INROADS

Real Analysis Exchange
Vol. 40(1), 2014/2015, pp. 209–218

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ABSOLUTE CONTINUITY IN PARTIAL DIFFERENTIAL EQUATIONS

Abstract

In this note, we study a function which frequently appears in partial differential equations. We prove that this function is absolutely continuous; hence it can be written as a definite integral. As a result, we obtain some estimates regarding solutions of the Hamilton-Jacobi systems.

1 Introduction

Let H be a differential operator of order $m \in \mathbb{N}$ and let $f \in L^p(D)$ be a positive function, where $p \in (1, \infty)$ and D is a smooth bounded domain in $\mathbb{R}^n$. Consider the equation:

$$H(u) = f, \quad \text{in } D \tag{1}$$

A function $u \in W^{m,p}(D) \cap C(\overline{D})$ is called a strong solution of (1) provided that $H(u) = f$ almost everywhere (a. e.) in D. We assume the operator H satisfies the following condition:

For any $u \in W^m(D)$ and $\gamma \in \mathbb{R}:$ $H(u) = 0$ a. e. in $E_\gamma := \{x \in D \mid u(x) = \gamma\}$. **(P)**

Mathematical Reviews subject classification: Primary: 28D05, 35F21
Key words: absolute continuity, weak convergence, rearrangements of functions, measure preserving maps, Hamilton-Jacobi systems
Received by the editors January 15, 2014
Communicated by: Viktor Kolyada
*Amin Farjudian's work on this article has been partially supported by the Natural Science Foundation of China (Grant No. 61070023) and Ningbo Natural Science Programme by Ningbo S&T bureau (Grant No. 2010A610104).

For a measurable function $h : D \to \mathbb{R}$, the distribution function of h, denoted $\lambda_h(\alpha)$, is defined as follows:

$$\lambda_h(\alpha) \coloneqq |\{x \in D \mid h(x) \geq \alpha\}| \equiv |\{h \geq \alpha\}|, \quad (\forall \alpha \in \mathbb{R})$$

where $|\cdot|$ denotes the n-dimensional Lebesgue measure. Clearly λ_h is decreasing, and if h is continuous, then λ_h will be strictly decreasing. Moreover, in case the graph of h has no significant flat sections (i.e. $\forall \gamma \in \mathbb{R} : |\{h = \gamma\}| = 0$), then λ_h will be continuous. The decreasing rearrangement of h, denoted $h^*(s)$, is defined as follows:

$$\begin{cases} h^* : [0, |D|] \to \mathbb{R} \\ h^*(s) = \inf\{\alpha \mid \lambda_h(\alpha) \leq s\} \end{cases}.$$

Note that if h is continuous and its graph has no significant flat sections, then

$$\lambda_h \circ h^*(s) = s \quad \text{and} \quad h^* \circ \lambda_h(\alpha) = \alpha.$$

We also need to recall some background from rearrangements of functions. Given $g_0 : D \subseteq \mathbb{R}^n \to \mathbb{R}$, the rearrangement class generated by g_0, denoted $\mathcal{R}(g_0)$, is the set of functions $g : D \to \mathbb{R}$ such that $\lambda_g(\alpha) = \lambda_{g_0}(\alpha)$ for every real α. If $g_0 \in L^p(D)$, then $\mathcal{R}(g_0) \subseteq L^p(D)$, and $\forall g \in \mathcal{R}(g_0) : \|g\|_p = \|g_0\|_p$. The weak closure of $\mathcal{R}(g_0)$ in $L^p(D)$ is denoted by $\overline{\mathcal{R}(g_0)}$ which, unlike $\mathcal{R}(g_0)$, enjoys some nice properties and characterizations that are stated in the following lemma. For the proof and further reading see [3, 4, 5, 9]:

Lemma 1. *Let $g_0 \in L^p(D)$ be a non-negative function, and $\mathcal{R}(g_0)$ be the rearrangement class generated by g_0. Then:*

(1) $\overline{\mathcal{R}(g_0)}$ is convex and weakly compact in $L^p(D)$.

