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HAUSDORFF AND PACKING MEASURES OF BALANCED CANTOR SETS

Abstract

We estimate the h-Hausdorff and h-packing measures of balanced Cantor sets, and characterize the corresponding dimension partitions. This generalizes results known for Cantor sets associated with positive decreasing summable sequences and central Cantor sets.

1 Introduction

The size of a non-empty set $E \subset \mathbb{R}$ can be characterized by its Hausdorff dimension or packing dimension, although the corresponding Hausdorff or packing measure of the set can be zero or infinity. In such a case, it is desirable to have a more refined description of the dimension. The more general notions of h-Hausdorff measure and h-packing measure were already considered by Hausdorff ([7], or see [8]) and Tricot ([9, 10]), with the power functions x^α replaced by more general dimension functions h. If there exists a function h such that $0 < H^h(E) \leq P^h(E) < \infty$, the set E is said to be h-regular, and h provides a more precise description of the dimension of E.

If E is a central Cantor set with ratio of dissection r_k at step k, then it is easy to see that E is h-regular if and only if $h(r_1 \cdots r_k) \equiv \frac{1}{2^k}$. The set of all dimension functions h which make a set h-regular was similarly characterized for the Cantor sets associated with positive decreasing summable sequences in

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[4] (called decreasing Cantor sets here). In this paper, we extend these results to a broader class of Cantor sets that we call balanced. In particular, we show that the balanced Cantor sets attain the maximal Hausdorff dimension within the collection of all cut-out sets associated with a given sequence.

2 Dimension functions and measures

A function h is said to be doubling if there exists $\tau > 0$ such that $h(2x) \leq \tau h(x)$ for all x. A function $h : [0, A) \to [0, \infty)$ is called a dimension function (or a gauge function) if h is continuous, increasing, doubling and $h(0) = 0$. Let $\mathbb{D}$ be the set of dimension functions. The power functions $h(x) = x^\alpha, \alpha > 0$, are typical examples of dimension functions.

The diameter of any set $A \subset \mathbb{R}$ is denoted by $|A|$. Let $h \in \mathbb{D}$. The h-Hausdorff measure of a set $E \subset \mathbb{R}$ is defined to be

$$H^h(E) := \lim_{\delta \to 0^+} \inf \left\{ \sum_{i=1}^{\infty} h(|E_i|) : E \subset \bigcup_{i=1}^{\infty} E_i, |E_i| \leq \delta \right\}.$$

A δ-packing of a set E is a countable, disjoint family of open balls $\{B_i\}_i$ centred at points in E with $|B_i| \leq \delta$. The h-packing pre-measure of E is

$$P_0^h(E) := \lim_{\delta \to 0^+} \sup \left\{ \sum_{i=1}^{\infty} h(|B_i|) : \{B_i\}_{i=1}^{\infty} \text{ is a } \delta\text{-packing of } E \right\},$$

and the h-packing measure of E is

$$P^h(E) := \inf \left\{ \sum_{i=1}^{\infty} P_0^h(E_i) : E = \bigcup_{i=1}^{\infty} E_i \right\}.$$

If $h(x) = h_\alpha(x) = x^\alpha$ for some $\alpha \geq 0$, this is the usual Hausdorff measure $H^\alpha(E)$, packing pre-measure $P_0^\alpha(E)$ and packing measure $P^\alpha(E)$.

When $h \in \mathbb{D}$, it is proved in [10] that

$$H^h(E) \leq P^h(E) \leq P_0^h(E)$$

for $E \subset \mathbb{R}$. A set E is called h-regular if $0 < H^h(E) \leq P^h(E) < \infty$ and α-regular if $0 < H^\alpha(E) \leq P^\alpha(E) < \infty$. In such cases we also call E an h-set or an α-set, respectively. If E is an α-set, then α is the Hausdorff and packing dimension of E.

3 Balanced Cantor sets

By a Cantor set, C, we mean a subset of $\mathbb{R}$ of Lebesgue measure 0 that is totally disconnected, compact and perfect. It will have the form

$$C = I \setminus \bigcup_{i=1}^{\infty} A_i,$$

where I is a closed and bounded interval and $\{A_i\}$ is a sequence of disjoint open subintervals $A_i \subset I$ with $|I| = \sum_{i=1}^{\infty} |A_i|$.

Next, we introduce a symbol space W. For each integer $k \geq 1$, let $n_k \geq 2$. Let $D_0 := \{e\}, D_k := \{w_1 \cdots w_k : 0 \leq w_l \leq n_l - 1 \text{ for } 1 \leq l \leq k\}$. Let

$$W := \bigcup_{k=0}^{\infty} D_k$$

be the set of all words with finite length. W is called a symbol space. If $w = w_1 \cdots w_k \in W$, its length is denoted as $|w| = k$.

If we fix a symbol space W, we can always obtain a representation of C corresponding to W, by which we mean we can find closed intervals I_w for $w \in D_k$ such that

$$C = \bigcap_{k=1}^{\infty} \bigcup_{w \in D_k} I_w. \tag{1}$$

One way to do this is as follows. Let $I_e := I$. For each $k \geq 1$ and $w \in W$ of length $|w| = k - 1$, we can find $n_k - 1$ largest gaps, $G_{w,i}$, in each $I_w \setminus C$ by the total disconnectedness and perfectness of C. Since C is perfect, the endpoints of the gaps will not touch one another and $I_w \setminus \bigcup_i G_{w,i}$ gives n_k closed subintervals I_{wj} of I_w. Inductively, we obtain a family of closed intervals $\mathcal{F} := \{I_w : w \in W\}$ with property (1). Of course there can be many other choices for the intervals I_w.

