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ON PARTITIONS OF THE REAL LINE INTO CONTINUUM MANY THICK SUBSETS

Abstract

Three classical constructions of Lebesgue nonmeasurable sets on the real line **R** are envisaged from the point of view of the thickness of those sets. It is also shown, within **ZF** & **DC** theory, that the existence of a Lebesgue nonmeasurable subset of **R** implies the existence of a partition of **R** into continuum many thick sets.

Very soon after Lebesgue's invention (in 1902) of his measure λ on the real line $\mathbf{R}$, the three constructions of extraordinary point sets in $\mathbf{R}$ have followed. They were done, respectively, by Vitali [17], Hamel [4], and Bernstein [1].

An important byproduct of each of those constructions is the statement of the existence of a Lebesgue nonmeasurable subset of $\mathbf{R}$. In this connection, it is reasonable to stress here that those constructions differ essentially from each other. Namely, recall that:

(a) in [17] Vitali takes a selector V of the quotient set $\mathbf{R}/\mathbf{Q}$, where $\mathbf{Q}$ denotes the field of all rational numbers, and shows that V cannot be measurable with respect to any measure on $\mathbf{R}$ which extends λ and is translation invariant;

(b) in [4] Hamel considers $\mathbf{R}$ as a vector space over $\mathbf{Q}$ and establishes the existence of a basis for this space; then he defines a nontrivial endomorphism of the additive group $(\mathbf{R}, +)$, which turns out to be nonmeasurable in the Lebesgue sense;

(c) in [1] Bernstein utilizes the method of transfinite recursion and defines a subset B of $\mathbf{R}$ such that both sets B and $\mathbf{R} \setminus B$ meet every nonempty perfect

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set in $\mathbf{R}$; so both B and $\mathbf{R} \setminus B$ turn out to be nonmeasurable with respect to λ.

It is needless to say that all the above mentioned constructions are based on appropriate uncountable forms of the Axiom of Choice (**AC**), which were radically rejected by Lebesgue in that time. Many years later, it was demonstrated by Solovay [16] that an uncountable version of **AC** is absolutely necessary for obtaining Lebesgue nonmeasurable point sets in $\mathbf{R}$, at least, if one believes that the theory

$$\mathbf{ZFC} \ \& \ (\text{there exists a (strongly) inaccessible cardinal})$$

is consistent. Some delicate and unexpected issues occur here (for more detailed explanation, see e.g. [2]). In this context, it is also reasonable to mention the papers [14] and [15] in which it is stated that the assumption of the existence of a (strongly) inaccessible cardinal cannot be removed from Solovay's result [16].

Denote by $\mathbf{c}$ the cardinality of the continuum.

In [11] Luzin and Sierpiński have extended Bernstein's construction for obtaining a partition of the unit interval $[0, 1]$ (or equivalently, of $\mathbf{R}$) into continuum many Lebesgue nonmeasurable sets. Actually, they proved the following statement.

Theorem 1. *The real line $\mathbf{R}$ admits a partition $\{B_i : i \in I\}$ such that:*

(1) $\mathrm{card}(I) = \mathbf{c}$*;*

(2) *every set B_i $(i \in I)$ meets any nonempty perfect subset of $\mathbf{R}$;*

In particular, all B_i $(i \in I)$ are Bernstein subsets of $\mathbf{R}$ and so are nonmeasurable in the Lebesgue sense.

Further generalization of Bernstein's construction looks as follows (cf. [8], [12]).

Theorem 2. *There exists a covering $\{B_j : j \in J\}$ of the real line $\mathbf{R}$ with its subsets, satisfying these three conditions:*

(1) $\mathrm{card}(J) > \mathbf{c}$*;*

(2) *every set B_j $(j \in J)$ meets each nonempty perfect set in $\mathbf{R}$;*

(3) *the family $\{B_j : j \in J\}$ is almost disjoint; i.e.,* $\mathrm{card}(B_j \cap B_{j'}) < \mathbf{c}$ *for any two distinct indices $j \in J$ and $j' \in J$.*

The conditions (2) and (3) of Theorem 2 readily imply that every set B_j $(j \in J)$ is a Bernstein subset of $\mathbf{R}$.

