RESEARCH

Real Analysis Exchange
Vol. 42(1), 2017, pp. 49–78

Michael Dymond,* Institut für Mathematik, Universität Innsbruck, 6020 Innsbruck, Austria. email: Michael.Dymond@uibk.ac.at

Beata Randrianantoanina, Department of Mathematics, Miami University, Oxford, OH 45056, USA. email: randrib@miamioh.edu

Huaqiang Xu, Department of Mathematics, Miami University, Oxford, OH 45056, USA. email: xuhuaqiang1990@gmail.com

ON INTERVAL BASED GENERALIZATIONS OF ABSOLUTE CONTINUITY FOR FUNCTIONS ON $\mathbb{R}^n$

Abstract

We study notions of absolute continuity for functions defined on $\mathbb{R}^n$ similar to the notion of α-absolute continuity in the sense of Bongiorno. We confirm a conjecture of Malý that 1-absolutely continuous functions do not need to be differentiable a.e., and we show several other pathological examples of functions in this class. We establish some containment relations of the class $1\text{-}AC_{\mathrm{WDN}}$ which consits of all functions in $1\text{-}AC$ which are in the Sobolev space $W^{1,2}_{loc}$, are differentiable a.e. and satisfy the Luzin (N) property, with previously studied classes of absolutely continuous functions.

1 Introduction

The classical Vitali's definition says that when $\Omega \subseteq \mathbb{R}$, a function $f : \Omega \longrightarrow \mathbb{R}$ is *absolutely continuous* if for all $\varepsilon > 0$, there exists $\delta > 0$ so that for every finite collection of disjoint intervals $\{[a_i, b_i]\}_{i=1}^k \subset \Omega$ we have (below $\mathcal{L}^n$ denotes the

Mathematical Reviews subject classification: Primary: 26A03, 26A04; Secondary: 26A05
Key words: absolute continuity, differentiability, Luzin (N) property
Received by the editors April 4, 2014
Communicated by: Luisa Di Piazza
*The first named author was supported by EPSRC funding as a PhD student at the University of Birmingham, UK.

Lebesgue measure on $\mathbb{R}^n$)

$$\sum_{i=1}^{k} \mathcal{L}^1[a_i, b_i] < \delta \Rightarrow \sum_{i=1}^{k} |f(a_i) - f(b_i)| < \varepsilon. \tag{1.1}$$

The study of the space of absolutely continuous functions on $[0, 1]$ and their generalizations to domains in $\mathbb{R}^n$ is connected to the problem of finding regular subclasses of Sobolev spaces which goes back to Cesari and Calderón [11, 9]. On the other hand, the Banach space of generalized absolutely continuous functions on $[0, 1]$ is closely related to the famous James space, and it is an example of a separable space not containing ℓ_1 but with a non-separable dual [22, 21]. Moreover it has a very rich subspace structure [1, 2], but several questions about the Banach space structure of this space remain open, see [1].

There are several natural ways of generalizing the definition of absolute continuity for functions of several variables (cf. [12, 20, 28, 34, 1]).

One approach is to replace the intervals in the antecedent of (1.1) by balls in $\mathbb{R}^n$ and differences in the conclusion of (1.1) by oscillations of f on the images of balls from (1.1). This approach goes back to Banach, Vitali and Tonelli [5, 31, 30] (cf. [20]). More recently Malý [26] suggested another fruitful approach which is to replace the intervals in the antecedent of (1.1) by balls of a selected norm in $\mathbb{R}^n$ and replace sums in the conclusion of (1.1) by sums of oscillations raised to the power equal to the dimension of the domain space. This generalized notion gives functions in the Sobolev space $W^{1,n}_{loc}(\Omega)$, when $\Omega \subset \mathbb{R}^n$, and it has been extensively studied by Malý, Csörnyei, Hencl and Bongiorno [13, 17, 18, 19, 8]. Csörnyei [13] proved that this notion does depend on the shape of the balls substituted for intervals in (1.1). Thus the incomparable classes $\mathcal{Q}$-AC and $\mathcal{B}$-AC are defined where cubes (i.e. balls in the ℓ^n_∞-norm) or Euclidean balls are used, respectively. Hencl [17] introduced a shape-independent class AC_H (see Definition 2.2) which contains both classes $\mathcal{Q}$-AC and $\mathcal{B}$-AC and so that AC_H is contained in the Sobolev space $W^{1,n}_{loc}$ and that all functions in AC_H are differentiable a.e. and satisfy the Luzin (N) property and the change of variable formula.

Bongiorno [6] introduced another generalization of Vitali's classical definition for functions of several variables, which is simultaneously similar to Arzelà's notion of bounded variation for functions on $\mathbb{R}^2$, cf. [12], and to Malý's definition [26].

Definition 1.1. *(Bongiorno [6]) Let $0 < \alpha < 1$. A function $f : \Omega \to \mathbb{R}^l$, where $\Omega \subset \mathbb{R}^n$ is open, is said to be* α**-absolutely continuous** *(denoted α-$AC^{(n)}(\Omega, \mathbb{R}^l)$ or α-AC) if for all $\varepsilon > 0$, there exists $\delta > 0$, such that for any* *finite collection of disjoint α-regular intervals* $\{[\mathbf{a}_i, \mathbf{b}_i] \subset \Omega\}_{i=1}^k$ *we have*

$$\sum_{i=1}^k \mathcal{L}^n([\mathbf{a}_i, \mathbf{b}_i]) < \delta \Rightarrow \sum_{i=1}^k |f(\mathbf{a}_i) - f(\mathbf{b}_i)|^n < \varepsilon. \tag{1.2}$$

Here, for $\mathbf{a} \in \mathbb{R}^l$, $|\mathbf{a}|$ denotes the Euclidean norm of $\mathbf{a}$, and, given $\alpha \in (0, 1]$, we say that an interval

$$[\mathbf{a}, \mathbf{b}] \stackrel{\text{def}}{=} \{\mathbf{x} = (x_\nu)_{\nu=1}^n \in \mathbb{R}^n \colon\ a_\nu \leq x_\nu \leq b_\nu, \nu = 1, \dots, n\}$$

is *α-regular* if

$$\frac{\mathcal{L}^n([\mathbf{a}, \mathbf{b}])}{(\max_\nu |a_\nu - b_\nu|)^n} \geq \alpha.$$

Note that the class of 1-regular intervals is precisely the class of cubes with sides parallel to the co-ordinate axes.

Bongiorno [6] showed that for all $0 < \alpha < 1$,

$$Q\text{-}AC^{(n)}(\Omega, \mathbb{R}^l) \subsetneq \alpha\text{-}AC^{(n)}(\Omega, \mathbb{R}^l) \subsetneq AC_H^{(n)}(\Omega, \mathbb{R}^l).$$

In 2012, Malý [24] asked us about the properties of absolutely continuous functions in a sense similar to Definition 1.1, but without restriction to α-regular intervals for a specified $0 < \alpha < 1$. This question led us to the following definitions:

Definition 1.2. *We say that a function $f : \Omega \to \mathbb{R}^l$ ($\Omega \subset \mathbb{R}^n$ open) is* **0-absolutely continuous***, denoted $0\text{-}AC^{(n)}(\Omega, \mathbb{R}^l)$ or $0\text{-}AC$, (resp. strongly 0-absolutely continuous, denoted strong-$0\text{-}AC^{(n)}(\Omega, \mathbb{R}^l)$ or strong-$0\text{-}AC$) if for every $\varepsilon > 0$, there exists $\delta > 0$, such that for any finite collection of disjoint arbitrary intervals $\{[\mathbf{a}_i, \mathbf{b}_i] \subset \Omega\}_{i=1}^k$ we have*

$$\sum_{i=1}^k |\mathbf{a}_i - \mathbf{b}_i|^n < \delta \Rightarrow \sum_{i=1}^k |f(\mathbf{a}_i) - f(\mathbf{b}_i)|^n < \varepsilon, \tag{1.3}$$

respectively,

$$\sum_{i=1}^k \mathcal{L}^n([\mathbf{a}_i, \mathbf{b}_i]) < \delta \Rightarrow \sum_{i=1}^k |f(\mathbf{a}_i) - f(\mathbf{b}_i)|^n < \varepsilon. \tag{1.4}$$

Note that the antecedent of implication (1.3) is equivalent to the antecedent of (1.2), since intervals in (1.2) are α-regular for a fixed $\alpha \in (0, 1]$, so the notion of the α-absolute continuity is naturally extended by the notion of the

0-absolute continuity, despite the visual similarity of the conditions (1.2) and (1.4). In fact, the antecedent of (1.4) is much weaker than that of (1.2), since there is no assumption of α-regularity of intervals.

We show that, when $n \geq 2$, the condition (1.4) characterizes constant functions, and (1.3) characterizes functions that are locally L-Lipschitz (Section 3).

The main goal of this paper is to study an analog of Bongiorno's notion for $\alpha = 1$.

Definition 1.3. *We say that a function $f : \Omega \to \mathbb{R}^l$ ($\Omega \subset \mathbb{R}^n$ open) is* **1-absolutely continuous**, *denoted $1\text{-}AC^{(n)}(\Omega, \mathbb{R}^l)$ or $1\text{-}AC$, if for every $\varepsilon > 0$, there exists $\delta > 0$, such that for any finite collection of disjoint 1-regular intervals $\{[\mathbf{a}_i, \mathbf{b}_i] \subset \Omega\}_{i=1}^k$ we have*

$$\sum_{i=1}^{k} \mathcal{L}^n([\mathbf{a}_i, \mathbf{b}_i]) < \delta \Rightarrow \sum_{i=1}^{k} |f(\mathbf{a}_i) - f(\mathbf{b}_i)|^n < \varepsilon.$$

We show that the class $1\text{-}AC$ is not contained in AC_H. We show that, similarly as the Sobolev space $W^{1,n}(\Omega)$, when $\Omega \subset \mathbb{R}^n$ and $n > 1$, cf. [15, 25], the class $1\text{-}AC$ contains functions with pathological properties such as:

(i) bounded but nowhere continuous,

(ii) continuous but nowhere differentiable,

(iii) differentiable a.e. but without the Luzin (N) property.

However we prove that every function in $1\text{-}AC$ has a directional derivative in the direction $(1, \dots, 1)$ at a.e. point of the domain (Theorem 5.8).

Moreover the class $1\text{-}AC$ is useful for the study of the Bongiorno's classes $\alpha\text{-}AC$. Namely, in [29] it is proved that

Theorem 1.4. *([29, Theorem 3.2]) For all $0 < \alpha < 1$,*

$$\alpha\text{-}AC = 1\text{-}AC \cap AC_H.$$

We finish the paper by showing where the class $1\text{-}AC_{\mathrm{WDN}}$, (consisting of all functions in $1\text{-}AC$ which are in the Sobolev space $W^{1,n}_{loc}$, are differentiable a.e. and satisfy the Luzin (N) property) fits in the hierarchy of previously studied classes. Namely we prove that (Theorem 5.9):

$$\begin{aligned}&\mathcal{Q}\text{-}AC \subsetneq \alpha\text{-}AC = 1\text{-}AC_{\mathrm{WDN}} \cap AC_H \subseteq 1\text{-}AC_{\mathrm{WDN}}\\ &\subsetneq 1\text{-}AC_{\mathrm{WDN}} \cup AC_H \subsetneq \operatorname{lin\,span}(1\text{-}AC_{\mathrm{WDN}} \cup AC_H) \subseteq 1\text{-}AC_{\mathrm{HWDN}},\end{aligned}$$

$$1\text{-}AC_{\mathrm{DN}} \setminus AC_H \neq \emptyset,$$

where $1\text{-}AC_{\mathrm{HWDN}}$ denotes the set of functions in $1\text{-}AC_H$ (see Defnition 4.1 and Remark 4.3) which are in the Sobolev space $W^{1,2}_{loc}$, are differentiable a.e., and satisfy the Luzin (N) property. We pose some open questions.