If $|w| = k$, I_w is called a Cantor interval of level k. Denote the number of intervals at level k by $N_k = |D_k| = n_1 \cdots n_k$ and the average length of Cantor intervals at level k by

$$s_k = \frac{1}{N_k} \sum_{w \in D_k} |I_w|.$$

Since $w \in D_k$ can be mapped bijectively to $1 \leq j \leq N_k$, we also label I_w as I_j^k, $j = 1, \cdots, N_k$ and with I_j^k placed to the left of I_{j+1}^k. We will use both notations interchangeably. The Cantor set then also has the form

$$C = \bigcap_{k=1}^{\infty} \bigcup_{j=1}^{N_k} I_j^k. \tag{2}$$

We identify a subcollection of the Cantor sets which satisfies a certain balancing property in this representation. The collection of Cantor sets under consideration will include all the central Cantor sets and the decreasing Cantor sets. In the rest of the paper, we assume $M := \sup_k n_k < \infty$.

Definition 1. Let W be a symbol space. A Cantor set C is said to be **W-balanced** if C has a W-representation as in (2), with the associated Cantor intervals satisfying the property that there exist some $K \geq 1$ and $L_1, L_2 \geq 0$ such that

$$s_{k+L_1} \leq |I_j^k| \leq s_{k-L_2}$$

for any $k \geq K$ and $1 \leq j \leq N_k$. C will be called **balanced** if it is W-balanced for some symbol space W. Let $\mathscr{C}$ denote the collection of all balanced Cantor sets.

Example 1 (Central Cantor sets and homogeneous Cantor sets)**.** Let $n_k = 2$ and $0 < r_k \leq b < \frac{1}{2}$ for all k. For each interval I_w of level $k-1$, let I_{w0} and I_{w1} be the left and right intervals of level k obtained by removing a centred, open interval from I_w so that $|I_{w0}| = |I_{w1}| = |I_w| r_k$. I_{w0} and I_{w1} share, respectively, the left and right endpoints with I_w. The Cantor set C formed is called a **central Cantor set** and has a representation corresponding to $W = \bigcup_{k=0}^{\infty} D_k$, where $D_k = \{0,1\}^k$.

More generally, let $n_k \geq 2$, $0 < r_k$ and $n_k r_k \leq b < 1$ for all k. For each interval I_w of level $k-1$, let I_{wj}, $0 \leq j \leq n_k - 1$, be n_k subintervals of equal length in I_w so that $|I_{w0}| = \cdots = |I_{w(n_k-1)}| = |I_w| r_k$. I_{w0} and $I_{w(n_k-1)}$ share, respectively, the left and right endpoints with I_w. Moreover, we require the subintervals to be equally spaced; i.e., the gap lengths between adjacent subintervals I_{wj} and $I_{w(j+1)}$ are all the same. The Cantor set formed is called a **homogeneous Cantor set**. An example is $C + C$ where C is the middle fourth Cantor set. Here $I = [0,2]$, $n_k = 3$ and $r_k = \frac{1}{4}$ for all k.

In both cases, the average length of the intervals of level k is

$$s_k = r_1 \cdots r_k = |I_w|$$

for any w with $|w| = k$, so the Cantor set is balanced.

Example 2 (Decreasing Cantor sets)**.** Let $a = (a_i)_{i=1}^{\infty}$ be a positive, decreasing, summable sequence, and let I be a closed interval with $|I| = \sum_{i=1}^{\infty} a_i$. Remove an open interval A_1 of length a_1 from I, leaving two closed non-trivial intervals I_1^1 on the left and I_2^1 on the right with lengths

$$|I_1^1| = \sum_{l=1}^{\infty} \sum_{p=0}^{2^{l-1}-1} a_{2^l+p} \text{ and } |I_2^1| = \sum_{l=1}^{\infty} \sum_{p=2^{l-1}}^{2^l-1} a_{2^l+p}.$$

Recursively, suppose we have constructed $\{I_j^k\}_{1\le j\le 2^k}$ at level k, ordered from left to right. Remove from each interval, I_j^k, an open interval of length a_{2^k+j-1} and obtain two closed intervals $I_{2j-1}^{k+1}, I_{2j}^{k+1}$ of step $k+1$, where

$$|I_{2j-1}^{k+1}| = \sum_{l=0}^{\infty} \sum_{p=(2j-2)2^l}^{(2j-1)2^l-1} a_{2^{l+k+1}+p} \text{ and } |I_{2j}^{k+1}| = \sum_{l=0}^{\infty} \sum_{p=(2j-1)2^l}^{(2j)2^l-1} a_{2^{l+k+1}+p}.$$

The positions of the gaps A_i removed and the intervals I_j^k are uniquely determined. We call the Cantor set

$$C_a := \bigcap_{k=1}^{\infty} \bigcup_{j=1}^{2^k} I_j^k$$

a **decreasing Cantor set**. It also corresponds to the symbol space $W = \bigcup_{k=0}^{\infty}\{0,1\}^k$.

