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J. Marshall Ash, Department of Mathematics, DePaul University, Chicago, IL 60614, U.S.A. email: mash@depaul.edu

Stefan Catoiu,* Department of Mathematics, DePaul University, Chicago, IL 60614, U.S.A. email: scatoiu@depaul.edu

DIRECTIONAL DIFFERENTIABILITY IN THE EUCLIDEAN PLANE

Abstract

Smoothness conditions on a function $f : \mathbb{R}^2 \to \mathbb{R}$ that are weaker than being differentiable or Lipschitz at a point are defined and studied.

1 Introduction

The definition of the ordinary derivative of a real valued function f of a real variable x can be written in two different ways. Fix x. We may either write

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}, \tag{1}$$

or say that $f'(x)$ is the number that must be substituted for d in the linear expression

$$L(h) = f(x) + dh$$

to make the line L approximate f to better than linear order, i.e. when $d = f'(x)$,

$$f(x+h) - L(h) = o(h) \text{ as } h \to 0. \tag{2}$$

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We consider two notions of smoothness for a function $f : \mathbb{R}^2 \to \mathbb{R}$. Fix a point $x = (x_1, x_2)$. From the first form of the one dimensional derivative comes the existence of the gradient function at x,

$$\nabla f(x) = \left(\frac{\partial f}{\partial x}(x), \frac{\partial f}{\partial y}(x) \right).$$

From the second form comes the property of being differentiable at x, namely the existence of a tangent plane

$$T(h) = f(x) + \nabla f(x) \cdot h = f(x) + \frac{\partial f}{\partial x_1}(x) h_1 + \frac{\partial f}{\partial x_2}(x) h_2,$$

where $h = (h_1, h_2)$, which approximates f to better than linear order, i.e.

$$f(x+h) - T(h) = o(\|h\|) \text{ as } \|h\| = \sqrt{h_1^2 + h_2^2} \to 0.$$

The existence of the gradient is a much weaker condition than differentiability, the existence of the tangent plane.

We define another smoothness condition, that of being (locally) Lipschitz at x. This means that

$$f(x+h) - f(x) = O(\|h\|) \text{ as } \|h\| \to 0,$$

i.e., we define f to be *Lipschitz at x* if

$$C = C(x) = \lim \sup_{\|h\| \to 0} \frac{|f(x+h) - f(x)|}{\|h\|} < \infty.$$

Differentiability at x immediately implies being Lipschitz at x, since

$$\begin{aligned} |f(x+h) - f(x)| &= |\nabla f(x) \cdot h + o(\|h\|)| \\ &\leq \|\nabla f(x)\| \, \|h\| + o(\|h\|) \\ &= O(\|h\|) + o(\|h\|) = O(\|h\|). \end{aligned}$$

The following theorem appeared in more general form in [3], where the proof of the special case that is our Theorem 1 is correctly reported to be part of the 1951 University of Chicago thesis of H. William Oliver [4]. A second proof of Theorem 1 that appeared in [2] needs to be slightly augmented by Theorem 3 of [1].

Theorem 1. *If a function f on $\mathbb{R}^d$ is Lipschitz of order n at each point of a (Lebesgue) measurable set E of $\mathbb{R}^d$, then there is a subset F of E such that $|E \backslash F| = 0$ and such that f is differentiable relative to F of order n at every point of F.*

This well known result says that the property of being Lipschitz is generically equivalent to being differentiable in the sense that for every measurable function, the set E of points where the function is Lipschitz (relative to E) but not differentiable (relative to E) has measure 0. This theorem provides a powerful tool for proving that various conditions that are pointwise weaker than differentiability are generically equivalent to differentiability. Its availability was the motivation for our defining three directional smoothness notions which, although obviously weaker than differentiability (see the simple Examples 2 and 3 below), would turn out to generically imply being Lipschitz, and thus, by Theorem 1, generically equivalent to differentiability. Theorem 4 below shows that these smoothness conditions do not generically imply being Lipschitz. Although this spoils the original plan, it does discover the existence of new, generically more general, smoothness conditions.