On the other hand, we observe that a small adjustment of the function constructed by Csörnyei in [13, Theorem 2] (see (5.12) below) shows that

$$1\text{-}AC_{\mathrm{WDN}} \setminus \mathcal{B}\text{-}AC \neq \emptyset, \quad \text{and} \quad \mathcal{B}\text{-}AC \setminus 1\text{-}AC_{\mathrm{WDN}} \neq \emptyset.$$

2 Preliminaries

Let $C_0(\mathbb{R}^n, \mathbb{R}^l)$ denote the set of all continuous functions $f : \mathbb{R}^n \to \mathbb{R}^l$ with compact support. For $f \in C_0(\mathbb{R}^n, \mathbb{R}^l)$, and a measurable set $A \subset \mathbb{R}^n$, let $\operatorname{osc}(f, A)$ denote the oscillation of f on A, i.e.

$$\operatorname{osc}(f, A) = \operatorname{diam} f(A).$$

Let $K_0 \subset \mathbb{R}^n$ be a fixed symmetric closed convex set with non-empty interior, and let $\mathcal{K}$ denote the set of all balls of $\mathbb{R}^n$ in the norm defined by set K_0, i.e.,

$$\mathcal{K} = \{a + rK_0 : a \in \mathbb{R}^n, r > 0\}.$$

Definition 2.1. *(Csörnyei [13]) We say that a function $f \in C_0(\mathbb{R}^n, \mathbb{R}^l)$ is* **absolutely continuous with respect to** $\mathcal{K}$ *(denoted $f \in \mathcal{K}\text{-}AC$) if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every finite collection of disjoint sets $\{K_i\}_{i=1}^k \subset \mathcal{K}$,*

$$\sum_{i=1}^{k} \mathcal{L}^n(K_i) < \delta \Rightarrow \sum_{i=1}^{k} \operatorname{osc}^n(f, K_i) < \varepsilon.$$

Malý [26] considered functions absolutely continuous with respect to the family $\mathcal{B}$ of Euclidean balls in $\mathbb{R}^n$ and showed that all functions in $\mathcal{B}\text{-}AC$ are differentiable a.e. and satisfy the change of variables formula, similarly as functions in $\mathcal{Q}\text{-}AC$, where $\mathcal{Q}$ denotes the family of cubes, i.e. balls in the ℓ^n_∞-norm. Csörnyei [13] and Hencl and Malý [19] showed that the classes $\mathcal{B}\text{-}AC$ and $\mathcal{Q}\text{-}AC$ are incomparable.

In 2002, Hencl [17] introduced the following shape-independent class of absolutely continuous functions which contains both classes $\mathcal{Q}\text{-}AC$ and $\mathcal{B}\text{-}AC$.

Definition 2.2. *(Hencl [17]) We say that a function $f : \Omega \to \mathbb{R}^l$ ($\Omega \subset \mathbb{R}^n$ open) is in $AC^{(n)}_H(\Omega, \mathbb{R}^l)$ (briefly AC_H) if there exists $\lambda \in (0, 1)$ (equivalently,* *for all $\lambda \in (0,1)$) so that for all $\varepsilon > 0$, there exists $\delta > 0$ so that for any finite collection of disjoint closed balls $\{B(\mathbf{x}_i, r_i) \subset \Omega\}_{i=1}^k$*

$$\sum_{i=1}^{k} \mathcal{L}^n(B(\mathbf{x}_i, r_i)) < \delta \Rightarrow \sum_{i=1}^{k} \operatorname{osc}^n(f, B(\mathbf{x}_i, \lambda r_i)) < \varepsilon.$$

Hencl [17] proved that $AC_H \subset W^{1,n}_{loc}$ and that all functions in AC_H are differentiable a.e. and satisfy the Luzin (N) property and the change of variables formula.

3 Classes 0-AC and strong-0-AC

The main result of this section is the following.

Theorem 3.1. *Let $n \geq 2$, $\Omega \subset \mathbb{R}^n$ be open and connected, and $f : \Omega \to \mathbb{R}^l$. Then:*

(a) *$f \in$ strong-0-AC if and only if f is constant on Ω,*

(b) *$f \in$ 0-AC if and only if f is locally L-Lipschitz on Ω, i.e. there exists $L > 0$ so that for all $\mathbf{a} \in \Omega$, there exists an open neighborhood $U_\mathbf{a}$ of $\mathbf{a}$ so that f is Lipschitz with constant L on $U_\mathbf{a}$.*

Remark 3.2. It is well known that if a function f is locally L-Lipschitz on a quasiconvex set Ω, then f is Lipschitz on Ω. Recall that a set Ω is called **quasiconvex** if there exists $C > 0$, so that every pair of points $\mathbf{a}, \mathbf{b} \in \Omega$ can be joined by a curve γ in Ω such that length$(\gamma) \leq C|\mathbf{a} - \mathbf{b}|$, where by the length of a curve we mean as usual the quantity,

$$\operatorname{length}(\gamma) \stackrel{\text{def}}{=} \sup \sum_{i=0}^{N-1} |\gamma(t_{i+1}) - \gamma(t_i)|,$$

where the supremum is taken over all partitions $0 = t_0 < t_1 < \cdots < t_N = 1$ for a curve $\gamma : [0,1] \to \Omega$.

For a simple example of a function $f : \Omega \longrightarrow \mathbb{R}^2$ which is locally 1-Lipschitz on $\Omega \subset \mathbb{R}^2$, and thus it is in 0-$AC^{(2)}(\Omega, \mathbb{R})$, but is not Lipschitz on Ω, take Ω to be the "slit plane" $\Omega = \{(r,\theta) : 0 < r < \infty, -\pi < \theta < \pi\} \subset \mathbb{R}^2$, in polar coordinates, and put $f(r,\theta) = (r, \theta/2)$.

Proof. (a) It is enough to show that every strong-0-$AC^{(n)}(\Omega, \mathbb{R}^l)$ function f is constant. Clearly, if $f \in$ strong-0-$AC^{(n)}(\Omega, \mathbb{R}^l)$ then f is continuous on Ω.

Since Ω is open and connected, it is enough to show that for every $\mathbf{a} \in \Omega$ there exists $r > 0$ such that $B(\mathbf{a}, r) \subset \Omega$ and

$$\forall \mathbf{b} \in B(\mathbf{a}, r), \qquad f(\mathbf{b}) = f(\mathbf{a}). \tag{3.1}$$

To see this, fix $\varepsilon > 0$ and $\mathbf{b} \in B(\mathbf{a}, r)$. For each $j = 0, 1, \ldots, n$, put

$$(\mathbf{c}^{(j)})_i = \begin{cases} b_i, & \text{if } i \leq j, \\ a_i, & \text{if } i > j. \end{cases}$$

Then $\mathbf{c}^{(0)} = \mathbf{a}$, $\mathbf{c}^{(n)} = \mathbf{b}$, for each $0 \leq j \leq n$, $\mathbf{c}^{(j)} \in B(\mathbf{a}, r)$, and $\mathbf{c}^{(j)}$ and $\mathbf{c}^{(j+1)}$ differ only on the $(j+1)$-th coordinate. Since f is continuous, there exists $\delta_1 > 0$, such that for all $0 \leq j \leq n$ and all $\mathbf{x}$ with $|\mathbf{x} - \mathbf{c}^{(j)}| < \delta_1$ we have $\mathbf{x} \in B(\mathbf{a}, r)$, and

$$|f(\mathbf{c}^{(j)}) - f(\mathbf{x})| < \frac{\varepsilon}{2n}. \tag{3.2}$$

To create non-degenerate intervals of small measure, for $t > 0$, we put

$$(\mathbf{x}^{(j)}(t))_i = \begin{cases} (\mathbf{c}^{(j)})_j, & \text{if } i = j, \\ (\mathbf{c}^{(j)})_i + t \operatorname{sgn}((\mathbf{c}^{(j)})_j - (\mathbf{c}^{(j)})_j), & \text{if } i \neq j. \end{cases}$$

Then, for $j = 1, \ldots, n$,

$$|\mathbf{x}^{(j)}(t) - \mathbf{c}^{(j)}| = \sqrt{n-1}\; t, \tag{3.3}$$

and $\mathcal{L}^n([\mathbf{x}^{(j)}, \mathbf{c}^{(j-1)}] = t^{n-1}|a_j - b_j|$. By (1.4), there exists $\delta_2 > 0$, such that if $t^{n-1}|a_j - b_j| \leq t^{n-1} r < \delta_2$, then

$$|f(\mathbf{c}^{(j-1)}) - f(\mathbf{x}^{(j)}(t))|^n < \left(\frac{\varepsilon}{2n}\right)^n. \tag{3.4}$$

Let $\delta = \min(\delta_1/\sqrt{n-1}, (\delta_2/r)^{1/(n-1)})$ and $0 < t < \delta$. Then, by (3.2), (3.3) and (3.4), we have

$$\begin{aligned} |f(\mathbf{c}^{(j)}) - f(\mathbf{c}^{(j-1)})| &\leq |f(\mathbf{c}^{(j)}) - f(\mathbf{x}^{(j)}(t))| + |f(\mathbf{x}^{(j)}(t)) - f(\mathbf{c}^{(j-1)})| \\ &< \frac{\varepsilon}{2n} + \frac{\varepsilon}{2n} = \frac{\varepsilon}{n}. \end{aligned}$$

Thus

$$|f(\mathbf{a}) - f(\mathbf{b})| \leq \sum_{j=1}^{n} |f(\mathbf{c}^{(j)}) - f(\mathbf{c}^{(j-1)})| < n\frac{\varepsilon}{n} = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this proves (3.1), and ends the proof of part (a).

(b) To see that locally L-Lipschitz functions are 0-absolutely continuous, suppose that $f : \Omega \to \mathbb{R}^l$ is locally L-Lipschitz for some $L > 0$. Then, by compactness, for any interval $[\mathbf{a}, \mathbf{b}] \subseteq \Omega$ there exists a finite set $\{t_i\}_{i=1}^N \subset [0, 1]$ so that $0 = t_1 < \cdots < t_N = 1$ and f is L-Lipschitz on each subinterval $\{\mathbf{a} + t(\mathbf{b} - \mathbf{a}) : t \in [t_i, t_{i+1}]\}$. Thus

$$\begin{aligned} |f(\mathbf{a}) - f(\mathbf{b})| &\le \sum_{i=1}^{N-1} |f(\mathbf{a} + t_{i+1}(\mathbf{b} - \mathbf{a})) - f(\mathbf{a} + t_i(\mathbf{b} - \mathbf{a}))| \\ &\le \sum_{i=1}^{N-1} L|(\mathbf{a} + t_{i+1}(\mathbf{b} - \mathbf{a})) - (\mathbf{a} + t_i(\mathbf{b} - \mathbf{a}))| = L|\mathbf{a} - \mathbf{b}|. \end{aligned} \tag{3.5}$$

By (3.5), for any $\varepsilon > 0$, and for any finite collection of non-overlapping arbitrary intervals $\{[\mathbf{a}_i, \mathbf{b}_i] \subset \Omega\}_{i=1}^k$ with $\sum_{i=1}^k |\mathbf{a}_i - \mathbf{b}_i|^n < \delta = \frac{\varepsilon}{L^n}$, we have

$$\sum_{i=1}^k |f(\mathbf{a}_i) - f(\mathbf{b}_i)|^n \le \sum_{i=1}^k L^n |\mathbf{a}_i - \mathbf{b}_i|^n < L^n \cdot \frac{\varepsilon}{L^n} = \varepsilon,$$

which completes the proof.

The proof in the other direction is smilar. Let $\delta > 0$ be such that (1.3) is satisfied with $\varepsilon = 1$. Let $\mathbf{a} \in \Omega$ and $r > 0$ be such that $B(\mathbf{a}, nr) \subseteq \Omega$. For any $\mathbf{b} \in B(\mathbf{a}, r)$, let $k \in \mathbb{N}$ be such that $\delta/2 \le |\mathbf{a} - \mathbf{b}|^n / k^{n-1} < \delta$. We split the interval $[\mathbf{a}, \mathbf{b}]$ into k subintervals of equal length, i.e. for $i = 0, \ldots, k$ we define points $\mathbf{a}_i = \mathbf{a} + i(\mathbf{b} - \mathbf{a})$. Then, for each i, $[\mathbf{a}_{i-1} - \mathbf{a}_i] \subseteq B(\mathbf{a}, nr)$, and

$$\sum_{i=1}^k |\mathbf{a}_i - \mathbf{a}_{i-1}|^n = k\Big(\frac{|\mathbf{b} - \mathbf{a}|}{k}\Big)^n < \delta.$$

Thus, by (1.3), the Jensen's inequality, and the definition of k, we have

$$|f(\mathbf{b}) - f(\mathbf{a})|^n \le k^{n-1} \sum_{i=1}^k |f(\mathbf{a}_i) - f(\mathbf{a}_{i-1})|^n \le k^{n-1} \le \frac{2}{\delta} |\mathbf{b} - \mathbf{a}|^n.$$

□

4 The Hencl type extension of the class 1-AC

We give an analog of Definition 2.2, and we prove that, similarly as for other classes of absolutely continuous functions, the classes 1-AC_λ do not depend on λ when $0 < \lambda < 1$.