The role of Bernstein sets in general topology, the theory of Boolean algebras, and measure theory is well known (see, for instance, [10], [12], [13]).

In classical measure theory, the significance of these sets is primarily caused by providing various counterexamples for seemingly valid statements in real analysis and by constructions of measures lacking various regularity properties (see e.g. [7], [8]).

Let E be a ground set, and let μ be a measure defined on some σ-algebra of subsets of E.

Recall that μ is said to be diffused (or continuous) if all singletons in E belong to the domain of μ and μ vanishes at all of them.

A set $Z \subset E$ is said to be μ-thick in E if the equality $\mu_*(E \setminus Z) = 0$ holds true, where μ_* denotes the inner measure associated with μ.

Example 1. Let $\mathcal{M}$ denote the class of the completions of all nonzero σ-finite diffused Borel measures on $\mathbf{R}$. It is not difficult to show that if B is any Bernstein set in $\mathbf{R}$ and μ is any measure from the class $\mathcal{M}$, then both B and $\mathbf{R} \setminus B$ are μ-thick subsets of $\mathbf{R}$ and, consequently, they are nonmeasurable with respect to μ. Actually, this measure-theoretical property completely characterizes Bernstein sets in $\mathbf{R}$ (see e.g. [3], [8]).

We have already mentioned three classical constructions, each of which gives an example of a λ-nonmeasurable set in $\mathbf{R}$. Moreover, Bernstein's construction directly yields the partition $\{B, \mathbf{R} \setminus B\}$ of $\mathbf{R}$ into two λ-thick subsets.

In this connection, let us demonstrate that Hamel's construction directly leads to a partition of $\mathbf{R}$ into countably many λ-thick subsets of $\mathbf{R}$.

For this purpose, let us consider $\mathbf{R}$ as a vector space over the field $\mathbf{Q}$.

Let $\{e_i : i \in I\}$ be a Hamel basis for this space containing 1; i.e., $e_{i_0} = 1$ for some index $i_0 \in I$. Denote by V the vector space over $\mathbf{Q}$ generated by the family $\{e_i : i \in I \setminus \{i_0\}\}$.

It is not difficult to check that V is a special kind of a Vitali set in $\mathbf{R}$.

Actually, V is a selector of $\mathbf{R}/\mathbf{Q}$ but the choice of this selector is done so carefully that V turns out to be able to carry the vector structure over $\mathbf{Q}$ induced by $\mathbf{R}$. We now assert that V is λ-thick in $\mathbf{R}$. Indeed, suppose otherwise, i.e., there exists a λ-measurable set $C \subset \mathbf{R}$ such that

$$\lambda(C) > 0, \quad C \cap V = \emptyset.$$

It is easy to see that V is everywhere dense in $\mathbf{R}$ (because any uncountable subgroup of $(\mathbf{R}, +)$ is necessarily everywhere dense in $\mathbf{R}$). Consequently, we may take a countable family $\{v_j : j \in J\} \subset V$ which is everywhere dense in $\mathbf{R}$, too. Obviously, for this family, we may write

$$V \cap (\{v_j : j \in J\} + C) = \emptyset.$$

Taking into account the metrical transitivity (ergodicity) of λ with respect to any everywhere dense subset of $\mathbf{R}$, we get

$$\lambda(\mathbf{R} \setminus (\{v_j : j \in J\} + C)) = 0.$$

Therefore, $\lambda(V) = 0$, which is impossible in view of the translation invariance of λ and of the relations

$$\mathbf{R} = \mathbf{Q} + V = \cup\{q + V : q \in \mathbf{Q}\}, \quad \lambda(\mathbf{R}) = +\infty.$$

The obtained contradiction yields the desired result.

We thus come to the countable partition $\{q+V : q \in \mathbf{Q}\}$ of $\mathbf{R}$ into λ-thick sets. It follows from this fact that, for any natural number $n \geq 2$, there exists a partition $\{A_1, A_2, \dots, A_n\}$ of $\mathbf{R}$ into λ-thick sets, and so all A_k $(1 \leq k \leq n)$ are nonmeasurable with respect to λ.

