RESEARCH

Real Analysis Exchange
Vol. 40(1), 2014/2015, pp. 141–156

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GENERALIZED KIESSWETTER'S FUNCTIONS

Abstract

In 1966, Kiesswetter found an interesting example of continuous everywhere but differentiable nowhere functions using base-4 expansion of real numbers. In this paper we show how Kiesswetter's function can be extended to general cases. We also provide an equivalent form for such functions via a recurrence relation.

1 Introduction

Finding functions that are continuous everywhere but differentiable nowhere is a classical problem in real analysis. While Weierstrass's famous example is the first such function in publication, the earliest example is believed to be found by Bolzano around 1830 (although Bolzano's example was published much later, see [11]). There had been a great deal of research in this area in the late 19th and early 20th centuries after Weierstrass's example was published. Several important examples including the well-known Takagi's function [10] and its generalizations were introduced. Many mathematicians have made their contributions to this problem, and there is a rich literature on this subject (see [1], [2], [6], [7] and [9]).

In 1966, Kiesswetter ([5], [8]) found a new example. His example is defined as follows. For $x \in [0, 1]$, let

$$x = \sum_{k=1}^{\infty} \frac{x_k}{4^k},$$

Mathematical Reviews subject classification: Primary: 26A06, 26A27; Secondary: 26A15
Key words: continuous functions, non-differentiable functions
Received by the editors May 26, 2014
Communicated by: Krzysztof Ciesielski

where $x_k = 0, 1, 2$ or 3, be a base-4 expansion of x. Let

$$K(x) = \sum_{k=1}^{\infty} \frac{(-1)^{\alpha_k} X(x_k)}{2^k}, \tag{1}$$

where $X(0) = 0$, $X(x_k) = x_k - 2$ if $x_k > 0$, $\alpha_1 = 0$, and α_k is the number of x_i's such that $x_i = 0$ for $i < k$ if $k \geq 2$. Kiesswetter's function is intriguing in the way it is defined, and it has an interesting property: its graph is invariant under a map consisting of four affine transformations in $\mathbb{R}^2$. This property provides geometric insight into the understanding of the function. Edgar [3] studied the Hausdorff dimension of Kiesswetter's fractal.

It is natural to ask whether Kiesswetter's function can be extended to general cases. In his paper, Kiesswetter predicted that if x is given in a base-a expansion ($a \geq 4$), $b = a - 2k \geq 2$ ($k = 1, 2, \ldots$), and if f is given as an infinite series similar to (1) by using b instead of 2, then f would be continuous everywhere but differentiable nowhere on $[0, 1]$ for appropriately chosen $X(x_k)$'s.

In this paper, we show that the construction of the generalized Kiesswetter's functions is only possible when $b = a - 2$.

2 The construction of generalized cases

Let $x \in [0, 1]$ and let $a \geq 2$ be an integer. Suppose x has the following base-a expansion:

$$x = \sum_{k=1}^{\infty} \frac{x_k}{a^k} = 0.x_1 x_2 \cdots x_n \cdots ,$$

where $x_k = 0, 1, 2, \ldots, a - 1$ for $k = 1, 2, \ldots$. Let $b > 1$ be a real number that is to be determined, and let

$$f(x) = \sum_{k=1}^{\infty} \frac{(-1)^{\alpha_k} u(x_k)}{b^k}, \tag{2}$$

where α_k is defined in the same way as for (1), and $u(x_k)$ is to be determined. It is easy to see the series (2) converges. We first consider conditions under which f is well defined, because some $x \in [0, 1]$ can have two base-a expansions as indicated in the following lemma.

Lemma 1. *Let $x \in [0, 1]$.*

(a) *If x is given by*

$$x = \frac{x_1}{a} + \frac{x_2}{a^2} + \cdots + \frac{x_n}{a^n},$$

where $x_n \geq 1$, then

$$x = \frac{x_1}{a} + \frac{x_2}{a^2} + \cdots + \frac{x_n - 1}{a^n} + \frac{a-1}{a^{n+1}} + \frac{a-1}{a^{n+2}} + \cdots .$$

(b) *Any x has either one or two base-a expansions.*

PROOF. It is easy to see that (a) follows from a simple computation, and (b) follows from (a) and the fact that if a base-a expansion of $x \in [0,1]$ does not end with all 0's (or equivalently with all $(a-1)$'s), then its expansion is unique. □

Now we give necessary and sufficient conditions for $u(x_k)$ such that the function given by (2) is well defined.