Following [7], we use the following notation: given an interval $[\mathbf{x},\mathbf{y}]$ and $0<\lambda<1$, we denote $|f([\mathbf{x},\mathbf{y}])| = |f(\mathbf{y})-f(\mathbf{x})|$, and by ${}^{\lambda}[\mathbf{x},\mathbf{y}]$ we mean the interval with center $(\mathbf{x}+\mathbf{y})/2$ and sides of length $\lambda(y_\nu - x_\nu)$, $\nu = 1,\dots,n$.

Definition 4.1. *(cf. [7]) Let $\alpha\in(0,1]$ and $\lambda\in(0,1)$. A function $f:\Omega\to\mathbb{R}^l$ ($\Omega\subset\mathbb{R}^n$ open) is said to be in α-$AC_\lambda^{(n)}(\Omega,\mathbb{R}^l)$ (briefly α-AC_λ) if for each $\varepsilon>0$, there exists $\delta>0$ such that for any finite collection of disjoint α-regular intervals $\{[\mathbf{a}_i,\mathbf{b}_i]\subset\Omega\}_{i=1}^k$ we have*

$$\sum_{i=1}^k \mathcal{L}^n([\mathbf{a}_i,\mathbf{b}_i]) < \delta \Rightarrow \sum_{i=1}^k |f({}^{\lambda}[\mathbf{a}_i,\mathbf{b}_i])|^n < \varepsilon. \tag{4.1}$$

Bongiorno [7] proved that for all $\alpha<1$, the class α-AC_λ is independent of λ. We prove the same result for 1-AC_λ.

Theorem 4.2. *Let $0<\lambda_1<\lambda_2<1$. Then*

$$1\text{-}AC_{\lambda_1}^{(n)}(\Omega,\mathbb{R}^l) = 1\text{-}AC_{\lambda_2}^{(n)}(\Omega,\mathbb{R}^l).$$

Proof. Clearly, 1-$AC_{\lambda_2}\subseteq$ 1-AC_{λ_1}. For the other direction, suppose $f\in$ 1-AC_{λ_1} and $\varepsilon>0$. Fix $p\in\mathbb{N}$ such that $p>\frac{2\lambda_2}{(1-\lambda_2)\lambda_1}$, and let $\delta>0$ be such that (4.1) holds with ε/p^n.

Let $\{[\mathbf{a}_i,\mathbf{b}_i]\subseteq\Omega\}$ be a finite family of disjoint 1-regular intervals such that $\sum_i\mathcal{L}^n([\mathbf{a}_i,\mathbf{b}_i])<\delta$, and denote ${}^{\lambda_2}[\mathbf{a}_i,\mathbf{b}_i] = [\mathbf{c}_i,\mathbf{d}_i]$. Then

$$|f({}^{\lambda_2}[\mathbf{a}_i,\mathbf{b}_i])| = |f([\mathbf{c}_i,\mathbf{d}_i])| \le \sum_{j=0}^{p-1}|f\big(\big[\mathbf{c}_i + \frac{(\mathbf{d}_i-\mathbf{c}_i)j}{p}, \mathbf{c}_i + \frac{(\mathbf{d}_i-\mathbf{c}_i)(j+1)}{p}\big]\big)|$$

Thus there exists $j_0\in\{0,\dots,p-1\}$ such that for

$$[\overline{\mathbf{a}_i},\overline{\mathbf{b}_i}] = {}^{\frac{1}{\lambda_1}}\big[\mathbf{c}_i + \frac{(\mathbf{d}_i-\mathbf{c}_i)j_0}{p}, \mathbf{c}_i + \frac{(\mathbf{d}_i-\mathbf{c}_i)(j_0+1)}{p}\big],$$

we have

$$|f({}^{\lambda_2}[\mathbf{a}_i,\mathbf{b}_i])| \le p|f({}^{\lambda_1}[\overline{\mathbf{a}_i},\overline{\mathbf{b}_i}])|. \tag{4.2}$$

Since $p>\frac{2\lambda_2}{(1-\lambda_2)\lambda_1}$, the intervals $[\overline{\mathbf{a}_i},\overline{\mathbf{b}_i}]\subseteq[\mathbf{a}_i,\mathbf{b}_i]$ and thus are disjoint, and $\sum_i\mathcal{L}^n([\overline{\mathbf{a}_i},\overline{\mathbf{b}_i}])\le\sum_i\mathcal{L}^n([\mathbf{a}_i,\mathbf{b}_i])<\delta$. Thus, by the choice of δ and (4.2), we get

$$\sum_i|f({}^{\lambda_2}[\mathbf{a}_i,\mathbf{b}_i])|^n \le p^n\sum_i|f({}^{\lambda_1}[\overline{\mathbf{a}_i},\overline{\mathbf{b}_i}])|^n < p^n\frac{\varepsilon}{p^n} = \varepsilon.$$

$\square$

Remark 4.3. By Theorem 4.2, in analogy with Definition 2.2 and [7], we will write 1-AC_H and 1-$AC_H^{(n)}(\Omega,\mathbb{R}^l)$, instead of 1-$AC_\lambda$ and 1-$AC_\lambda^{(n)}(\Omega,\mathbb{R}^l)$.

Corollary 4.4.

$$AC_H^{(n)}(\Omega,\mathbb{R}^l) \subseteq 1\text{-}AC_H^{(n)}(\Omega,\mathbb{R}^l).$$

Proof. By the definition of the α-regularity of intervals, if $\alpha < 1$ then α-$AC_H \subseteq$ 1-AC_H. Bongiorno [7] proved that for all $\alpha < 1$, α-$AC_H = AC_H$. Thus $AC_H \subseteq$ 1-AC_H. □

Remark 4.5. It follows directly from the definitions that 1-$AC \subseteq$ 1-AC_H. We will prove in the next section that 1-$AC_H \setminus AC_H \supseteq$ 1-$AC \setminus AC_H \neq \emptyset$, and that $AC_H \setminus$ 1-$AC \neq \emptyset$, see Theorem 5.9.

5 Class 1-AC

To simplify notation, the results of this section, except Theorem 5.8, are stated for functions defined on subsets of $\mathbb{R}^2$ with range in $\mathbb{R}$. However they can be easily generalized to functions from $\Omega \subset \mathbb{R}^n$ to $\mathbb{R}^l$ for any $n \geq 2, l \in \mathbb{N}$.

For $d \in (0,\infty)$ let $S_d \subset \mathbb{R}^2$ denote the open square with vertices $(\pm 2d, 0)$, $(0, \pm 2d)$; and for $d = \infty$ we set $S_d = S_\infty = \mathbb{R}^2$. Let $d \in (0,\infty]$. First we restrict our attention to measurable functions $f : S_d \to \mathbb{R}$ with separated variables in the sense that there exist functions $h, g : (-d,d) \to \mathbb{R}$ so that for all $s,t \in (-d,d)$,

$$f(s\mathbf{x}_1 + t\mathbf{x}_2) = h(s)g(t), \tag{5.1}$$

where $\mathbf{x}_1 = (-1,1), \mathbf{x}_2 = (1,1)$, or, equivalently, for all $(x,y) \in S_d$,

$$f(x,y) = h\Big(\frac{y-x}{2}\Big)g\Big(\frac{y+x}{2}\Big).$$

Theorem 5.1. *Let $d \in (0,\infty]$ and $f : S_d \to \mathbb{R}$ be a measurable function that satisfies* (5.1). *Then*

(a) if g is constant and h is any measurable function, then $f \in$ 1-$AC^2(S_d,\mathbb{R})$.

(b) if h is bounded and g is Lipschitz, then $f \in$ 1-$AC^2(S_d,\mathbb{R})$.

(c) If $f \in$ 1-$AC_H^2(S_d,\mathbb{R})$, $f \neq 0$ a.e., and $\operatorname{supp}(g) \subsetneq (-d,d)$*, then g is $\frac{1}{2}$-Hölder.*

Proof of Theorem 5.1. (a) Since f is constant on segments of slope 1, for any 1-regular interval $[\mathbf{a},\mathbf{b}] \subseteq S_d$ we have $|f(\mathbf{a}) - f(\mathbf{b})|^2 = 0$. Thus $f \in$ 1-$AC^2(S_d,\mathbb{R})$.

(b) Let $M \in \mathbb{R}$ be a bound of $h(s)$ and $L > 0$ be a Lipschitz constant for $g(t)$. Then for all $s, t_1, t_2 \in (-d, d)$ we have

$$|f(s\mathbf{x}_1 + t_1\mathbf{x}_2) - f(s\mathbf{x}_1 + t_2\mathbf{x}_2)| = |h(s)| \cdot |g(t_1) - g(t_2)| \leq ML|t_1 - t_2|. \quad (5.2)$$

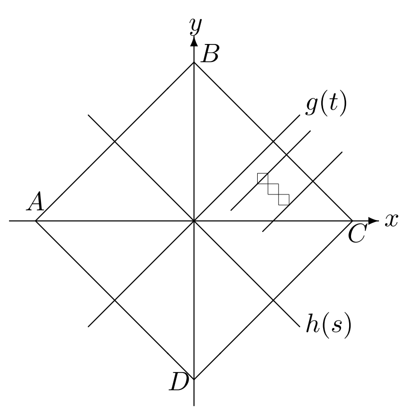
Let $\varepsilon > 0$. Put $\delta = \frac{\varepsilon}{M^2L^2}$. Let $\{[\mathbf{a}_i, \mathbf{b}_i] \subset S_d\}_{i=1}^k$ be any finite collection of disjoint 1-regular intervals with $\sum_i^k \mathcal{L}^2([\mathbf{a}_i, \mathbf{b}_i]) < \delta$. By 1-regularity of intervals $[\mathbf{a}_i, \mathbf{b}_i]$, there exist $s_i, t_{i,1}, t_{i,2} \in (-d, d)$ so that $\mathbf{a}_i = s_i\mathbf{x}_1 + t_{i,1}\mathbf{x}_2$, $\mathbf{b}_i = s_i\mathbf{x}_1 + t_{i,2}\mathbf{x}_2$ and $\mathcal{L}^2([\mathbf{a}_i, \mathbf{b}_i] = |t_{i,1} - t_{i,2}|^2$. Thus, by (5.2), we have

$$\begin{aligned}\sum_{i=1}^k |f(\mathbf{a}_i) - f(\mathbf{b}_i)|^2 &\leq \sum_{i=1}^k (ML|t_{i,1} - t_{i,2}|)^2 \\ &= M^2L^2 \sum_{i=1}^k \mathcal{L}^2([\mathbf{a}_i, \mathbf{b}_i]) < M^2L^2 \cdot \delta = \varepsilon.\end{aligned}$$

So $f \in 1\text{-}AC^2(S_d, \mathbb{R})$.

