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BOWEN'S FORMULA FOR SHIFT-GENERATED FINITE CONFORMAL CONSTRUCTIONS

Abstract

We study shift-generated finite conformal constructions; i.e. conformal constructions generated by a general shift (shift of finite type, sofic shift and non-sofic shift alike) over a finite alphabet. These constructions are not restricted to shifts of finite type or sofic shifts as in the classical limit set constructions. In particular, we prove that the limit sets of such constructions satisfy Bowen's formula, which gives the Hausdorff dimension of the limit set as the zero of the topological pressure. We look at several examples, including a one-dimensional construction generated by the so-called context-free shift.

1 Introduction

Finite iterated functions systems (IFSs) have been studied for more than 30 years now (among others, see [6, 2], as well as the textbooks [4, 1]). Of major

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interest is their limit set and its Hausdorff dimension. Bowen's formula characterizes the Hausdorff dimension of the limit set as the zero of the topological pressure function of the system (in a more general context, see also [3]).

Finite IFSs are generated by a full shift over a finite alphabet. Systems generated by subshifts of finite type were later studied (for instance, see [13, 14]). These systems are nowadays called graph directed Markov systems (GDMSs) and have even been examined when generated by countably infinite alphabets (see [9, 10, 16, 12]). It turns out that Bowen's formula holds for every finite conformal GDMS (see [5]). However, there are many IFS- or GDMS-like constructions that are generated by subshifts which are not of finite type. An example is given in this paper of a construction generated by the context free shift (cf. section 4). In this case, the underlying space of the construction is neither of finite type nor sofic (see [7] for a general presentation of shifts).

In section 2 we give a general overview of shift-generated constructions that are just an extension of the standard IFSs to the case where a full shift is replaced by a subshift. In section 3 we first define the topological pressure function for such a construction. Then we prove Bowen's formula by constructing a measure on the symbolic space and pushing it downwards to the phase space. The fact that the shift space is compact permits such a construction. Many techniques from the theory of IFSs and GDMSs will be used as well as many results previously proven. Finally, in section 4 we look at several examples including the one-dimensional conformal construction generated by the context-free shift. In the particular case where the generators are similarities, the topological pressure function of the construction is closely related to the topological entropy of the context-free shift, and thus the Hausdorff dimension of the limit set of the induced constructions can be expressed in terms of the topological entropy of the context-free shift.

2 Preliminaries

Let E be a finite set with at least two elements. Let A be a subshift of the (one-sided) full shift E^∞ on E. This is equivalent to A being shift invariant and closed, and therefore compact. Every $\omega \in A$ is said to be an infinite admissible word. Let $\mathcal{B}_n(A)$ be the set of all subwords of length $n \geq 1$ that appear in words of A. Hence $\mathcal{B}_n(A)$ contains all the admissible words (a.k.a. blocks) of length n. Let $\mathcal{B}(A) = \cup_{n\geq 1}\mathcal{B}_n(A)$ be the set of all finite admissible blocks. For every $\omega \in \mathcal{B}(A)$, we denote by $|\omega|$ the length of ω. For every $\omega \in \mathcal{B}(A) \cup A$ and $1 \leq n \leq |\omega|$, we denote by $\omega|_n$ the word $\omega_1\omega_2 \ldots \omega_n$.

Let X be a non-empty compact metric space. A construction generated by the subshift A is based upon a set of generators $\Phi = \{\varphi_e : X_e \to X\}_{e\in E}$, where the φ_e's are one-to-one contractions and the X_e's are non-empty compact subsets of X such that $\varphi_f(X_f) \subseteq X_e$ whenever $ef \in \mathcal{B}_2(A)$. Let $0 < s < 1$ be such that all these generators have a contraction ratio that does not exceed s. For every $\omega \in \mathcal{B}(A)$, set $X_\omega = X_{\omega_{|\omega|}}$ and

$$\varphi_\omega : X_\omega \to X, \qquad \varphi_\omega := \varphi_{\omega_1} \circ \varphi_{\omega_2} \circ \ldots \circ \varphi_{\omega_{|\omega|}}.$$

