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RELATION BETWEEN L_p-DERIVATES AND PEANO, APPROXIMATE PEANO AND BOREL DERIVATES OF HIGHER ORDER

Abstract

The definition of the L_p-derivative is such that it involves only the absolute value of the function and therefore the definition of L_p-derivates is not possible from the definition of L_p-derivative. Therefore, a special technique is used to define them and relations between L_p-derivates and Peano, approximate Peano and Borel derivates are studied.

1 Introduction

Let $f : \mathbb{R} \to \mathbb{R}$ and let $x \in \mathbb{R}$. If there exist polynomials $P(t)$, $Q(t)$ and $R(t)$, depending on f and x, of degree at most k such that

$$f(x+t) - P(t) = o(t^k) \ \text{ as } \ t \to 0, \tag{1}$$

$$\int_0^h \frac{1}{t^k} \left[f(x+t) - Q(t)\right] dt = o(h) \ \text{ as } \ h \to 0, \tag{2}$$

$$\left(\frac{1}{h}\int_0^h |f(x+t) - R(t)|^p dt\right)^{\frac{1}{p}} = o(h^k) \ \text{ as } \ h \to 0, p > 0, \tag{3}$$

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then f is said to have respectively Peano derivative, Borel derivative, and L_p-derivative at x of order k where, we have assumed integrability of f in (2) and $f \in L_p$ in (3) in some neighbourhood of x. If $\frac{a_k}{k!}$ is the coefficient of t^k in $P(t)$, (respectively $Q(t)$ and $R(t)$) then a_k is called Peano derivative (respectively Borel derivative and L_p-derivative) of f at x of order k which is denoted by $f_{(k)}(x)$ (respectively $BD_k f(x)$ and $f_{(k),p}(x)$). The definition of approximate Peano derivative of order k is the same as in (1), but in this case we take the approximate limit instead of the ordinary limit and the notation in this case is $f_{(k),a}(x)$. It is clear from the definition that in each case if the derivative of order $k \geq 2$ exists, then the derivative of order i, $0 \leq i < k$ also exists.

The L_p-derivative is defined by Calderon and Zygmund [4] for studying local properties of partial differential equations. Since then, this derivative is extended to higher order, and several authors have discussed various properties of this derivative [1] [2] [3] [4] [5] [6] [7] [8] [10] [11] [12] [13] [14] [15] [16]. It is known that the L_p-derivative of order k is more general than the Peano derivative of order k in the sense that there is a set E of positive Lebesgue measure and a function having no limit at each point of E which has an L_p-derivative of order k for any k and for every positive p at each point of E [2]. Therefore it is natural to ask, what are the relations between L_p-derivates and Peano derivates of order k? Before answering this question, one needs the definitions of L_p- derivates of order k. Assuming the existence of $f_{(k)}(x)$, $BD_k f(x)$, and $f_{(k),a}(x)$ one can define the four derivates of order $k+1$ corresponding to $f_{(k)}(x)$, $BD_k f(x)$, and $f_{(k),a}(x)$ respectively. But assuming the existence of $f_{(k),p}(x)$, the definition of the L_p-derivates of order $k+1$ of f at x is not possible in this manner since the definition of $f_{(k),p}(x)$ involves only the absolute value. So we are to adopt a different approach. The purpose of this paper is to define the L_p-derivates and establish the relation between L_p-derivates and Peano, approximate Peano and Borel derivates. It may be noted that the relation between Peano and Borel derivates is already known [10]. The definition of L_p-derivates will also enable us to define infinite L_p-derivatives.

In what follows we shall use the following notations: for any function $A : \mathbb{R} \to \mathbb{R}$, its positive and negative parts are defined as $[A]_+ = \max[A, 0], [A]_- = \max[-A, 0]$ respectively. Clearly,

$$A = [A]_+ - [A]_- \tag{4}$$

$$|A| = [A]_+ + [A]_- \tag{5}$$

If $A : \mathbb{R} \to \mathbb{R}$ and $B : \mathbb{R} \to \mathbb{R}$, then

$$[A + B]_+ \leq [A]_+ + [B]_+ \text{ and } [A - B]_- \leq [A]_- + [B]_+ \tag{6}$$

and if $A \le B$, then

$$[A]_+ \le [B]_+ \ \text{ and } \ [B]_- \le [A]_- \tag{7}$$

2 Peano and Borel derivates

Lemma 2.1. *Let $\psi(x,t)$ be a function of $x, t \in \mathbb{R}, t \neq 0$. Then the right hand upper limit of ψ at x as $t \to 0_+$ is given by*

$$\psi^+(x) = inf\ S$$

where

$$\psi^+(x) = \limsup_{t\to 0_+} \psi(x,t)$$

and

$$S = \{a : a \in \mathbb{R}, [\psi(x,t) - a]_+ = o(1), \quad as \ \ t \to 0_+\}$$

PROOF. Let x be fixed. Suppose $\psi^+(x) = \infty$. We show that S is empty. If possible, let $a \in S$. Then

$$\lim_{t\to 0_+} [\psi(x,t) - a]_+ = 0.$$

Since $\psi(x,t) - a \le [\psi(x,t) - a]_+$, $\limsup_{t\to 0_+}(\psi(x,t) - a) \le 0$ and so $\limsup_{t\to 0_+} \psi(x,t) \le a$ which is a contradiction since $\psi^+(x) = \infty$. So, S is empty. Next, suppose that $\psi^+(x)$ is finite and let $\psi^+(x) < M$. Then there is a $\delta > 0$ such that $\psi(x,t) < M$ for $0 < t < \delta$. So, $[\psi(x,t) - M]_+ = 0$ for $0 < t < \delta$ and hence $M \in S$. This shows that every $a > \psi^+(x)$ is a member of S. Again let $m < \psi^+(x)$. Then there is a sequence $\{t_n\}$ such that $t_n \to 0_+$ as $n \to \infty$ and $\psi(x,t_n) > m+\epsilon$ for all n where $m < m+\epsilon < \psi^+(x)$. Hence $[\psi(x,t_n) - m]_+ > \epsilon$ for all n and so $m \notin S$. This shows that if $a < \psi^+(x)$, then $a \notin S$. Therefore, $\psi^+(x) = \inf S$. Finally, suppose $\psi^+(x) = -\infty$. Then $\lim_{t\to 0_+} \psi(x,t) = -\infty$. Let $a \in \mathbb{R}$. Then there is a $\delta > 0$ such that $\psi(x,t) < a$ for $0 < t < \delta$. so $[\psi(x,t) - a]_+ = 0$ for $0 < t < \delta$. Hence $a \in S$. Thus, every member of $\mathbb{R}$ is a member of S and hence $\inf S = -\infty$. □

Corollary 2.2. *Let $f : \mathbb{R} \to \mathbb{R}$ and $x \in \mathbb{R}$ be fixed. If the Peano derivative $f_{(r-1)}(x)$, r being a fixed positive integer, exists finitely, then the right hand upper Peano derivate $\overline{f}^+_{(r)}(x)$ is given by*

$$\overline{f}^+_{(r)}(x) := \limsup_{t\to 0_+} \frac{r!}{t^r} \left[f(x+t) - \sum_{i=0}^{r-1} \frac{t^i}{i!} f_{(i)}(x) \right]$$

$$= \inf\left\{a : a \in \mathbb{R}; \left[f(x+t) - \sum_{i=0}^{r-1}\frac{t^i}{i!}f_{(i)}(x) - a\frac{t^r}{r!}\right]_+ = o(t^r) \quad as \quad t \to 0_+\right\}$$