(c) Since f is measurable and $f \neq 0$ a.e., there exists $c > 0$ and a set $A \subseteq \mathbb{R}$ of positive measure so that $|h(s)| \geq c$ for all $s \in A$. Fix $\lambda \in (0, 1)$, and let $\delta > 0$ be such that (4.1) is satisfied for $\varepsilon = 1$. Let $\sigma > 0$ be such that $\operatorname{supp}(g) \subseteq [-d + \sigma, d - \sigma]$, and fix $\eta < \min\{\delta, \mathcal{L}^1(A), 1, 2\sigma\}$. For any $t, t' \in \operatorname{supp}(g)$ with $|t - t'| \leq \lambda\eta$, there exists $k \in \mathbb{N}$, so that

$$k|t - t'| < \lambda\eta \leq (k + 1)|t - t'|. \quad (5.3)$$



Since $\eta < \mathcal{L}^1(A)$, there exist $\{s_i\}_{i=1}^k \subset A$ so such that $|s_i - s_j| > \eta/k$ for all $i \neq j$. Put

$$\begin{aligned} \mathbf{a}_i &= s_i\mathbf{x}_1 + \Big(\frac{t+t'}{2} - \frac{1}{\lambda}\frac{|t-t'|}{2}\Big)\mathbf{x}_2, \\ \mathbf{b}_i &= s_i\mathbf{x}_1 + \Big(\frac{t+t'}{2} + \frac{1}{\lambda}\frac{|t-t'|}{2}\Big)\mathbf{x}_2. \end{aligned} \tag{5.4}$$

Then for all $i = 1, \dots, k$, $\{[\mathbf{a}_i, \mathbf{b}_i]\}_{i=1}^k$ is a 1-regular interval with the side of length $|t-t'|/\lambda < \eta/k < |s_i - s_j|$. Thus intervals $\{[\mathbf{a}_i, \mathbf{b}_i]\}_{i=1}^k$ are disjoint and $\sum_{i=1}^k \mathcal{L}^2[\mathbf{a}_i, \mathbf{b}_i] = k(\frac{1}{\lambda}|t-t'|)^2 < \frac{\eta k}{k^2} < \delta$, and ${}^\lambda[\mathbf{a}_i, \mathbf{b}_i] = [s_i\mathbf{x}_1 + t\mathbf{x}_2, s_i\mathbf{x}_1 + t'\mathbf{x}_2]$, for all i. for all i. Since $t, t' \in \operatorname{supp}(g)$ and $|t - t'| \leq \lambda\eta < \lambda\sigma$, each $[\mathbf{a}_i, \mathbf{b}_i]$ is contained in S_d. Thus, by (4.1), we have

$$\begin{aligned} 1 &> \sum_{i=1}^k |f(s_i\mathbf{x}_1 + t\mathbf{x}_2) - f(s_i\mathbf{x}_1 + t'\mathbf{x}_2)|^2 \\ &= \sum_{i=1}^k |h(s_i)|^2 |g(t) - g(t')|^2 \\ &\geq kc^2 |g(t) - g(t')|^2. \end{aligned}$$

By (5.3) we get

$$|g(t) - g(t')|^2 \leq \frac{1}{c^2 k} \leq \frac{2}{c^2(k+1)} \leq \frac{2}{c^2\lambda\eta}|t - t'|.$$

Thus g is $\frac{1}{2}$-Hölder for all $t, t' \in (-d, d)$ with $|t-t'| < \lambda\eta$. Since g is uniformly continuous, it is also $\frac{1}{2}$-Hölder for distances larger than $\lambda\eta$. $\square$

Remark 5.2. It is possible to supplement part (c), by stating that when $f \in 1\text{-}AC_H^2(S_d, \mathbb{R})$, $f \neq 0$ a.e., and $\operatorname{supp}(g) \subsetneq (-d, d)$, then g is in the class V^2 of generalized absolutely continuous functions, based on the 2nd power total variation of functions on $\mathbb{R}$ introduced by Wiener [32] and Young [33]. For any $p \geq 1$, the class V^p, is defined in [23] as the class of functions $g : (a, b) \to \mathbb{R}$, (here $(a, b) \subseteq \mathbb{R}$) such that for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all finite sets of disjoint intervals $(t_\nu, t'_\nu) \subseteq (a, b)$

$$\Big(\sum_{\nu=1}^k |t_\nu - t'_\nu|^p\Big)^{\frac{1}{p}} < \delta \implies \Big(\sum_{\nu=1}^k |g(t_\nu) - g(t'_\nu)|^p\Big)^{\frac{1}{p}} < \varepsilon. \tag{5.5}$$

Indeed, we can proceed in a way analogous to the proof of part (c), and define points $\mathbf{a}_{i,\nu}, \mathbf{b}_{i,\nu}$ so that for all i, ν the intervals $[\mathbf{a}_{i,\nu}, \mathbf{b}_{i,\nu}] \subset S_d$ are disjoint and ${}^\lambda[\mathbf{a}_{i,\nu}, \mathbf{b}_{i,\nu}] = [s_i\mathbf{x}_1 + t_\nu\mathbf{x}_2, s_i\mathbf{x}_1 + t'_\nu\mathbf{x}_2]$. Then the same computation as in part (c) shows that g satisfies (5.5) with $p = 2$.

Love [23, Theorems 2 and 4] proved that a measurable function f is in the class V^p if and only if

$$\lim_{h\to 0^+} V_p(f(x+h)-f(x); a<x<b-h) = 0, \tag{5.6}$$

where $V_p(f(x); a<x<b)$ denotes the Wiener-Young p-th power generalization of total variation, defined by

$$V_p(f(x); a<x<b) = \sup\Big\{\big(\sum_{i=1}^{k} |f(x_i)-f(x_{i-1})|^p\big)^{\frac{1}{p}}\Big\},$$

where the supremum is taken over all finite partitions of (a,b).

We note, that the condition (5.6) is similar, but not the same as the classical characterization of functions in the Sobolev space $W^{1,p}$ through the p-th power estimate of the L_p-modulus of continuity of f, i.e. if $f\in L_p$, then $f\in W^{1,p}$ iff there exists $M>0$ so that for all $\delta>0$,

$$\omega^{(p)}(\delta, f) = \sup_{|\xi|\le\delta}\int_{(a,b)} |f(x+\xi)-f(x)|^p dx \le M\delta^p.$$

It would be interesting to determine whether there are some conditions on the function h in (5.1), so that if $f\in 1\text{-}AC^2$ then $g\in W^{1,2}$, and some, possibly different, mild conditions on h, so that if $g\in W^{1,2}$ then $f\in 1\text{-}AC^2$.

On the other hand, it also would be interesting to determine if $f\in 1\text{-}AC^2$ provided that h is measurable, and bounded, and g is any function in V^2.

Remark 5.3. Since $\alpha\text{-}AC\subset 1\text{-}AC$, $1\text{-}AC$ and $1\text{-}AC_H$ also contain functions which do not satisfy (5.1). We do not know whether, in general, $f_s(t)\overset{\text{def}}{=} f(s\mathbf{x}_1+t\mathbf{x}_2)$ is $\frac{1}{2}$-Hölder whenever $f\in 1\text{-}AC^2_H(S_d,\mathbb{R})$ and $s\in(-d,d)$. However, in Theorem 5.8 below, we prove that for a.e. s in the domain, f_s is differentiable a.e.

Using Theorem 5.1(c) we can obtain examples of functions with "nice" properties, but not in $1\text{-}AC_H$. In particular we have:

Corollary 5.4. *There exists a function differentiable everywhere which does not belong to* $1\text{-}AC_H$.

Proof. It is enough to take a function satisfying (5.1) with h constant and g differentiable but not $\frac{1}{2}$-Hölder, e.g. $g(t)=t^2\sin 1/t^4$. $\square$

On the other hand, using Theorem 5.1(a) and (b) with g constant or Lipschitz and appropriately chosen h, we can construct functions in $1\text{-}AC^2(\mathbb{R}^2,\mathbb{R})$

with various pathological properties. We note that to obtain examples that do not separate variables in the sense of (5.1), it is enough to use a sum of described examples and any function in 1-AC. Thus we obtain:

Corollary 5.5. *There exist examples of functions f in 1-AC such that*

(*a*) *f is bounded and nowhere continuous;*

(*b*) *f is continuous, but nowhere differentiable in any direction other than $\mathbf{v} = (1, 1)$.*

PROOF. The proofs rely on Theorem 5.1. For (a), take g constant and h discontinuous at every point of $(-d, d)$. For (b) take $g : (-1, 1) \to \mathbb{R}$ defined by $g(t) = 1-|t|$, and let $h : (-1, 1) \to \mathbb{R}$ be any continuous, nowhere differentiable function. It is easy to see that f is not differentiable at any point $\mathbf{p} = s_0\mathbf{x}_1 + t_0\mathbf{x}_2 \in \operatorname{supp} f$ in any direction $\mathbf{v} = v_1\mathbf{x}_1 + v_2\mathbf{x}_2$ other than $\mathbf{x}_2 = (1, 1)$. Indeed, we have

$$\begin{aligned}
\lim_{\lambda\to 0}\frac{f(\mathbf{p}+\lambda\mathbf{v})-f(\mathbf{p})}{\lambda} &= \lim_{\lambda\to 0}\frac{h(s_0+\lambda v_1)g(t_0+\lambda v_2)-h(s_0)g(t_0)}{\lambda}\\
&= \lim_{\lambda\to 0}\frac{(1-(|t_0+\lambda v_2|)h(s_0+\lambda v_1)-(1-|t_0|)h(s_0)}{\lambda}\\
&= \lim_{\lambda\to 0}(1-|t_0|)\Big[\frac{h(s_0+\lambda v_1)-h(s_0)}{\lambda}\Big] \mp \lim_{\lambda\to 0} v_2 h(s_0+\lambda v_1)\\
&= \lim_{\lambda\to 0}(1-|t_0|)\Big[\frac{h(s_0+\lambda v_1)-h(s_0)}{\lambda}\Big] \mp v_2 h(s_0).
\end{aligned}$$

Since $v_1 \neq 0$ and h is nowhere differentiable, $\lim_{\lambda\to 0}\frac{h(s_0+\lambda v_1)-h(s_0)}{\lambda}$ does not exist anywhere. Therefore $D_{\mathbf{v}}f$ does not exist at any $\mathbf{p} \in \operatorname{supp} f$. □

Next we show that there exists a function $f \in$ 1-AC so that f is differentiable almost everywhere, but f does not satisfy the Luzin (N) property.

Recall that a function $f : \mathbb{R}^n \to \mathbb{R}^l$ is said to satisfy *the Luzin (N) property* if $\mathcal{H}^n(f(E)) = 0$ whenever $E \subseteq \mathbb{R}^n$ and $\mathcal{L}^n(E) = 0$, where $l \geq n$ and $\mathcal{H}^n$ denotes the n-dimensional Hausdorff measure on $\mathbb{R}^l$, see e.g. [34].

It is known that all absolutely continuous functions on $\mathbb{R}$ and all functions in AC_H satisfy the Luzin (N) property. However, when $n > 1$, we have the following:

Theorem 5.6. *Suppose $n > 1$ and $l \geq 1$ are integers. Then there exists an almost everywhere differentiable function f in* 1-$AC(\mathbb{R}^n, \mathbb{R}^l)$ *and a set $U \subseteq \mathbb{R}^n$ with $\mathcal{L}^n(U) = 0$ and $\mathcal{L}^l(f(U)) > 0$.*

Remark 5.7. Note that when $l \geq n$, any subset of $\mathbb{R}^l$ with positive l-dimensional Lebesgue measure, has positive n-dimensional Hausdorff measure. In fact, if $l > n$, the n-dimensional Hausdorff measure of such a set is necessarily infinite. Hence, Theorem 5.6 shows that functions in the class 1-$AC_D(\mathbb{R}^n, \mathbb{R}^l)$ (that consists of all almost eveywhere differentiable functions in 1-$AC(\mathbb{R}^n, \mathbb{R}^l)$) may be severely expanding; when $l \geq n$, the conclusion of Theorem 5.6 is stronger than the assertion that f does not have the Luzin (N) property.

Proof of Theorem 5.6. Let $\varphi : \mathbb{R} \to [0,1]$ be an extension of the Cantor function (see [10], [14]) on $[0,1]$ such that φ is constant on $\mathbb{R} \setminus [0,1]$. Let C denote the standard ternary Cantor set. Recall that $\varphi(C) = [0,1]$ and φ is constant on each connected component of $[0,1] \setminus C$. Hence, φ is differentiable with derivative zero almost everywhere.

Let $p : [0,1] \to \mathbb{R}^{l-1}$ denote a space filling curve with $\mathcal{L}^{l-1}(p([0,1])) > 0$, (see [3]). Note that the function $p \circ \varphi : [0,1] \to \mathbb{R}^{l-1}$ is differentiable with derivative zero almost everywhere.

Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ be an orthonormal basis of $\mathbb{R}^n$ so that $\mathbf{x}_n = (1/\sqrt{n})(\mathbf{e}_1 + \ldots + \mathbf{e}_n)$. Define a set $U \subseteq \mathbb{R}^n$ by

$$U = \{t_1\mathbf{x}_1 + \ldots + t_n\mathbf{x}_n : t_1 \in C, t_2 \ldots, t_n \in \mathbb{R}\}.$$

We note that U has Lebesgue measure zero, since it is an isomorphic image of $C \times \mathbb{R}^{n-1}$. Let $f : \mathbb{R}^n \to \mathbb{R}^l$ be the function defined by

$$f(t_1\mathbf{x}_1 + \ldots + t_n\mathbf{x}_n) = \begin{cases} \varphi(t_1) & \text{if } l = 1, \\ (p \circ \varphi(t_1), t_2) & \text{if } l > 1. \end{cases}$$

Then

$$f(U) = \begin{cases} [0,1] & \text{if } l = 1, \\ p([0,1]) \times \mathbb{R} & \text{if } l > 1. \end{cases}$$

Hence, $\mathcal{L}^l(f(U)) > 0$, and f does not have the Luzin (N) property. Further, note that f is differentiable almost everywhere.

It only remains to verify that $f \in$ 1-$AC(\mathbb{R}^n, \mathbb{R}^l)$. One can check that each component of the function f has the form given by the generalization of Theorem 5.1(b) (for $\mathbb{R}^n$ rather than $\mathbb{R}^2$). By an adaptation of the proof of Theorem 5.1(b), we get that $f \in$ 1-$AC(\mathbb{R}^n, \mathbb{R}^l)$. □

The next theorem contains a positive result about properties of functions in 1-AC_H.