2 Definitions and examples

Let $\mathbb{U} = \left\{x \in \mathbb{R}^2 : \|x\| = 1\right\}$ be the unit circle. Let $x \in \mathbb{R}^2$ and $u \in \mathbb{U}$. Say that a function $f : \mathbb{R}^2 \to \mathbb{R}$ is *differentiable at x in the direction u*, if there is a constant vector $(a(x,u), b(x,u))$ such that, as $\eta \to 0^+$,

$$f(x + \eta u) = f(x) + (a, b) \cdot \eta u + o(\eta);$$

and that f is *Lipschitz at x in the direction u with Lipschitz constant $C(x,u)$, if*

$$f(x + \eta u) = f(x) + O(\eta) \text{ as } \eta \to 0^+,$$

where

$$C(x,u) = \lim_{\eta \to 0^+} \sup \frac{|f(x + \eta u) - f(x)|}{\eta}.$$

Further define f to be *directional Lipschitz at x with directional Lipschitz constant $C_d(x)$* if

$$C_d(x) = \sup_{u \in \mathbb{U}} C(x,u) < \infty.$$

Example 2. *Let $u_\theta = (\cos\theta, \sin\theta)$, for $\theta \in [0, 2\pi)$, and let g be any real function defined on $[0, 2\pi)$. For all $x = \rho u_\theta \in \mathbb{R}^2$, let*

$$f(x) = \begin{cases} g(\theta)\rho & \text{, if } x \neq 0, \\ 0 & \text{, if } x = \mathbf{0}. \end{cases}$$

Then

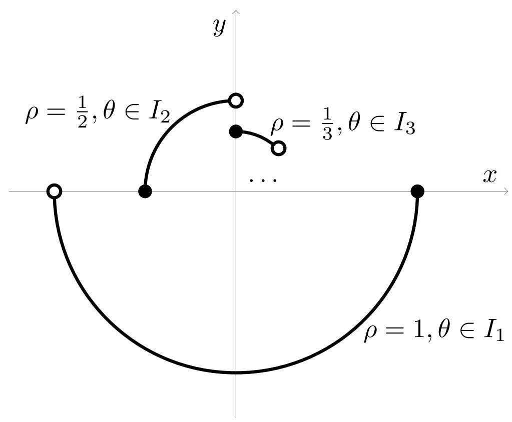
$$f(\mathbf{0} + \eta u) = f(\mathbf{0}) + O(\eta) \text{ as } \eta \to 0^+.$$

For a fixed θ, f *is Lipschitz at* $x = \mathbf{0}$ *in the direction* u_θ *with Lipschitz constant*

$$\lim_{\eta \to 0+} \sup \frac{|f(\eta u)|}{\eta} = \lim_{\eta \to 0^+} \frac{|g(\theta)\,\eta|}{\eta} = |g(\theta)|\,;$$

f *is actually differentiable in every direction* u_θ. *However, if* g *is unbounded,* f *is not directional Lipschitz at* $x = \mathbf{0}$.

The next example involves the characteristic function of the two-dimensional set represented in the picture below.



Example 3. *Decompose the interval* $(0, 2\pi]$ *into a disjoint union of intervals*

$$(0, 2\pi] = \bigcup_{n=1}^{\infty} \left(\frac{2\pi}{2^n}, \frac{2\pi}{2^{n-1}} \right] = \bigcup_{n=1}^{\infty} I_n.$$

Let $\{g_n\}_{n \geq 1}$ *be a sequence functions* $g_n : [0, \infty) \to \mathbb{R}$ *defined as*

$$g_n = \chi_{\left\{\frac{1}{n}\right\}}$$

the characteristic function of the single element set $\left\{\frac{1}{n}\right\}$. *Note that* g_n *is identically equal to zero in a neighborhood of* 0, *so* g_n *is differentiable at zero and* $g_n'(0) = 0$.