Theorem 2. *The function given by* (2) *is well defined if and only if*

$$\begin{cases} u(1) = -\dfrac{u(a-1)}{b-1} + \dfrac{b\,u(0)}{b+1}, \\ u(x_n) = u(x_n - 1) + \dfrac{u(a-1)}{b-1} - \dfrac{u(0)}{b+1}, & \text{if } 2 \leq x_n \leq a-2 \\ \dfrac{b-a+2}{b-1} u(a-1) = \dfrac{b-a+2}{b+1} u(0). \end{cases} \tag{3}$$

PROOF. By Lemma 1, if f is well defined, then two base-a expansions of $x \in [0,1]$ produce the same function value. More specifically, if

$$x = \frac{x_1}{a} + \frac{x_2}{a^2} + \cdots + \frac{x_n}{a^n},$$

where $x_n \geq 1$, or equivalently

$$x = \frac{x_1}{a} + \frac{x_2}{a^2} + \cdots + \frac{x_n - 1}{a^n} + \frac{a-1}{a^{n+1}} + \frac{a-1}{a^{n+2}} + \cdots , \tag{4}$$

then we need to choose $u(x_k)$ such that

$$\begin{aligned} &\sum_{k=1}^{n-1} \frac{(-1)^{\alpha_k} u(x_k)}{b^k} + \frac{(-1)^{\alpha_n} u(x_n)}{b^n} + \sum_{k=n+1}^{\infty} \frac{(-1)^{\alpha_k} u(0)}{b^k} \\ &\quad = \sum_{k=1}^{n-1} \frac{(-1)^{\alpha_k} u(x_k)}{b^k} + \frac{(-1)^{\alpha_n} u(x_n - 1)}{b^n} + \sum_{k=n+1}^{\infty} \frac{(-1)^{\alpha'_k} u(a-1)}{b^k}, \end{aligned}$$

where α'_k denotes the number of x_i's such that $x_i = 0$ for $i < k$ with respect to (4). The equation above is equivalent to

$$\begin{aligned}\frac{(-1)^{\alpha_n}u(x_n)}{b^n} &+ u(0)\sum_{k=n+1}^{\infty}\frac{(-1)^{\alpha_k}}{b^k}\\ &= \frac{(-1)^{\alpha_n}u(x_n-1)}{b^n} + u(a-1)\sum_{k=n+1}^{\infty}\frac{(-1)^{\alpha'_k}}{b^k}.\end{aligned} \tag{5}$$

Case 1. Suppose $x_n = 1$. Then for $k \geq n+1$, $\alpha_k = \alpha_n + k - n - 1$, $\alpha'_k = \alpha_n + 1$. Then (5) becomes

$$\begin{aligned}\frac{(-1)^{\alpha_n}u(1)}{b^n} &+ u(0)\sum_{k=n+1}^{\infty}\frac{(-1)^{\alpha_n+k-n-1}}{b^k}\\ &= \frac{(-1)^{\alpha_n}u(0)}{b^n} + u(a-1)\sum_{k=n+1}^{\infty}\frac{(-1)^{\alpha_n+1}}{b^k},\end{aligned}$$

which is equivalent to

$$\begin{aligned}\frac{(-1)^{\alpha_n}u(1)}{b^n} &+ u(0)(-1)^{\alpha_n-n-1}\left(-\frac{1}{b}\right)^{n+1}\frac{1}{1+\frac{1}{b}}\\ &= \frac{(-1)^{\alpha_n}u(0)}{b^n} + u(a-1)(-1)^{\alpha_n+1}\frac{1}{b^{n+1}}\frac{1}{1-\frac{1}{b}},\end{aligned}$$

or

$$u(1) + \frac{u(0)}{b+1} = u(0) - \frac{u(a-1)}{b-1};$$

i.e.,

$$u(1) = -\frac{u(a-1)}{b-1} + \frac{b\,u(0)}{b+1}.$$

Case 2. Suppose $2 \leq x_n \leq a-1$. Then for $k \geq n+1$, $\alpha_k = \alpha_n + k - n - 1$, $\alpha'_k = \alpha_n$. It follows from (5) that