Since $a=(a_i)$ is decreasing, $\{|I_j^k|\}_{(k,j)}$ is lexicographically decreasing. In consequence,

$$s_{k+1} \le |I_1^{k+1}| \le |I_j^k| \le |I_{2^{k-1}}^{k-1}| \le s_{k-1}$$

for all j, and this C is balanced.

Example 3. Let $a=(a_i)_{i=1}^{\infty}$, let I be defined as in Example 2 and let $W = \bigcup_{k=0}^{\infty} D_k$ be an arbitrary symbol space. Recall that $N_k = |D_k| = n_1 \cdots n_k$. Let $\sigma : \mathbb{N} \to \mathbb{N}$ be a permutation of natural numbers such that for all $k \ge 1$, if $N_k \le i \le N_{k+1}-1$, then $N_k \le \sigma(i) \le N_{k+1}-1$. Define a sequence $b=(b_i)$ by $b_i := a_{\sigma(i)}$. At the first step, we remove $n_1 - 1$ open intervals B_i with length b_i, $1 \le i \le n_1 - 1$, from I and obtain n_1 closed intervals $I_j^1, 1 \le j \le n_1$. Repeat as above and define

$$C_b^W := \bigcap_{k=1}^{\infty} \bigcup_{j=1}^{N_k} I_j^k.$$

We can check that $s_{k+1} \le |I_j^k| \le s_{k-1}$ for all j, so C_b^W is balanced as well. If σ is the identity map, then $b=a$; i.e., $b_i = a_i$ for all $i \ge 1$. We call C_a^W a **general decreasing Cantor set**.

4 Hausdorff and packing measure of a balanced Cantor set

First, we estimate the Hausdorff and packing pre-measure of a balanced Cantor set. This generalizes the results in [1] and [3].

Theorem 1. *Let C be a balanced Cantor set with the number of Cantor intervals $N_k = n_1 \cdots n_k$ and the average length s_k at level k. There exist positive constants A, B such that for any $h \in \mathbb{D}$, we have:*

1. $A \liminf_{k\to\infty} N_k h(s_k) \leq H^h(C) \leq B \liminf_{k\to\infty} N_k h(s_k)$,
2. $A \limsup_{k\to\infty} N_k h(s_k) \leq P_0^h(C) \leq B \limsup_{k\to\infty} N_k h(s_k)$.

PROOF. (1) The proof is based on the idea in [1, Lemma 4].

(1) (a) With the balanced property, we have

$$\sum_{j=1}^{N_k} h(|I_j^k|) \leq N_k h(s_{k-L_2}) \leq M^{L_2} N_{k-L_2} h(s_{k-L_2})$$

when k is large enough. The upper bound is obtained by taking the liminf and putting $B := M^{L_2}$.

(1) (b) Let $\lambda = \liminf_{k\to\infty} N_k h(s_k)$. If $\lambda = 0$, then the lower bound is trivial, so assume $\lambda > 0$. For any $\varepsilon > 0$, there exists K_0 such that for any $k \geq K_0$, we have $(1-\varepsilon)\lambda < N_k h(s_k)$ and also $s_{k+L_1} \leq |I_j^k|$ by the balanced property.

Let $0 < \delta < \min_j |I_j^{K_0}|$. Let $\{B_i\}_i$ be a δ-covering of C by open intervals and let $R = \bigcup_i B_i$. As C is compact, we can assume the covering consists of finitely many intervals, say $\{B_i\}_{i=1}^M$. There exists $K > 1$ such that $\bigcup_{j=1}^{N_K} I_j^K \subset R$, by the finite intersection property of a compact set.

We can also assume the intervals B_i in the covering are disjoint. Otherwise, the intersection of two intervals will be open and must contain some gap of the Cantor set. We can then shrink down the intervals to make them disjoint and get a lower estimate of $H^h(C)$.

In order to obtain a further lower bound, we replace each B_i by the smallest possible (single) closed interval, V_i, containing $B_i \cap \bigcup_{j=1}^{N_K} I_j^K$. Then $\sum h(|V_i|) \leq \sum h(|B_i|)$. If $V_i = \emptyset$, then we simply discard it.

Let τ_i be the number of intervals of level K contained in V_i. Then $\tau_i \geq 1$ and $\sum_i \tau_i = N_K = n_1 \cdots n_K$. For each i, let p_i be the non-negative integer such that

$$\frac{N_K}{N_{K-p_i}} \leq \tau_i < \frac{N_K}{N_{K-p_i-1}}. \tag{3}$$

For any integer j, let $Q(j) = \frac{N_K}{N_j}$.