Consider the map $f : \mathbb{R}^2 \to \mathbb{R}$, *defined at* $x = \rho u_\theta$, *for* $\rho \geq 0$, *by*

$$f(x) = g_n(\rho)\,,\ \textit{if } \theta \in I_n,\ \textit{for } n = 1, 2, \ldots.$$

This implies that $f(0)$ is unambiguously equal to 0. The directional derivative of f satisfies

$$\begin{aligned} D_{\mathbf{u}_\theta} f(\mathbf{0}) &= \lim_{\rho\to 0^+} \frac{f(\mathbf{0}+\rho\mathbf{u}_\theta) - f(\mathbf{0})}{\rho} \\ &= g_n'(0), \text{ if } \theta \in I_n, \text{ for some } n, \\ &= 0. \end{aligned}$$

This implies that f has directional derivative 0 in every direction at the origin and thus is Lipschitz at the origin with Lipschitz constant $C(u) = 0$ in any direction u. This makes f directional Lipschitz at $x = \mathbf{0}$ with directional Lipschitz constant $C_d = 0$.

On the other hand, for any sequence $\{\theta_n\}_{n\geq 1}$, with $\theta_n \in I_n$, for all n,

$$\lim_{n\to\infty} f\left(\frac{1}{n}u_{\theta_n}\right) = \lim_{n\to\infty} 1 = 1 \neq 0 = f(0),$$

so f is not continous at $x = 0$; a fortiori, f is neither Lipschitz nor differentiable at $x = 0$.

3 Main result

This section is devoted to proving the following theorem, which is the main result of this note.

Theorem 4. *There is a function χ and a set $E \subset \mathbb{R}^2$ of positive measure such that χ is differentiable in almost every direction at each point of E, but χ is not Lipschitz a.e. on E.*

To prove the theorem, we construct χ and E using several steps.

3.1

Let $B(x,r)$ denote an open disc of radius r centered at a point $x \in \mathbb{R}^2$. Before making the construction, we point out two facts.

Lemma 5. *Let $y \in \mathbb{R}^2$ be an exterior point to the ball $B = B(x,r)$, and let $D = |\overline{xy}|$. Then the angle θ that B subtends when viewed from y satisfies $\theta < \pi\dfrac{r}{D}$.*

PROOF. Let $\overline{yz}$ be one of the two tangents from y to B. Then triangle $\triangle xyz$ has angles $\frac{\pi}{2}$ at z, and $\frac{\theta}{2}$ at y. The sine function is concave-down in the first quadrant, so its graph stays above the segment from $(0,0)$ to $\left(\frac{\pi}{2},1\right)$. This means that

$$\frac{2}{\pi}t < \sin t, \text{ for } t \in \left(0, \frac{\pi}{2}\right).$$

In particular, for $t = \frac{\theta}{2}$, this leads to $\frac{2}{\pi}\left(\frac{\theta}{2}\right) < \sin\left(\frac{\theta}{2}\right) = \frac{|\overline{xz}|}{|\overline{xy}|} = \frac{r}{D}$. □

Lemma 6 (Borel-Cantelli)**.** *If $\{A_i\}$ are measurable sets satisfying $\sum_{i=1}^{\infty}|A_i| < \infty$, then $|\{x : x \in$ infinitely many $A_i\}| = 0$.*

Here is the very well known, very short proof of this.

Proof. For every natural number k,

$$|\{x \ : \ x \in \text{infinitely many } A_i\}| = \left|\bigcap_{k=1}^{\infty}\bigcup_{i=k}^{\infty} A_i\right| \le \left|\bigcup_{i=k}^{\infty} A_i\right| \le \sum_{i=k}^{\infty}|A_i|.$$