$$\begin{aligned}\frac{(-1)^{\alpha_n}u(x_n)}{b^n} &+ u(0)\sum_{k=n+1}^{\infty}\frac{(-1)^{\alpha_n+k-n-1}}{b^k}\\ &= \frac{(-1)^{\alpha_n}u(x_n-1)}{b^n} + u(a-1)\sum_{k=n+1}^{\infty}\frac{(-1)^{\alpha_n}}{b^k},\end{aligned}$$

which is equivalent to

$$\begin{aligned}\frac{(-1)^{\alpha_n}u(x_n)}{b^n} &+ u(0)(-1)^{\alpha_n-n-1}\left(-\frac{1}{b}\right)^{n+1}\frac{1}{1+\frac{1}{b}}\\ &= \frac{(-1)^{\alpha_n}u(x_n-1)}{b^n} + u(a-1)(-1)^{\alpha_n}\frac{1}{b^{n+1}}\frac{1}{1-\frac{1}{b}},\end{aligned}$$

or

$$u(x_n) + \frac{u(0)}{b+1} = u(x_n-1) + \frac{u(a-1)}{b-1};$$

i.e.,

$$u(x_n) = u(x_n-1) + \frac{u(a-1)}{b-1} - \frac{u(0)}{b+1}.$$

Then

$$u(2) = u(1) + \frac{u(a-1)}{b-1} - \frac{u(0)}{b+1},$$

and

$$u(3) = u(2) + \frac{u(a-1)}{b-1} - \frac{u(0)}{b+1} = u(1) + 2\left[\frac{u(a-1)}{b-1} - \frac{u(0)}{b+1}\right].$$

Repeating this process, we obtain

$$\begin{aligned}u(a-1) &= u(1) + (a-2)\left[\frac{u(a-1)}{b-1} - \frac{u(0)}{b+1}\right]\\ &= -\frac{u(a-1)}{b-1} + \frac{b\,u(0)}{b+1} + (a-2)\left[\frac{u(a-1)}{b-1} - \frac{u(0)}{b+1}\right].\end{aligned}$$

Therefore,

$$\frac{b-a+2}{b-1}u(a-1) = \frac{b-a+2}{b+1}u(0).$$

Combining the results above, we conclude that if f is well defined, then all three conditions of (3) are satisfied.

Conversely, suppose all conditions of (3) are satisfied. Then the proof above indicates that (5) is true, and therefore, f is well defined. □

Next we take a further look at the functions satisfying (3).

Theorem 3. *Let f be defined by* (2) *and satisfy all conditions of* (3)*. Then f is a constant if and only if*

$$\frac{u(a-1)}{b-1} = \frac{u(0)}{b+1}.$$

PROOF. Suppose f is a constant. Then $f(0) = f(1)$. Write

$$1 = \frac{a-1}{a} + \frac{a-1}{a^2} + \frac{a-1}{a^3} + \cdots .$$

Then

$$\frac{u(0)}{b} - \frac{u(0)}{b^2} + \frac{u(0)}{b^3} - \cdots = \frac{u(a-1)}{b} + \frac{u(a-1)}{b^2} + \frac{u(a-1)}{b^3} + \cdots .$$

Hence

$$\frac{u(0)}{b}\frac{1}{1+\frac{1}{b}} = \frac{u(a-1)}{b}\frac{1}{1-\frac{1}{b}},$$

so that

$$\frac{u(0)}{b+1} = \frac{u(a-1)}{b-1}.$$

Conversely, suppose

$$\frac{u(0)}{b+1} = \frac{u(a-1)}{b-1}.$$

Substituting this equation into the equations of (3), we have

$$u(1) = u(2) = \cdots = u(a-1) = \frac{b-1}{b+1}u(0).$$

For any $x \in [0,1]$, let $x = 0.x_1x_2\cdots x_n\cdots$, and let

$$c_k = \frac{(-1)^{\alpha_k}u(x_k)}{b^k}.$$

If $x_k = 0$, then

$$c_k = \frac{(-1)^{\alpha_k}u(0)}{b^k} = \frac{u(0)}{b+1}\frac{(-1)^{\alpha_k}(b+1)}{b^k} = \frac{u(0)}{b+1}\left[\frac{(-1)^{\alpha_k}}{b^{k-1}} - \frac{(-1)^{\alpha_{k+1}}}{b^k}\right].$$