PROOF. Putting $\psi(x,t) = \dfrac{r!}{t^r}\left[f(x+t) - \sum_{i=0}^{r-1}\dfrac{t^i}{i!}f_{(i)}(x)\right]$ in Lemma 2.1 we have

$$\begin{aligned}\overline{f}^{+}_{(r)}(x) &= \inf\left\{a : a \in \mathbb{R}; \left[\frac{r!}{t^r}\left(f(x+t) - \sum_{i=0}^{r-1}\frac{t^i}{i!}f_{(i)}(x)\right) - a\right]_+ = o(1) \text{ as } t \to 0_+\right\}\\ &= \inf\left\{a : a \in \mathbb{R}; \left[f(x+t) - \sum_{i=0}^{r-1}\frac{t^i}{i!}f_{(i)}(x) - a\frac{t^r}{r!}\right]_+ = o(t^r) \text{ as } t \to 0_+\right\}\end{aligned}$$

□

Corollary 2.3. *Let $f : \mathbb{R} \to \mathbb{R}$ and $x \in \mathbb{R}$ be fixed. Let f be integrable in some neighborhood of x. If the Borel derivative $BD_{r-1}f(x)$, exists finitely, then the right hand upper Borel derivate $\overline{BD}^{+}_{r}f(x)$ is given by*

$$\begin{aligned}\overline{BD}^{+}_{r}f(x) &:= \limsup_{h\to 0_+}\frac{1}{h}\int_0^h \frac{r!}{t^r}\left[f(x+t) - \sum_{i=0}^{r-1}\frac{t^i}{i!}BD_i f(x)\right]dt\\ &= \inf\left\{a : a \in \mathbb{R}; \left[\frac{1}{h}\int_0^h \frac{r!}{t^r}\left(f(x+t) - \sum_{i=0}^{r-1}\frac{t^i}{i!}BD_i f(x) - a\frac{t^r}{r!}\right)dt\right]_+ \right.\\ &\qquad\qquad \left. = o(1) \;\; as \;\; h \to 0_+\right\}\end{aligned}$$

PROOF. Putting $\psi(x,h) = \dfrac{1}{h}\int_0^h \dfrac{r!}{t^r}\left[f(x+t) - \sum_{i=0}^{r-1}\dfrac{t^i}{i!}BD_i f(x)\right]dt$ in Lemma

2.1, we have

$$\overline{BD}_r^+ f(x) = \inf\left\{a : a \in \mathbb{R}; \left[\frac{1}{h}\int_0^h \frac{r!}{t^r}\left(f(x+t) - \sum_{i=0}^{r-1}\frac{t^i}{i!}BD_i f(x)\right)dt - a\right]_+ = o(1) \text{ as } h \to 0_+\right\}$$

$$= \inf\left\{a : a \in \mathbb{R}; \left[\frac{1}{h}\int_0^h \frac{r!}{t^r}\left(f(x+t) - \sum_{i=0}^{r-1}\frac{t^i}{i!}BD_i f(x) - a\frac{t^r}{r!}\right)dt\right]_+ = o(1) \text{ as } h \to 0_+\right\}$$

□

3 L_p-derivates

Theorem 3.1. *Let $f : \mathbb{R} \to \mathbb{R}$ and let $x \in \mathbb{R}$. Let $f \in L_p$, $1 \le p < \infty$, in some neighbourhood of x and let the L_p-derivative $f_{r-1,p}(x)$ exist where r is a positive integer. If*

$$E_+(f) := \left\{a : a \in \mathbb{R}; \left(\frac{1}{h}\int_0^h ([\Phi(t) - a\frac{t^r}{r!}]_+)^p dt\right)^{\frac{1}{p}} = o(h^r) \ \ as\ \ h \to 0_+\right\} \quad (8)$$

and

$$E_-(f) := \left\{a : a \in \mathbb{R}; \left(\frac{1}{h}\int_0^h ([\Phi(t) - a\frac{t^r}{r!}]_-)^p dt\right)^{\frac{1}{p}} = o(h^r) \ \ as\ \ h \to 0_+\right\} \quad (9)$$

where

$$\Phi(t) = f(x+t) - \sum_{i=0}^{r-1}\frac{t^i}{i!}f_{(i),p}(x)$$

then

$$\inf E_+(f) \ge \sup E_-(f). \quad (10)$$

Moreover, if

$$\inf E_+(f) = \sup E_-(f) = \lambda \ say,\ \lambda \ is\ finite, \quad (11)$$

then

$$\left(\frac{1}{h}\int_0^h |\Phi(t) - \lambda t^r/r!|^p\, dt\right)^{\frac{1}{p}} = o(h^r) \;\; as \;\; h \to 0_+ \tag{12}$$

and conversely, if (12) holds for some λ then (11) holds.

Proof. If the set $E_+(f)$ is empty, we take $\inf E_+(f) = \infty$ and if $E_-(f)$ is empty, we take $\sup E_-(f) = -\infty$. So, we prove (10) if $E_+(f)$ and $E_-(f)$ are non empty.

We write for $t > 0$, $W(a,t) := \Phi(t) - at^r/r!$. Let $\alpha = \inf E_+(f)$ and $\beta = \sup E_-(f)$. Let $a_1 \in E_+(f)$ and $a_1 < a_2$. Then $W(a_2,t) < W(a_1,t)$ and so by (7) $[W(a_2,t)]_+ \le [W(a_1,t)]_+$ and hence for $h > 0$ we have

$$\int_0^h ([W(a_2,t)]_+)^p dt \le \int_0^h ([W(a_1,t)]_+)^p dt$$

which shows that $a_2 \in E_+(f)$. From this we conclude that if $a > \alpha$ then $a \in E_+(f)$. Again if $b_1 \in E_-(f)$ and $b_2 < b_1$ then $W(b_2,t) > W(b_1,t)$ and so by (7) $[W(b_2,t)]_- \le [W(b_1,t)]_-$ and hence

$$\int_0^h ([W(b_2,t)]_-)^p dt \le \int_0^h ([W(b_1,t)]_-)^p dt,$$

which shows that $b_2 \in E_-(f)$. So, if $b < \beta$, then $b \in E_-(f)$.

