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BASIC INTRODUCTION TO EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Abstract

This article discusses the definitions and properties of exponential and logarithmic functions. The treatment is based on the basic properties of real numbers, sequences and continuous functions. This treatment avoids the use of definite integrals.

1 Introduction

Exponential and logarithmic functions are elementary and they are taught to all science and economics students. However, their rigorous definitions and some of their properties are not straightforward. In most calculus textbooks they are introduced by defining the logarithmic function as an indefinite integral as in [4]. This approach is a nice application of powerful tools like the *Fundamental Theorem of Calculus.* If we follow this approach, a simple expression like $2^{\sqrt{3}}$ will have no meaning until we learn definite integrals. This is the main disadvantage of this approach. There are other different approaches to introduce these functions. Most elementary real analysis textbooks define the exponential function as a power series [2].

In some cases, first year students need to learn about these functions in an early stage before the introduction of integration. The definition in this case will be intuitive rather than rigorous. The number $2^{\sqrt{3}}$, as an example, can be introduced as the limit of sequence $(2^{q_n})_{n\geq 1}$, where $(q_n)_{n\geq 1}$ is a sequence of rational numbers converging to $\sqrt{3}$. The general case will be understood by analogy. This approach is demonstrated in many calculus textbooks, see [1].

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The latter approach gives an intuitive definition to exponential functions. It is used in Rudin's book [2, page 22]. However, the introduction of the number e and the derivation of differentiation properties of these functions in this book were established using power series [2, page 178].

This article establishes the differentiation and other basic properties for the exponential and logarithmic functions using the intuitive definition of the exponential function. The proofs are basic (but not elementary) without any use of definite integrals. However, we need some properties of continuous functions and sequences. The main step is showing that the following limit

$$\lim_{h\to 0}\frac{a^h-1}{h}$$

exists for any $a>0$. After studying this limit all the properties of exponential and logarithmic functions follow easily.

I should add that after finishing the first draft, I found the paper [3] by H. Samelson and I found that my work is almost identical with Samelson's work. The only difference is the use of convexity in his proof. What surprises me is that Samelson ideas had gone unnoticed for 40 years. This approach deserves to be known and I use it as an excuse for finishing and submitting an article demonstrating these ideas in some detail. The note [5] by J. P. Tull gives a similar approach in a very brief discussion.

2 Exponential Functions

In this section we introduce exponential functions and their basic properties. The main theorem is the differentiability of exponential functions. We will need the following remark in some of the following arguments.

Remark 1. If $a>1$ and $m\in\mathbb{N}$, then

$$\frac{a-1}{ma}<a^{\frac{1}{m}}-1<\frac{a-1}{m}.$$

To show this note that $a-1=(a^{1/m})^m-1=(a^{1/m}-1)\sum_{k=0}^{m-1}a^{k/m}$. Since $1<a^{k/m}<a$, we find that $m<\sum_{k=0}^{m-1}a^{k/m}<ma$. By multiplying both sides of the latter inequality by $(a^{1/m}-1)$, we get the required result.

Lemma 2. *If $a>0$, there is a constant C_a such that $|a^q-1|\le C_a\,|q|$ for any rational number q with $|q|<1$.*

PROOF. Assume first that $a>1$. Let q be a rational number such that $0<q<1$. Then $q=\frac{n}{m}$ where n,m are positive integers such that $n<m$.

Hence,

$$a^q - 1 = \left(a^{\frac{1}{m}}\right)^n - 1 = \left(a^{\frac{1}{m}} - 1\right) \sum_{k=0}^{n-1} a^{\frac{k}{m}}.$$

Note that $a^{\frac{k}{m}} < a^q < a$ whenever $0 \leq k < n$. Thus, $\sum_{k=0}^{n-1} a^{\frac{k}{m}} < na^q$, and this implies

$$a^q - 1 < na^q \left(a^{\frac{1}{m}} - 1\right).$$

Using Remark 1, we get

$$a^q - 1 < qa^q \left(a - 1\right) < qa \left(a - 1\right). \tag{2.1}$$

Now, if $0 > q > -1$, then $0 < -q < 1$. Hence, by the last inequality $a^{-q} - 1 < -qa^{-q} \left(a - 1\right)$. Multiplying by a^q yields

$$1 - a^q < -q \left(a - 1\right) < -qa \left(a - 1\right). \tag{2.2}$$

By both of inequalities (2.1) and (2.2), we get

$$|a^q - 1| < |q| a \left(a - 1\right),$$

whenever $a > 1$ and $|q| < 1$.