If $x_k \neq 0$, then

$$c_k = \frac{u(0)}{b+1}\frac{(-1)^{\alpha_k}(b-1)}{b^k} = \frac{u(0)}{b+1}\left[\frac{(-1)^{\alpha_k}}{b^{k-1}} - \frac{(-1)^{\alpha_{k+1}}}{b^k}\right].$$

Therefore,

$$S_n(x) = \sum_{k=1}^{n} c_k$$

is a telescoping sum. Since

$$c_1 = \frac{u(0)}{b+1}\left[1 - \frac{(-1)^{\alpha_2}}{b}\right],$$

$$S_n(x) = \frac{u(0)}{b+1}\left[1 - \frac{(-1)^{\alpha_{n+1}}}{b^n}\right] \to \frac{u(0)}{b+1}$$

as $n \to \infty$. This shows that

$$f(x) = \frac{u(0)}{b+1};$$

i.e., f is a constant. □

If a function f given by (2) is well defined, i.e., satisfies all conditions of (3), then it is a constant if $b \neq a-2$ by the third equation of (3) and Theorem 3. So in order for a function given by (2) to be well defined and non-constant, it is necessary that $b = a-2$. If $b = a-2$, then the third equation of (3) is always true, hence for the function to be well defined, only the first two equations of (3) need to hold. Since $b \geq 2$ is an integer, $a = b+2$ needs to be an integer greater than or equal to 4. Therefore, we have the following result on the construction of the generalized Kiesswetter's functions.

Theorem 4. *If $a \geq 4$ is an integer, $b = a-2$ and*

$$\frac{u(a-1)}{b-1} \neq \frac{u(0)}{b+1},$$

then the function given by (2) satisfying the first two conditions of (3) is well defined and non-constant. If $a = 4$, $b = 2$, $u(0) = 0$ and $u(3) = 1$, then the function becomes the original Kiesswetter's function given by (1).

In the next two theorems, we show that the generalized Kiesswetter's functions are continuous everywhere but differentiable nowhere on $[0,1]$.

Theorem 5. *Let f be a function defined in Theorem 4. Then*

$$|f(x) - f(y)| \leq C\,|x-y|^{\ln b/\ln a} \tag{6}$$

for some positive constant C and all $x, y \in [0,1]$. In particular, f is continuous on $[0,1]$.

PROOF. We only need to prove the inequality for $x \neq y$. We may assume $y > x$. Then there is some nonnegative integer n such that

$$\frac{1}{a^{n+1}} \leq y - x \leq \frac{1}{a^n}.$$

Hence,

$$|x-y|^{\ln b/\ln a} \geq \frac{1}{(a^{n+1})^{\ln b/\ln a}} = \frac{1}{e^{(n+1)(\ln a)(\ln b/\ln a)}} = \frac{1}{e^{(n+1)\ln b}} = \frac{1}{b^{n+1}}.$$

Let $[0,1]$ be partitioned into a^n subintervals of equal length of $1/a^n$, and let t_0, t_1, …, and t_{a^n} be the endpoints of these subintervals. Then both x and y either lie in the same subinterval or in adjacent subintervals.

Suppose x and y lie in the same subinterval. There exists t_m for some $m = 0, 1, \ldots, a^n - 1$ such that

$$t_m \leq x < y \leq t_m + \frac{1}{a^n}.$$

Thus,

$$x = t_m + \sum_{k=n+1}^{\infty} \frac{x_k}{a^k} \qquad \text{and} \qquad y = t_m + \sum_{k=n+1}^{\infty} \frac{y_k}{a^k},$$

which leads to

$$f(x) - f(y) = \sum_{k=n+1}^{\infty} \frac{(-1)^{\alpha_k} u(x_k)}{b^k} - \sum_{k=n+1}^{\infty} \frac{(-1)^{\alpha'_k} u(y_k)}{b^k}.$$