Remark 1. In general, Vitali's construction does not lead to a λ-thick subset of $\mathbf{R}$. Indeed, fix a real $\varepsilon > 0$ and take an arbitrary nonempty open interval Δ in $\mathbf{R}$ with $\lambda(\Delta) < \varepsilon$.

For any $x \in \mathbf{R}$, the set $x + \mathbf{Q}$ is everywhere dense in $\mathbf{R}$, so has nonempty intersection with Δ. This circumstance immediately implies that there exists a Vitali set W entirely contained in Δ and, consequently, $\lambda^*(W) < \varepsilon$, where λ^* denotes the outer measure associated with λ. We thus conclude that there are Vitali sets in $\mathbf{R}$ with arbitrarily small outer Lebesgue measure.

Some other unexpected and extraordinary properties of Vitali sets are discussed in [9].

We shall work in **ZF** & **DC** theory, where **DC** stands for, as usual, the Principle of Dependent Choices (see [5], [6], [16]). This principle is stronger than the Axiom of Countable Choice (**CC**) and much weaker than **AC**. Moreover, as we have already mentioned earlier, Solovay's famous result [16] states that assuming the existence of a model of **ZFC** with a (strongly) inaccessible cardinal, there is a model of **ZF** & **DC** in which all sets of reals are Lebesgue measurable (more formally, Con(**ZFC** & **I**) implies Con(**ZF** & **DC** & **LM**)).

Our goal now is to obtain (within **ZF** & **DC** theory) a partition of $\mathbf{R}$ into continuum many λ-thick sets by supposing only the existence of a two-element partition $\{A, A'\}$ of $\mathbf{R}$ such that both sets A and A' are λ-thick in $\mathbf{R}$.

As demonstrated above, Bernstein's and Hamel's constructions give such a partition $\{A, A'\}$ within **ZFC**.

We need the following two auxiliary propositions which belong to **ZF** & **DC** theory.

Lemma 1. *Let E_1 and E_2 be two Polish spaces, let μ_1 be a Borel probability diffused measure on E_1, and let μ_2 be a Borel probability diffused measure on E_2. Then there exists a Borel isomorphism $\phi : E_1 \to E_2$ which is simultaneously an isomorphism between μ_1 and μ_2; i.e., we have $\mu_2(\phi(X)) = \mu_1(X)$ for every Borel subset X of E_1.*

This lemma is well known (for a proof, within **ZF** & **DC** theory, see e.g. [3] or [7]).

Lemma 2. *Let $\{E_n : n = 1, 2, \dots, n, \dots\}$ be a countable family of separable metric spaces and let, for each natural number $n \geq 1$, the space E_n be equipped with a probability Borel measure μ_n. Further, let us denote:*

$$E = \prod\{E_n : n = 1, 2, \dots, n, \dots\}, \quad \mu = \otimes\{\mu_n : n = 1, 2, \dots, n, \dots\}.$$

Suppose also that a sequence of sets $X_n \subset E_n$ $(n = 1, 2, \dots, n, \dots)$ is given. The following two assertions are equivalent:

(1) *the product set $X = \prod\{X_n : n = 1, 2, \dots, n, \dots\}$ is μ-thick in E;*

(2) *the set X_n is μ_n-thick in E_n for each index $n = 1, 2, \dots$.*

Proof. The implication (1) $\Rightarrow$ (2) is almost trivial. So we will focus our attention on the converse implication (2) $\Rightarrow$ (1). Suppose that (2) is satisfied.

Since E is a separable metric space, the probability product measure μ is defined on the Borel σ-algebra of E and, in addition to this, μ is inner regular. The latter means that, for each Borel set $Z \subset E$, the equality

$$\mu(Z) = \sup\{\mu(F) : F \subset Z,\ F \text{ is closed in } E\}$$

is valid. Therefore, it suffices to demonstrate that $X \cap P \neq \emptyset$ for any closed set $P \subset E$ with $\mu(P) > 0$.