Theorem 5.8. *Every function $f \in$ 1-$AC_H^{(n)}(\mathbb{R}^n, \mathbb{R}^l)$ is differentiable a.e. in the direction $\mathbf{e}_1 + \mathbf{e}_2 + \cdots + \mathbf{e}_n$.*

PROOF. Let $n \geq 1$, $\mathbf{x}_n = \mathbf{e}_1 + \mathbf{e}_2 + \cdots + \mathbf{e}_n$ and choose vectors $\mathbf{x}_1, \ldots, \mathbf{x}_{n-1} \in \mathbb{R}^n$ so that $\mathbf{x}_1, \ldots, \mathbf{x}_n$ is an orthogonal basis. In what follows we will identify $\mathbb{R}^{n-1}$ with the $n-1$ dimensional subspace of $\mathbb{R}^n$ spanned by $\mathbf{x}_1, \ldots, \mathbf{x}_{n-1}$ via the correspondence $\mathbf{s} \leftrightarrow s_1\mathbf{x}_1 + \ldots + s_{n-1}\mathbf{x}_{n-1}$. For $\mathbf{u} \in \mathbb{R}^n$, we define a function $f_{\mathbf{u}} : \mathbb{R} \to \mathbb{R}^l$ by $f_{\mathbf{u}}(t) = f(\mathbf{u} + t\mathbf{x}_n)$. Moreover, given a point $t_0 \in \mathbb{R}$ and a function $g : \mathbb{R} \to \mathbb{R}^l$ we define

$$\operatorname{Lip}(g, t_0) = \limsup_{t \to t_0} \frac{|g(t) - g(t_0)|}{|t - t_0|}. \tag{5.7}$$

We will show that the set

$$E := \{\mathbf{u} \in \mathbb{R}^n : \operatorname{Lip}(f_{\mathbf{u}}, 0) = \infty\}$$

has n-dimensional Lebesgue measure zero.

We claim that this suffices: Indeed, if E has measure zero then, using Fubini's Theorem, we get that for almost every $\mathbf{s} \in \mathbb{R}^{n-1}$, $\operatorname{Lip}(f_{\mathbf{s}+t\mathbf{x}_n}, 0) < \infty$ for almost every $t \in \mathbb{R}$. Observe that $\operatorname{Lip}(f_{\mathbf{s}}, t) = \operatorname{Lip}(f_{\mathbf{s}+t\mathbf{x}_n}, 0)$ for all $\mathbf{s} \in \mathbb{R}^{n-1}$ and $t \in \mathbb{R}$. It follows that $\operatorname{Lip}(f_{\mathbf{s}}, t) < \infty$ for almost every $s \in \mathbb{R}^{n-1}$ and almost every $t \in \mathbb{R}$. Now, applying the Stepanov Theorem [4], we conclude that for almost every $\mathbf{s} \in \mathbb{R}^{n-1}$, $f_{\mathbf{s}}$ is differentiable almost everywhere (in $\mathbb{R}$). Clearly f is differentiable at $\mathbf{s} + t\mathbf{x}_n$ in the direction $\mathbf{x}_n$ if and only if $f_{\mathbf{s}}$ is differentiable at t. Hence f is differentiable in the direction $\mathbf{x}_n$ almost everywhere.

We now prove that E has measure zero. Choose $\delta > 0$ such that

$$\sum_{k=1}^{N} \mathcal{L}^n([\mathbf{a}_k, \mathbf{b}_k]) < \delta \Rightarrow \sum_{k=1}^{N} \left| f\Big(\frac{\mathbf{a}_k + 3\mathbf{b}_k}{4}\Big) - f\Big(\frac{3\mathbf{a}_k + \mathbf{b}_k}{4}\Big)\right|^n < 1, \tag{5.8}$$

whenever $\{[a_k, b_k]\}_{k=1}^N$ is a finite collection of disjoint, 1-regular intervals in $\mathbb{R}^n$. Let $\{A_m\}_{m=1}^\infty$ be a countable collection of closed intervals with pairwise disjoint, non-empty interiors such that $\mathbb{R}^n = \bigcup_{m=1}^\infty A_m$ and $\mathcal{L}^n(A_m) < \delta$ for all m. It suffices to show that the outer measure $\mathcal{L}^{n*}(E \cap A_m) = 0$ for all m.

Fix $m \geq 1$. For each $p \in \mathbb{N}$, we define a collection of intervals $\mathcal{V}_p$ by

$$\mathcal{V}_p = \left\{[\mathbf{a}, \mathbf{b}] \subseteq A_m : \frac{\left|f\big(\frac{\mathbf{a}+3\mathbf{b}}{4}\big) - f\big(\frac{3\mathbf{a}+\mathbf{b}}{4}\big)\right|}{\frac{\|\mathbf{b}-\mathbf{a}\|}{2}} \geq p\right\}. \tag{5.9}$$

Note that $\mathcal{V}_p$ is a Vitali cover of $E \cap \operatorname{Int}(A_m)$. Hence, by the Vitali Covering Theorem, we can find a collection $\{I_k = [\mathbf{a}_k, \mathbf{b}_k]\}_{k=1}^\infty$ of pairwise disjoint intervals from $\mathcal{V}_p$ such that

$$\mathcal{L}^{n*}\Big((E \cap \operatorname{Int}(A_m)) \setminus \bigcup_{k=1}^{\infty} I_k\Big) = 0.$$

Choose an integer $K \geq 1$ so that $\sum_{k=1}^{K} \mathcal{L}^n(I_k) > \mathcal{L}^{n*}(E \cap A_m) - \frac{1}{p}$. Since the intervals I_k are pairwise disjoint and contained in A_m we have that $\sum_{k=1}^{K} \mathcal{L}^n(I_k) < \mathcal{L}^n(A_m) < \delta$. Thus, using (5.8) and (5.9), we get

$$\begin{aligned}
1 &> \sum_{k=1}^{K} \Big| f\Big(\frac{\mathbf{a}_k + 3\mathbf{b}_k}{4}\Big) - f\Big(\frac{3\mathbf{a}_k + \mathbf{b}_k}{4}\Big)\Big|^n \\
&\geq 2^{-n} p^n \sum_{k=1}^{K} \big\|\mathbf{b}_k - \mathbf{a}_k\big\|^n \\
&\geq 2^{-n} p^n \sum_{k=1}^{K} \mathcal{L}^n(I_k) \\
&\geq 2^{-n} p^n \left(\mathcal{L}^{n*}(E \cap A_m) - \frac{1}{p}\right).
\end{aligned}$$

Hence,

$$\mathcal{L}^{n*}(E \cap A_m) \leq \frac{2^n}{p^n} + \frac{1}{p}.$$

Letting $p \to \infty$, we deduce that $\mathcal{L}^{n*}(E \cap A_m) = 0$. □

In view of the presented above pathological examples of functions in 1-AC, it becomes of interest to define the restricted class 1-AC_{WDN} which consists of all functions in 1-AC which are in the Sobolev space $W^{1,n}_{loc}$, are differentiable a.e. and satisfy the Luzin (N) property (we will sometimes restrict 1-AC to only some of these good properties).

Our final result shows where the classes 1-AC_{WDN} fits in the hierarchy of previously studied classes.

Theorem 5.9. *The following holds*

$$\begin{aligned}
&\mathcal{Q}\text{-}AC \subsetneq \alpha\text{-}AC = 1\text{-}AC_{\mathrm{WDN}} \cap AC_H \subseteq 1\text{-}AC_{\mathrm{WDN}} \\
&\subsetneq 1\text{-}AC_{\mathrm{WDN}} \cup AC_H \subsetneq \operatorname{lin\,span}(1\text{-}AC_{\mathrm{WDN}} \cup AC_H) \subseteq 1\text{-}AC_{\mathrm{HWDN}},
\end{aligned} \tag{5.10}$$

$$1\text{-}AC_{\mathrm{DN}} \setminus AC_H \neq \emptyset, \tag{5.11}$$

and

$$1\text{-}AC_{\mathrm{WDN}} \setminus \mathcal{B}\text{-}AC \neq \emptyset, \quad \text{and} \quad \mathcal{B}\text{-}AC \setminus 1\text{-}AC_{\mathrm{WDN}} \neq \emptyset. \tag{5.12}$$

Here $1\text{-}AC_{\mathrm{HWDN}}$ *denotes the set of functions in* $1\text{-}AC_H$ *which are in the Sobolev space* $W^{1,2}_{loc}$*, are differentiable a.e. and satisfy the Luzin (N) property.*

Remark 5.10. We note an open problem which we did not resolve. In view of all other examples, one would expect that the set $1\text{-}AC_{\mathrm{DN}} \setminus 1\text{-}AC_{\mathrm{WDN}}$ is nonempty. However we do not have an example to illustrate this, see also Remark 5.2. If the function that we construct below to prove (5.11) belongs to $1\text{-}AC_{\mathrm{WDN}}$, it would also clarify the nature of the third relation in (5.10).

Remark 5.11. Bongiorno [8] introduced another class of absolute continuity denoted $AC^n_\Lambda(\Omega, \mathbb{R}^l)$, or simply AC_Λ, so that $AC_H \subsetneq AC_\Lambda$ and all functions in AC_Λ are differentiable a.e. and satisfy the Luzin (N) property, but $AC^n_\Lambda(\Omega,\mathbb{R}^l) \not\subset W^{1,n}_{loc}(\Omega,\mathbb{R}^l)$. It would be interesting to determine what are the classes $1\text{-}AC \cap AC_\Lambda$, $1\text{-}AC_H \cap AC_\Lambda$ and $1\text{-}AC_{\mathrm{HWDN}} \cap AC_\Lambda$.

It also would be interesting to determine what are the relations between $AC_\Lambda \cap W^{1,n}_{loc}$, $1\text{-}AC_{\mathrm{HWDN}}$ and the linear span of $1\text{-}AC_{\mathrm{WDN}} \cup AC_H$.

Proof of Theorem 5.9. The first containment of (5.10) is due to Bongiorno [6]. The next equality follows from Theorem 1.4, since all functions in AC_H are in the Sobolev space $W^{1,2}_{loc}$, are differentiable a.e. and satisfy the Luzin (N) property. All following containments are clear.

The proof of (5.11) is technical and we postpone it till the end. The Bongiorno's example [6, Example 4.(1)] shows that

$$AC_H \setminus 1\text{-}AC_{\mathrm{WDN}} \neq \emptyset. \tag{5.13}$$

Since both AC_H and $1\text{-}AC_{\mathrm{WDN}}$ are linearly closed it follows from (5.13) that lin span$(1\text{-}AC_{\mathrm{WDN}} \cup AC_H) \setminus (1\text{-}AC_{\mathrm{WDN}} \cup AC_H) \neq \emptyset$.

The first part of (5.12) follows from (5.11) since $\mathcal{B}\text{-}AC \subset AC_H$.

The other part of (5.12) follows from a small adjustment of the function constructed by Csörnyei in [13, Theorem 2].

Indeed, let f be the function defined in [13, Theorem 2]. We rotate f clockwise by 90° to obtain the function g. Csörnyei showed that $f \notin Q\text{-}AC^2(\Omega,\mathbb{R})$. By a similar argument, using the same notation, since the right-upper corner of each $Q_{mk} = [\mathbf{a}_{mk}, \mathbf{b}_{mk}]$ is a 1-regular interval for $m \in \mathbb{N}$ and $k = 1, 2, \ldots, r_m$, and since the collection $\{Q_{mk}\}$ forms a pairwise disjoint system of 1-regular intervals, we have for the new function g

$$|g(\mathbf{b}_{mk}) - g(\mathbf{a}_{mk})| = \omega_m.$$

Thus,

$$\sum_{m=1}^{\infty}\sum_{k=1}^{r_m} |g(\mathbf{b}_{mk}) - g(\mathbf{a}_{mk})|^2 = \sum_{m=1}^{\infty}\sum_{k=1}^{r_m} \omega_m^2 = \sum_{m=1}^{\infty} r_m\omega_m^2 = \sum_{m=1}^{\infty} \frac{1}{4m} = \infty.$$

Therefore, $g \notin 1\text{-}AC^2(\Omega, \mathbb{R}^l)$. However, by the same argument as in [13, Theorem 2], g is in $\mathcal{B}\text{-}AC^2(\Omega, \mathbb{R})$.