Given $\omega \in A$, the compact sets $\varphi_{\omega|_n}(X_{\omega|_n})$, $n \geq 1$, are decreasing and their diameters converge to zero. More precisely,

$$\operatorname{diam}\big(\varphi_{\omega|_n}(X_{\omega|_n})\big) \leq s^n \operatorname{diam}(X).$$

This implies that the set

$$\bigcap_{n\geq 1} \varphi_{\omega|_n}(X_{\omega|_n})$$

is a singleton. We define the coding map $\pi : A \to X$ by

$$\{\pi(\omega)\} = \bigcap_{n\geq 1} \varphi_{\omega|_n}(X_{\omega|_n})$$

and by

$$J_A = \pi(A)$$

the limit set associated to the construction generated by A.

We call a shift-generated construction *conformal* if the following conditions are satisfied:

(1) X_e is a connected compact subset of a Euclidean space $\mathbf{R}^d$ and $X_e = \overline{\operatorname{Int}_{\mathbb{R}^d}(X_e)}$ for every $e \in E$, where d is common to all e.

(2) (Open Set Condition (OSC)) For every $e, f \in E$, $e \neq f$,

$$\varphi_e(\operatorname{Int}(X_e)) \cap \varphi_f(\operatorname{Int}(X_f)) = \emptyset.$$

(3) For every $f \in E$, there exists a connected open set W_f with $X_f \subseteq W_f \subseteq \mathbb{R}^d$ so that the map φ_f extends to a C^1 conformal diffeomorphism of W_f into $\bigcap_{e\in E: ef\in\mathcal{B}_2(A)} W_e$.

(4) There are two constants $L \geq 1$ and $\alpha > 0$ so that

$$\big||\varphi_e'(x)| - |\varphi_e'(y)|\big| \leq L\|(\varphi_e')^{-1}\|^{-1} \cdot |x-y|^\alpha$$

for every $e \in E$ and for every pair of points $x, y \in W_e$, where $|\varphi_e'(x)|$ represents the norm of the derivative.

Remark 1. *If $d \geq 2$ and a construction satisfies conditions (1) and (3), then it also satisfies condition (4) with $\alpha = 1$ according to Proposition 4.2.1 in [11].*

As a straightforward consequence of (4) we get the following:

(4') (Bounded Distortion Property (BDP)) There exists $K \geq 1$ such that for all $\omega \in \mathcal{B}(A)$ and for all $x, y \in W_\omega$,

$$|\varphi'_\omega(y)| \leq K|\varphi'_\omega(x)|.$$

We shall now list some basic geometric consequences of conditions (1)—(4′). They directly extend the properties found on pages 73 and 74 of [11]. We included their proofs for the sake of completeness.

We first obtain a metric upper bound on the size of the image of any convex set, as well as a set-theoretic upper bound on the image of any ball. For every $\omega \in \mathcal{B}(A)$, set $\|\varphi'_\omega\| = \|\varphi'_\omega\|_{X_\omega} = \sup_{x \in X_\omega} |\varphi'_\omega(x)|$ and $\|\varphi'_\omega\|_{W_\omega} = \sup_{x \in W_\omega} |\varphi'_\omega(x)|$.

Lemma 2. *For all $\omega \in \mathcal{B}(A)$ and all convex subsets C of W_ω,*

$$diam(\varphi_\omega(C)) \leq \|\varphi'_\omega\|_{W_\omega} diam(C) \leq K\|\varphi'_\omega\| diam(C). \tag{1}$$

Moreover, for all $\omega \in \mathcal{B}(A)$, all $x \in X_\omega$ and all radii $0 \leq r \leq dist(X_\omega, \partial W_\omega)$,

$$\varphi_\omega(B(x,r)) \subseteq B\big(\varphi_\omega(x), r\|\varphi'_\omega\|_{W_\omega}\big) \subseteq B\big(\varphi_\omega(x), Kr\|\varphi'_\omega\|\big). \tag{2}$$

Proof. These results are simple consequences of the Mean Value Inequality and BDP. □

We now give an upper bound on the size of the images of the sets X_e.