These facts will be used in the following arguments. If possible let $\alpha < \beta$. Choose $\alpha < \gamma_1 < \gamma_2 < \beta$. Then $\gamma_1, \gamma_2 \in E_+(f) \bigcap E_-(f)$. Since $\gamma_1 \in E_+(f) \bigcap E_-(f)$, by (8) and (9)

$$\left(\frac{1}{h}\int_0^h ([W(\gamma_1,t)]_+)^p dt\right)^{\frac{1}{p}} = o(h^r) \;\; \text{as} \;\; h \to 0_+ \tag{13}$$

and

$$\left(\frac{1}{h}\int_0^h ([W(\gamma_1,t)]_-)^p dt\right)^{\frac{1}{p}} = o(h^r) \;\; \text{as} \;\; h \to 0_+ \tag{14}$$

So, by (5) and by Minkowski's inequality we get using (13) and (14)

$$\begin{aligned}
&\Big(\frac{1}{h}\int_0^h (|W(\gamma_1,t)|)^p dt\Big)^{\frac{1}{p}} \\
&\qquad = \left(\frac{1}{h}\int_0^h ([W(\gamma_1,t)]_+ + [W(\gamma_1,t)]_-)^p dt\right)^{\frac{1}{p}} \\
&\qquad \le \left(\frac{1}{h}\int_0^h ([W(\gamma_1,t)]_+)^p dt\right)^{\frac{1}{p}} + \left(\frac{1}{h}\int_0^h ([W(\gamma_1,t)]_-)^p dt\right)^{\frac{1}{p}} \\
&\qquad = o(h^r) \text{ as } h \to 0_+.
\end{aligned}$$

This shows that γ_1 is the L_p-derivative of f at x of order r. Similarly γ_2 is the L_p-derivative of f at x of order r. But this is a contradiction, since the L_p-derivative, if it exists, is unique [[10] ; p55]. So, $\alpha \ge \beta$ and (10) is proved.

For the second part, suppose that (11) holds. Let $\epsilon > 0$ be arbitrary. Let $h > 0$ and $0 \le t \le h$. Then

$$W(\lambda,t) = \Phi(t) - \lambda\frac{t^r}{r!} = \Phi(t) - (\lambda+\epsilon)\frac{t^r}{r!} + \epsilon\frac{t^r}{r!} = W(\lambda+\epsilon,t) + \epsilon\frac{t^r}{r!}.$$

So, by (6) if $t > 0$ then

$$[W(\lambda,t)]_+ \le [W(\lambda+\epsilon,t)]_+ + [\epsilon\frac{t^r}{r!}]_+ = [W(\lambda+\epsilon,t)]_+ + \epsilon\frac{t^r}{r!}.$$

Applying Minkowski's inequality, and since $\lambda+\epsilon \in E_+(f)$ this gives

$$\begin{aligned}
\left(\frac{1}{h}\int_0^h ([W(\lambda,t)]_+)^p dt\right)^{\frac{1}{p}} &\le \left(\frac{1}{h}\int_0^h ([W(\lambda+\epsilon,t)]_+)^p dt\right)^{\frac{1}{p}} \\
&\qquad + \left(\frac{1}{h}\int_0^h (\epsilon\frac{t^r}{r!})^p dt\right)^{\frac{1}{p}} \\
&= o(h^r) + \frac{\epsilon}{r!}\frac{h^r}{(rp+1)^{\frac{1}{p}}}.
\end{aligned}$$

So

$$\lim_{h\to 0_+} \frac{1}{h^r}\left(\frac{1}{h}\int_0^h ([W(\lambda,t)]_+)^p dt\right)^{\frac{1}{p}} \le \frac{\epsilon}{r!}\frac{1}{(rp+1)^{\frac{1}{p}}} \tag{15}$$

Again

$$W(\lambda,t) = \Phi(t) - \lambda\frac{t^r}{r!} = \Phi(t) - (\lambda-\epsilon)\frac{t^r}{r!} - \epsilon\frac{t^r}{r!} = W(\lambda-\epsilon,t) - \epsilon\frac{t^r}{r!}$$

and so by (6) if $t > 0$ then

$$[W(\lambda,t)]_- \le [W(\lambda-\epsilon,t)]_- + [\epsilon\frac{t^r}{r!}]_+ = [W(\lambda-\epsilon,t)]_- + \epsilon\frac{t^r}{r!}.$$

Applying Minkowski's inequality, since $\lambda-\epsilon \in E_-(f)$

$$\begin{aligned}\left(\frac{1}{h}\int_0^h ([W(\lambda,t)]_-)^p dt\right)^{\frac{1}{p}} &\le \left(\frac{1}{h}\int_0^h ([W(\lambda-\epsilon,t)]_-)^p dt)\right)^{\frac{1}{p}} \\ &\quad + \left(\frac{1}{h}\int_0^h (\epsilon\frac{t^r}{r!})^p dt\right)^{\frac{1}{p}} \\ &= o(h^r) + \frac{\epsilon}{r!}\frac{h^r}{(rp+1)^{\frac{1}{p}}}.\end{aligned}$$

So

$$\lim_{h\to 0_+}\frac{1}{h^r}\left(\frac{1}{h}\int_0^h ([W(\lambda,t)]_-)^p dt\right)^{\frac{1}{p}} \le \frac{\epsilon}{r!}\frac{1}{(rp+1)^{\frac{1}{p}}}. \tag{16}$$

Applying (5) and Minkowski's inequality and using (15) and (16) we have

$$\begin{aligned}&\lim_{h\to 0_+}\frac{1}{h^r}\left(\frac{1}{h}\int_0^h |W(\lambda,t)|^p dt\right)^{\frac{1}{p}} \\ &= \lim_{h\to 0_+}\frac{1}{h^r}\left(\frac{1}{h}\int_0^h ([W(\lambda,t)]_+ + [W(\lambda,t)]_-)^p dt\right)^{\frac{1}{p}} \\ &\le \lim_{h\to 0_+}\frac{1}{h^r}\left[\left(\frac{1}{h}\int_0^h ([W(\lambda,t)]_+)^p dt\right)^{\frac{1}{p}} + \left(\frac{1}{h}\int_0^h ([W(\lambda,t)]_-)^p dt\right)^{\frac{1}{p}}\right] \\ &\le \frac{2\epsilon}{r!}\frac{1}{(rp+1)^{\frac{1}{p}}}.\end{aligned}$$

Since ϵ is arbitrary,

$$\lim_{h\to 0_+}\frac{1}{h^r}\left(\frac{1}{h}\int_0^h |W(\lambda,t)|^p dt\right)^{\frac{1}{p}} = 0$$

and since $W(\lambda,t) = \Phi(t) - \lambda\frac{t^r}{r!}$ this gives

$$\left(\frac{1}{h}\int_0^h \left(|\Phi(t) - \lambda\frac{t^r}{r!}|\right)^p dt\right)^{\frac{1}{p}} = o(h^r) \text{ as } h \to 0_+$$

completing the proof of (12).
To prove the converse, suppose that (12) holds, then by (5)

$$\left(\frac{1}{h}\int_0^h ([W(\lambda,t)]_+)^p dt\right)^{\frac{1}{p}} \le \left(\frac{1}{h}\int_0^h |W(\lambda,t)|^p dt\right)^{\frac{1}{p}} = o(h^r) \qquad (17)$$

as $h \to 0_+$ and

$$\left(\frac{1}{h}\int_0^h ([W(\lambda,t)]_-)^p dt\right)^{\frac{1}{p}} \le \left(\frac{1}{h}\int_0^h |W(\lambda,t)|^p dt\right)^{\frac{1}{p}} = o(h^r) \qquad (18)$$

as $h \to 0_+$. From (17) and (18) $\lambda \in E_+(f) \bigcap E_-(f)$. So applying (10), (11) is proved. This completes the proof of Theorem 3.1. □

Theorem 3.1 now helps us to define upper and lower L_p-derivates.