(2) $\overline{\mathcal{R}(g_0)} = \overline{co\,(\mathcal{R}(g_0))}$, the closed convex hull of $\mathcal{R}(g_0)$.

(3) The following characterization stands:

$$\overline{\mathcal{R}(g_0)} = \Big\{ g \mid \forall s \in (0, |D|) : \\ \int_0^s g^*(t)\,dt \leq \int_0^s g_0^*(t)\,dt \ \text{ and } \int_0^{|D|} g^*(t)\,dt = \int_0^{|D|} g_0^*(t)\,dt \Big\}.$$

The set of measure-preserving maps from D onto $[0, |D|]$ is a non-empty set (e.g. see [12, Chapter 11]) which will be denoted by $\mathcal{M}(D, [0, |D|])$. By a result attributed to Ryff [13], given $g : D \to \mathbb{R}$, there exists $\phi \in \mathcal{M}(D, [0, |\, D\,|])$ such that $g = g^* \circ \phi$ almost everywhere in D.

We now introduce the function that is the main drive behind writing this note. To this end, we assume $u \in W^{m,p}(D) \cap C(\overline{D})$ is a strong solution of (1). We are interested in the function $\xi : [0, |\, D\,|] \to \mathbb{R}$ defined by:

$$\xi(s) = \int_{\{u \geq u^*(s)\}} f(x)\, dx. \tag{2}$$

Thanks to property **(P)** on page 209, and of course the fact that f is positive, the level sets $\{u = \gamma\}$ must have zero measure; hence ξ is well-defined. This function is frequently referred to in partial differential equations, particularly when one is interested in comparing the solution of a boundary value problem to that of a symmetrized problem, the latter being readily solved. There are many references in this regard, e.g. [2, 6, 14], to mention a few. In this note we prove that ξ is absolutely continuous; hence it can be represented by a definite integral of the form $\int_0^s F(\tau)d\tau$. Then, we will prove that the integrand F composed with any measure-preserving map $\phi \in \mathcal{M}(D, [0, |\, D\,|])$ belongs to $\overline{\mathcal{R}(f)}$. Using these two results, we point out a couple of applications.

Throughout this paper, we use some standard notations. For example, $W^{m,p}(D)$ and $W^m(D)$ denote the usual Sobolev spaces. The space $L^p(D)$ comprises functions whose p-th powers are integrable, and the norm in this space is defined by $\|f\|_p = \left(\int_D |f|^p\, dx\right)^{1/p}$. Moreover, $C(D)$ and $C(\overline{D})$ denote the spaces of continuous functions over D and its closure $\overline{D}$, respectively, and the corresponding norm is denoted by $\|\cdot\|_\infty$. The arrow "$\to$" indicates strong convergence, whilst "$\rightharpoonup$" indicates weak convergence in spaces under discussion.

2 Main results

Our first main result is the following:

Theorem 2. *The function ξ, as defined in (2), is absolutely continuous on $[0, |\, D\,|]$.*

Proof. Let $\epsilon > 0$, and consider a finite sequence $\{(\alpha_i, \beta_i) \mid 1 \leq i \leq N\}$ of non-overlapping subintervals of $[0, |\, D\,|]$ such that $\sum_{i=1}^N (\beta_i - \alpha_i) < \delta$, where δ is a positive number to be determined later. By setting $t(\alpha_i) = u^*(\alpha_i)$ and $t(\beta_i) = u^*(\beta_i)$, we will have:

$$\sum_{i=1}^N |\xi(\beta_i) - \xi(\alpha_i)| = \sum_{i=1}^N \left| \int_{\{t(\beta_i) < u < t(\alpha_i)\}} f(x)\, dx \right| = \int_E f(x)\, dx, \tag{3}$$

where $E = \bigcup_{i=1}^{N} \{x : u^*(\beta_i) < u(x) < u^*(\alpha_i)\}$. By applying the Hölder inequality we obtain:

$$\int_E f(x)\,dx \le |E|^{\frac{1}{q}}\, \|f\|_p, \tag{4}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Note that $|E| = \sum_{i=1}^{N}(\beta_i - \alpha_i)$. This, along with (3) and (4), will give the desired result, provided that $\delta < \left(\frac{\epsilon}{\|f\|_p}\right)^q$. □

Corollary 3. *The function ξ, as defined in (2), satisfies*

$$\xi(s) = \int_0^s F(\tau)\,d\tau \tag{5}$$

for some integrable function F.