Lemma 3. *For each $0 \leq r \leq \min\{\mathrm{dist}(X_e, \partial W_e) : e \in E\}$, there is a constant $D = D(r) \geq 1$ such that*

$$diam\big(\varphi_\omega(B(X_\omega, r))\big) \leq D\|\varphi'_\omega\|_{W_\omega} \leq KD\|\varphi'_\omega\|, \ \forall\, \omega \in \mathcal{B}(A). \tag{3}$$

In particular, there exists $D = D(0) \geq 1$ such that

$$diam\big(\varphi_\omega(X_\omega)\big) \leq D\|\varphi'_\omega\|_{W_\omega} \leq KD\|\varphi'_\omega\|, \ \forall\, \omega \in \mathcal{B}(A). \tag{4}$$

Proof. Take $0 \leq r < \Delta := \min\{\mathrm{dist}(X_e, \partial W_e) : e \in E\}$ and let $r' = \Delta - r$. Then $\varphi_f(B(X_f, r)) \subseteq B(X_e, sr)$ for all $ef \in \mathcal{B}_2(A)$, which implies that we may take $B(X_e, r)$ as W_e for all $e \in E$. Since the set $\overline{B(X_e, r)}$ is compact and connected, we may cover it by a finite chain of balls $B(x_e^1, r'), \ldots, B(x_e^q, r')$

with centers $x_e^1, \ldots, x_e^q$ in $\overline{B(X_e, r)}$; i.e. $B(x_e^i, r') \cap B(x_e^{i+1}, r') \neq \emptyset$ for each $1 \le i < q$ and $\bigcup_{i=1}^q B(x_e^i, r') \supseteq X_e$. Using (1), we then conclude that

$$\mathrm{diam}\big(\varphi_\omega(B(X_\omega, r))\big) \le q\|\varphi_\omega'\|_{W_\omega} 2r' \le D\|\varphi_\omega'\|_{W_\omega}$$

for all $\omega \in \mathcal{B}(A)$, where $D = 2\Delta q$. □

We shall now obtain lower bounds. First, we establish the counterpart of (2).

Lemma 4. *For all $\omega \in \mathcal{B}(A)$, all $x \in X_\omega$ and all $0 \le r \le dist(X_\omega, \partial W_\omega)$,*

$$\varphi_\omega(B(x, r)) \supseteq B\big(\varphi_\omega(x), K^{-1} r\|\varphi_\omega'\|_{W_\omega}\big). \tag{5}$$

Proof. First, using BDP observe that for any $\omega \in \mathcal{B}(A)$ and $z \in W_\omega$,

$$\big|(\varphi_\omega^{-1})'(\varphi_\omega(z))\big|^{-1} = \Big|\big((\varphi_\omega^{-1})'(\varphi_\omega(z))\big)^{-1}\Big| = |\varphi_\omega'(z)| \ge K^{-1}\|\varphi_\omega'\|_{W_\omega}.$$

So

$$\|(\varphi_\omega^{-1})'\|_{\varphi_\omega(W_\omega)} \le K\|\varphi_\omega'\|_{W_\omega}^{-1}. \tag{6}$$

Now, fix ω, x and r as in the statement. Let $R > 0$ be the maximal radius such that

$$B(\varphi_\omega(x), R) \subseteq \varphi_\omega(B(x, r)). \tag{7}$$

Then $\partial\big(B(\varphi_\omega(x), R)\big) \cap \partial\big(\varphi_\omega(B(x, r))\big) \neq \emptyset$, and in view of the Mean Value Inequality and (6), we have

$$\varphi_\omega^{-1}\big(B(\varphi_\omega(x), R)\big) \subseteq B\big(x, R\|(\varphi_\omega^{-1})'\|_{\varphi_\omega(W_\omega)}\big) \subseteq B\big(x, KR\|\varphi_\omega'\|_{W_\omega}\big),$$

which implies that $B(\varphi_\omega(x), R) \subseteq \varphi_\omega(B(x, KR\|\varphi_\omega'\|_{W_\omega}))$. It ensues from the openness of map φ_ω that $KR\|\varphi_\omega'\|_{W_\omega}^{-1} \ge r$. Using (7), we finally obtain (5). □

We shall now prove the counterpart of (4).