If $p_i = 0$, then $1 \leq \tau_i < n_k$ and V_i contains some I_j^K. In this case, $|I_j^K| \leq |V_i| \leq |B_i| < \delta$, and hence $K > K_0$. Thus,

$$\frac{1}{N_{K-p_i+L_1+1}}(1-\varepsilon)\lambda < \frac{1}{N_{K+L_1}}(1-\varepsilon)\lambda < h(s_{K+L_1}) \leq h(|I_j^K|) \leq h(|V_i|).$$

If $p_i \geq 1$, then $2 \leq Q(K-p_i) \leq \tau_i < Q(K-p_i-1)$. Note that V_i contains at least $Q(K-p_i)$ consecutive intervals of level K and

$$Q(K-p_i) \geq 2Q(K-p_i+1).$$

Consider the level $K-p_i+1$. Each interval $I_j^{K-p_i+1}$ contains $Q(K-p_i+1)$ subintervals of level K. It follows that V_i must contain an interval $I_j^{K-p_i+1}$ for some j, and the length of V_i must be at least $|I_j^{K-p_i+1}|$. So we have

$$|I_j^{K-p_i+1}| \leq |V_i| < \delta.$$

In particular, this forces $K-p_i+1 > K_0$, and hence,

$$\frac{1}{N_{K-p_i+L_1+1}}(1-\varepsilon)\lambda < h(s_{K-p_i+L_1+1}) \leq h(|I_j^{K-p_i+1}|) \leq h(|V_i|) \tag{4}$$

by the balanced property.

Using (3) and (4) we obtain

$$\begin{aligned}\frac{1}{M^{L_1+2}}(1-\varepsilon)\lambda = \frac{(1-\varepsilon)\lambda}{M^{L_1+2}}\sum_i \frac{\tau_i}{N_k} &< (1-\varepsilon)\lambda \sum_i \frac{1}{N_{K-p_i-1}M^{L_1+2}} \\ &\leq (1-\varepsilon)\lambda\sum_i \frac{1}{N_{K-p_i+L_1+1}} < \sum_i h(|V_i|) \leq \sum_i h(|B_i|).\end{aligned}$$

Since $\{B_i\}_i$ is any δ-covering of C and $\varepsilon > 0$ is arbitrary, we get

$$\frac{1}{M^{L_1+2}} \liminf_{k\to\infty} N_k h(s_k) \leq H^h(C).$$

(2) We modify the proof of [3, Theorem 4.2].

(2) (a) Let $d < \limsup_{k\to\infty} N_k h(s_k)$. There exists a subsequence $\{k_p\}_{p\geq 1}$ such that $d < N_{k_p} h(s_{k_p}) \leq N_{k_p} h(|I_j^{k_p-L_1}|)$, where the second inequality follows from the balanced property.

For any $\delta > 0$, take k_p large enough that $|I_j^{k_p-L_1}| < \delta$ for all j. Let us take the family of intervals $\{B_i := B(x_i, r)\}_{i=1}^{N_{k_p-L_1-1}}$, where $r = s_{k_p}/2$ and x_i is the left endpoint of $I_{in}^{k_p-L_1}$, $n = n_{k_p-L_1}$. The balls are centred in C. Since $|I_j^{k_p-L_1}| \geq s_{k_p} > r$ for any j, we have the inclusion $B_i \subset I_i^{k_p-L_1-1}$ and the balls are pairwise disjoint.

As $|B_i| = s_{k_p} < \delta$, this is a δ-packing and

$$\sum_i h(|B_i|) = \sum_{i=1}^{N_{k_p-L_1-1}} h(s_{k_p}) = N_{k_p-L_1-1} h(s_{k_p}) > \frac{d}{M^{L_1+1}}.$$

Thus,

$$\frac{1}{M^{L_1+1}}\limsup_{k\to\infty} N_k h(s_k) \le P_0^h(C).$$

(2) (b) Let $\varepsilon > 0$. There exists k_0 such that

$$\sup_{k\ge k_0} N_k h(s_k) \le \limsup_{k\to\infty} N_k h(s_k) + \varepsilon.$$

Choose δ small enough that $2\delta < |I_j^{k_0+L_2+2}|$ for all j.

Let $\{B_i\}_i$ be a δ-packing of C and take

$$k_i := \min\{k : I_j^k \subset B_i \text{ for some } 1 \le j \le N_k\}.$$

Then $k_i \ge k_0 + L_2 + 2$ and B_i is centred at a point of an interval of level $k_i - 1$, but does not contain the interval. Therefore, $|B_i|/2 < |I_{j_i}^{k_i-1}|$, where $I_{j_i}^{k_i-1}$ is the interval of level $k_i - 1$ containing the center of B_i. As $n_k \ge 2$,

$$|B_i| < 2|I_{j_i}^{k_i-1}| \le n_{k_i-L_2-1} s_{k_i-L_2-1} \le s_{k_i-L_2-2}$$

from the balanced property, and therefore,

$$\sum_i h(|B_i|) \le \sum_i h(s_{k_i-L_2-2}).$$

Let $l_1 < \cdots < l_m$ be the distinct k_i's, and let θ_p be the number of repetitions of l_p; i.e., θ_p is the number of B_i's containing an interval of level l_p, but none of those at level $l_p - 1$. Each ball B_i of the packing associated to l_p contains at least $\frac{N_{l_m}}{N_{l_p}}$ intervals of step l_m. Since $\{B_i\}_i$ is a disjoint family, $\sum_{p=1}^{m-1}\theta_p\frac{N_{l_m}}{N_{l_p}}$ intervals of level l_m are already covered by the B_i's corresponding to $l_1, \cdots, l_{m-1}$. θ_m can only be at most the number of the remaining intervals at level l_m:

$$\theta_m \le N_{l_m} - \sum_{p=1}^{m-1}\theta_p\frac{N_{l_m}}{N_{l_p}} = N_{l_m}\Big(1 - \sum_{p=1}^{m-1}\frac{\theta_p}{N_{l_p}}\Big).$$

This implies $\sum_{p=1}^{m}\frac{\theta_p}{N_{l_p}} \le 1$.