Let $M = \max\{|u(0)|, |u(1)|, \ldots, |u(a-1)|\}$. Then

$$|f(x) - f(y)| \leq 2M \sum_{k=n+1}^{\infty} \frac{1}{b^k} = \frac{2M}{b^n(b-1)} \leq \frac{2Mb}{b-1}|x-y|^{\ln b/\ln a}.$$

If x and y lie in the adjacent subintervals, then the argument above and the triangle inequality easily imply that

$$|f(x) - f(y)| \leq \frac{4Mb}{b-1}|x-y|^{\ln b/\ln a}.$$

This completes the proof of the theorem. □

Theorem 6. *Let f be a function defined in Theorem 4. Then f is differentiable nowhere on $[0,1]$.*

Proof. Suppose a base-a expansion of x does not end with all 0's (or equivalently with all $(a-1)$'s). Let

$$x = \sum_{k=1}^{\infty} \frac{x_k}{a^k} = 0.x_1x_2\cdots .$$

We construct two sequences $\{a_n\}$ and $\{b_n\}$ as follows:

$$a_n = \sum_{k=1}^{n} \frac{x_k}{a^k} = 0.x_1x_2\cdots x_n,$$

$$b_n = \sum_{k=1}^{n} \frac{x_k}{a^k} + \sum_{k=n+1}^{\infty} \frac{a-1}{a^k} = 0.x_1x_2\cdots x_n(a-1)(a-1)\cdots .$$

Then $a_n < x < b_n$, $a_n \to x$, $b_n \to x$ and

$$0 < b_n - a_n = \sum_{k=n+1}^{\infty} \frac{a-1}{a^k} = \frac{1}{a^n}.$$

From

$$f(a_n) = \sum_{k=1}^{n} \frac{(-1)^{\alpha_k} u(x_k)}{b^k} + \sum_{k=n+1}^{\infty} \frac{(-1)^{\alpha_{n+1}+k-n-1} u(0)}{b^k},$$

$$f(b_n) = \sum_{k=1}^{n} \frac{(-1)^{\alpha_k} u(x_k)}{b^k} + \sum_{k=n+1}^{\infty} \frac{(-1)^{\alpha_{n+1}} u(a-1)}{b^k},$$

we have

$$|f(b_n) - f(a_n)| = \left| u(a-1) \sum_{k=n+1}^{\infty} \frac{1}{b^k} - u(0) \sum_{k=n+1}^{\infty} \frac{(-1)^{k-n-1}}{b^k} \right|.$$

A simple computation gives

$$|f(b_n) - f(a_n)| = \left| u(a-1)\frac{1}{b^n(b-1)} - u(0)\frac{1}{b^n(b+1)} \right| = \frac{C}{b^n},$$

where by the hypothesis of Theorem 4,

$$C = \left| \frac{u(a-1)}{b-1} - \frac{u(0)}{b+1} \right| > 0.$$

Thus,

$$\left|\frac{f(b_n)-f(a_n)}{b_n-a_n}\right| = C\left(\frac{a}{b}\right)^n = C\left(\frac{b+2}{b}\right)^n \to \infty \tag{7}$$

as $n\to\infty$. It easily follows that either

$$\frac{f(b_n)-f(x)}{b_n-x} \quad \text{or} \quad \frac{f(a_n)-f(x)}{a_n-x}$$

is unbounded as $n\to\infty$. Therefore, f is not differentiable at x.

A similar argument shows that f is not differentiable at x if a base-a expansion of x ends with all 0's. □

3 An equivalent form

In this section, we describe an equivalent form for the generalized Kiesswetter's functions using affine transformations. Let $a \geq 4$ be an integer, and let $u(i)$ $(i = 0, 1, \ldots, a-1)$ be real numbers that satisfy the conditions in Theorems 2 and 4.

Let

$$v(i) = \begin{cases} -1, & i = 0, \\ 1, & i = 1, 2, \ldots, a-1. \end{cases}$$

Let F_i $(i = 1, 2, \ldots, a)$ be affine transformations in $\mathbb{R}^2$ defined by

$$F_i\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1/a & 0 \\ 0 & v(i-1)/(a-2) \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} (i-1)/a \\ u(i-1)/(a-2) \end{bmatrix}.$$

It is easy to see that each transformation F_i maps the graph of a function on an interval I onto the graph of another function on the interval $(1/a)I + (i-1)/a$.

