).

Definition 47. We say that a packing

$$\pi = \{([u_i, v_i], w_i) : i = 1, 2, 3, \ldots, n\}$$

is compatible with a composite pair $(\mathcal{E}, \delta)$ for a set E provided that $w_i = u_i$ or $w_i = v_i$ and either

1. w_i belongs to some $S \in \mathcal{E}$ and the interval $[u_i, v_i]$ has both endpoints in the same set $\overline{S}$, or else

2. $w_i \in \mathrm{SIS}(\mathcal{E})$ and $|v_i - u_i| < \delta(w_i)$.

This definition is enough for a Cousin lemma and useful for a characterization of the Denjoy-Khintchine integral.

Lemma 48 (Cousin lemma)**.** *Suppose that $(\mathcal{E}, \delta)$ is an arbitrary composite pair for the interval $[a,b]$. Then there is a partition*

$$\pi = \{([u_i, v_i], w_i) : i = 1, 2, 3, \ldots, n\}$$

of $[a,b]$ that is compatible with the composite pair $(\mathcal{E}, \delta)$.

Proof. This follows from Lemma 15. □

5.5 Characterization of the Denjoy-Khintchine integral

For a characterization of the Denjoy-Khintchine integral, one might not be entirely happy with Theorem 46. The reason is that the property that F is continuous does not follow from the variational estimate (11) but is assumed *ab initio*. Henstock would certainly not have been satisfied with this since his program was always to construct a division space that could be used to exactly describe all or most known integrals, preferably as a limit of a Riemann sums.

The analyses [13] and [31] might have met his requirements but it is not clear how they fit exactly into the Henstock formalities. Our version is just a simplification of these. Note that this theorem just repeats Theorem 46 without the assumption of continuity but with the extra assumptions needed to guarantee the existence of partitions. Those extra assumptions impose continuity on F as we shall see.

Theorem 49. *Let f, F be real functions defined on a closed bounded interval $[a,b]$. The following two statements are equivalent:*

1. *F is ACG on $[a,b]$ and $F'_{ap}(x) = f(x)$ at almost every point $x \in (a,b)$.*
2. *For each $\epsilon > 0$ there is an $[a,b]$-form $\mathcal{E}_0$ such that for every $[a,b]$-form $\mathcal{E}$ for which $\mathcal{E}_0 \prec \mathcal{E}$ one can choose a gauge δ so that $(\mathcal{E}, \delta)$ is a composite pair and every partition $\{([u_i, v_i], w_i)\}$ of $[a,b]$ that is compatible with this pair must satisfy*

$$\sum_{i=1}^{n} |F(v_i) - F(u_i) - f(w_i)(v_i - u_i)| < \epsilon \tag{15}$$

and also (hence)

$$\left| F(b) - F(a) - \sum_{i=1}^{n} f(w_i)(v_i - u_i) \right| < \epsilon. \tag{16}$$

PROOF. Note that (16) follows from (15) so that it is only the latter that we need to consider for the proof.

The first statement of the theorem assumes that F is continuous (it is part of the definition of ACG) so we can appeal to Theorem 46. Using that theorem. we choose an $[a,b]$-form $\mathcal{E}_0$ so that

$$\sum_{i=1}^{n} |F(v_i) - F(u_i) - f(w_i)(v_i - u_i)| < \epsilon/2 \tag{17}$$

for every packing $\{([u_i, v_i], w_i) : i = 1, 2, 3, \dots, n\}$ that is compatible with the $[a,b]$-form $\mathcal{E}_0$. Suppose that $\mathcal{E}_0 \prec \mathcal{E}$. Let c_1, c_1, c_1, ... be an enumeration of all the points of SIS($\mathcal{E}$). We define a gauge δ on this set by requiring that

$$|F(v) - F(u)| + |f(c_i)|(v - u) < \epsilon 2^{-i-1} \tag{18}$$

whenever $v = c_i$ or $u = c_i$ and $v - u < \delta(c_i)$.

Consider any partition $\{([u_i, v_i], w_i)\}$ of $[a,b]$ that is compatible with the composite pair $(\mathcal{E}, \delta)$. (Such a partition does exist by Lemma 48.) Those elements of the partition that are compatible with $\mathcal{E}$ are also compatible with $\mathcal{E}_0$ and so form a packing to which the inequality (17) applies. The remaining elements form a packing finer than δ to which the inequality (18) applies. Totaling all of these terms, we see that the sum in (15) is smaller than ϵ.

The other direction in the theorem immediately follows from Theorem 46. Well not quite! We do not know yet if F is continuous. Thus, we need to derive continuity from the statement itself.