Lemma 5. *There exists a constant $D \ge 1$ such that*

$$diam(\varphi_\omega(X_\omega)) \ge D^{-1}\|\varphi_\omega'\|_{W_\omega}, \ \ \forall\, \omega \in \mathcal{B}(A). \tag{8}$$

Proof. Let $\Delta = \mathrm{dist}(X_\omega, \partial W_\omega)$. Fix $x \in X_\omega$ and $y \in (X_\omega \backslash \{x\}) \cap B(x, K^{-1}\Delta)$. For every $\omega \in \mathcal{B}(A)$ we have by (5) that

$$\varphi_\omega(B(x, \Delta)) \supseteq B\big(\varphi_\omega(x), K^{-1}\Delta\|\varphi_\omega'\|_{W_\omega}\big),$$

while we have by (2) that

$$\varphi_\omega(y) \in B\big(\varphi_\omega(x), K^{-1}\Delta\|\varphi_\omega'\|_{W_\omega}\big).$$

Applying the Mean Value Inequality to φ_ω^{-1} restricted to the convex set $B\big(\varphi_\omega(x), K^{-1}\Delta\|\varphi_\omega'\|_{W_\omega}\big)$ followed by (6), we obtain

$$\begin{aligned}|y-x| &= |\varphi_\omega^{-1}(\varphi_\omega(y)) - \varphi_\omega^{-1}(\varphi_\omega(x))|\\ &\le \|(\varphi_\omega^{-1})'\|_{\varphi_\omega(W_\omega)}|\varphi_\omega(y)-\varphi_\omega(x)|\\ &\le K\|\varphi_\omega'\|_{W_\omega}^{-1}|\varphi_\omega(y)-\varphi_\omega(x)|.\end{aligned}$$

Thus,

$$\mathrm{diam}(\varphi_\omega(X_\omega)) \ge |\varphi_\omega(y)-\varphi_\omega(x)| \ge K^{-1}\|\varphi_\omega'\|_{W_\omega}|y-x|.$$

□

Finally, we make a simple geometric observation which follows from the OSC.

Lemma 6. *For all $0<\kappa_1<\kappa_2<\infty$, all $r>0$ and all $x\in X$, the cardinality of any collection of mutually incomparable words $\omega\in\mathcal{B}(A)$ that satisfy the conditions*

$$\varphi_\omega(X_\omega)\cap B(x,r)\ne\emptyset$$

and

$$\kappa_1 r \le diam(\varphi_\omega(X_\omega)) < \kappa_2 r$$

is bounded above by the number

$$((1+\kappa_2)KD(R\kappa_1)^{-1})^d,$$

where R is the minimum between the radius of the largest ball that can be inscribed in all the sets X_e and $\min\{\mathrm{dist}(X_e,\partial W_e): e\in E\}$.

Proof. Let $V_d=\lambda_d(B(0,1))$ be the Lebesgue measure of the unit ball in $\mathbb{R}^d$. Fix $0<\kappa_1<\kappa_2<\infty$, $r>0$ and $x\in X$. Let W be a collection of incomparable words as described in the statement. Then for every $\omega\in W$, we have

$$\varphi_\omega(X_\omega)\subseteq B\big(x, r+\mathrm{diam}(\varphi_\omega(X_\omega))\big)\subseteq B(x,(1+\kappa_2)r).$$