Let $G(g, I)$ denote the graph of a function g on I; i.e.,

$$G(g, I) = \{(x, g(x)) : \ x \in I\}.$$

Definition 1. *Let g_0 denote a bounded function on $[0,1]$. For $n = 1, 2, \ldots$, let g_n be a function on $[0,1]$ defined as follows. For $i = 1, 2, 3, \ldots, a-1$,*

$$G\left(g_n, \left[\frac{i-1}{a}, \frac{i}{a}\right)\right) = F_i(G(g_{n-1}, [0,1))),$$

and

$$G\left(g_n, \left[\frac{a-1}{a}, 1\right]\right) = F_n(G(g_{n-1}, [0,1])).$$

The functions g_n defined above may have discontinuities in $[0, 1]$. We have the following recurrence relation for g_n.

Lemma 7. *Let $g_n(x)$ be given in the definition above. Then for any positive integer n,*

$$g_n(x) = \frac{v(i-1)}{a-2}\, g_{n-1}(ax - i + 1) + \frac{u(i-1)}{a-2}, \qquad \frac{i-1}{a} \le x < \frac{i}{a},$$

for $i = 1, 2, 3, \ldots, a-1$, and

$$g_n(x) = \frac{1}{a-2}\, g_{n-1}(ax - a + 1) + \frac{u(a-1)}{a-2}, \qquad \frac{a-1}{a} \le x \le 1.$$

Proof. For $i = 1, 2, \ldots, a-1$, the transformation F_i shrinks the graph of g_{n-1} over $[0, 1)$ horizontally by a ratio of $1/a$ and vertically by a ratio of $v(i-1)/(a-2)$. Then it moves that graph to the right by $(i-1)/a$ and up by $u(i-1)/(a-2)$. The resulting graph is placed on the interval $[(i-1)/a, i/a)$. Thus,

$$\begin{aligned} g_n(x) &= \frac{v(i-1)}{a-2} g_{n-1}\left(a\left(x - \frac{i-1}{a}\right)\right) + \frac{u(i-1)}{a-2} \\ &= \frac{v(i-1)}{a-2} g_{n-1}(ax - i + 1) + \frac{u(i-1)}{a-2} \end{aligned}$$

for $(i-1)/a \le x < i/a$. The same argument holds for the case $i = a$. $\square$

Now we establish the following equivalence result.

Theorem 8. *Let f be a function defined in Theorem 4, let g_0 be any bounded function on $[0, 1]$, and let g_n be given in the definition above. Then $f(x) = \lim_{n\to\infty} g_n(x)$ on $[0, 1]$.*

Proof. Let $b = a - 2$. If $x = 1$, then by Lemma 7,

$$g_n(1) = \frac{g_{n-1}(1)}{b} + \frac{u(a-1)}{b}.$$

Since

$$g_{n-1}(1) = \frac{g_{n-2}(1)}{b} + \frac{u(a-1)}{b},$$

we have

$$g_n(1) = \frac{g_{n-2}(1)}{b^2} + \frac{u(a-1)}{b^2} + \frac{u(a-1)}{b}.$$

Continuing this, we obtain

$$g_n(1) = \frac{g_0(1)}{b^n} + \frac{u(a-1)}{b^n} + \frac{u(a-1)}{b^{n-1}} + \cdots + \frac{u(a-1)}{b}.$$

Since g_0 is bounded and $1/b^n \to 0$ as $n \to \infty$, we conclude

$$\lim_{n\to\infty} g_n(1) = \sum_{k=1}^{\infty} \frac{u(a-1)}{b^k} = f(1).$$

We next consider $0 \le x < 1$. Since x has possibly two expansions, we let $x = 0.x_1x_2\cdots x_n\cdots$ denote the expansion that does not end with all $(a-1)$'s. (If an expansion of x ends with all $(a-1)$'s, then it has an expansion that ends with all 0's by Lemma 1.) From

$$x_1 \le a \cdot x = x_1 + 0.x_2x_3\cdots < x_1 + 1,$$

we get

$$\frac{x_1}{a} \le x < \frac{x_1+1}{a}.$$

In the recursive formulas in Lemma 7, let $i = x_1 + 1$, and we obtain

$$g_n(x) = \frac{v(x_1)}{b} g_{n-1}(ax - x_1) + \frac{u(x_1)}{b} = \frac{v(x_1)}{b} g_{n-1}(0.x_2x_3\cdots) + \frac{u(x_1)}{b}.$$