We now prove (5.11). The construction is an adjustment of [13, Theorem 2] and [6, Example 4.(2)]. Since the construction is very technical we provide all details for the convenience of the reader. We note that the presented below example does not satisfy (5.1). We do not know whether an example satisfying (5.1) exists.

We define f as the sum of an absolutely convergent series of non-negative continuous functions f_m so that the support of each f_m is covered by the union of

$$r_m = 4^{m-1}(m-1)!m!$$

pairwise disjoint squares

$$Q_{m1}, Q_{m2}, \ldots, Q_{mr_m},$$

with

$$\mathcal{L}^2\left(\bigcup_{k=1}^{r_{m+1}} Q_{(m+1)k}\right) < \frac{1}{8}\mathcal{L}^2\left(\bigcup_{k=1}^{r_m} Q_{mk}\right) < \left(\frac{1}{8}\right)^m \mathcal{L}^2(Q_{11}), \tag{5.14}$$

and such that

$$\max f_m = \omega_m \stackrel{\text{def}}{=} \frac{1}{2^m m!}.$$

The square Q_{11} is arbitrarily chosen in Ω. Assume that, for a given m, the functions $f_1, f_2, \ldots, f_{m-1}$ and the squares $Q_{hk}, 1 \le h \le m, 1 \le k \le r_m$, have been defined. We define f_m and the squares $\mathcal{Q}_{(m+1)j}$, for $j = 1, \ldots, r_{m+1}$ as follows: for a fixed $k \in 1, \ldots, r_m$, we put a horizontal and a vertical line through the midpoint $O = (o_1, o_2)$ of the square Q_{mk}. Denote by $2d_1$ the length of the side of Q_{mk} and by

$$\begin{aligned} A_1 &= (o_1 + d_1, o_2 + d_1),\ A_2 = (o_1 + d_1, o_2 - d_1),\\ A_3 &= (o_1 - d_1, o_2 - d_1),\ A_4 = (o_1 - d_1, o_2 + d_1), \end{aligned}$$

the verices of the square Q_{mk}.

Let $d = \frac{1}{2}d_1$. We construct a smaller square $A_1B_1C_1D_1$ in the upper right corner of the square $A_1A_2A_3A_4$, where

$$B_1 = (o_1 + d_1, o_2 + d), C_1 = (o_1 + d, o_2 + d), D_1 = (o_1 + d_1, o_2 + d). \tag{5.15}$$

The length of side $A_1B_1 = d_1 - d = d$. Note that the interval $T_{mk} = [O, B_1] \subset Q_{mk}$ is $\frac{1}{2}$-regular since

$$\frac{\mathcal{L}^2(T_{mk})}{d_1^2} = \frac{\mathcal{L}^2[O, B_1]}{d_1^2} = \frac{d \cdot d_1}{d_1^2} = \frac{1}{2}. \tag{5.16}$$

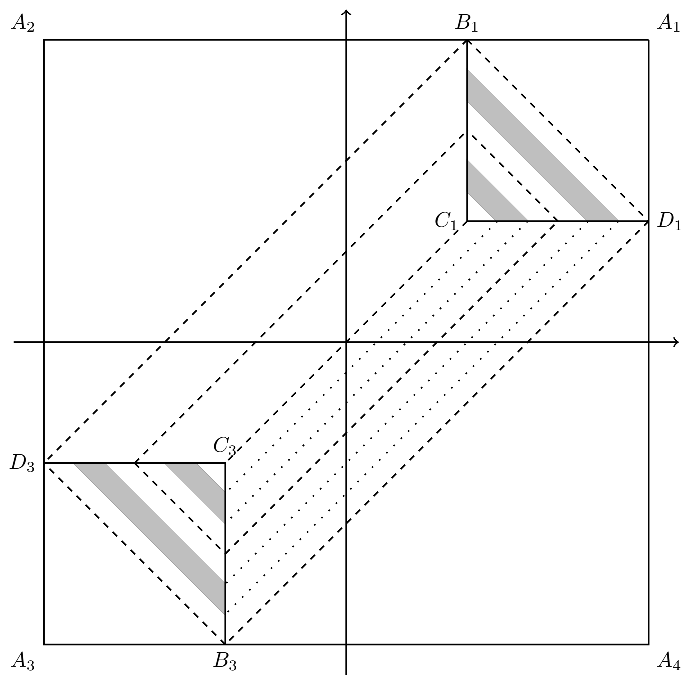


Figure 1: Q_{mk} when $m = 2$

Similarly we construct a square $A_3B_3C_3D_3$ in the lower left corner of the square $A_1A_2A_3A_4$, where

$$B_3 = (o_1 - d_1, o_2 - d), C_3 = (o_1 - d, o_2 - d), D_3 = (o_1 - d, o_2 - d_1).$$

Look at the square $A_1B_1C_1D_1$ with the side length equal to d, cf. Figure 1. For every interval

$$[a_i, b_i] = \begin{cases} [\frac{d}{2^i}, \frac{d}{2^{i-1}}], i = 1, 2, ..., m-1, \\ [0, \frac{d}{2^{m-1}}], i = m, \end{cases}$$

we put $2(m+1)$ small disjoint squares $Q_{(m+1)j}$ inside each strip

$$M_i = \left\{ (x, y) \in \mathbb{R}^2, \rho(x, y) \in \left[\frac{2a_i + b_i}{3}, \frac{a_i + 2b_i}{3}\right] \cap \Box A_1B_1C_1D_1 \right\},$$

where $\rho(x, y) = |x - o_1 - d| + |y - o_2 - d|$ (as shaded in Figure 1); and we also put $2(m+1)$ small disjoint squares $Q_{(m+1)j}$ inside each strip

$$M_i' = \left\{(x,y) \in \mathbb{R}^2, \rho'(x,y) \in \left[\frac{2a_i+b_i}{3}, \frac{a_i+2b_i}{3}\right] \cap \square A_3B_3C_3D_3\right\},$$

where $\rho'(x,y) = |x-o_1+d| + |y-o_2+d|$.

Thus all squares $Q_{(m+1)j}$ are contained in the union of of squares $A_1B_1C_1D_1$ and $A_3B_3C_3D_3$, whose union has measure equal to $\frac{1}{8}$ of the measure of square Q_{mk}, so that (5.14) is satisfied.

We require that the distribution of the squares inside the strips is the following (we will describe the situation for the strip M_i, and the strip M_i' will be symmetric):

Case 1: if m is even, then we put

① $m/2+1$ squares $Q_{(m+1)j}$ inside each of the following strips:

$$M_i \cap \left\{(x,y) : x > o_1 + d, 0 < y - o_2 - d < \frac{d}{2^{i+2}}\right\};$$
$$M_i \cap \left\{(x,y) : 0 < x - o_1 - d < \frac{d}{2^{i+2}}, y > o_2 + d\right\};$$

② $m/2$ squares $Q_{(m+1)j}$ into the interval $[\mathbf{p}_1, \mathbf{p}_2]$ with endpoints $\mathbf{p}_1$,$\mathbf{p}_2$ on the line $y = o_2+d+(x-o_1-d)/2$ and so that $\rho(\mathbf{p}_1) = \frac{4}{3}\cdot\frac{d}{2^i}, \rho(\mathbf{p}_2) = \frac{5}{3}\cdot\frac{d}{2^i}$;

③ $m/2$ squares $Q_{(m+1)j}$ into the interval $[\mathbf{q}_1, \mathbf{q}_2]$ with endpoints $\mathbf{q}_1$,$\mathbf{q}_2$ on the line $y = o_2+d+2(x-o_1-d)$ and so that $\rho(\mathbf{q}_1) = \frac{4}{3}\cdot\frac{d}{2^i}, \rho(\mathbf{q}_2) = \frac{5}{3}\cdot\frac{d}{2^i}$.

Case 2: if m is odd, then we put

④ $(m+1)/2$ squares $Q_{(m+1)j}$ inside each of the strips in ①;

⑤ $(m+1)/2$ squares $Q_{(m+1)j}$ into the interval $[\mathbf{p}_1, \mathbf{p}_2]$ with endpoints $\mathbf{p}_1$,$\mathbf{p}_2$ on the line $y = o_2+d+(x-o_1-d)/2$ and so that $\rho(\mathbf{p}_1) = \frac{4}{3}\cdot\frac{d}{2^i}, \rho(\mathbf{p}_2) = \frac{5}{3}\cdot\frac{d}{2^i}$;

⑥ $(m+1)/2$ squares $Q_{(m+1)j}$ into the interval $[\mathbf{q}_1, \mathbf{q}_2]$ with endpoints $\mathbf{q}_1$,$\mathbf{q}_2$ on the line $y = o_2+d+2(x-o_1-d)$ and so that $\rho(\mathbf{q}_1) = \frac{4}{3}\cdot\frac{d}{2^i}, \rho(\mathbf{q}_2) = \frac{5}{3}\cdot\frac{d}{2^i}$.

On the square $A_1B_1C_1D_1 \subset Q_{mk}$, we define the function f_m by $f_m(\mathbf{x}) = \tilde{f}_m(\rho(\mathbf{x}))$, where

$$\tilde{f}_m(\rho) = \begin{cases} 0, & \text{if } \rho \geq d; \\ \frac{i\omega_m}{m}, & \text{if } \rho = d/2^i, i = 1, 2, \dots, m-1; \\ \omega_m, & \text{if } \rho = 0; \end{cases}$$

Moreover, on the intervals $[a_i, b_i]$, $i = 1, 2, ..., m$, we define $\tilde{f}_m$ by

$$\tilde{f}_m(\rho) = \begin{cases} \frac{(\tilde{f}_m(a_i)+\tilde{f}_m(b_i))}{2}, & \text{if } \rho \in [\frac{2a_i+b_i}{3}, \frac{a_i+2b_i}{3}]; \\ \text{linear}, & \text{if } \rho \in [a_i, \frac{2a_i+b_i}{3}] \cup [\frac{a_i+2b_i}{3}, b_i]. \end{cases}$$

We perform the same construction in the square $A_3B_3C_3D_3$.

On the polygon $B_1C_1D_1B_3C_3D_3$ the function f_m is constant along the lines with slope 1. Outside the polygon $B_1A_1D_1B_3A_3D_3$ the function f_m is 0. We define the function

$$f = \sum_{m=1}^{\infty} f_m.$$

For every $x \in \Omega$, there exists the $m_x \in \mathbb{N}$ such that for all $m \geq m_x$, $f_m(x) = 0$. Thus f is well defined.

Claim 1: f is differentiable almost everywhere.

By the Rademacher-Stepanov theorem (see e.g. [16, Theorem 3.1.9], a short proof in [27]), it is enough to prove that $\mathrm{Lip}(f, x) < \infty$ for a.e. $x \in \Omega$, where $\mathrm{Lip}(f, x)$ was defined in (5.7).

For every $x \in \Omega$, there exists the smallest $m_x \in \mathbb{N}$ such that for all $m \geq m_x$, $f_m(x) = 0$ and without loss of generality $m_x > 1$. Moreover for every x there exists a neighborhood $B(x, r)$ such that for any $y \in B(x, r)$ and any $m \neq m_x$, we have $f_m(x) = f_m(y)$. Thus $\mathrm{Lip}(f, x) = \mathrm{Lip}(f_{m_x}, x)$. Since functions f_m are Lipschitz for every m, we conclude that $\mathrm{Lip}(f_{m_x}, x)$, and thus also $\mathrm{Lip}(f, x)$, is finite.

Claim 2: f satisfies the Luzin (N) property.

This is clear since f scalar valued.

Claim 3: $f \notin \frac{1}{2}\text{-}AC^2(\Omega, \mathbb{R})$.

By (5.16) for each $m \in \mathbb{N}$, and $k = 1, 2, \ldots, r_m$, the disjoint intervals $T_{mk} \subset Q_{mk}$ are $\frac{1}{2}$-regular, and by (5.14), for every $\delta > 0$, there exists $m_0 \in \mathbb{N}$ so that

$$\mathcal{L}^2\left(\bigcup_{m=m_0}^{\infty}\bigcup_{k=1}^{r_m} T_{mk}\right) < \frac{1}{7 \cdot 8^{m_0-2}}\mathcal{L}^2(Q_{11}) < \delta.$$

Since, for each $m \in \mathbb{N}$, and $k = 1, 2, \ldots, r_m$, $|f(T_{mk})| = \omega_m$, we have

$$\sum_{m=m_0}^{\infty}\sum_{k=1}^{r_m} |f(T_{mk})|^2 = \sum_{m=m_0}^{\infty}\sum_{k=1}^{r_m} \omega_m^2 = \sum_{m=m_0}^{\infty} r_m\omega_m^2 = \sum_{m=m_0}^{\infty} \frac{1}{4m} = \infty.$$

Therefore, $f \notin \frac{1}{2}\text{-}AC^2(\Omega, \mathbb{R})$.