Since all the sets $\{\varphi_\omega(\mathrm{Int}(X_\omega))\}_{\omega\in W}$ are mutually disjoint by the OSC, using (5) and (4) successively we obtain that

$$\begin{aligned}
(1+\kappa_2)^d r^d V_d &= \lambda_d\big(B(x,(1+\kappa_2)r)\big)\\
&\geq \lambda_d\Big(\bigcup_{\omega\in W}\varphi_\omega(\mathrm{Int}(X_\omega))\Big)\\
&= \sum_{\omega\in W}\lambda_d\big(\varphi_\omega(\mathrm{Int}(X_\omega))\big)\\
&\geq \sum_{\omega\in W}\lambda_d\big(\varphi_\omega(B(x,R))\big)\\
&\geq \sum_{\omega\in W}\lambda_d\big(B\big(\varphi_\omega(x),K^{-1}R\|\varphi'_\omega\|_{W_\omega}\big)\big)\\
&\geq \sum_{\omega\in W}\lambda_d\big(B\big(\varphi_\omega(x),K^{-1}RD^{-1}\mathrm{diam}(\varphi_\omega(X_\omega))\big)\big)\\
&\geq \sum_{\omega\in W}\lambda_d\big(B\big(\varphi_\omega(x),(KD)^{-1}R\kappa_1 r\big)\big)\\
&= \#W((KD)^{-1}R\kappa_1 r)^d V_d.
\end{aligned}$$

Hence

$$\#W \leq ((1+\kappa_2)(KD)(R\kappa_1)^{-1})^d.$$

□

3 Bowen's formula for finite conformal constructions

Next, we define the topological pressure function which will play a central role in studying shift-generated conformal constructions.

Given $t \geq 0$ and $n \geq 1$, we denote the nth-level partition function $Z_{n,A}(t)$ by

$$Z_{n,A}(t) = \sum_{\omega\in\mathcal{B}_n(A)}\|\varphi'_\omega\|^t.$$

For every $t \geq 0$, the sequence $(Z_{n,A}(t))_{n\geq 1}$ is submultiplicative and thus we can define the topological pressure function $P_A(t)$ of the construction by

$$P_A(t) = \lim_{n\to\infty}\frac{1}{n}\log Z_{n,A}(t) = \inf_{n\geq 1}\frac{1}{n}\log Z_{n,A}(t).$$

The topological pressure function $P_A : [0,\infty) \to \mathbb{R}$ is strictly decreasing to negative infinity, convex and hence continuous. Indeed, the strictly decreasing behavior of the pressure can be more precisely described as follows. Let $0 \leq t_1 < t_2$. Then $Z_{n,A}(t_2) \leq s^{n(t_2-t_1)} Z_{n,A}(t_1)$ for all $n \geq 1$. Therefore $P_A(t_2) \leq (t_2-t_1)\log s + P_A(t_1)$. The convexity of the pressure follows from the convexity of its partition functions $Z_{n,A}$. The continuity is a direct consequence of the convexity.

Let h be the unique zero of the topological pressure function. The following proposition affirms that the Hausdorff dimension of the limit set is less than or equal to that zero.

Proposition 7. *Let Φ be a finite conformal construction generated by a subshift A, and let h be the zero of its topological pressure function. Then*

$$HD(J_A) \leq h,$$

where $HD(J_A)$ is the Hausdorff dimension of the limit set J_A. In particular, if $h = 0$, then $HD(J_A) = h = 0$.

Proof. Let $t > h$. Then $P_A(t) < 0$. Using (4), for every sufficiently large $n \geq 1$, we get that

$$\sum_{\omega \in \mathcal{B}_n(A)} \left[\operatorname{diam}(\varphi_\omega(X_\omega))\right]^t \leq (KD)^t \sum_{\omega \in \mathcal{B}_n(A)} \|\varphi'_\omega\|^t \leq (KD)^t e^{\frac{1}{2}nP_A(t)}.$$

Since the families $\{\varphi_\omega(X_\omega)\}_{\omega \in \mathcal{B}_n(A)}$, $n \geq 1$, are covers of J_A whose diameters tend to 0 as $n \to \infty$, we conclude that $\mathcal{H}^t(J_A) = 0$, where $\mathcal{H}^t$ represents the t-dimensional Hausdorff measure. Thus, $\mathrm{HD}(J_A) \leq t$ and since $t > h$ was arbitrarily chosen, we conclude that $\mathrm{HD}(J_A) \leq h$. $\square$