For this purpose, we shall construct by recursion an element

$$y = (y_1, y_2, \dots,\ y_n, \dots) \in X \cap P.$$

Suppose that, for a natural number n, the finite sequence

$$(y_1, y_2, \dots, y_n) \in X_1 \times X_2 \times \dots \times X_n$$

has already been defined so that the inequality

$$\nu_n(P(y_1, y_2, \dots, y_n)) > 0$$

holds true, where

$$\nu_n = \otimes\{\mu_m : m = n+1, n+2, \dots\},$$
$$P(y_1, y_2, \dots, y_n) = \{(x_{n+1}, x_{n+2}, \dots) : (y_1, y_2, \dots, y_n, x_{n+1}, x_{n+2}, \dots) \in P\}.$$

According to classical Fubini's theorem, the set of all those elements x_{n+1} from E_{n+1} which satisfy the inequality

$$\nu_{n+1}(P(y_1, y_2, \dots, y_n, x_{n+1})) > 0$$

is μ_{n+1}-measurable and has strictly positive μ_{n+1}-measure in E_{n+1}.

Since the set X_{n+1} is μ_{n+1}-thick in E_{n+1}, there exists a point $y_{n+1} \in X_{n+1}$ such that

$$\nu_{n+1}(P(y_1, y_2, \dots, y_n, y_{n+1})) > 0.$$

We thus see that our recursion works and, after countably many steps, yields the sequence

$$y = (y_1, y_2, \dots, y_n, \dots) \in X.$$

Observe now that, by virtue of the definition of y, every neighborhood of y has common elements with the set P. Since P is closed, we immediately conclude that $y \in P$, so $y \in P \cap X$.

This completes the proof of Lemma 2 (let us underline once more that the argument presented above is done within **ZF** & **DC** theory).

□

Remark 2. In Lemma 2, the assumption that all spaces E_n are separable and metrizable is not necessary. The conclusion of this lemma remains valid under much weaker assumptions, but the above formulation suffices for our further purposes.

Remark 3. Preserving the notation of Lemma 2, let Z be an arbitrary μ-thick set in E.

Then it is easy to verify that, for every natural number $n \geq 1$, the set $\mathrm{pr}_n(Z)$ is μ_n-thick in E_n, where $\mathrm{pr}_n(Z)$ denotes the n-th projection of Z. The converse assertion is not true, in general.

Indeed, simple examples show that the equalities $\mathrm{pr}_n(Z) = E_n$ may be valid simultaneously for all natural numbers $n \geq 1$ but, at the same time, the set Z may be of μ-measure zero.

Remark 4. Let $k \geq 1$ be a natural number, let $\{E_n : n = 1, 2, \dots, k\}$ be a finite family of ground sets and let, for each natural number $n \in \{1, 2, \dots, k\}$, the set E_n be equipped with a probability measure μ_n. Further, let us denote:

$$E = \prod\{E_n : n = 1, 2, \dots, k\}, \quad \mu = \otimes\{\mu_n : n = 1, 2, \dots, k\}.$$

Suppose also that a finite sequence of sets $X_n \subset E_n$ $(n = 1, 2, \dots, k)$ is given. Then the following two assertions are equivalent:

(a) the product set $X = \prod\{X_n : n = 1, 2, \dots, k\}$ is μ-thick in E;

(b) the set X_n is μ_n-thick in E_n for each index $n \in \{1, 2, \dots, k\}$.

We thus see that in the case of a finite sequence of probability measure spaces (or, more generally, of nonzero σ-finite measure spaces), the analogue of Lemma 2 is valid in **ZF** & **DC** theory without assuming any regularity properties of the measures.

Now, we are ready to present the main result of this paper (in what follows we will denote by λ the restriction of the Lebesgue measure on $\mathbf{R}$ to the unit interval $[0, 1]$).