Proof. By Theorem 2, ξ is absolutely continuous. Hence, we can apply Corollary 14 in [12], together with the fact that $\xi(0) = 0$, to deduce that

$$\xi(s) = \int_0^s \xi'(\tau)\,d\tau$$

almost everywhere in $[0, |\,D\,|]$. So by setting $F(s) = \xi'(s)$, we get the desired result. □

We now state our second main result:

Theorem 4. *Let F be the function in Corollary 3 and $\phi \in \mathcal{M}(D, [0, |\,D\,|])$. Then $F \circ \phi \in \overline{\mathcal{R}(f)}$.*

Proof. Note that $\lambda_{F\circ\phi}(\alpha) = \lambda_F(\alpha)$ for every $\alpha \in \mathbb{R}$. Thus, $(F \circ \phi)^*(s) = F^*(s)$ for almost every $s \in [0, |\,D\,|]$. Hence, in view of item (3) of Lemma 1, it suffices to prove:

(i) $\int_0^{|\,D\,|} F^*(s)\,ds = \int_0^{|\,D\,|} f^*(s)\,ds$.

(ii) $\int_0^s F^*(t)\,dt \le \int_0^s f^*(t)\,dt, \quad \forall s \in (0, |\,D\,|)$.

Proving (i) is straightforward as

$$\begin{aligned}\int_0^{|\,D\,|} F^*(t)\,dt &= \int_0^{|\,D\,|} F(t)\,dt = \xi(|\,D\,|)\\ &= \int_{\{\,u\,\ge\, t(\,|\,D\,|)\,\}} f\,dx = \int_{\{u\ge 0\}} f\,dx = \int_D f\,dx = \int_0^{|\,D\,|} f^*(t)\,dt,\end{aligned}$$

where we have used Corollary 3. To prove (ii), we consider the following steps:

Step 1. Let $\mathcal{U}$ be an open subset of $(0, |\,D\,|)$. Then, we can write $\mathcal{U} = \bigcup_{i=1}^{\infty}(A_i, B_i)$, where (A_i, B_i) are mutually disjoint. Hence,

$$\begin{aligned}
\int_{\mathcal{U}} F(\tau)d\tau &= \sum_{i=1}^{\infty} \int_{A_i}^{B_i} F(\tau)d\tau = \sum_{i=1}^{\infty} \left(\int_0^{B_i} F(\tau)d\tau - \int_0^{A_i} F(\tau)d\tau \right) \\
&= \sum_{i=1}^{\infty} \left(\int_{\{u \geq t(B_i)\}} f\,dx \ - \int_{\{u \geq t(A_i)\}} f\,dx \right) = \sum_{i=1}^{\infty} \int_{\{t(B_i) \leq u < t(A_i)\}} f\,dx \\
&= \int_{\bigcup\{t(B_i) \leq u < t(A_i)\}} f\,dx \ \leq \int_0^{|\bigcup\{t(B_i) \leq u < t(A_i)\,|} f^*(s)\,ds \\
&= \int_0^{\sum(B_i - A_i)} f^*(s)\,ds \ = \int_0^{|\,\mathcal{U}\,|} f^*(s)\,ds.
\end{aligned}$$