As a result,

$$\begin{aligned}\sum_i h(|B_i|) &\le \sum_i h(s_{k_i-L_2-2}) = \sum_{p=1}^{m}\theta_p h(s_{l_p-L_2-2})\\ &\le M^{L_2+2}\sum_{p=1}^{m}\frac{\theta_p}{N_{l_p}} N_{l_p-L_2-2} h(s_{l_p-L_2-2})\\ &\le M^{L_2+2}(\limsup_{k\to\infty} N_k h(s_k) + \varepsilon),\end{aligned}$$

since $l_p - L_2 - 2 \geq k_0$. Hence,

$$P_0^h(C) \leq M^{L_2+2} \limsup_{k\to\infty} N_k h(s_k).$$

$\square$

In Theorem 1, only the packing pre-measure is estimated, but not the packing measure. In general, we only know that $P^h(E) \leq P_0^h(E)$ for $E \subset \mathbb{R}$, and the strict inequality can happen. However, the packing measure $P^h(C)$ and the packing pre-measure $P_0^h(C)$ are finite and positive simultaneously for a balanced Cantor set C. To prove this, we will make use of the following version of the mass distribution principle.

Lemma 2 ([10, 4])**.** *Let $E \subset \mathbb{R}$. Let μ be a finite regular Borel measure, and let $h \in \mathbb{D}$ be a dimension function. If*

$$\liminf_{r\to 0} \frac{\mu(B(x_0,r))}{h(r)} < c \tag{5}$$

for all $x_0 \in E$, then

$$P^h(E) \geq \frac{\mu(E)}{c}.$$

Theorem 3. *Let C be a balanced Cantor set and $h \in \mathbb{D}$. If $P_0^h(C) = \infty$, then $P^h(C) = \infty$. If $P_0^h(C) > 0$, then $P^h(C) > 0$.*

Proof. Let μ be the uniform Cantor measure defined by $\mu(I_j^k) = \frac{1}{N_k}$. Let $x_0 \in C$ and $r > 0$. The balanced property tells us that there exist L_1, L_2 such that $s_{k+L_1} \leq |I_j^k| \leq s_{k-L_2}$ for large enough k.

Suppose k is the minimal integer such that $B(x_0, r)$ contains an interval of level k. The minimality of k ensures that $B(x_0, r)$ can intersect at most $2n_k$ intervals of level k, which implies $\mu(B(x_0,r)) \leq 2n_k \frac{1}{N_k} = \frac{2}{N_{k-1}}$. Let I_j^k be a level k interval contained in $B(x_0, r)$, so $|I_j^k| \leq 2r$. Since h is doubling, there exists some $\tau > 0$ such that

$$h(s_{k+L_1}) \leq h(|I_j^k|) \leq h(2r) \leq \tau h(r).$$

Then

$$\frac{\mu(B(x_0,r))}{h(r)} \leq \frac{2\tau}{N_{k-1} h(s_{k+L_1})} = \frac{2\tau M^{L_1+1}}{N_{k+L_1} h(s_{k+L_1})},$$

so that

$$c_0 := \liminf_{r\to 0} \frac{\mu(B(x_0,r))}{h(r)} \leq \frac{2\tau M^{L_1+1}}{\limsup_{k\to\infty} N_k h(s_k)}.$$

By the inequalities in Theorem 1, $P_0^h(C) > 0$ implies $\limsup N_k h(s_k) > 0$, while $P_0^h(C) = \infty$ implies $\limsup N_k h(s_k) = \infty$. If $P_0^h(C) > 0$, then $c_0 < \infty$ and $P^h(C) \geq \frac{\mu(C)}{c_0} > 0$ by the lemma. Correspondingly, if $P_0^h(C) = \infty$, then $c_0 = 0$. Hence $P^h(C) \geq \frac{\mu(C)}{c} > 0$ for every $c > 0$ and $P^h(C) = \infty$. □

Corollary 4. *Let C be a balanced Cantor set and $h \in \mathbb{D}$. Then*

1. $P_0^h(C) = 0$ *if and only if* $P^h(C) = 0$,
2. $P_0^h(C) = \infty$ *if and only if* $P^h(C) = \infty$, *and*
3. $0 < P_0^h(C) < \infty$ *if and only if* $0 < P^h(C) < \infty$.