Definition 3.2. Let $f_{(r-1),p}(x)$ exist. Then the right upper and right lower L_p derivates of f at x of order r, denoted by $\overline{f}^{+}_{(r),p}(x)$ and $\underline{f}^{+}_{(r),p}(x)$ respectively, are defined by

$$\overline{f}^{+}_{(r),p}(x) := \inf\left\{a \in \mathbb{R} : \left(\frac{1}{h}\int_0^h \left(\left[f(x+t) - \sum_{i=0}^{r-1}\frac{t^i}{i!}f_{(i),p}(x) - a\frac{t^r}{r!}\right]_+\right)^p dt\right)^{\frac{1}{p}} = o(h^r) \text{ as } h \to 0_+\right\}$$

and

$$\underline{f}^{+}_{(r),p}(x) := \sup\left\{a \in \mathbb{R} : \left(\frac{1}{h}\int_0^h \left(\left[f(x+t) - \sum_{i=0}^{r-1}\frac{t^i}{i!}f_{(i),p}(x) - a\frac{t^r}{r!}\right]_-\right)^p dt\right)^{\frac{1}{p}} = o(h^r) \text{ as } h \to 0_+\right\}$$

The definitions of the left upper and left lower L_p-derivates of f at x of order r are now obtained by considering $f(t) = g(-t)$ for all $t \in \mathbb{R}$ and applying the above definitions for g. In fact, it can be verified that $f_{(i),p}(x) = (-1)^i g_{(i),p}(-x)$ for $i = 0, 1, 2, ..., r-1$ and so the left upper and left lower L_p-derivates of f at x of order r are defined by

$$\begin{aligned}\overline{f}^{-}_{(r),p}(x) &= \overline{g}^{+}_{(r),p}(-x) \quad \text{if } r \text{ is even}\\ &= -\underline{g}^{+}_{(r),p}(-x) \quad \text{if } r \text{ is odd}\end{aligned}$$

and

$$\underline{f}^{-}_{(r),p}(x) = \overline{g}^{+}_{(r),p}(-x) \quad \text{if } r \text{ is even}$$
$$= -\overline{g}^{+}_{(r),p}(-x) \quad \text{if } r \text{ is odd}$$

The both sided upper and lower derivates are

$$\overline{f}_{(r),p}(x) = \max[\overline{f}^{+}_{(r),p}(x), \overline{f}^{-}_{(r),p}(x)]$$

and

$$\underline{f}_{(r),p}(x) = \min[\underline{f}^{+}_{(r),p}(x), \underline{f}^{-}_{(r),p}(x)]$$

If $\overline{f}_{(r),p}(x) = \underline{f}_{(r),p}(x)$, the common value is the L_p-derivative of f at x of order r, possibly infinite. In view of Theorem 3.1 it is clear that this definition agrees with the previous one given in [[10], p. 55].

4 Relation between approximate Peano derivates , L_p-derivates and Peano derivates

Theorem 4.1. *If the L_p-derivative $f_{(r-1),p}(x)$ exists , $1 \le p < \infty$, then the approximate Peano derivative $f_{(r-1),a}(x)$ also exists and they are equal. Moreover*

$$\underline{f}^{+}_{(r),p}(x) \le \underline{f}^{+}_{(r),a}(x) \le \overline{f}^{+}_{(r),a}(x) \le \overline{f}^{+}_{(r),p}(x) \tag{19}$$

with similar relations for left derivates.

Proof. If r=1 then the theorem is true [[8], Theorem 2]. We suppose that the theorem is true for $r = i$ and prove it for $r = i+1$. Let $r = i+1$. Suppose that $f_{(i),p}(x)$ exists. Then $f_{(i-1),p}(x)$ exists [[10], p 56]. Since the result is true for $r = i$, $f_{(i-1),a}(x)$ exists and

$$\underline{f}^{+}_{(i),p}(x) \le \underline{f}^{+}_{(i),a}(x) \le \overline{f}^{+}_{(i),a}(x) \le \overline{f}^{+}_{(i),p}(x). \tag{20}$$

Since $f_{(i),p}(x)$ exists , (20) shows that $f_{(i),a}(x)$ exists and $f_{(i),p}(x) = f_{(i),a}(x)$. We are to prove that

$$\underline{f}^{+}_{(i+1),p}(x) \le \underline{f}^{+}_{(i+1),a}(x) \le \overline{f}^{+}_{(i+1),a}(x) \le \overline{f}^{+}_{(i+1),p}(x) \tag{21}$$

Let $\overline{f}^{+}_{(i+1),a}(x) = \alpha, \overline{f}^{+}_{(i+1),p}(x) = \beta$. If possible, suppose $\alpha > \beta$. Choose $\alpha > \gamma > \beta$. Then by definition of α the set

$$E = \left\{ t : t > 0; \frac{(i+1)!}{t^{i+1}} \left[f(x+t) - \sum_{k=0}^{i} \frac{t^k}{k!} f_{(k),a}(x) \right] > \gamma \right\}$$

has positive upper density in the right of the point $t = 0$. Hence there is $\delta > 0$ and a sequence $\{h_n\}$ such that $h_n \to 0^+$ as $n \to \infty$ and

$$\frac{\mu(E \cap [0, h_n])}{h_n} > \delta \text{ for all } n.$$

Hence

$$\mu(E \cap [0, h_n]) > \delta h_n \text{ for all } n. \tag{22}$$

Also, by the definition of β, there is $\sigma, \beta \leq \sigma < \gamma$ such that

$$\left(\frac{1}{h}\int_0^h \left(\left[f(x+t) - \sum_{k=0}^{i} \frac{t^k}{k!} f_{(k),p}(x) - \sigma \frac{t^{i+1}}{(i+1)!}\right]_+\right)^p dt\right)^{\frac{1}{p}} = o(h^{i+1})$$

as $h \to 0_+$. So

$$\left(\frac{1}{h_n}\int_0^{h_n} \left(\left[f(x+t) - \sum_{k=0}^{i} \frac{t^k}{k!} f_{(k),p}(x) - \sigma \frac{t^{i+1}}{(i+1)!}\right]_+\right)^p dt\right)^{\frac{1}{p}} = o(h_n^{i+1}) \tag{23}$$

as $n \to \infty$. Also, for fixed n we have by (7)

$$\begin{aligned} &\int_0^{h_n} \left(\left[f(x+t) - \sum_{k=0}^{i} \frac{t^k}{k!} f_{(k),p}(x) - \sigma \frac{t^{i+1}}{(i+1)!}\right]_+\right)^p dt \\ &\geq \int_0^{h_n} \left(\left[f(x+t) - \sum_{k=0}^{i} \frac{t^k}{k!} f_{(k),p}(x) - \gamma \frac{t^{i+1}}{(i+1)!}\right]_+\right)^p dt \qquad (24) \\ &\geq \int_{E \cap [0,h_n]} \left(\left[f(x+t) - \sum_{k=0}^{i} \frac{t^k}{k!} f_{(k),p}(x) - \gamma \frac{t^{i+1}}{(i+1)!}\right]_+\right)^p dt = C, \end{aligned}$$

say. Then $C > 0$. For, if $C = 0$, then by (5) and the property of Lebesgue integral, the integrand of the last expression in (24) would vanish a.e. on $E \cap [0, h_n]$. This is a contradiction since E has positive upper density in the right of the point $t = 0$. Therefore

$$\begin{aligned} &\frac{1}{h_n^{i+1}}\left(\frac{1}{h_n}\int_o^{h_n} \left(\left[f(x+t) - \sum_{k=0}^{i} \frac{t^k}{k!} f_{(k),p}(x) - \sigma \frac{t^{i+1}}{(i+1)!}\right]_+\right)^p dt\right)^{\frac{1}{p}} \\ &\qquad \geq \frac{1}{h_n^{i+1}} C^{\frac{1}{p}} \left(\frac{1}{h_n}\right)^{\frac{1}{p}} = C^{\frac{1}{p}} \frac{1}{h_n^{i+1+\frac{1}{p}}} \to \infty \text{ as } n \to \infty \end{aligned}$$

which contradicts (23). This proves the last inequality in (21). The proof of the first inequality in (21) is similar. Thus, (21) is proved and so (19) is proved by induction. □