□

3.2

Let $Q = \{q_j : j = 1, 2, \dots\}$ be a countable dense subset of $\mathbb{R}^2$. We first inductively construct a countable union $A = \bigcup_{i=1}^{\infty} A_i$ of pairwise disjoint open discs $A_i = B_i(q_{n_i}, r_i)$, so that A is dense in $\mathbb{R}^2$ and the radii are decreasing so rapidly that

$$\sum_{i=1}^{\infty}\left(2^i r_i\right)^2 < \infty. \tag{3}$$

Choose $n_1 = 1$ and $r_1 = 3^{-1}$. Choose n_2 to be the index of the first $q \in Q$ such that $q \notin \overline{A}_1$. the closure of A_1, and $r_2 = \min\left\{3^{-2}, \text{ dist } (q_{n_2}, A_1)\right\}$. Choose n_3 to be the index of the first $q \in Q$ such that $q \notin \overline{A}_1 \cup \overline{A}_2$ and $r_3 = \min\left\{3^{-3}, \text{ dist } (q_{n_3}, A_1 \cup A_2)\right\}$. Proceed inductively. It is clear that $Q \subset \overline{A}$ so that $\mathbb{R}^2 = \overline{Q} \subset \overline{A}$, and A is dense. Also $\sum_{i=1}^{\infty}\left(2^i r_i\right)^2 \le \sum_{i=1}^{\infty}\left(2^i 3^{-i}\right)^2 < \infty$.

Let χ be the characteristic function of A. Since A is open, at each $x \in A$, χ is identically 1 in a neighborhood of x, thus has derivative 0 at x, and *a fortiori* has directional derivative 0 in a.e. direction at x.

3.3

Let $R_i = 2^i r_i$, for $i = 1, 2, \dots$, and form

$$B = \bigcup_{i=1}^{\infty} B_i, \text{ where each } B_i = B_i\left(q_{n_i}, R_i\right).$$

The discs here may no longer be disjoint, and each B_i is a superset of A_i that is much bigger than A_i; nevertheless, because of condition (3), $|B|$ is finite. Thus $C = \mathbb{R}^2 \setminus B$ has positive (in fact, infinite) measure. Fix $x \in C$. For each i, $x \notin B_i$, so the distance D_i from x to the center q_{n_i} of B_i satisfies $D_i \geq R_i$. Since q_{n_i} is also the center of A_i, by Lemma 5, the measure of the set T_i of directions from x that point toward A_i satisfies

$$|T_i| \leq \pi \frac{r_i}{D_i} \leq \pi \frac{r_i}{R_i} = \pi 2^{-i}.$$

Thus $\sum_{i=1}^{\infty} |T_i|$ is finite, so by Lemma 6 the set of directions seeing infinitely many A_i has measure 0 and the set of directions seeing finitely many A_i has full measure 2π. For any A_i, the distance from x to A_i is strictly positive, and the infimum of a finite number of positive numbers is positive. So the distance from x to A is positive in a.e. direction. Then χ is identically 0 for a positive distance in a.e. direction and, in particular, χ has a directional derivative 0 in a.e. direction at every point of C.

3.4

Let $E = A \cup C$. Although χ has directional derivative 0 in a.e. direction at every $x \in E$, on a subset of E of positive measure, namely C, χ is not Lipschitz relative to E. In fact it is not even continuous relative to E. For let $x \in C$. Approach x along any direction in which χ has a directional derivative to see that

$$\liminf_{y \to x, y \in E} \chi(y) = 0.$$

But since A is dense, there are points of A arbitrarily close to x. These are points of E where $\chi = 1$, so

$$\limsup_{y \to x, y \in E} \chi(y) = 1.$$

4 Conjecture

The assumption of Theorem 4 allows bad behavior in a thin (measure zero) set of directions. The example takes full advantage of this misbehavior. It therefore gives no direct insight into the following conjecture.

Conjecture 7. *Let $E = \{x \in \mathbb{R}^2 : f$ is Lipschitz in every direction at $x\}$. Then f is Lipschitz a.e. on E.*

Evidence in favor of the conjecture. If a function of one real variable has two Peano derivatives a.e., it does not follow that it has two ordinary derivatives a.e. But if it has two Peano derivatives everywhere and is bounded above, it does have two ordinary derivatives everywhere; see [5]. This is a pretty weak argument, perhaps only an analogy.

Evidence against the conjecture. If the hypothesis "every direction" is weakened to "a.e. direction," the result is false, as shown by Theorem 4 above. As we just mentioned, the proof of Theorem 4 makes critical use of a hypothesis that exempts a measure zero set of directions from consideration at each point, so this argument is also not very strong.

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