Fix a point x_0 in $[a,b]$. We can illustrate by taking it as an interior point of the interval. The same argument can be altered to handle an endpoint. Let $\mathcal{E}_0$ be as given in the second statement of the theorem. Define $\mathcal{E}_1$ to be the collection of all the sets $[a, x_0 - 1/n]$, $[x_0 + 1/n, b]$ for $n = 1, 2, 3, \dots$ together with the singleton set $\{x_0\}$. This is an $[a,b]$-form. Choose any $[a,b]$-form $\mathcal{E}$ with

$$\mathcal{E}_0 \wedge \mathcal{E}_1 \prec \mathcal{E}$$

and find a gauge δ so that $(\mathcal{E}, \delta)$ is a composite pair satisfying the inequality (15). Notice that $\delta(x_0)$ must be defined since x_0 belongs to SIS($\mathcal{E}$).

Consider any interval $[x_0, t]$ with $x_0 < t < x_0 + \delta(x_0)$. There is a partition that is compatible with the pair $(\mathcal{E}, \delta)$ that includes the element $([x_0, t], x_0)$. Consequently

$$|F(t) - F(x_0) - f(x_0)(t - x_0)| < \epsilon.$$

As this is true for all $x_0 < t < x_0 + \delta(x_0)$ we have that

$$\limsup_{t \to x_0+} |F(t) - F(x_0)| \leq \epsilon.$$

It follows that F is continuous on the right at x_0. Similarly, it is continuous on the left. Having established continuity we are finished. □

Riemann sums characterization This characterization of the Denjoy-Khintchine integral is of the variational type. A version that uses Riemann sums directly would assert something like this: f is Denjoy-Khintchine integrable on $[a,b]$ if and only if, for every $\epsilon > 0$,

$$\left| \int_a^b f(x)\,dx - \sum_{i=1}^{n} f(w_i)(v_i - u_i) \right| < \epsilon$$

for partitions satisfying some appropriate condition. We already have one direction for this statement. The other direction requires proving a lemma of the "Saks-Henstock" type so that the connection with the variational characterization can be made. See Ene [13] and Sworowski [31] for a discussion and further details.

6 Weak derivatives

We are obliged to add one more topic to this discussion. In theories of this type there are three closely connected constructs: the variational measures, the integral, and the derivative. The reason the theory is relatively direct and economical is that all three of these concepts are defined directly by the covers or covering relations themselves.

We can illustrate with the ordinary derivative and its relation to full covers (one aspect of which we already have seen in the introduction to Section 2.6).

The following are equivalent for functions F and f:

1. $F'(x) = f(x)$ at every point x of a set E.

2. For every $\epsilon > 0$ there is a gauge $\delta : E \to \mathbb{R}^+$ so that the covering relation

$$\beta_\epsilon = \left\{ ([u,v],w) : \left| \frac{F(v)-F(u)}{v-u} - f(w) \right| < \epsilon \right\}$$

   contains all pairs $([u,v],w)$ for which $w \in E$, $w = u$ or $w = v$, and $v - u < \delta(w)$.

This is just a reformulation of the definition of a derivative using our covering language. It is evident, then, how to define a "weak" version of the derivative.

Definition 50. Let F, $f : \mathbb{R} \to \mathbb{R}$ and let $E \subset \mathbb{R}$. Then f is said to be a *weak derivative* of F on the set E provided, for every $\epsilon > 0$, there is an E-form $\mathcal{E}$ so that the covering relation

$$\beta_\epsilon = \left\{([u,v],w) : \left|\frac{F(v)-F(u)}{v-u} - f(w)\right| < \epsilon\right\}$$

contains all pairs $([u,v],w)$ for which $w \in S$ for some set $S \in \mathcal{E}$, $w = u$ or $w = v$, and both u and v belong to $\overline{S}$.

This weak derivative is related to the derivative introduced by Tolstov [35] and [36] that he used to obtain a Perron-type of characterization of the Denjoy-Khintchine integral. His definition, however, uses exclusively E-forms consisting of perfect sets and he expresses the idea more directly and narrowly in terms of relative derivatives. In Section 6.2 we present a variant of his ideas.

6.1 Properties of the weak derivative

The series of lemmas we now prove explore this concept.

Lemma 51. *Let F, f_1, $f_2 : \mathbb{R} \to \mathbb{R}$ and let $E \subset \mathbb{R}$. If both f_1 and f_2 are weak derivatives of F on E then $f_1(x) = f_2(x)$ nearly everywhere on E.*

PROOF. By definition there is, for each positive integer n, a pair of E-forms $\mathcal{E}_{n1}$ and $\mathcal{E}_{n2}$ so that the pair of covering relations

$$\beta_{ni} = \left\{([u,v],w) : \left|\frac{F(v)-F(u)}{v-u} - f_i(w)\right| < \frac{1}{n}\right\}$$

satisfies the definition of a weak derivative for the E-forms $\mathcal{E}_{ni}$ for $i = 1$ and $i = 2$. Let $\mathcal{E}_n = \mathcal{E}_{n1} \wedge \mathcal{E}_{n2}$. Note that $\beta_{n1} \cap \beta_{n2}$ is a covering relation that contains all pairs $([u,v],w)$ for which $w \in S$ for some set $S \in \mathcal{E}_n$, $w = u$ or $w = v$, and both u and v belong to $\overline{S}$.