Now, if $0 < a < 1$, then $a^{-1} > 1$. Applying the previous inequality we get:

$$|a^q - 1| = \left|\left(a^{-1}\right)^{-q} - 1\right| < |-q| a^{-1} \left(a^{-1} - 1\right) = |q| a^{-1} \left(a^{-1} - 1\right),$$

whenever $0 < a < 1$ and $|q| < 1$. □

Let $a, x \in \mathbb{R}$ where $a > 0$. Then, the number a^x is defined as

$$a^x = \lim_n a^{q_n}$$

where $\left(q_n\right)_{n \geq 1}$ is an increasing sequence of rational numbers such that $q_n \to x$. This limit always exists since $\left(a^{q_n}\right)_{n \geq 1}$ is bounded and monotone (increasing if $a > 1$ and decreasing if $a < 1$). Moreover, Lemma 2 implies that this limit is the same if $\left(q_n\right)_{n \geq 1}$ is replaced by any sequence of rational numbers $\left(p_n\right)_{n \geq 1}$ (not necessarily monotone) with $p_n \to x$. Thus, the number a^x is well defined. Moreover, using elementary convergence argument, this definition establishes some of the basic lows of exponents as: $a^{x+y} = a^x a^y$, $\left(ab\right)^x = a^x b^x$.

For any $a > 0$, define the function $E_a : \mathbb{R} \to \mathbb{R}$, $x \mapsto a^x$. This function is called *the exponential function with base a*. It has the following basic properties for any $a > 0, x, y \in \mathbb{R}$:

1. $a > 1, x < y \Rightarrow E_a(x) < E_a(y)$ (E_a is strictly increasing when $a > 1$);
2. $a < 1, x < y \Rightarrow E_a(x) > E_a(y)$ (E_a is strictly decreasing when $a < 1$);
3. $a > 1 \Rightarrow \lim_{x\to-\infty} E_a(x) = 0,\ \lim_{x\to+\infty} E_a(x) = +\infty$;
4. $a < 1 \Rightarrow \lim_{x\to-\infty} E_a(x) = +\infty,\ \lim_{x\to+\infty} E_a(x) = 0$;
5. The function E_a is continuous on $\mathbb{R}$ for any $a > 0$;
6. $a^{xy} = (a^x)^y$.

Let $a > 0$ and $a \neq 1$. By the properties of the function E_a and its continuity, we find that E_a is a one-to-one correspondence from $\mathbb{R}$ onto the interval $(0, +\infty)$. Thus, E_a has an inverse function which will be denoted by $\log_a : (0, +\infty) \to \mathbb{R}$. We call $\log_a$ *the logarithmic function with base a.*

Let a be a positive number. Define the function

$$\ell_a : \mathbb{R} \setminus \{0\} \to \mathbb{R},\ \ell_a(x) = \frac{a^x - 1}{x}.$$

It is obvious that ℓ_a is continuous. Now, we prove that $\lim_{t\to 0} \ell_a(t)$ exists. To do this we need the following two lemmas.

Lemma 3. *Let a be a positive number such that $a \neq 1$. If m_1, m_2 are non-zero integers such that $m_1 < m_2$, then $\ell_a(m_1) < \ell_a(m_2)$.*

PROOF. Let m be a positive integer. Then,

$$\begin{aligned}\ell_a(m+1) - \ell_a(m) &= \frac{ma^m(a-1) - (a^m - 1)}{m(m+1)} \\ &= \frac{(a-1)\sum_{k=0}^{m-1}\left(a^m - a^k\right)}{m(m+1)} \\ &= \frac{(a-1)^2\sum_{k=0}^{m-1}\sum_{j=0}^{m-k-1} a^{k+j}}{m(m+1)} > 0.\end{aligned}$$

Thus, $\ell_a(m) < \ell_a(m+1)$. By induction, we can show that $\ell_a(m_1) < \ell_a(m_2)$ whenever $0 < m_1 < m_2$.