Remark 4.2. In [5], Evans proved that if f is the L_p-derivative of a function F of order n and if ϕ is a primitive of F, then f is the Peano derivative of ϕ of order $n+1$ and hence concluded that f has all the properties of a Peano derivative. The above theorem gives directly that every L_p-derivative is the approximate Peano derivative of the same function and of the same order and hence satisfies all the properties of approximate Peano derivative [9].

Theorem 4.3. *If $f \in L_p$ in some neighbourhood of x and if the Peano derivative $f_{(r-1)}(x)$ exists finitely then for any p , $1 \le p < \infty$*

$$\underline{f}^{+}_{(r)}(x) \le \underline{f}^{+}_{(r),p}(x) \le \overline{f}^{+}_{(r),p}(x) \le \overline{f}^{+}_{(r)}(x). \tag{25}$$

Proof. Since $f_{(r-1)}(x)$ exists, the L_p-derivative $f_{(r-1),p}(x)$ exists and $f_{(i)}(x) = f_{(i),p}(x)$ for $0 \le i \le r-1$ (see [[10], p 130]). Let

$$E_+(f) = \left\{ a \in \mathbb{R} : \left(\frac{1}{h} \int_0^h \left(\left[f(x+t) - \sum_{i=0}^{r-1} \frac{t^i}{i!} f_{(i),p}(x) - a\frac{t^r}{r!} \right]_+ \right)^p dt \right)^{\frac{1}{p}} \right.$$
$$\left. = o(h^r) \text{ as } h \to 0_+ \right\}$$

and

$$F_+(f) = \left\{ a \in \mathbb{R} : \left[f(x+t) - \sum_{i=0}^{r-1} \frac{t^i}{i!} f_{(i)}(x) - a\frac{t^r}{r!} \right]_+ = o(t^r) \text{ as } t \to 0_+ \right\}$$

We show that $F_+(f) \subset E_+(f)$ let $a \in F_+(f)$. Let

$$V(t) = f(x+t) - \sum_{i=0}^{r-1} \frac{t^i}{i!} f_{(i)}(x) - a\frac{t^r}{r!}$$

and $\epsilon > 0$. Then since $a \in F_+(f)$ there is $\delta > 0$ such that $\frac{1}{t^r}[V(t)]_+ < \epsilon$ for $0 < t < \delta$, and so $[V(t)]_+ < \epsilon t^r$ for $0 < t < \delta$. Hence

$$\left(\frac{1}{h} \int_0^h ([V(t)]_+)^p dt \right)^{\frac{1}{p}} < \epsilon \frac{h^r}{(rp+1)^{\frac{1}{p}}} \text{ for } 0 < h < \delta.$$

Since ϵ is arbitrary

$$\left(\frac{1}{h}\int_0^h ([V(t)]_+)^p dt\right)^{\frac{1}{p}} = o(h^r) \text{ as } h \to 0_+. \tag{26}$$

Since $f_{(i)}(x) = f_{(i),p}(x)$ for $0 \le i \le r-1$, (26) shows that $a \in E_+(f)$. So, $F_+(f) \subset E_+(f)$. Hence from the definition of $\overline{f}^+_{(r),p}(x)$ and from Corollary 2.2

$$\overline{f}^+_{(r),p}(x) = \inf E_+(f) \le \inf F_+(f) = \overline{f}^+_{(r)}(x)$$

proving the last inequality in (25). The proof of the first inequality in (25) is similar. □

Theorem 4.4. *If $f_{(r-1),p}(x)$ exists and $1 \le q < p < \infty$ then $f_{(r-1),q}(x)$ exists and*

$$\underline{f}^+_{(r),p}(x) \le \underline{f}^+_{(r),q}(x) \le \overline{f}^+_{(r),q}(x) \le \overline{f}^+_{(r),p}(x). \tag{27}$$

Proof. Since $f_{(r-1),p}(x)$ exists, $f_{(r-1),q}(x)$ exists and $f_{(i),p}(x) = f_{(i),q}(x)$ for $0 \le i \le r-1$ (see [[10]; p 58]) and so for any $a \in \mathbb{R}$

$$\sum_{i=0}^{r-1} \frac{t^i}{i!} f_{(i),q}(x) + a\frac{t^r}{r!} = \sum_{i=0}^{r-1} \frac{t^i}{i!} f_{(i),p}(x) + a\frac{t^r}{r!} = \psi(t), \quad \text{say.}$$

Since $f \in L_p$, $f(x+t) - \psi(t) \in L_p$ for fixed x and so $[f(x+t) - \psi(t)]_+ \in L_p$. Hence $([f(x+t) - \psi(t)]_+)^q \in L_{\frac{p}{q}}$. Since $1 \in L_{\frac{p}{p-q}}$, by Holder's inequality

$$\int_0^h ([f(x+t) - \psi(t)]_+)^q dt \le \left(\int_0^h ([f(x+t) - \psi(t)]_+)^p \, dt\right)^{\frac{q}{p}} h^{\frac{p-q}{p}}.$$

Hence

$$\left(\frac{1}{h}\int_0^h ([f(x+t) - \psi(t)]_+)^q \, dt\right)^{\frac{1}{q}} \le \left(\frac{1}{h}\int_0^h ([f(x+t) - \psi(t)]_+)^p \, dt\right)^{\frac{1}{p}}$$

which shows that

$$\left\{a \in \mathbb{R} : \left(\frac{1}{h}\int_0^h \left(\left[f(x+t) - \sum_{i=0}^{r-1} \frac{t^i}{i!} f_{(i),p}(x) - a\frac{t^r}{r!}\right]_+\right)^p dt\right)^{\frac{1}{p}} \right.$$
$$\left. = o(h^r) \text{ as } h \to 0_+ \right\}$$

is a subset of

$$\left\{a \in \mathbb{R} : \left(\frac{1}{h}\int_0^h \left(\left[f(x+t) - \sum_{i=0}^{r-1} \frac{t^i}{i!} f_{(i),q}(x) - a\frac{t^r}{r!}\right]_+\right)^q dt\right)^{\frac{1}{q}} = o(h^r) \text{ as } h \to 0_+\right\}.$$

Hence from the definition $\overline{f}^+_{(r),q}(x) \leq \overline{f}^+_{(r),p}(x)$, this completes the proof of the last inequality of (27). The proof of the first inequality of (27) is similar. $\square$

Theorem 4.5. *If the Peano derivative $f_{(r-1)}(x)$ exists and $1 \leq q < p < \infty$ then*

$$\underline{f}^+_r(x) \leq \underline{f}^+_{(r),p}(x) \leq \underline{f}^+_{(r),q}(x) \leq \underline{f}^+_{(r),a}(x) \leq \overline{f}^+_{(r),a}(x) \leq \overline{f}^+_{(r),q}(x) \leq \overline{f}^+_{(r),p}(x) \leq \overline{f}^+_{(r)}(x).$$

The proof follows from Theorem 4.1, Theorem 4.3, and Theorem 4.4.