Let N denote the countable set of points that are isolated in any set $S \in \mathcal{E}_n$ for any n. Fix a point $w \in E \setminus N$. Note that, for each positive integer n,

$$|f_1(w) - f_2(w)| < 2/n$$

since we can always find at least one element $([u,v],w)$ from $\beta_{n1} \cap \beta_{n2}$ for such points w. Thus, f_1 and f_2 agree on E except at countably many points. $\square$

Lemma 52. *Let F, $f : \mathbb{R} \to \mathbb{R}$ and let $E \subset \mathbb{R}$. Suppose that F is continuous and that f is a weak derivative of F on E. Then F is VBG on E and $F'_{ap}(x) = f(x)$ at almost every point x of E.*

Proof. We can assume that E is bounded. Suppose that $E \subset [a,b]$ for some interval. Define the covering relation

$$\beta = \left\{ ([u,v],w) : \left| \frac{F(v)-F(u)}{v-u} - f(w) \right| < \frac{\epsilon}{b-a} \right\}$$

and choose an E-form $\mathcal{E}$ so that β contains all pairs $([u,v],w)$ for which $w \in S$ for some set $S \in \mathcal{E}$, $w = u$ or $w = v$, and both u and v belong to $\overline{S}$.

Take any packing $\{([u_i,v_i],w_i)\}$ that is compatible with $\mathcal{E}$. Such a packing must be a subset of β. Check that

$$\sum_{i=1}^{n} |F(v_i) - F(u_i) - f(w_i)(v_i - u_i)| < \sum_{i=1}^{n} \epsilon(v_i - u_i)/(b-a) \le \epsilon.$$

The proof of Theorem 46 can be repeated to show that F is VBG and that $F'_{ap}(x) = f(x)$ at almost every point x of E. Although that theorem is stated for $[a,b]$-forms, the same methods with minor changes would suffice to prove this. □

Lemma 53. *Let F, $f : \mathbb{R} \to \mathbb{R}$ and let $\{E_n\}$ be a sequence of sets whose union contains a set E. Suppose that F is continuous and that the derivative of F relative to the set E_n is equal to $f(x)$ at every point of E_n. Then f is a weak derivative of F on E.*

Proof. We can assume that the $\{E_n\}$ are pairwise disjoint. Then for each n there is a gauge $\delta_n : E_n \to \mathbb{R}^+$ so that

$$\left| \frac{F(v)-F(u)}{v-u} - f(w) \right| < \epsilon$$

for all u, $v \in E_n$, $w = u$ or $w = v$ and $v - u < \delta_n(w)$. Since F is continuous, we must also have that

$$\left| \frac{F(v)-F(u)}{v-u} - f(w) \right| \le \epsilon$$

for all $w = u$ or $w = v$, $w \in E_n$ and both u and v in $\overline{E_n}$ provided also $v - u < \delta_n(w)$.

By our usual methods, there is a choice of E_n-form $\mathcal{E}_n$ so that the collection of all pairs $([u,v],w)$ satisfying the conditions that $w \in S$ for some set $S \in \mathcal{E}_n$, $w = u$ or $w = v$, and both u and v belong to $\overline{S}$, must also satisfy the requirement that $v - u < \delta_n(w)$.

Take $\mathcal{E}$ as the collection of all sets $E \cap S$ for S in some $\mathcal{E}_n$. It follows that the covering relation

$$\beta = \left\{([u,v],w) : \left|\frac{F(v)-F(u)}{v-u} - f(w)\right| \leq \epsilon\right\}$$

has the required property relative to the E-form $\mathcal{E}$ to verify that f is a weak derivative of F on E. □

Lemma 54. *Suppose that $F : \mathbb{R} \to \mathbb{R}$ is a continuous function that is VBG on a set E. Then there is a set of measure zero N so that $F'_{ap}(x)$ exists at every point of $E \setminus N$ and $F'_{ap}(x)$ is a weak derivative of F on $E \setminus N$.*

Proof. By the familiar theory of VBG functions we know that we can find a sequence of sets $\{E_n\}$ and a set N of measure zero so that the union of the sequence contains all of $E \setminus N$ and so that the relative derivative of F relative to each set E_n exists and is equal to $F'_{ap}(x)$. Thus the lemma follows from the preceding one. □

6.2 Weak Perron integral

Let us, briefly, give a version of the Perron integral following the ideas that Tolstov proposed. He defined his major and minor functions as to be continuous and satisfying a differentiation property relative to a sequence of perfect sets. Ours is similar, perhaps identical, and perhaps may characterize the Denjoy-Khintchine integral. We will not, however, impose any extensive development on the reader.