5 Dimension partition of a balanced Cantor set

The dimension partition of a set $E \subset \mathbb{R}$ (as defined in [4]) is a partition of the set $\mathbb{D}$ of dimension functions into six sets, $\mathbb{H}_\beta^E \cap \mathbb{P}_\gamma^E$, for $\beta \leq \gamma \in \{0, 1, \infty\}$, where

$$\mathbb{H}_1^E = \{h \in \mathbb{D} : 0 < H^h(E) < \infty\},\ \mathbb{P}_1^E = \{h \in \mathbb{D} : 0 < P^h(E) < \infty\},$$

and for $\eta = 0, \infty$,

$$\mathbb{H}_\eta^E = \{h \in \mathbb{D} : H^h(E) = \eta\},\ \mathbb{P}_\eta^E = \{h \in \mathbb{D} : P^h(E) = \eta\}.$$

Assume C is a balanced Cantor set with the number of Cantor intervals $N_k = n_1 \cdots n_k$ and the average length s_k at level k. Define $h_C(s_k) = \frac{1}{N_k}$ and extend h_C to a piecewise linear function on $[0, \infty)$ with $h_C(0) := \lim_{x \to 0^+} h_C(x) = 0$. Then h_C can be shown to be a dimension function, and C is h_C-regular by Theorem 1 and Corollary 4. We call h_C an **associated dimension function** of C.

This means that $\mathbb{H}_1^C \cap \mathbb{P}_1^C$ is always non-empty for the balanced Cantor sets. Indeed, we get immediately from Theorem 1 and Corollary 4 the following description of the dimension partition for the balanced Cantor sets, which was previously shown for the decreasing Cantor sets in [4].

Corollary 5. *Let C be a balanced Cantor set with the number of Cantor intervals $N_k = n_1 \cdots n_k$ and the average length s_k at level k. Then*

$$\mathbb{H}_1^C = \{h \in \mathbb{D} : 0 < \liminf_{k\to\infty} N_k h(s_k) < \infty\},\ \mathbb{H}_\beta^C = \{h \in \mathbb{D} : \liminf_{k\to\infty} N_k h(s_k) = \beta\},$$
$$\mathbb{P}_1^C = \{h \in \mathbb{D} : 0 < \limsup_{k\to\infty} N_k h(s_k) < \infty\},\ \mathbb{P}_\beta^C = \{h \in \mathbb{D} : \limsup_{k\to\infty} N_k h(s_k) = \beta\},$$

where $\beta = 0, \infty$.

Corollary 6. *Let C be a balanced Cantor set with the number of Cantor intervals $N_k = n_1 \cdots n_k$ and the average length s_k at level k. Then*

$$\dim_H C = \liminf_{k\to\infty} \frac{-\log N_k}{\log s_k} \text{ and } \dim_P C = \limsup_{k\to\infty} \frac{-\log N_k}{\log s_k}.$$

In fact, we can further describe the dimension partition of a balanced Cantor set in terms of its associated dimension function h_C.

Proposition 7. *Let C be a balanced Cantor set with an associated dimension function h_C and $g \in \mathbb{D}$.*

1. *If $\liminf_{x\to 0^+} \frac{g(x)}{h_C(x)} > 0$ (or finite), then $H^g(C) > 0$ (or $H^g(C) < \infty$). In particular, if $\liminf_{x\to 0^+} \frac{g(x)}{h_C(x)} = \infty$ (or 0), then $H^g(C) = \infty$ (respectively $H^g(C) = 0$).*

2. *If $\limsup_{x\to 0^+} \frac{g(x)}{h_C(x)} > 0$ (or finite), then $P_0^g(C) > 0$ (or $P_0^g(C) < \infty$). In particular, if $\limsup_{x\to 0^+} \frac{g(x)}{h_C(x)} = \infty$ (or 0), then $P_0^g(C) = \infty$ (respectively $P_0^g(C) = 0$).*

Proof. (1) Let $\lambda_* := \liminf_{x\to 0^+} \frac{g(x)}{h_C(x)} > 0$. For any $0 < \alpha < \lambda_*$, there is a $\delta > 0$ such that $g(x) \geq \alpha h_C(x)$ for all $0 < x < \delta$. Then $H^g(C) \geq \alpha H^{h_C}(C) > 0$ by the definition of Hausdorff measure. If $\lambda_* = \infty$, then α can be arbitrarily large, and hence, $H^g(C) = \infty$.

Suppose $\lambda_* < \infty$. For any $\alpha > \lambda_*$, there exists a positive decreasing sequence $\{\delta_m\}_m$ such that $\lim_{m\to\infty} \delta_m = 0$ and $g(\delta_m) \leq \alpha h_C(\delta_m)$. Let $k = k(m)$ be the integer such that $s_k \leq \delta_m < s_{k-1}$. Then $|I_j^{k+L_2}| \leq s_k \leq \delta_m$, and $\{I_j^{k+L_2}\}_j$ is a δ_m-covering of C. Therefore,

$$\begin{aligned} H_{\delta_m}^g(C) &\leq \sum_{j=1}^{N_{k+L_2}} g(|I_j^{k+L_2}|) \leq N_{k+L_2} g(\delta_m) \\ &< N_{k+L_2} \alpha h_C(\delta_m) \leq \alpha M^{L_2+1} N_{k-1} h_C(s_{k-1}) = \alpha M^{L_2+1}. \end{aligned}$$

Taking limits gives

$$H^g(C) \leq \lambda_* M^{L_2+1} < \infty.$$

(2) Let $\lambda^* := \limsup_{x\to 0^+} \frac{g(x)}{h_C(x)} > 0$. For any $0 < \alpha < \lambda^*$, there exists a positive decreasing sequence $\{\delta_m\}_m$ such that $\lim_{m\to\infty} \delta_m = 0$ and $g(\delta_m) \geq \alpha h_C(\delta_m)$. Let k be the integer such that $s_k \leq \delta_m < s_{k-1}$. For $1 \leq j \leq$