Similarly,

$$\begin{aligned} g_n(x) &= \frac{v(x_1)}{b}\left[\frac{v(x_2)}{b} g_{n-2}(0.x_3x_4\cdots) + \frac{u(x_2)}{b}\right] + \frac{u(x_1)}{b} \\ &= \frac{v(x_1)v(x_2)}{b^2} g_{n-2}(0.x_3x_4\cdots) + \frac{v(x_1)u(x_2)}{b^2} + \frac{u(x_1)}{b}. \end{aligned}$$

Repeating this process, we have

$$\begin{aligned} g_n(x) = {} & \frac{v(x_1)v(x_2)\cdots v(x_n)}{b^n} g_0(0.x_{n+1}x_{n+2}\cdots) \\ & + \frac{v(x_1)v(x_2)\cdots v(x_{n-1})}{b^n} u(x_n) \\ & + \frac{v(x_1)v(x_2)\cdots v(x_{n-2})}{b^{n-1}} u(x_{n-1}) + \cdots + \frac{u(x_1)}{b}. \end{aligned}$$

By the definition of $v(i)$, we have

$$v(x_1)v(x_2)\ldots v(x_n) = (-1)^{\alpha_{n+1}},$$

and thus,

$$\begin{aligned} g_n(x) &= \frac{(-1)^{\alpha_{n+1}}}{b^n} g_0(0.x_{n+1}x_{n+2}\cdots) + \frac{(-1)^{\alpha_n} u(x_n)}{b^n} \\ &\quad + \frac{(-1)^{\alpha_{n-1}} u(x_{n-1})}{b^{n-1}} + \cdots + \frac{u(x_1)}{b} \\ &= \frac{(-1)^{\alpha_{n+1}}}{b^n} g_0(0.x_{n+1}x_{n+2}\cdots) + \sum_{k=1}^{n} \frac{(-1)^{\alpha_k} u(x_k)}{b^k}. \end{aligned}$$

Therefore,

$$\lim_{n\to\infty} g_n(x) = \sum_{k=1}^{\infty} \frac{(-1)^{\alpha_k} u(x_k)}{b^k} = f(x).$$

This completes the proof of the theorem. □

We make a comment here. Let X denote the complete metric space consisting of all bounded functions on $[0,1]$ with the metric

$$d(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|.$$

Let φ denote the map from X into X defined by

$$\varphi(g)(x) = \frac{v(i-1)}{a-2} g(ax - i + 1) + \frac{u(i-1)}{a-2}, \qquad \frac{i-1}{a} \le x < \frac{i}{a},$$

for $i = 1, 2, \ldots, a-1$, and

$$\varphi(g)(x) = \frac{1}{a-2} g(ax - a + 1) + \frac{u(a-1)}{a-2}, \qquad \frac{a-1}{a} \le x \le 1.$$

Then it is easy to see

$$d(\varphi(f), \varphi(g)) \le \frac{1}{a-2}\, d(f,g).$$

Thus, φ is a contraction. The Contraction Mapping Principle claims that there is a unique $f \in X$ such that $\varphi(f) = f$; i.e., the graph of f is invariant under φ. This function is one of the generalized Kiesswetter's functions.

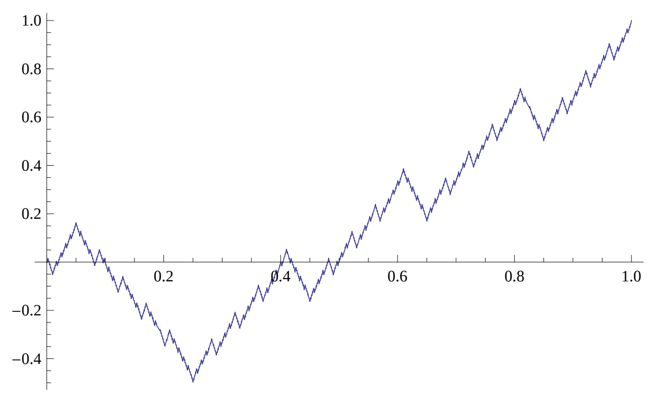
Theorem 8 provides a way of visualizing the graph of f from the graph of g_n for large n. We consider two special cases. For simplicity, we let $g_0(x) = x$.