5 Relation between Borel derivates and L_p-derivates

Theorem 5.1. *If $f_{(r-1),p}(x)$, $1 \leq p < \infty$, exists then $BD_{r-1}f(x)$ exists and they are equal. Moreover*

$$\underline{f}^+_{(r),p}(x) \leq \underline{BD}^+_r f(x) \leq \overline{BD}^+_r f(x) \leq \overline{f}^+_{(r),p}(x). \tag{28}$$

Proof. The first part of the theorem is proved in [[10], p 140]. We prove (28). Let $f_{(r-1),p}(x)$ exists. Then $BD_{r-1}f(x)$ exists and $BD_{r-1}f(x) = f_{(r-1),p}(x)$. Let

$$G_+(f) = \left\{a \in \mathbb{R} : \frac{1}{h}\int_0^h \frac{r!}{t^r}\left[f(x+t) - \sum_{i=0}^{r-1} \frac{t^i}{i!} BD_i f(x) - a\frac{t^r}{r!}\right]_+ dt = o(1) \text{ as } h \to 0_+\right\}$$

and

$$E_+(f) = \left\{a \in \mathbb{R} : \left(\frac{1}{h}\int_0^h \left(\left[f(x+t) - \sum_{i=0}^{r-1} \frac{t^i}{i!} f_{(i),p}(x) - a\frac{t^r}{r!}\right]_+\right)^p dt\right)^{\frac{1}{p}} = o(h^r) \text{ as } h \to 0_+\right\}.$$

We show that $E_+(f) \subset G_+(f)$. Let $a \in E_+(f)$. We write

$$U(t) = f(x+t) - \sum_{i=0}^{r-1} \frac{t^i}{i!} f_{(i),p}(x) - a\frac{t^r}{r!}.$$

Then since $a \in E_+(f)$

$$\left(\int_0^h ([U(t)]_+)^p \, dt \right)^{\frac{1}{p}} = o(h^r) h^{\frac{1}{p}} \quad \text{as} \quad h \to 0_+. \tag{29}$$

Applying Holder's inequality we get from (29)

$$\int_0^h [U(t)]_+ dt \le \left(\int_0^h ([U(t)]_+)^p \, dt \right)^{\frac{1}{p}} h^{1-\frac{1}{p}} = o(h^r)h \quad \text{as} \quad h \to 0_+. \tag{30}$$

Hence there is a $\delta > 0$ such that

$$\frac{1}{t^{r+1}} \int_0^t [U(\xi)]_+ d\xi < 1 \quad \text{for} \quad 0 < t < \delta.$$

Integrating this we have

$$\int_0^h \frac{1}{t^{r+1}} \int_0^t [U(\xi)]_+ d\xi dt < h \quad \text{for} \quad 0 < h < \delta. \tag{31}$$

Let $h, 0 < h < \delta$, be fixed . By (30), for any $\epsilon > 0$

$$\int_0^\epsilon [U(\xi)]_+ d\xi = o(\epsilon^{r+1}) \quad \text{as} \quad \epsilon \to 0_+.$$

Hence

$$\frac{1}{\epsilon^{r+1}} \int_0^\epsilon [U(\xi)]_+ d\xi \to 0 \quad \text{as} \quad \epsilon \to 0_+.$$

So, there is $\delta_1, 0 < \delta_1 < h$, such that

$$\frac{1}{\epsilon^{r+1}} \int_0^\epsilon [U(\xi)]_+ d\xi < 1 \quad \text{for} \quad 0 < \epsilon < \delta_1,$$

which gives

$$\int_0^\epsilon [U(\xi)]_+ d\xi < \epsilon^{r+1} \quad \text{for} \quad 0 < \epsilon < \delta_1.$$

Hence

$$\int_\epsilon^h \frac{1}{t^{r+1}} \int_0^\epsilon [U(\xi)]_+ d\xi dt \le \int_\epsilon^h \frac{\epsilon^{r+1}}{t^{r+1}} dt = \frac{\epsilon^{r+1}}{r} \left(\frac{1}{\epsilon^r} - \frac{1}{h^r} \right)$$

for $0 < \epsilon < \delta_1$. So, letting $\epsilon \to 0_+$

$$\lim_{\epsilon \to 0_+} \int_\epsilon^h \frac{1}{t^{r+1}} \int_0^\epsilon [U(\xi)]_+ d\xi dt = 0. \tag{32}$$

Also by (31)

$$\int_0^\epsilon \frac{1}{t^{r+1}} \int_0^t [U(\xi)]_+ d\xi dt < \epsilon \quad \text{for} \quad 0 < \epsilon < \delta.$$

Hence letting $\epsilon \to 0_+$

$$\lim_{\epsilon \to 0_+} \int_0^\epsilon \frac{1}{t^{r+1}} \int_0^t [U(\xi)]_+ d\xi dt = 0. \tag{33}$$

Now

$$\begin{aligned} \int_0^h \frac{1}{t^{r+1}} \int_0^t [U(\xi)]_+ d\xi dt &= \left(\int_0^\epsilon + \int_\epsilon^h \right) \frac{1}{t^{r+1}} \int_0^t [U(\xi)]_+ d\xi dt \\ &= \int_0^\epsilon \frac{1}{t^{r+1}} \int_0^t [U(\xi)]_+ d\xi dt + \int_\epsilon^h \frac{1}{t^{r+1}} \left(\int_0^\epsilon + \int_\epsilon^t \right) [U(\xi)]_+ d\xi dt \\ &= \int_0^\epsilon \frac{1}{t^{r+1}} \int_0^t [U(\xi)]_+ d\xi dt + \int_\epsilon^h \frac{1}{t^{r+1}} \int_0^\epsilon [U(\xi)]_+ d\xi dt \\ &\qquad + \int_\epsilon^h \frac{1}{t^{r+1}} \int_\epsilon^t [U(\xi)]_+ d\xi dt. \end{aligned} \tag{34}$$

Letting $\epsilon \to 0_+$, we get from (32), (33) and (34)

$$\int_0^h \frac{1}{t^{r+1}} \int_0^t [U(\xi)]_+ d\xi dt = \lim_{\epsilon \to 0_+} \int_\epsilon^h \frac{1}{t^{r+1}} \int_\epsilon^t [U(\xi)]_+ d\xi dt. \tag{35}$$

For $0 < \epsilon < h$, integrating by parts

$$\int_\epsilon^h \frac{1}{t^r} [U(t)]_+ dt = \frac{1}{h^r} \int_\epsilon^h [U(t)]_+ dt + r \int_\epsilon^h \frac{1}{t^{r+1}} \int_\epsilon^h [U(\xi)]_+ d\xi dt. \tag{36}$$