Claim 4: $f \in 1\text{-}AC^2(\Omega, \mathbb{R}) \setminus AC^2_H(\Omega, \mathbb{R})$.

By Theorem 1.4 and Claim 3, it is enough to show that f is in $1\text{-}AC^2(\Omega, \mathbb{R})$.

To see this, first note that, by an adaptation of [13, Lemma 3], for every 1-regular interval $I = [\mathbf{a}, \mathbf{b}]$, there exists an index $m = m(I)$, so that,

$$|f(\mathbf{a}) - f(\mathbf{b})|^2 \leq 16|f_{m(I)}(\mathbf{a}) - f_{m(I)}(\mathbf{b})|^2.$$

Let

$$\mathcal{D}_1 = \left\{ I = [\mathbf{a}, \mathbf{b}] : I \text{ is 1-regular and } |f_{m(I)}(\mathbf{a}) - f_{m(I)}(\mathbf{b})| \leq 9\frac{\omega_{m(I)}}{m(I)} \right\},$$

$$\mathcal{D}_2 = \left\{ I = [\mathbf{a}, \mathbf{b}] : I \text{ is 1-regular and } |f_{m(I)}(\mathbf{a}) - f_{m(I)}(\mathbf{b})| > 9\frac{\omega_{m(I)}}{m(I)} \right\}.$$

We will prove that there exist two measures μ_1 and μ_2, absolutely continuous with respect to the Lebesgue measure, such that

$$|f_{m(I)}(\mathbf{a}) - f_{m(I)}(\mathbf{b})|^2 \leq \mu_1([\mathbf{a}, \mathbf{b}]) \tag{5.17}$$

for each $[\mathbf{a}, \mathbf{b}] \in \mathcal{D}_1$, and

$$|f_{m(I)}(\mathbf{a}) - f_{m(I)}(\mathbf{b})|^2 \leq \mu_2([\mathbf{a}, \mathbf{b}]) \tag{5.18}$$

for each $[\mathbf{a}, \mathbf{b}] \in \mathcal{D}_2$.

If such measures exist, then the absolute continuity of μ_1 and μ_2 implies that for all $\varepsilon > 0$, there exists $\delta > 0$, such that for each finite collection of non-overlapping 1-regular intervals $\{[\mathbf{a}_j, \mathbf{b}_j]\}$ with $\mathcal{L}^2(\bigcup_j [\mathbf{a}_j, \mathbf{b}_j]) < \delta$,

$$\mu_1(\bigcup_j [\mathbf{a}_j, \mathbf{b}_j]) < \frac{\varepsilon}{32}, \quad \mu_2(\bigcup_j [\mathbf{a}_j, \mathbf{b}_j]) < \frac{\varepsilon}{32}.$$

Hence, we obtain

$$\begin{aligned}
\sum_j |f(\mathbf{a}_j) - f(\mathbf{b}_j)|^2 &\leq 16 \sum_j |f_{m([\mathbf{a}_j, \mathbf{b}_j])}(\mathbf{a}_j) - f_{m([\mathbf{a}_j, \mathbf{b}_j])}(\mathbf{b}_j)|^2 \\
&\leq 16 \sum_j \mu_1([\mathbf{a}_j, \mathbf{b}_j]) + 16 \sum_j \mu_2([\mathbf{a}_j, \mathbf{b}_j]) \\
&= 16\mu_1\Big(\bigcup_j [\mathbf{a}_j, \mathbf{b}_j]\Big) + 16\mu_2\Big(\bigcup_j [\mathbf{a}_j, \mathbf{b}_j]\Big) < \varepsilon,
\end{aligned}$$

which proves that $f \in 1\text{-}AC^2(\Omega, \mathbb{R})$.

Thus, to complete the proof, it is enough to prove the existence of measures μ_1 and μ_2.

Existence of the measure μ_1

For a fixed 1-regular interval $I = [\mathbf{a}, \mathbf{b}] \in \mathcal{D}_1$, let $m = m(I)$ be such that $|f(\mathbf{a}) - f(\mathbf{b})| \leq 4|f_{m(I)}(\mathbf{a}) - f_{m(I)}(\mathbf{b})|$. If $|f_m(\mathbf{a}) - f_m(\mathbf{b})| = 0$, then we can remove this interval $[\mathbf{a}, \mathbf{b}]$ without affecting our results. If $|f_m(\mathbf{a}) - f_m(\mathbf{b})| > 0$, then let $I' = [\mathbf{a}', \mathbf{b}']$ be the smallest 1-regular sub-interval of I such that $|f_m(\mathbf{a}') - f_m(\mathbf{b}')| = |f_m(\mathbf{a}) - f_m(\mathbf{b})|$. Then $I' \subset Q_{mk}$ for some k, and there are three cases:

ⓐ Both points $\mathbf{a}'$ and $\mathbf{b}'$ are in $\triangle B_1C_1D_1$.

ⓑ Both points $\mathbf{a}'$ and $\mathbf{b}'$ are in $\triangle B_3C_3D_3$.

ⓒ Point $\mathbf{b}'$ is in $\triangle B_3C_3D_3$, and $\mathbf{a}'$ is in $\triangle B_1C_1D_1$.

In case ⓒ, let F_1 be the intersection point of line $\mathbf{a}'\mathbf{b}'$ and the side B_1C_1 (or C_1D_1), and $I^* = [F_1, \mathbf{b}'] \subset I'$. If $|f_m(F_1) - f_m(\mathbf{b}')| > 9\frac{\omega_m}{m}$, then we put this interval I^* into set $\mathcal{D}_2$. If $|f_m(F_1) - f_m(\mathbf{b}')| \leq 9\frac{\omega_m}{m}$, then we set $I' = I^*$. Therefore, combining these cases, without loss of generality, we can assume that $I' \subset \square A_1B_1C_1D_1$.

For $1 \leq i \leq m - 9$, we set

$$S_i = \Big\{\mathbf{x} \in \mathbb{R}^2 : \rho(\mathbf{x}) \in \bigcup_{j=i}^{i+9} [a_j, b_j]\Big\},$$

Since $|f_m(\mathbf{a}') - f_m(\mathbf{b}')| \leq 9\frac{\omega_m}{m}$, we have $\mathbf{a}', \mathbf{b}' \in S_i$ for some integer $1 \leq i \leq m-9$. Now notice that f_m is Lipschitz on $\bigcup_{j=i}^{i+9}[a_j, b_j]$ with Lipschitz constant

$$\begin{aligned} K &= \frac{\omega_m/2m}{\frac{1}{3} \cdot \frac{\sqrt{2}}{2} \cdot (b_{i+9} - a_{i+9})} \leq \frac{\omega_m/2m}{\frac{1}{3} \cdot \frac{\sqrt{2}}{2} \cdot (d/2^{i+9-1} - d/2^{i+9})} \\ &= 3 \cdot 2^{9-1} \cdot \sqrt{2} \cdot \frac{\omega_m}{m} \cdot \frac{1}{d/2^i}. \end{aligned}$$

Therefore

$$|f_m(\mathbf{a}') - f_m(\mathbf{b}')|^2 \leq 9 \cdot 2^{17} \cdot \frac{\omega_m^2}{m^2} \cdot \frac{(\mathrm{diam} I')^2}{(d/2^i)^2}.$$

Since $(\mathrm{diam} I')^2 \le 2\mathcal{L}^2(I')$ and $\mathcal{L}^2(S_i) < 8 \cdot (d/2^i)^2$, we have

$$|f_m(\mathbf{b}') - f_m(\mathbf{a}')|^2 \le 9 \cdot 2^{21} \cdot \frac{\omega_m^2}{m^2} \cdot \frac{\mathcal{L}^2(I')}{\mathcal{L}^2(S_i)} = \int_{I'} 9 \cdot 2^{21} \cdot \frac{\omega_m^2}{m^2} \cdot \frac{1}{\mathcal{L}^2(S_i)}.$$

Thus if we set $\mu_1 = \int g$, where

$$g(\mathbf{x}) = 9 \cdot 2^{21} \cdot \sum_{m=1}^{\infty} \sum_{k=1}^{r_m} \sum_{i=1}^{m-9} \left(\frac{\omega_m^2}{m^2} \cdot \frac{\chi_{s_i}(\mathbf{x})}{\mathcal{L}^2(S_i)} \right) \in L^1(\mathbb{R}^2),$$

then μ_1 satisfies (5.17) for each interval $[\mathbf{a}, \mathbf{b}] \in \mathcal{D}_1$ (cf. [13, p. 154] for details).

Existence of the measure μ_2

Let μ_2 be an absolutely continuous measure for which

$$\mu_2(Q_{mk}) = \frac{4}{m \cdot r_m}, \quad m \in \mathbb{N}, k = 1, \dots, r_m.$$

This measure exists because

$$\sum_{j=1}^{r_{m+1}/r_m} \mu_2(Q_{(m+1)j}) = \frac{4}{(m+1) \cdot r_m} < \frac{4}{m \cdot r_m} = \mu_2(Q_{mk}),$$

and

$$\sum_{j=1}^{r_m} \mu_2(Q_{mk}) = \frac{4}{m} \to 0.$$

As before, for a fixed 1-regular interval $I = [\mathbf{a}, \mathbf{b}] \in \mathcal{D}_2$, let $m = m(I)$ be such that $|f(\mathbf{a}) - f(\mathbf{b})| \le 4|f_{m(I)}(\mathbf{a}) - f_{m(I)}(\mathbf{b})|$. Let $I' = [\mathbf{a}', \mathbf{b}']$ be the smallest 1-regular sub-interval of I such that $|f_m(\mathbf{a}') - f_m(\mathbf{b}')| = |f_m(\mathbf{a}) - f_m(\mathbf{b})|$. Then $I' \subset Q_{mk}$ for some k, and by the argument above, without loss of generality, we can assume that $I' \subset \square A_1 B_1 C_1 D_1$.

Let

$$\beta = \frac{|f(\mathbf{a}) - f(\mathbf{b})| \cdot m}{\omega_m}.$$

Since $I \in \mathcal{D}_2$, we have $\beta \ge 9$. For simplicity, we can assume that $C_1 = (0, 0)$. Let j be the smallest integer with $\rho(\mathbf{b}') \ge \frac{d}{2^j}$, and i be the biggest integer with $\rho(\mathbf{a}') \geq \frac{d}{2^{j+i}}$. We denote $\mathbf{a}' = (a_1, a_2), \mathbf{b}' = (b_1, b_2)$, and, for $1 \leq v \leq i$, we set

$$S_v = \Big\{\mathbf{x} \in \mathbb{R}^n : \rho(\mathbf{x}) \in \Big[\frac{d}{2^{j+v}}, \frac{d}{2^{j+v-1}}\Big]\Big\}.$$

Let $\mathbf{a}_{1t}$ and $\mathbf{a}_{2t}$ be the points on the lines $y = 2x$ and $y = x/2$, respectively, so that

$$\rho(\mathbf{a}_{1t}) = \rho(\mathbf{a}_{2t}) = \frac{t}{3} \cdot \frac{d}{2^{j+i}}, \text{ with } 4 \leq t \leq 5.$$

Let $\mathbf{b}_{1t}$, $\mathbf{b}_{2t}$ and $\mathbf{c}_{1t}$, $\mathbf{c}_{2t}$ be the images of the orthogonal projections of $\mathbf{a}_{1t}$, $\mathbf{a}_{2t}$ onto the horizontal line and onto the vertical line through C_1, respectively. Finally let $\mathbf{d}_t = (\frac{t}{3} \cdot \frac{d}{2^{j+v}}, 0)$, $\mathbf{f}_t = (0, \frac{t}{3} \cdot \frac{d}{2^{j+v}})$. Since

$$|C_1 - \mathbf{c}_{1t}| = |\mathbf{a}_{1t} - \mathbf{b}_{1t}| = 2|C_1 - \mathbf{b}_{1t}| = 2|\mathbf{a}_{1t} - \mathbf{c}_{1t}|,$$

$$|\mathbf{b}_{1t} - \mathbf{d}_t| = |\mathbf{a}_{1t} - \mathbf{b}_{1t}|, \text{ and } |\mathbf{a}_{1t} - \mathbf{c}_{1t}| = |\mathbf{f}_t - \mathbf{c}_{1t}|,$$

we have

$$\frac{t}{3} \cdot \frac{d}{2^{j+v}} = |C_1 - \mathbf{d}_t| = 3|C_1 - \mathbf{b}_{1t}|, \tag{5.19}$$

and

$$\frac{t}{3} \cdot \frac{d}{2^{j+v}} = |C_1 - \mathbf{f}_t| = 3|\mathbf{a}_{1t} - \mathbf{c}_{1t}|. \tag{5.20}$$