First we let $a = 5$, $u(0) = 0$ and $u(4) = 2$. Then by Theorem 2, $u(1) = -1$, $u(2) = 0$, $u(3) = 1$. The recursive formulas for g_n that are given in Lemma 7 are

$$g_n(x) = \begin{cases} -(1/3)g_{n-1}(5x), & 0 \le x < 1/5, \\ (1/3)g_{n-1}(5x-1) - 1/3, & 1/5 \le x < 2/5, \\ (1/3)g_{n-1}(5x-2), & 2/5 \le x < 3/5, \\ (1/3)g_{n-1}(5x-3) + 1/3, & 3/5 \le x < 4/5, \\ (1/3)g_{n-1}(5x-4) + 2/3, & 4/5 \le x \le 1. \end{cases}$$

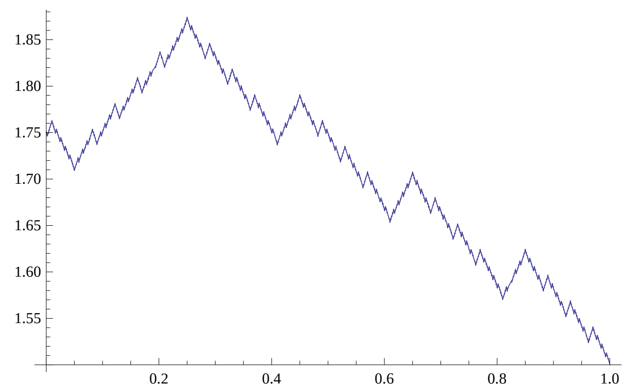
We use the recursive formulas above and *Mathematica* to graph g_{100} (see Figure 1).

Figure 1: The graph of g_{100} for case $a = 5$, $u(0) = 0$, and $u(4) = 2$



Let us consider another case where $a = 5$, $u(0) = 7$, and $u(4) = 3$. By Theorem 2, $u(1) = 15/4$, $u(2) = 7/2$, $u(3) = 13/4$. Using the recursive formulas given in Lemma 7, we obtain the graph of g_{100} (see Figure 2).

Figure 2: The graph of g_{100} for case $a = 5$, $u(0) = 7$, and $u(4) = 3$



References

[1] P. C. Allaart and K. Kawamura, *The Takagi function: a survey*, Real Anal. Exchange, **37** (2011), 1–54.

[2] A. Bruckner, *Differentiation of Real Functions*, CRM Monograph Series, Amer. Math. Soc. and Centre de Recherches Mathématiques, 1994.

[3] G. A. Edgar, *Kiesswetter's fractal has Hausdorff dimension 3/2*, Real Anal. Exchange, **14** (1989), 215–223.

[4] G. A. Edgar, *Measure, Topology, and Fractal Geometry*, Springer-Verlag, New York, 1990.

[5] G. A. Edgar, *Classics on Fractals*, Addison-Wesley, Massachusetts, 1993.

[6] B. Gelbaum and J. Olmsted, *Counterexamples in Analysis*, Holden Day, Amsterdam, 1964.

[7] A. B. Kharazishvili, *Strange Functions in Real Analysis*, Chapman and Hall/CRC, New York, 2006.

[8] K. Kiesswetter, *Ein einfaches Beispiel fur eine Funktion, welche uberall stetig und nicht differenzierbar ist*, Math. Phys. Semesterber., **13** (1966), 216–221 (German).

[9] K. G. Spurrior, *Continuous Nowhere Differentiable Functions*, South Carolina Honors College, 2004.

[10] T. Takagi, *A simple example of the continuous function without derivative*, Proc. Math. Phys. Soc. Tokyo, Ser. II, **1** (1903), 176–177.

[11] J. Thim, *A Continuous Nowhere Differentiable Function*, Master Thesis, Department of Mathematics, Lulea University of Technology, 2003.