N_{k-L_1-2}, take as x_j the left endpoint of the interval $I_{jn}^{k-L_1-1}$, where $n = n_{k-L_1-1}$. The collection $\{B(x_j, \delta_m/2)\}_j$ will be disjoint. Thus,

$$P_{\delta_m}^g(C) \geq \sum_{j=1}^{N_{k-L_1-2}} g(\delta_m) \geq \alpha N_{k-L_1-2} h_C(s_k) \geq \frac{\alpha}{M^{L_1+2}} N_k h_C(s_k),$$

and therefore $P_0^g(C) > 0$.

The proof that $P_0^g(C) < \infty$ if $\limsup_{x\to 0^+} \frac{g(x)}{h_C(x)} < \infty$ is similar to the first part of (1).

□

The following corollary can be compared with the result for the self-similar sets with the open set condition in [11].

Corollary 8. *Let C be a balanced Cantor set with an associated dimension function h_C. Then for $\beta = 0, \infty$,*

$$\mathbb{H}_1^C = \{g \in \mathbb{D} : 0 < \liminf_{x\to 0^+} \frac{g(x)}{h_C(x)} < \infty\},\ \mathbb{H}_\beta^C = \{g \in \mathbb{D} : \liminf_{x\to 0^+} \frac{g(x)}{h_C(x)} = \beta\},$$
$$\mathbb{P}_1^C = \{g \in \mathbb{D} : 0 < \limsup_{x\to 0^+} \frac{g(x)}{h_C(x)} < \infty\},\ \mathbb{P}_\beta^C = \{g \in \mathbb{D} : \limsup_{x\to 0^+} \frac{g(x)}{h_C(x)} = \beta\}.$$

Corollary 9. *Let C_1 and C_2 be balanced Cantor sets with h_{C_1} and h_{C_2} as their respective associated dimension functions. Then the following are equivalent.*

(a) $h_{C_1} \equiv h_{C_2}$; i.e., there are $A, B > 0$ such that $Ah_{C_2}(x) \leq h_{C_1}(x) \leq Bh_{C_2}(x)$ for all small $x > 0$.

(b) $\mathbb{H}_\beta^{C_1} \cap \mathbb{P}_\gamma^{C_1} = \mathbb{H}_\beta^{C_2} \cap \mathbb{P}_\gamma^{C_2}$ for all $\beta \leq \gamma \in \{0, 1, \infty\}$.

(c) $\mathbb{H}_1^{C_1} \cap \mathbb{P}_1^{C_1} = \mathbb{H}_1^{C_2} \cap \mathbb{P}_1^{C_2}$.

Note that C_1 and C_2 can be balanced with respect to different symbol spaces.

Remark 1. Similar arguments to those given in [6] can be used to show that if C_1 and C_2 are both W-balanced, then $h_{C_1} \equiv h_{C_2}$ if and only if there exists an integer L such that $s_{k+L}^{C_2} \leq s_k^{C_1} \leq s_{k-L}^{C_2}$ for all $k > L$. Here $\{s_k^{C_i}\}_k$ are the average interval lengths of C_i.

6 Balanced Cantor sets within the collection of cut-out sets

Let $a = (a_i)$ be a positive, decreasing, summable sequence, $|I| = \sum_{i=1}^{\infty} a_i$, and let $A_i \subset I$ be a sequence of disjoint open subintervals with $|A_i| = a_i$. Then $E := I \setminus \bigcup_{i=1}^{\infty} A_i$ is called a **cut-out set** associated with the sequence $a = (a_i)$. The collection of all these cut-out sets E is denoted by $\mathscr{C}_a$.

It is known that among all the sets in $\mathscr{C}_a$, the decreasing Cantor set C_a associated with the sequence a has the maximal Hausdorff dimension and maximal Hausdorff measure up to a constant [1, 5]. On the other hand, the prepacking dimensions of all the sets in $\mathscr{C}_a$ are the same and equal to the upper box dimension [2]. Since the packing and prepacking dimensions of C_a coincide [4], it follows that $\dim_P E \leq \dim_{P_0} E = \dim_{P_0} C_a = \dim_P C_a$ for any $E \in \mathscr{C}_a$. However, it is shown in [5] that C_a has the least packing premeasure up to a constant among the sets in $\mathscr{C}_a$. In the following we will prove similar results for the balanced Cantor sets in $\mathscr{C}_a$.

For any $E \subset \mathbb{R}$ and $r > 0$, let

$$N(E, r) = \min\{k : E \subset \bigcup_{i=1}^{k} B(x_i, r)\}$$

and

$$P(E, r) = \max\{k : \{B(x_i, r)\}_{1 \leq i \leq k} \text{ is a } 2r\text{-packing of } E\}.$$

They can be compared by following lemma.