Now, suppose that $m_1 < m_2 < 0$. Then $0 < -m_2 < -m_1$. By the previous argument, $\ell_{a^{-1}}(-m_2) < \ell_{a^{-1}}(-m_1)$. Note that $\ell_a(x) = -\ell_{a^{-1}}(-x)$ for any $x \neq 0$. Thus, $\ell_a(m_1) = -\ell_{a^{-1}}(-m_1) < -\ell_{a^{-1}}(-m_2) = \ell_a(m_2)$.

Note that $\ell_a(1) - \ell_a(-1) = \dfrac{(a-1)^2}{a} > 0$. Thus, $\ell_a(1) > \ell_a(-1)$.

Now, if $m_1 < 0 < m_2$, then $\ell_a(m_1) \leq \ell_a(-1) < \ell_a(1) \leq \ell_a(m_2)$. This completes the proof. $\square$

Lemma 4. *If a is a positive number such that $a \neq 1$, then ℓ_a is strictly increasing.*

Proof. Let p_1, p_2 be non-zero rational numbers with $p_1 < p_2$. Then, we can write $p_1 = m_1/n$, $p_2 = m_2/n$, where m_1, m_2, n are integers such that $m_1 < m_2$ and $n > 0$. Hence,

$$\ell_a(p_1) = n\ell_{a^{\frac{1}{n}}}(m_1) < n\ell_{a^{\frac{1}{n}}}(m_2) = \ell_a(p_2).$$

Now, let x, p be two non-zero real numbers and suppose that p is a rational number. If $p < x$, then $\ell_a(p) < \ell_a(x)$. To show this, choose a strictly increasing sequence of non–zero rational numbers $(r_n)_{n\geq 1}$ such that $r_1 = p$ and $r_n \to x$. Note that $(\ell_a(r_n))_{n\geq 1}$ is strictly increasing. By continuity, $\ell_a(r_n) \to \ell_a(x)$, thus, $\ell_a(p) < \ell_a(x)$. Similarly, if $x < p$, then $\ell_a(x) < \ell_a(p)$.

Now, if $x_1, x_2 \in \mathbb{R} \setminus \{0\}$ with $x_1 < x_2$, then we can choose a non-zero rational number p such that $x_1 < p < x_2$. Thus, $\ell_a(x_1) < \ell_a(p) < \ell_a(x_2)$. $\square$

Theorem 5. *The limit $\lim_{x\to 0} \ell_a(x) = \lim_{x\to 0} \dfrac{a^x - 1}{x}$ exists for any $a > 0$.*

Proof. Since ℓ_a is increasing, both of its one-sided limits at 0 exist. Let $l_\pm = \lim_{x\to 0^\pm} \ell_a(x)$. It is easy to check that $\ell_a(x) = a^x \ell_a(-x)$. Thus,

$$l_- = \lim_{x\to 0^-} \ell_a(x) = \lim_{x\to 0^-} a^x \ell_a(-x) = \lim_{t\to 0^+} a^{-t} \ell_a(t) = l_+.$$

The equality $l_- = l_+$ implies that the limit exists. $\square$

For any $a > 0$, we define $\ln a = \lim_{x\to 0} \ell_a(x) = \lim_{x\to 0} \dfrac{a^x - 1}{x}$. Note that $\ln : (0, +\infty) \to \mathbb{R}$ is a well-defined function. This function is called the *natural logarithm*. Note that

$$\lim_{h\to 0} \frac{E_a(x+h) - E_a(x)}{h} = \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} = a^x \lim_{h\to 0} \frac{a^h - 1}{h} = a^x \ln a.$$

Thus, the following result is immediate.

Theorem 6. *The exponential function E_a is differentiable at any $x \in \mathbb{R}$ with $E_a'(x) = a^x \ln a$.*

3 The Natural Logarithm

In this section we study the natural logarithm and its relation with exponential and logarithmic functions.

Theorem 7. *Let x, y be positive numbers. The following statements hold:*

1. $\ln(xy) = \ln x + \ln y$;
2. $\ln x^r = r \ln x$ *for any* $r \in \mathbb{R}$;
3. $\dfrac{x-1}{x} \leq \ln x \leq x - 1$;
4. $\lim_{x\to 0} \dfrac{\ln(x+1)}{x} = 1$;
5. $\ln$ *is differentiable at any* $x > 0$ *with* $\dfrac{\mathrm{d}}{\mathrm{d}x} \ln x = \dfrac{1}{x}$;
6. $\ln$ *is strictly increasing and* $\lim_{x\to 0^+} \ln x = -\infty,\ \lim_{x\to+\infty} \ln x = +\infty$.