Proposition 8. *Let Φ be a finite conformal construction generated by a subshift A, and let h be the zero of its pressure function. There exists a constant $S \geq 1$ so that for each $0 \leq t \leq h$ there exists a Borel probability measure μ on A such that*

$$\mu([\omega]) \leq S\|\varphi'_\omega\|^t, \qquad \forall\ \omega \in \mathcal{B}(A).$$

Proof. Assume momentarily that $h > 0$. Fix $0 \leq t < h$. For every $k \geq 1$ and every $\omega \in \mathcal{B}_k(A)$, choose an arbitrary $\zeta_\omega \in [\omega]$ and thereafter define

$$\mu_k(B) = \frac{\sum_{\omega \in \mathcal{B}_k(A)} \|\varphi'_\omega\|^t \delta_{\zeta_\omega}}{Z_{k,A}(t)}$$

for every Borel subset B of A, where δ_ζ is the Dirac measure concentrated at ζ. In particular,

$$\mu_k([\omega]) = \frac{\|\varphi'_\omega\|^t}{Z_{k,A}(t)}$$

for every $\omega \in \mathcal{B}_k(A)$. Since $t < h$, we know that $P_A(t) > 0$ and hence $\sup_{k\geq 1} Z_{k,A}(t) = \infty$. Thus, there exists a strictly increasing subsequence $(k_j)_{j\geq 1}$ of natural numbers so that for every $j \geq 1$ and for every $1 \leq p \leq k_j$,

$$Z_{k_j,A}(t) \geq Z_{p,A}(t).$$

Since E is finite, the sets E^∞ and A are compact and therefore the sequence $(\mu_{k_j})_{j\geq 1}$ has a convergent subsequence in the weak* topology. Without loss of generality, we may denote such a subsequence by the same notation as the sequence itself and let μ be its weak* limit. Let $n \geq 1$ and $\omega \in \mathcal{B}_n(A)$. Choose j so that $k_j > n$. We then have

$$\begin{aligned}
\mu_{k_j}([\omega]) &= \mu_{k_j}\Big(\bigcup_{\alpha\in\mathcal{B}_{k_j-n}(A)}[\omega\alpha]\Big) = \sum_{\alpha\in\mathcal{B}_{k_j-n}(A)}\mu_{k_j}([\omega\alpha]) \\
&\leq \frac{\sum_{\alpha\in\mathcal{B}_{k_j-n}(A)}\|\varphi'_\omega\|^t\|\varphi'_\alpha\|^t}{Z_{k_j,A}(t)} = \|\varphi'_\omega\|^t\frac{Z_{k_j-n,A}(t)}{Z_{k_j,A}(t)} \leq \|\varphi'_\omega\|^t.
\end{aligned}$$

Since $[\omega]$ is clopen, we deduce from the Portmanteau theorem that

$$\mu([\omega]) \leq \|\varphi'_\omega\|^t.$$

If $t = h$ (whether $h = 0$ or not), a dichotomy exists: either $\sup_{k\geq 1} Z_{k,A}(h) = \infty$ or $\sup_{k\geq 1} Z_{k,A}(h) < \infty$. In the former case, the above argument applies. In the latter case, since $P_A(h) \geq 0$ we know that $\inf_{k\geq 1} Z_{k,A}(h) \geq 1$. Therefore, the argument above yields

$$\mu([\omega]) \leq S\|\varphi'_\omega\|^h,$$

where μ is the weak* limit of a convergent subsequence of $(\mu_k)_{k\geq 1}$ and $1 \leq S := \sup_{k\geq 1} Z_{k,A}(h)$. □

This measure μ on A induces a measure m on X via the coding map. For every Borel set $Y \subseteq X$, let

$$m(Y) = \mu(\pi^{-1}(Y)).$$

Note that m is supported on J_A.