Moreover, since

$$|\mathbf{a}_{2t} - \mathbf{c}_{2t}| = |C_1 - \mathbf{b}_{2t}| = 2|\mathbf{a}_{2t} - \mathbf{b}_{2t}| = 2|C_1 - \mathbf{c}_{2t}|,$$

$$|\mathbf{a}_{2t} - \mathbf{b}_{2t}| = |\mathbf{b}_{2t} - \mathbf{d}_t|, \text{ and } |\mathbf{c}_{2t} - \mathbf{f}_t| = |\mathbf{a}_{2t} - \mathbf{c}_{2t}|,$$

we have

$$\frac{t}{3} \cdot \frac{d}{2^{j+v}} = |C_1 - \mathbf{d}_t| = 3|\mathbf{a}_{2t} - \mathbf{b}_{2t}|, \tag{5.21}$$

and

$$\frac{t}{3} \cdot \frac{d}{2^{j+v}} = |C_1 - \mathbf{f}_t| = 3|C_1 - \mathbf{c}_{2t}|. \tag{5.22}$$

Therefore, if $v \leq i - 1$, by (5.19),

$$|\mathbf{a}_{1t} - \mathbf{b}_{1t}| = 2 \cdot |C_1 - \mathbf{b}_{1t}| = \frac{t}{3^2} \cdot \frac{d}{2^{j+v-1}} > \frac{4}{3^2} \cdot \frac{d}{2^{j+v-1}} > \frac{d}{2^{j+v-1}} \geq \frac{d}{2^{j+i}} > a_2,$$

and, by (5.22),

$$|\mathbf{a}_{2t} - \mathbf{c}_{2t}| = 2 \cdot |C_1 - \mathbf{c}_{2t}| = \frac{t}{3^2} \cdot \frac{d}{2^{j+v-1}} > \frac{4}{3^2} \cdot \frac{d}{2^{j+v-1}} > \frac{d}{2^{j+v-1}} \geq \frac{d}{2^{j+i}} > a_1.$$

Moreover, if $v \le i-2$, by (5.21)

$$|\mathbf{a}_{2t} - \mathbf{b}_{2t}| = \frac{1}{3} \cdot |C_1 - \mathbf{d}_t| = \frac{t}{3^2} \cdot \frac{d}{2^{j+v}} > \frac{4}{3^2} \cdot \frac{d}{2^{j+v}} > \frac{d}{2^{j+v+2}} \ge \frac{d}{2^{j+i}} > a_2,$$

and, by (5.20),

$$|\mathbf{a}_{1t} - \mathbf{c}_{1t}| = \frac{1}{3} \cdot |C_1 - \mathbf{f}_t| = \frac{t}{3^2} \cdot \frac{d}{2^{j+v}} > \frac{4}{3^2} \cdot \frac{d}{2^{j+v}} > \frac{d}{2^{j+v}} \ge \frac{d}{2^{j+i}} > a_1.$$

Thus, to see that the points $\mathbf{a}_{1t}, \mathbf{a}_{2t}$ belong to the interval I', it suffices to show that

$$|\mathbf{a}_{1t} - \mathbf{b}_{1t}| < b_2 \text{ and } |\mathbf{a}_{2t} - \mathbf{b}_{2t}| < b_2;$$

$$|\mathbf{a}_{1t} - \mathbf{c}_{1t}| < b_1 \text{ and } |\mathbf{a}_{2t} - \mathbf{c}_{2t}| < b_1.$$

First of all, we have

$$b_1 > \frac{d}{2^{j+3}} \text{ and } b_2 > \frac{d}{2^{j+3}}.$$

Indeed, if $b_1 \le \frac{d}{2^{j+3}}$, then $b_2 > \frac{d}{2^{j+1}}$. Thus

$$b_2 - a_2 > \frac{d}{2^{j+1}} - \frac{d}{2^{j+i}} > \frac{d}{2^{j+3}} \ge b_1 > b_1 - a_1,$$

which contradicts 1-regularity of $[\mathbf{a}', \mathbf{b}']$. Similarly $b_2 \le \frac{d}{2^{j+3}}$.

Therefore, if $v \ge 4$, we have

$$\begin{aligned}
|\mathbf{a}_{1t} - \mathbf{b}_{1t}| &< \frac{5}{3^2} \cdot \frac{d}{2^{j+v-1}} < \frac{d}{2^{j+v-1}} \le \frac{d}{2^{j+3}} < b_2; \\
|\mathbf{a}_{2t} - \mathbf{b}_{2t}| &< \frac{5}{3^2} \cdot \frac{d}{2^{j+v}} < \frac{d}{2^{j+v}} \le \frac{d}{2^{j+3}} < b_2; \\
|\mathbf{a}_{1t} - \mathbf{c}_{1t}| &< \frac{5}{3^2} \cdot \frac{d}{2^{j+v}} < \frac{d}{2^{j+v}} \le \frac{d}{2^{j+3}} < b_1; \\
|\mathbf{a}_{1t} - \mathbf{c}_{1t}| &< \frac{5}{3^2} \cdot \frac{d}{2^{j+v-1}} < \frac{d}{2^{j+v-1}} \le \frac{d}{2^{j+3}} < b_1.
\end{aligned}$$

In conclusion, we have proved that, if $4 \le v \le i-2$, then the interval I' contains the points $\mathbf{a}_{1t}, \mathbf{a}_{2t}$ for all $4 \le t \le 5$. Thus I' contains the intervals $[\mathbf{a}_{14}, \mathbf{a}_{15}]$ and $[\mathbf{a}_{24}, \mathbf{a}_{25}]$. Therefore, I' covers at least $(i-5)m/2$ of the r_{m+1} squares $Q_{(m+1)h}$, $1 \le h \le r_{m+1}$. Thus

$$\mu_2([\mathbf{a}', \mathbf{b}']) \ge (i-5)m \cdot \frac{4}{(m+1)r_{m+1}} = (i-5) \cdot \frac{4m}{(m+1)^2} \cdot \omega_m^2,$$

and, since $\beta \le m$ and $\beta \le i+2$, we have

$$|f(\mathbf{a}') - f(\mathbf{b}')|^2 = \frac{\beta^2}{m^2} \cdot \omega_m^2 \le \frac{\beta^2}{m^2} \cdot \frac{(m+1)^2}{4m(i-5)} \cdot \mu_2([\mathbf{a}', \mathbf{b}']) < \frac{(i+2)(m+1)^2}{4(i-5)m^2} \cdot \mu_2([\mathbf{a}', \mathbf{b}']).$$

Moreover, since $m \ge 9$, we have $5m^2 - 22m + 7 > 0$. Since $i \ge 9$, we have $i(3m^2 - 2m + 1) > 22m^2 + 4m + 2$. Thus, $(i+2)(m+1)^2 < 2(i-5)m^2$, and hence

$$|f(\mathbf{a}') - f(\mathbf{b}')|^2 < \mu_2([\mathbf{a}', \mathbf{b}']).$$

□

Acknowledgment. We thank Jan Malý for bringing this problem to our attention, Olga Maleva for helpful discussions, and an anonymous referee for helpful comments which led to improvements of the paper.

References

[1] D. Apatsidis, S. A. Argyros, and V. Kanellopoulos, *Hausdorff measures and functions of bounded quadratic variation*, Trans. Amer. Math. Soc., **363** (2011), 4225–4262.

[2] S. A. Argyros, A. Manoussakis, and M. Petrakis, *Function spaces not containing l_1*, Israel J. Math., **135** (2003), 29–81.

[3] R. Ayala, M.J. Chávez, and A. Quintero, *A Hahn-Mazurkiewicz Theorem for generalized Peano continua* , Archiv der Math. **71(4)** (1998), 325-330.

[4] Z.M. Balogh, K.Rogovin and T. Zürcher, *The Stepanov differentiability theorem in metric measure spaces*, J. Geom. Anal., **14(3)** (2004), 405–422.

[5] S. Banach, *Sur les lignes rectifiables et les surfaces dont l'aire est finie*, Fund. Math., **7** (1925), 225–236.

[6] D. Bongiorno, *Absolutely continuous functions in $\mathbb{R}^n$*, J. Math. Anal. Appl., **303** (2005), 119–134.

[7] D. Bongiorno, *On the Hencl's notion of absolute continuity*, J. Math. Anal. Appl., **350** (2009), 562–567.

[8] D. Bongiorno, *On the problem of regularity in the Sobolev space* $W^{1,n}_{\mathrm{loc}}$, Topology Appl., **156** (2009), 2986–2995.

[9] A. P. Calderón, *On the differentiability of absolutely continuous functions*, Rivista Mat. Univ. Parma, **2** (1951), 203–213.

[10] G. Cantor, *De la puissance des ensembles parfaits de points* , Acta Math., **4(1)** (1884), 381–392.

[11] L. Cesari, *Sulle funzioni assolutamente continue in due variabili*, Ann. Scuola Norm. Super. Pisa (2), **10** (1941), 91–101.

[12] J. A. Clarkson and C. R. Adams, *On definitions of bounded variation for functions of two variables*, Trans. Amer. Math. Soc., **35** (1933), 824–854.

[13] M. Csörnyei, *Absolutely continuous functions of Rado, Reichelderfer, and Malý*, J. Math. Anal. Appl., **252** (2000), 147–166.

[14] O. Dovgoshey, O. Martio, V. Ryazanov and M. Vuorinen *The Cantor function* , Expo. Math., **24.1** (2006), 1–37.

[15] L. C. Evans and R. F. Gariepy, *Measure theory and fine properties of functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1992.

[16] H. Federer, *Geometric measure theory*, Die Grundlehren der mathematischen Wissenschaften, Band 153, Springer-Verlag New York Inc., New York, 1969.

[17] S. Hencl, *On the notions of absolute continuity for functions of several variables*, Fund. Math., **173** (2002), 175–189.

[18] S. Hencl, *Absolutely continuous functions of several variables and quasiconformal mappings*, Z. Anal. Anwendungen, **22** (2003), 767–778.

[19] S. Hencl and J. Malý, *Absolutely continuous functions of several variables and diffeomorphisms*, Cent. Eur. J. Math., **1** (2003), 690–705 (electronic).

[20] E. W. Hobson, *The theory of functions of a real variable and the theory of Fourier's series. Vol. I*, Dover Publications Inc., New York, N.Y., 1958.

[21] S. V. Kisliakov, *A remark on the space of functions of bounded p-variation*, Math. Nachr., **119** (1984), 137–140.

[22] J. Lindenstrauss and C. Stegall, *Examples of separable spaces which do not contain ℓ_1 and whose duals are non-separable*, Studia Math., **54** (1975), 81–105.

[23] E. R. Love, *A generalization of absolute continuity*, J. London Math. Soc., **26** (1951), 1–13.

[24] J. Malý. Private communication (2012).

[25] J. Malý, *Lusin's condition (N) and mappings of the class $W\hat{}*\ n$*, J. Reine Angew. Math, **458** (1995), 19–36.

[26] J. Malý, *Absolutely continuous functions of several variables*, J. Math. Anal. Appl., **231** (1999), 492–508.

[27] J. Malý, *A simple proof of the Stepanov theorem on differentiability almost everywhere*, Exposition. Math., **17** (1999), 59–61.

[28] T. Rado and P. V. Reichelderfer, *Continuous transformations in analysis. With an introduction to algebraic topology*, Springer-Verlag, Berlin, 1955.

[29] B. Randrianantoanina and H. Xu, *On the Bongiorno's notion of absolute continuity*, preprint. arXiv:1404.0064

[30] L. Tonelli, *Sulla quadratura delle superficie*, Rend. della Reale Accademia dei Lincei, Ser. VI, **3** (1926), 357–362 and 633–638.

[31] G. Vitali, *Sulle funzioni continue*, Fund. Math., **8** (1926), 175–188.

[32] N. Wiener, J. Massachusetts Institute Technology, **3** (1924), 73–94.

[33] L. C. Young, *An inequality of the Hölder type, connected with Stieltjes integration*, Acta Math., **67(1)** (1936), 251–282.

[34] W. P. Ziemer, *Weakly differentiable functions. Sobolev spaces and functions of bounded variation.*, vol. 120 of Graduate Texts in Mathematics, Springer-Verlag, New York, 1989.