Step 2. Let $\mathcal{V}$ be a measurable subset of $(0, |\,D\,|)$ and let $\epsilon > 0$. By Theorem 3.6 in [15], there exists an open set G containing $\mathcal{V}$ such that $|G \setminus \mathcal{V}| < \epsilon$. Whence

$$\begin{aligned}
\int_{\mathcal{V}} F(t)\,dt \leq \int_G F(t)\,dt &\leq \int_0^{|G|} f^*(s)\,ds \\
&= \int_0^{|\mathcal{V}|} f^*(s)\,ds + \int_{|\mathcal{V}|}^{|G|} f^*(s)\,ds \qquad (6) \\
&\leq \int_0^{|\mathcal{V}|} f^*(s)\,ds + \|f\|_p(|G| - |\mathcal{V}|)^{1/q},
\end{aligned}$$

using Step 2 and Hölder's inequality. Since $|G| - |\mathcal{V}| = |G \setminus \mathcal{V}| < \epsilon$, from (6) we infer

$$\int_{\mathcal{V}} F(t)\,dt \leq \int_0^{|\,\mathcal{V}\,|} f^*(s)\,ds + \epsilon^{1/q}\|f\|_p. \qquad (7)$$

Since ϵ is arbitrary, (7) implies

$$\int_{\mathcal{V}} F(t)\,dt \leq \int_0^{|\,\mathcal{V}\,|} f^*(s)\,ds.$$

Step 3. We recall the following maximization from [1] where the sup is taken over $\{\omega \subseteq [0, |\,D\,|] : |\,\omega\,| = \gamma\}$:

$$\sup \int_{\omega} F(t)\,dt = \int_0^{|\,\omega\,|} F^*(s)\,ds.$$

Now, fix $s \in (0, |D|)$, and apply Step 2 to obtain

$$\sup \int_\omega F(t)\,dt = \int_0^s F^*(t)\,dt, \tag{8}$$

with the sup taken over $\{\omega \subseteq [0, |D|] : |\omega| = s\}$. On the other hand, from Step 2, we have:

$$\int_\omega F(t)\,dt \leq \int_0^{|\omega|} f^*(s)\,ds. \tag{9}$$

From (8) and (9) we deduce

$$\int_0^s F^*(t)\,dt \leq \int_0^s f^*(t)\,dt,$$

as desired. □

Corollary 5. *Suppose the hypotheses of Theorem 4 hold. Then there exists a sequence of functions $\{F_n\}$ such that $F_n^*(s) = f^*(s)$ and $F_n \rightharpoonup F$ in $L^p(0, |D|)$.*

Proof. By Ryff's result, $f = f^* \circ \phi$ for some $\phi \in \mathcal{M}(D, [0, |D|])$. From Theorem 4, we infer $F \circ \phi \in \overline{\mathcal{R}(f)}$. So, there exists a sequence $\{f_n\} \subseteq \mathcal{R}(f)$ such that $f_n \rightharpoonup F \circ \phi$ in $L^p(D)$. Therefore, $f_n \circ \phi^{-1} \rightharpoonup F$ in $L^p(0, |D|)$. Clearly, $\lambda_{f_n \circ \phi^{-1}}(\alpha) = \lambda_f(\alpha)$, and so $(f_n \circ \phi^{-1})^*(s) = f^*(s)$. This completes the proof. □

3 Applications

In this section we will present a couple of applications of the results of the previous section. Throughout we will assume the extra condition $f \in C(\overline{D})$. Let us consider the following Hamilton-Jacobi system:

$$\begin{cases} |\nabla u| = f(x) & \text{in } D \\ u = 0 & \text{on } \partial D. \end{cases} \tag{10}$$

Lemma 6. *The system (10) has a strong positive solution $u \in W^{1,\infty}(D)$.*

Proof. From [10] we know that the system (10) has a strong solution $u \in W^{1,\infty}(D)$. Replacing u by $|u| \in W^{1,\infty}(D)$ if necessary, taking into account that $|\nabla(|u|)| = |\nabla u|$, we can assume u is non-negative. On the other hand, since f is positive, we can apply Lemma 7.7 in [7] to ensure that the level sets $\{u = \gamma\}$ have zero measure. Thus, u is essentially positive, as desired. □

Remark 1. For f and u as in Lemma 6, the function

$$\xi(s) = \int_{\{u \geq t\}} f(x)\, dx \quad (\text{where } s = \lambda_u(t))$$

is well defined. As a result, the function F from Corollary 3 is also well defined. Moreover, the conclusions of Theorem 2 and Theorem 4 hold.