Proposition 9. *Let Φ be a finite conformal construction generated by a subshift A, and let h be the zero of its pressure function. There exists a constant $C \geq 1$ such that for all $0 \leq t \leq h$, all $x \in J_A$ and all $0 < r < \frac{1}{2}diam(X)$ we have*

$$\frac{m(B(x,r))}{r^t} \leq C.$$

In particular, $HD(J_A) \geq h$ and $\mathcal{H}^h(J_A) > 0$.

Proof. Let $0 \le t \le h$. Fix $x \in J_A$ and $0 < r < \frac{1}{2}\mathrm{diam}(X)$. Then $x = \pi(\omega)$ for some $\omega \in A$. Let W be the family of all minimal (in the sense of length) words $\tau \in \mathcal{B}(A)$ such that

$$\varphi_\tau(X_\tau) \cap B(x,r) \neq \emptyset \quad \text{and} \quad \mathrm{diam}(\varphi_\tau(X_\tau)) < 2r.$$

Let $\tau \in W$. Then $\mathrm{diam}(\varphi_{\tau|_{|\tau|-1}}(X_{\tau|_{|\tau|-1}})) \ge 2r$, and it successively follows from (8), BDP and (4) that

$$\begin{aligned}\mathrm{diam}(\varphi_\tau(X_\tau)) &\ge D^{-1}\|\varphi'_\tau\|_{W_\tau} \ge (KD)^{-1}\|\varphi'_{\tau|_{|\tau|-1}}\|_{W_{\tau|_{|\tau|-1}}}\|\varphi'_{\tau_{|\tau|}}\|_{W_\tau}\\ &\ge (KD)^{-1}D^{-1}\mathrm{diam}(\varphi_{\tau|_{|\tau|-1}}(X_{\tau|_{|\tau|-1}}))\xi\\ &\ge 2(KD^2)^{-1}\xi r,\end{aligned}$$

where $\xi := \min\{\|\varphi'_e\| : e \in E\} > 0$. Given that the family W consists of mutually incomparable words and in virtue of Lemma 6 with $\kappa_1 = 2(KD^2)^{-1}\xi$ and $\kappa_2 = 2$, we get

$$\#W \le \big(3KD(R2(KD^2)^{-1}\xi)^{-1}\big)^d = \big(3K^2D^3(2R\xi)^{-1}\big)^d.$$

Since $\pi^{-1}(B(x,r)) \subseteq \bigcup_{\tau\in W}[\tau]$, we obtain using Proposition 8 and (8) that

$$\begin{aligned}m(B(x,r)) &= \mu\circ\pi^{-1}(B(x,r)) \le \mu(\cup_{\tau\in W}[\tau]) = \sum_{\tau\in W}\mu([\tau])\\ &\le \sum_{\tau\in W}S\|\varphi'_\tau\|^t \le S\sum_{\tau\in W}\big(D\mathrm{diam}(\varphi_\tau(X_\tau))\big)^t < S(\#W)(D\cdot 2r)^t\\ &\le S\big(3K^2D^3(2R\xi)^{-1}\big)^d(2D)^d r^t.\end{aligned}$$

This finishes the proof. □

Using Propositions 7 and 9, we obtain Bowen's formula.

Proposition 10. *Let Φ be a finite conformal construction generated by a subshift A, and let h be the zero of its pressure function. Then $HD(J_A) = h$ and $\mathcal{H}^h(J_A) > 0$.*

If we additionally assume that the strong separation condition is fulfilled, then we further have the following.