Letting $\epsilon \to 0_+$, in (36) we get from (35),

$$\int_0^h \frac{1}{t^r} [U(t)]_+ dt = \frac{1}{h^r} \int_0^h [U(t)]_+ dt + r \int_0^h \frac{1}{t^{r+1}} \int_0^t [U(\xi)]_+ d\xi dt. \tag{37}$$

From (30) and (37)

$$\int_0^h \frac{1}{t^r}[U(t)]_+ dt = \frac{1}{h^r}\cdot o(h^r)\cdot h + r\int_0^h \frac{1}{t^{r+1}}\cdot o(t^{r+1})dt$$
$$= o(h) + ro(h) = o(h) \ \ as \ \ h \to 0_+. \tag{38}$$

Since $f_{(i),p}(x) = BD_i f(x)$ for $i = 0, 1, ...r-1$, (38) gives

$$\int_0^h \frac{1}{t^r}\left[f(x+t) - \sum_{i=0}^{r-1}\frac{t^i}{i!}BD_i f(x) - a\frac{t^r}{r!}\right]_+ dt = o(h) \text{ as } h \to 0_+.$$

This shows that $a \in G_+(f)$ and therefore $E_+(f) \subset G_+(f)$. Hence $\inf G_+(f) \leq \inf E_+(f)$. Hence, from Corollary 2.3 and from the definition of $\overline{f}^+_{(r),p}(x)$, $\overline{BD}^+_r f(x) \leq \overline{f}^+_{(r),p}(x)$, proving the last inequality in (28). The proof of the first inequality is similar. $\square$

Now we show that the theorem analogous to Theorem 4.1 does not hold when a L_p-derivative is replaced by a Borel derivative.

Theorem 5.2. *For every $r \geq 1$ and for every $x \in \mathbb{R}$ there is a function $f : \mathbb{R} \to \mathbb{R}$ such that the Borel derivative $BD_r f(x)$ exists finitely, but the approximate Peano derivative $f_{(r),a}$ does not exist.*

Proof. Let r and x be fixed. Without loss of generality we may take $x = 0$. Divide the interval $[0, 1)$ by the points $1, \frac{1}{2}, \frac{1}{2^2}, ..., \frac{1}{2^n}, ...$ to get the collection of intervals $\{I_n = [\frac{1}{2^{n+1}}, \frac{1}{2^n}) : n = 0, 1, 2, ...\}$. Choose n and fix it. Divide the interval I_n by the points $\frac{1}{2^{n+1}} = a_1 < b_1 = a_2 < b_2 = ... = a_{2n} < b_{2n} = \frac{1}{2^n}$ into $2n$ equal subintervals $J_i = [a_i, b_i),\ \ i = 1, 2, ..., 2n$. Define $f_n(t) = \frac{1}{2^{nr}}$ for $t \in J_{2i}$ and $f_n(t) = -y_{2i-1},\ y_{2i-1} > 0$, for $t \in J_{2i-1}$ such that

$$\int_{J_{2i-1}} \frac{f_n(t)}{t^r}dt + \int_{J_{2i}} \frac{f_n(t)}{t^r}dt = 0, \ \ i = 1, 2, \cdots, n. \tag{39}$$

Clearly this gives

$$\int_{I_n} \frac{f_n(t)}{t^r}dt = 0 \ \ for \ \ n = 0, 1, 2, \ldots . \tag{40}$$

Let $f_n(t) = 0$ for $t \notin I_n$. Now that f_n is defined on $\mathbb{R}$ for each n, define $f(t) = \sum_{n=0}^{\infty} f_n(t)$. From (39)

$$-y_{2i-1}\int_{a_{2i-1}}^{b_{2i-1}} \frac{dt}{t^r} + \frac{1}{2^{nr}}\int_{a_{2i}}^{b_{2i}} \frac{dt}{t^r} = 0$$

and so

$$y_{2i-1} = \frac{\frac{1}{2^{nr}} \int_{a_{2i}}^{b_{2i}} \frac{dt}{t^r}}{\int_{a_{2i-1}}^{b_{2i-1}} \frac{dt}{t^r}}. \tag{41}$$

Now

$$1 < \frac{b_{2i}}{a_{2i-1}} = \frac{a_{2i-1} + 2\frac{1}{2n}\frac{1}{2^{n+1}}}{a_{2i-1}} = 1 + \frac{1}{a_{2i-1}} \frac{1}{n2^{n+1}} \tag{42}$$

and

$$\begin{aligned} a_{2i-1} &= \sum_{k=n+1}^{\infty} \mid I_k \mid + \frac{1}{2n} \cdot \frac{1}{2^{n+1}}(2i-2) \\ &= \sum_{k=n+1}^{\infty} \frac{1}{2^{k+1}} + \frac{i-1}{n2^{n+1}} = \frac{1}{2^{n+1}} + \frac{i-1}{n2^{n+1}} = \frac{n+i-1}{n2^{n+1}}. \end{aligned} \tag{43}$$

From (43)

$$\frac{1}{a_{2i-1}} \cdot \frac{1}{n2^{n+1}} = \frac{n2^{n+1}}{n+i-1} \cdot \frac{1}{n2^{n+1}} = \frac{1}{n+i-1} \to 0 \text{ as } n \to \infty,$$

and therefore we get from (42)

$$\log \frac{b_{2i}}{a_{2i-1}} \to 0 \text{ as } n \to \infty. \tag{44}$$

Let $r = 1$, Then from (41)

$$y_{2i-1} = \frac{\frac{1}{2^n} \log \frac{b_{2i}}{a_{2i}}}{\log \frac{b_{2i-1}}{a_{2i-1}}}. \tag{45}$$

Since $b_{2i} = a_{2i} + \frac{1}{2n} \cdot \frac{1}{2^{n+1}}$, $\frac{b_{2i}}{a_{2i}} = 1 + \frac{1}{a_{2i}} \cdot \frac{1}{2n} \cdot \frac{1}{2^{n+1}}$. Similarly, $\frac{b_{2i-1}}{a_{2i-1}} = 1 + \frac{1}{a_{2i-1}} \cdot \frac{1}{2n} \cdot \frac{1}{2^{n+1}}$. Hence, $1 < \frac{b_{2i}}{a_{2i}} < \frac{b_{2i-1}}{a_{2i-1}}$. and so by (45) $y_{2i-1} < \frac{1}{2^n}$. Hence

$$\mid f_n(t) \mid \leq \frac{1}{2^n} \text{ for all } n \text{ and for all } t \in J_{2i-1} \cup J_{2i}. \tag{46}$$

Let $0 < h < 1$. Then $h \in I_n$ for some n. If $h \in J_{2i-1} \cup J_{2i}$, then by (39), (40), (44), (46)

$$\begin{aligned} \left| \frac{1}{h} \int_0^h \frac{f(t)}{t} dt \right| &= \left| \frac{1}{h} \int_{a_{2i-1}}^h \frac{f_n(t)}{t} dt \right| \leq \frac{1}{2^n} \cdot \frac{1}{h} \int_{a_{2i-1}}^{b_{2i}} \frac{dt}{t} \\ &\leq \frac{1}{2^n} \cdot 2^{n+1} \cdot \log \frac{b_{2i}}{a_{2i-1}} \to 0 \text{ as } n \to \infty \end{aligned}$$

Now as $h \to 0_+$, $n \to \infty$ and so this gives

$$\int_0^h \frac{f(t)}{t} dt = o(h) \;\; as \;\; h \to o+$$

Since $f(0) = 0$, this shows that $BD_1 f(0)$ exists and $BD_1 f(0) = 0$.