Our first application is as follows:

Theorem 7. *Let $u \in W^{1,\infty}(D)$ be a strong positive solution of the Hamilton-Jacobi system (10), and let v be the unique solution of the following system*

$$\begin{cases} |\nabla Z| = F(\omega_n |x|^n) & \text{in } B \\ Z = 0 & \text{on } \partial B, \end{cases} \tag{11}$$

in which:

- *B is the ball centred at the origin with radius $(|D|/\omega_n)^{1/n}$, and ω_n indicates the volume of the unit n-dimensional ball.*
- *The function F is as in Corollary 3, which is well defined by Remark 1.*

Also, let $u^\sharp(x) \equiv u^*(\omega_n |x|^n)$, which in the literature is referred to as the Schwarz symmetrization of u. Then, $u^\sharp(x) \leq v(x)$ for $x \in B$.

PROOF. The proof is a consequence of Corollary 3, along the same lines as in the proof of Lemma 2.2 in [6]. □

Example 1. Choosing $f(x) = 1$ in Theorem 7 yields $F(t) = 1$. Thus, the conclusion of Theorem 7 states:

$$u^\sharp(x) \leq v(x) = R - |x|, \qquad x \in B,$$

where $R = (|D|/\omega_n)^{1/n}$. This estimate can be obtained directly as follows:

$$\begin{aligned} \lambda_u(t) = \int_{\{u \geq t\}} dx &= \int_{\{u \geq t\}} |\nabla u|\, dx \\ &= \int_t^{\|u\|_\infty} \left(\int_{\{u=\tau\}} dH^{n-1} \right) d\tau = \int_t^{\|u\|_\infty} P(\{u \geq \tau\})\, d\tau, \end{aligned} \tag{12}$$

where we have used the co-area formula (e.g. see [11]). Here, $P(E)$ stands for the perimeter of E in the sense of De Giorgi. By differentiating (12), and applying the classical Isoperimetric Inequality (e.g. see [8]), we derive

$$\lambda_u'(t) = -P(\{u \geq t\}) \leq -n\omega_n^{\frac{1}{n}} \lambda_u^{1-\frac{1}{n}}(t).$$

Thus, we obtain

$$1 \leq -\frac{\lambda_u'(t)}{n\omega_n^{\frac{1}{n}} \lambda_u^{1-\frac{1}{n}}(t)}. \tag{13}$$

Integrating (13) from 0 to t leads to

$$\begin{aligned} t \leq -\frac{1}{n\omega_n^{1/n}} \int_0^t \frac{\lambda_u'(\tau)}{\lambda_u^{1-\frac{1}{n}}(\tau)}\, d\tau &= -\frac{1}{n\omega_n^{1/n}} \int_{|\,D\,|}^{\lambda_u(t)} \frac{ds}{s^{1-\frac{1}{n}}} \\ &= \frac{1}{\omega_n^{1/n}} (|\,D\,|^{1/n} - \lambda_u^{1/n}(t)) = R - \left(\frac{\lambda_u(t)}{\omega_n}\right)^{1/n}. \end{aligned} \tag{14}$$

By letting $t = u^*(\omega_n |\,x\,|^n)$ in (14) and recalling $\lambda_u(u^*(\omega_n |\,x\,|^n)) = \omega_n |\,x\,|^n$, we obtain $u^\sharp(x) \leq R - |\,x\,|$ for $x \in B$, as expected.

The second application is stated in the following Theorem:

Theorem 8. *Let u be as in Theorem 7. Then*

$$\|u\|_\infty \leq C |\,D\,|^{1/n} \|f\|_\infty.$$

Proof. The proof is a consequence of Corollary 5, along the same lines as in the proof of Corollary 2.1 in [6]. □

Acknowledgment. The authors wish to thank the referees for their constructive critique of the first draft.

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