Proposition 11. *If Φ satisfies the strong separation condition, i.e. if*

$$\varphi_e(X_e) \cap \varphi_f(X_f) = \emptyset, \qquad \forall\, e,f \in E, e \neq f,$$

then the measure m supported on J_A is such that for every $0 \le t \le h$ and $\omega \in \mathcal{B}(A)$,

$$m(\varphi_\omega(X_\omega)) \le S\|\varphi'_\omega\|^t.$$

PROOF. Since $\pi([\omega]) \subseteq \varphi_\omega(X_\omega)$, we obtain

$$m(\varphi_\omega(X_\omega)) = \mu(\pi^{-1}(\varphi_\omega(X_\omega))) \geq \mu([\omega]).$$

Since both m and μ are probability measures and the same-level sets are mutually disjoint, we deduce that

$$m(\varphi_\omega(X_\omega)) = \mu([\omega]) \leq S\|\varphi'_\omega\|^t$$

by Proposition 8. □

4 Examples

Context-free shift-generated constructions

In this section we look at some examples including the limit set inside the unit interval generated by the context-free subshift on a three-letter alphabet and three similarities that have the same ratio $0 < a \leq \frac{1}{3}$. We give the exact value of the Hausdorff dimension of the limit set of such constructions, using the value of the topological entropy of the context-free shift (see [8]). So let $X = [0,1]$ and $0 < a \leq \frac{1}{3}$. Let $E = \{0,1,2\}$. For every $e \in E$, let $\varphi_e : X \to X$ be defined by $\varphi_e(x) = ax + \frac{e}{3}$. Let A be the context-free subshift of E^∞; i.e. the subshift which has for a forbidden set of words $\mathcal{F} = \{01^k2^l0 : 0 < k \neq l\}$. Let $\Phi = \{\varphi_e\}_{0 \leq e \leq 2}$. Let's have a look at the topological pressure function associated to this construction. For every $n \geq 1$, we have

$$Z_{n,A}(t) = \sum_{\omega \in \mathcal{B}_n(A)} \|\varphi'_\omega\|^t = \sum_{\omega \in \mathcal{B}_n(A)} (a^n)^t = \#\mathcal{B}_n(A) \cdot a^{nt}.$$

Therefore,

$$\begin{aligned} P_A(t) &= \lim_{n\to\infty} \frac{1}{n} \log Z_{n,A}(t) \\ &= \lim_{n\to\infty} \frac{1}{n} \log \#\mathcal{B}_n(A) + \lim_{n\to\infty} \frac{nt \log a}{n} \\ &= h_{top}(A) + t \log a, \end{aligned}$$

where $h_{top}(A)$ is the topological entropy of the context-free shift as a symbolic system. Consequently, the Hausdorff dimension of the limit set, which is the zero of the topological pressure function P_A according to Bowen's formula, is equal to

$$\mathrm{HD}(J_A) = -\frac{h_{top}(A)}{\log a} = -\frac{\log\bigl(1 + \sqrt{1+\sqrt{3}}\bigr)}{\log a}.$$

Families of one-dimensional systems generated by any shift-generated conformal construction

Let $E = \{1, ..., k\}$, $k \geq 2$, and let A be any subshift of the one-sided full shift E^∞. Consider also $0 < a \leq \frac{1}{k}$. Let $X = [0,1]$. For every $e \in E$ let $\varphi_e : X \to X$; $\varphi_e(x) = ax + \frac{e}{k}$. As before, for every $n \geq 1$ we have

$$Z_{n,A}(t) = \sum_{\omega \in \mathcal{B}_n(A)} \|\varphi'_\omega\|^t = \sum_{\omega \in \mathcal{B}_n(A)} (a^n)^t = \#\mathcal{B}_n(A) \cdot a^{nt},$$

and so

$$\begin{aligned} P_A(t) &= \lim_{n\to\infty} \frac{1}{n} \log Z_{n,A}(t) \\ &= \lim_{n\to\infty} \frac{1}{n} \log \#\mathcal{B}_n(A) + \lim_{n\to\infty} \frac{nt \log a}{n} \\ &= h_{top}(A) + t \log a. \end{aligned}$$

Thus,

$$\mathrm{HD}(J_A) = -\frac{h_{top}(A)}{\log a}.$$

Remark 12. *When $0 \leq a < \frac{1}{k}$, the strong separation condition is satisfied and we also have $0 \leq HD(J_A) < \frac{1}{k}$.*

Remark 13. *The only case in which the limit set has positive Lebesgue measure is when $a = \frac{1}{k}$ and A is the full shift.*

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