Now let $r \geq 2$. Then $\frac{1}{t^{r-1}}$ is a strictly convex function in $(0,1)$. Therefore, for any n and any i, since $a_{2i-1} < b_{2i-1} = a_{2i} < b_{2i}$, we have $\frac{1}{2}(\frac{1}{a_{2i-1}^{r-1}} + \frac{1}{b_{2i}^{r-1}}) > \frac{1}{a_{2i}^{r-1}}$ and hence

$$\frac{1}{a_{2i-1}^{r-1}} - \frac{1}{b_{2i-1}^{r-1}} > \frac{1}{a_{2i}^{r-1}} - \frac{1}{b_{2i}^{r-1}}. \tag{47}$$

From (47) and (41)

$$y_{2i-1} = \frac{1}{2^{nr}} \cdot \frac{\frac{1}{a_{2i}^{r-1}} - \frac{1}{b_{2i}^{r-1}}}{\frac{1}{a_{2i-1}^{r-1}} - \frac{1}{b_{2i-1}^{r-1}}} < \frac{1}{2^{nr}}$$

So

$$\mid f_n(t) \mid \leq \frac{1}{2^{nr}} \;\text{ for all } n \text{ and for all } \; t \in J_{2i-1} \cup J_{2i}. \tag{48}$$

If $0 < h < 1$, then $h \in J_{2i-1} \cup J_{2i}$ for some n and some i. So applying (39), (40) and (48)

$$\begin{aligned} \left| \frac{1}{h} \int_0^h \frac{f(t)}{t^r} dt \right| = \left| \frac{1}{h} \int_{a_{2i-1}}^h \frac{f_n(t)}{t^r} dt \right| &\leq \frac{1}{2^{nr}} \cdot \frac{1}{h} \int_{a_{2i-1}}^{b_{2i}} \frac{dt}{t^r} \\ &\leq \frac{1}{2^{nr}} \cdot 2^{n+1} \frac{1}{r-1} \cdot \left(\frac{1}{a_{2i-1}^{r-1}} - \frac{1}{b_{2i}^{r-1}} \right) \end{aligned} \tag{49}$$

From (43), $a_{2i-1} = \frac{n+i-1}{n2^{n+1}}$. Similarly $b_{2i} = \frac{n+i}{n2^{n+1}}$ and so

$$\begin{aligned} \frac{1}{a_{2i-1}^{r-1}} - \frac{1}{b_{2i}^{r-1}} &= \left(\frac{n2^{n+1}}{n+i-1} \right)^{r-1} - \left(\frac{n2^{n+1}}{n+i} \right)^{r-1} \\ &= (n2^{n+1})^{r-1} \left(\frac{1}{(n+i-1)^{r-1}} - \frac{1}{(n+i)^{r-1}} \right) \\ &= (n2^{n+1})^{r-1} \frac{(n+i)^{r-1} - (n+i-1)^{r-1}}{(n+i-1)^{r-1}(n+i)^{r-1}} \\ &= \frac{(n2^{n+1})^{r-1}}{(n+i-1)^{r-1}} \left(1 - \left(\frac{n+i-1}{n+i} \right)^{r-1} \right) \end{aligned} \tag{50}$$

From (49) and (50)

$$\begin{aligned}\left|\frac{1}{h}\int_0^h \frac{f(t)}{t^r}dt\right| &\le \frac{1}{2^{nr}}\cdot 2^{n+1}\cdot\frac{1}{r-1}\cdot(n2^{n+1})^{r-1}\frac{1-(\frac{n+i-1}{n+i})^{r-1}}{(n+i-1)^{r-1}}\\ &= \frac{(2^{n+1})^r}{2^{nr}}\cdot\frac{1}{r-1}\cdot n^{r-1}\cdot\frac{1-(\frac{n+i-1}{n+i})^{r-1}}{(n+i-1)^{r-1}}\\ &= 2^r\cdot\frac{1}{r-1}\left(\frac{n}{n+i-1}\right)^{r-1}\left(1-\left(\frac{n+i-1}{n+i}\right)^{r-1}\right)\to 0\end{aligned}$$

as $n\to\infty$. Therefore

$$\int_0^h \frac{f(t)}{t^r}dt = o(h) \text{ as } h\to 0_+$$

and so $BD_rf(0)$ exists and $BD_rf(0)=0$.

Now we are to show that the approximate Peano derivative $f_{(r),a}(x)$ of f at x of order r does not exist. Let $r=1$. If $t\in(0,1)$ then there is an n such that $t\in I_n$. If $t\in J_{2i}$ for some i, $1\le i\le n$, then $f(t)=f_n(t)=\frac{1}{2^n}$ and since $\frac{1}{2^{n+1}}\le t<\frac{1}{2^n}$, $\frac{f(t)}{t}>\frac{1}{2^n}\cdot 2^n=1$ and hence

$$\bigcup_{n=0}^{\infty}\bigcup_{i=1}^{n} J_{2i}\subset\left\{t: t\in(0,1);\frac{f(t)}{t}>1\right\}. \tag{51}$$

If $t\in J_{2i-1}$ for some i, $1\le i\le n$, then $f(t)=f_n(t)=-y_{2i-1}<0$ and so $\frac{f(t)}{t}<0$ and hence

$$\bigcup_{n=0}^{\infty}\bigcup_{i=1}^{n} J_{2i-1}\subset\left\{t: t\in(0,1);\frac{f(t)}{t}<0\right\} \tag{52}$$

Both the sets in the left hand side of (51) and (52) have positive right upper density at 0. So $\overline{f}^{+}_{(1),a}(0)\ge 1$ and $\underline{f}^{+}_{(1),a}(0)\le 0$, and therefore, $f_{(1),a}(0)$ does not exist.

Now suppose that $r\ge 2$. Let $1\le k<r$ and $0<t<1$. Then $t\in I_n$ for some n and so by (48), since $\frac{1}{2^{n+1}}\le t<\frac{1}{2n}$,

$$\left|\frac{f(t)}{t^k}\right| = \left|\frac{f_n(t)}{t^k}\right|\le\frac{1}{2^{nr}}\cdot 2^{(n+1)k}=\frac{2^k}{2^{n(r-k)}}\to 0 \text{ as } n\to\infty$$

and so $\lim_{t\to 0_+} \dfrac{f(t)}{t^k} = 0$. This shows that $f_{(k)}(0)$ exists and is zero for $k = 1, 2, ..., r-1$. As in the case of $r = 1$, $\frac{f(t)}{t^r} > \frac{1}{2^{nr}} \cdot 2^{nr} = 1$ for $t \in J_{2i}$ and $\frac{f(t)}{t^r} = \frac{-y_{2i-1}}{t^r} < 0$ for $t \in J_{2i-1}$ and so $\overline{f}^{+}_{(r),a}(0) \geq 1$ and $\underline{f}^{+}_{(r),a}(0) \leq 0$. Therefore, $f_{(r),a}(0)$ does not exist. $\square$

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