PROOF. (1) For any $x > 0, y > 0$, we have

$$\begin{aligned}
\ln(xy) &= \lim_{t\to 0} \frac{(xy)^t - 1}{t} = \lim_{t\to 0} \frac{x^t y^t - y^t + y^t - 1}{t} \\
&= \left(\lim_{t\to 0} y^t\right) \left(\lim_{t\to 0} \frac{x^t - 1}{t}\right) + \lim_{t\to 0} \frac{y^t - 1}{t} \\
&= 1 \cdot \ln x + \ln y = \ln x + \ln y.
\end{aligned}$$

(2) It is obvious when $r = 0$. If $r \neq 0$, then

$$\begin{aligned}
\ln x^r &= \lim_{t\to 0} \frac{(x^r)^t - 1}{t} = r \lim_{t\to 0} \frac{x^{rt} - 1}{rt} \\
&= r \lim_{u\to 0} \frac{x^u - 1}{u} = r \ln x.
\end{aligned}$$

(3) Let $x > 1$. Then Remark 1 yields

$$\frac{x-1}{x} < \frac{x^{1/m} - 1}{1/m} < x - 1,$$

for any positive integer m. By Theorem 5, $\ln x = \lim_{m\to\infty} \dfrac{x^{1/m} - 1}{1/m}$." Hence,

$$\frac{x-1}{x} < \ln x < x - 1$$

for $x > 1$. Now, if $0 < x < 1$, then $x^{-1} > 1$, and we use the previous inequality to obtain the desired inequality.

(4) We substitute x by $x+1$ in the inequality in 3, and we get

$$\frac{x}{x+1} < \ln(x+1) < x.$$

If $x > 0$, then $\frac{1}{x+1} < \frac{\ln(x+1)}{x} < 1$, and this implies $\lim_{x\to 0^+} \frac{\ln(x+1)}{x} = 1$. In the case $x < 0$, we have $\frac{1}{x+1} > \frac{\ln(x+1)}{x} > 1$, which implies $\lim_{x\to 0^-} \frac{\ln(x+1)}{x} = 1$.

(5) Let $x > 0$. Then,

$$\lim_{h\to 0} \frac{\ln(x+h) - \ln x}{h} = \frac{1}{x} \lim_{h\to 0} \frac{\ln(1+h/x)}{h/x} = \frac{1}{x} \lim_{t\to 0} \frac{\ln(1+t)}{t} = \frac{1}{x}.$$

Thus, $\frac{\mathrm{d}}{\mathrm{d}x}(\ln(x)) = \frac{1}{x}$.

(6) Since its derivative is positive, ln is strictly increasing. Let $x > 0$. By property (3), $\ln 2 > \frac{1}{2}$. Thus, $\ln 2^x > \frac{x}{2}$ for any $x > 0$. This implies that ln is unbounded. Since it is increasing, we have $\lim_{x\to+\infty} \ln x = +\infty$. On the other hand, $\lim_{x\to 0^+} \ln x = \lim_{t\to+\infty} \ln t^{-1} = -\lim_{t\to+\infty} \ln t = -\infty$. $\square$

For $x > 0,\ x \neq 1$, we define $e_x = x^{1/\ln x}$. Note that $\ln e_x = 1$. Since ln is one-to-one, we find that e_x is constant for any $x > 0,\ x \neq 1$. Thus, the function $x \mapsto e_x$ is constant. We denote this constant by e. This constant is called the *base of the natural logarithms*. Note that $e^{\ln x} = x$.

We denote the function E_e by exp. Note that $\ln(\exp(x)) = \ln(e^x) = x$ and $\exp(\ln x) = e^{\ln x} = x$. Therefore, ln is the inverse of $E_e = \exp$, in other words $\ln = \log_e$. It follows immediately that $a^x = e^{x \ln a}$ and $\log_a y = \frac{\ln y}{\ln a}$. From these equalities we can derive all the properties of exponential and logarithmic functions.

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