Lemma 10 ([5])**.** *For any $E_1, E_2 \in \mathscr{C}_a$ and $r > 0$,*

$$P(E_2, r) \leq 2N(E_1, r) \leq 2P(E_1, r/2) \leq 4N(E_2, r/2).$$

Theorem 11. *Let C be a balanced Cantor set in $\mathscr{C}_a$ for some $a = (a_i)$. If $h \in \mathbb{D}$ and E is any set in $\mathscr{C}_a$, then $H^h(E) \leq A\, H^h(C)$ and $P_0^h(C) \leq B\, P_0^h(E)$ for some constants A and B, which depend only on h and C.*

Proof. Recall that $h \in \mathbb{D}$ is doubling; i.e., $h(2x) \leq \tau h(x)$ for some τ. The previous lemma implies that for any cut-out set $E \in \mathscr{C}_a$, we have

$$H^h(E) \leq \liminf_{r \to 0} N(E, r) h(2r) \leq 2\tau^2 \liminf_{r \to 0} N(C, r) h(r).$$

If $r > 0$ is small, then there exists $k \in \mathbb{N}$ such that $s_k \leq r \leq s_{k-1}$. Consider the Cantor intervals $\{I_j^{k+L_2}\}_{j=1}^{N_{k+L_2}}$ at level $k + L_2$. Take their left endpoints as centres and form N_{k+L_2} balls with radius r (which is at least the length of any Cantor interval of level $k+L_2$). This is an r-covering of C. So $N(C,r) \leq N_{k+L_2}$.

Taking into account Theorem 1, we obtain that for a suitable constant A,

$$\begin{aligned} H^h(E) &\leq 2\tau^2 \liminf_{r\to 0} N(C,r)h(r) \\ &\leq 2\tau^2 M^{L_2+1} \liminf_{k\to\infty} N_k h(s_k) \leq AH^h(C). \end{aligned}$$

For the pre-packing measure, the previous lemma also gives that for any $E \in \mathscr{C}_a$,

$$\frac{1}{2}P(C,r)h(r) \leq P(E,r/2)h(r) \leq P_r^h(E).$$

As above, let $r>0$ be small and take $k\in\mathbb{N}$ such that $s_{k+1}\leq r\leq s_k$. Take the subset of Cantor intervals $\{I_{jn}^{k-L_1}\}_{j=1}^{N_{k-L_1-1}}$ at level $k-L_1$, where $n=n_{k-L_1}$. Take their left endpoints as centres and form N_{k-L_1-1} balls with radii r. Then $N_{k-L_1-1}\leq P(C,r)$ and again, applying Theorem 1, we deduce that there are constants B_1, B_2, B such that

$$P_0^h(C) \leq B_1 \limsup_{k\to\infty} N_k h(s_k) \leq B_2 \limsup_{r\to 0} P(C,r)h(r) \leq BP_0^h(E).$$

□

Corollary 12. *Let C be a balanced Cantor set in $\mathscr{C}_a$ for some $a=(a_i)$. If $h\in\mathbb{D}$ and E is any set in $\mathscr{C}_a$, then $\dim_H E \leq \dim_H C$.*

PROOF. This is because $H^\alpha(E) \leq A\, H^\alpha(C)$ for any $\alpha \geq 0$. □

Corollary 13. *If $C_1, C_2 \in \mathscr{C}_a$ are both balanced Cantor sets, then there exist positive constants A, B such that for all $h\in\mathbb{D}$,*

1. *$AH^h(C_2) \leq H^h(C_1) \leq BH^h(C_2)$,*
2. *$AP_0^h(C_2) \leq P_0^h(C_1) \leq BP_0^h(C_2)$.*

Hence, $\mathbb{H}_\beta^{C_1} = \mathbb{H}_\beta^{C_2}$ and $\mathbb{P}_\beta^{C_1} = \mathbb{P}_\beta^{C_2}$ for $\beta = 0, 1$ and ∞. In particular, $\dim_H C_1 = \dim_H C_2$ and $\dim_P C_1 = \dim_P C_2$.

It follows that for any balanced Cantor set C in $\mathscr{C}_a$, C and C_a have the same dimension partition. Note that once the sequence $a=(a_i)$ is fixed, we can construct many general decreasing Cantor sets C_a^W as in Example 3, with respect to different symbol spaces W. Since they are balanced Cantor sets, they all have the maximal Hausdorff dimension within the collection $\mathscr{C}_a$, namely $\dim_H C_a$.

Notice, also, that if $E\in\mathscr{C}_a$ and $\dim_H E < \dim_H C_a$, then E is not W-balanced for any symbol space W.

Remark 2. The self-similar Cantor sets with the open set condition are known to be α-regular and have the same dimension partition as the α-regular balanced Cantor sets ([11]). But we do not know whether they are balanced. Moreover, note that the criterion in the last paragraph of not being balanced does not apply in this case. In fact, let $\underline{\dim}_B$ denote the lower box dimension and let E be a self-similar Cantor set. It is well known that $\dim_H E = \underline{\dim}_B E$ ([2]). On the other hand, if a is the sequence of gap lengths of E arranged decreasingly, then $\underline{\dim}_B E = \underline{\dim}_B C_a$, since the lower box dimensions of any pair of sets in $\mathscr{C}_a$ coincide ([2]), and also $\underline{\dim}_B C_a = \dim_H C_a$ ([3]). Therefore, E attains the maximal Hausdorff dimension of the sets in $\mathscr{C}_a$.

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