The definition is just a modification of Definition 50.

Definition 55. Let $G,\ f : [a,b] \to \mathbb{R}$. Then G is said to be a *weak major function* for f on the interval $[a,b]$ if G is continuous, $G(a) = 0$, and, for every $\epsilon > 0$, there is an $[a,b]$-form $\mathcal{E}$ so that the covering relation

$$\beta_\epsilon = \left\{([u,v],w) : \frac{G(v)-G(u)}{v-u} > f(w) - \epsilon\right\}$$

contains all pairs $([u,v],w)$ satisfying the conditions that $w \in S$ for some set $S \in \mathcal{E}$, $w = u$ or $w = v$, and both u and v belong to $\overline{S}$.

A minor function would be similarly defined (or else one takes H as a minor function for f if $-H$ is a major function for $-f$).

Lemma 56. *Suppose that G and H are, respectively, weak major and weak minor functions for f on $[a,b]$. Then $G(b) \geq H(b)$.*

Proof. Let $\epsilon > 0$. Choose an $[a,b]$-form $\mathcal{E}_1$ for G and an $[a,b]$-form $\mathcal{E}_2$ for H as in the definition. Let $\mathcal{E} = \mathcal{E}_1 \wedge \mathcal{E}_2$ and write $F(x) = G(x) - H(x) + 2\epsilon x$. Then the covering relation

$$\begin{aligned}\beta &= \left\{([u,v],w) : \frac{G(v)-G(u)}{v-u} - \frac{H(v)-H(u)}{v-u} > -2\epsilon\right\} \\ &= \left\{([u,v],w) : \frac{F(v)-F(u)}{v-u} > 0\right\}\end{aligned}$$

has this property relative to the $[a,b]$-form $\mathcal{E}$: β must contain all pairs $([u,v],w)$ satisfying the conditions that $w \in S$ for some set $S \in \mathcal{E}$, $w = u$ or $w = v$, and both u and v belong to $\overline{S}$.

Let N be the collection of all points in $[a,b)$ that are isolated on the right in some set S belonging to $\mathcal{E}$. This set N is countable. If $x \in [a,b) \setminus N$, then there must be a sequence of points y_n decreasing to x for which each $([x,y_n],x)$ is in β and so

$$\frac{F(y_n)-F(x)}{y_n - x} > 0.$$

Hence, the upper right Dini derivative of F at x is nonnegative. This is so at every point of $[a,b)$ with at most the countably many exceptions in the set N. By an old theorem on the Dini derivatives of continuous functions (due to Dini himself), the function F is nondecreasing on $[a,b]$ (see [27, p. 204]). We deduce that

$$G(b) - H(b) = G(b) - G(a) - (H(b) - H(a)) \geq -2\epsilon(b-a).$$

Since ϵ is arbitrary, $G(b) \geq H(b)$. □

The integral is defined by the usual Perron requirement on major and minor functions, namely that the infimum of $G(b)$ over all choices of weak major functions for f and the supremum of $H(b)$ over all choices of weak minor functions H for f should agree. Evidently, if f is the weak derivative of a continuous function F on an interval $[a,b]$, then f is weak Perron integrable in this sense as well as Denjoy-Khintchine integrable on $[a,b]$ and F is an indefinite integral in both senses.

6.3 Some remarks on weak derivatives

The weak derivative and the approximate derivative of a continuous function can differ only on a set of measure zero. Thus, one might claim that the former need play no role in the theory since the better known, and more useful, approximate derivative can be employed instead.

Even so, a case can be made that the weak derivatives have a more intimate connection with the concepts VBG, ACG, and the Denjoy-Khintchine integral. A comparison between Definition 50 and the inequality (11) of Theorem 46 shows an immediate relationship.

For example, both of the following statements are correct:

> *If f is the weak derivative of a continuous function F on an interval $[a, b]$, then f is Denjoy-Khintchine integrable on $[a, b]$ and F is an indefinite integral.*

and

> *If f is the approximate derivative of a continuous function F at nearly every point of an interval $[a, b]$, then f is Denjoy-Khintchine integrable on $[a, b]$ and F is an indefinite integral.*

One can argue that the former statement reveals more of the essence of this integral than does the latter. Also, the connection between the two concepts (derivative and integral) is rather more immediate in the former.

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