Theorem 3. *Working in* **ZF** & **DC** *theory, suppose that there exists a partition* $\{A, A'\}$ *of the unit interval* $[0, 1]$ *into two subsets such that*

$$\lambda^*(A) = \lambda^*(A') = 1.$$

Then there exists a partition $\{Z_i : i \in I\}$ *of the same interval, which satisfies the following two conditions:*

(1) $\mathrm{card}(I) = \mathbf{c}$*;*

(2) $\lambda^*(Z_i) = 1$ *for each index* $i \in I$*.*

PROOF. Let $\mathbf{N}$ denote the set of all natural numbers. Consider the Hilbert cube $E = [0, 1]^{\mathbf{N}}$ equipped with the probability product measure

$$\mu = \lambda \otimes \lambda \otimes \cdots \otimes \lambda \otimes \cdots .$$

Take any subset K of $\mathbf{N}$ and put:

$$A_{n,K} = A \text{ if } n \in K, \text{ and } A_{n,K} = A' \text{ if } n \in \mathbf{N} \setminus K.$$

Further, for the same K, introduce the corresponding product set

$$Y_K = \prod\{A_{n,K} : n \in \mathbf{N}\}.$$

Proceeding in this manner, we come to the partition $\{Y_K : K \subset \mathbf{N}\}$ of the Hilbert cube E.

By virtue of Lemma 2, all members Y_K ($K \subset \mathbf{N}$) of this partition are μ-thick in E.

Let $\phi : E \to [0,1]$ be a Borel isomorphism which simultaneously is an isomorphism of μ onto λ (the existence of ϕ follows from Lemma 1).

Obviously, $\{\phi(Y_K) : K \subset \mathbf{N}\}$ is a partition of $[0,1]$ into continuum many λ-thick subsets of $[0,1]$. So we may put

$$\{Z_i : i \in I\} = \{\phi(Y_K) : K \subset \mathbf{N}\}.$$

This finishes the proof of Theorem 3. □

As we mentioned at the beginning, nontrivial endomorphisms of the additive group $(\mathbf{R}, +)$ were first exhibited in [4] and all of them turned out to be nonmeasurable in the Lebesgue sense. In connection with this fact, it is worth noticing that some of those endomorphisms can be measurable with respect to certain measures belonging to the class $\mathcal{M}$ introduced in Example 1.

Example 2. There exists a function $f : \mathbf{R} \to \mathbf{R}$ satisfying the following three conditions:

(a) the range $\mathrm{ran}(f)$ of f is contained in the field $\mathbf{Q}$ (consequently, $\mathrm{ran}(f)$ is at most countable);

(b) f is measurable with respect to some measure from the class $\mathcal{M}$;

(c) f is a nontrivial endomorphism of the additive group $(\mathbf{R}, +)$.

To obtain such an f, consider a nonempty perfect subset P of $\mathbf{R}$ linearly independent over the field $\mathbf{Q}$ (the existence of P is a well-known fact of classical point set theory; cf. [7], [12]).

Let $\{e_i : i \in I\}$ stand for some Hamel basis of $\mathbf{R}$ containing P.

We define $f : \mathbf{R} \to \mathbf{Q}$ as follows. Every real number x admits a unique representation in the form

$$x = q_{i_1} e_{i_1} + q_{i_2} e_{i_2} + \cdots + q_{i_n} e_{i_n},$$

where $n = n(x)$ is a natural number, $\{i_1, i_2, \ldots, i_n\}$ is a finite injective family of indices from I, and $\{q_{i_1}, q_{i_2}, \ldots, q_{i_n}\}$ is a finite family of nonzero rational numbers. We put

$$f(x) = q_{i_1} + q_{i_2} + \cdots + q_{i_n}.$$

Obviously, f is an additive function acting from $\mathbf{R}$ into $\mathbf{Q}$, so conditions (a) and (c) are valid. Further, the restriction $f|P$ is identically equal to 1. Let μ be a Borel diffused probability measure on $\mathbf{R}$ whose support is P, i.e., $\mu(\mathbf{R} \setminus P) = 0$, and let μ' denote the completion of μ. It is clear that $\mu' \in \mathcal{M}$ and f turns out to be μ'-measurable. Thus condition (b) is satisfied, too.

Remark 5. It can be shown that:

(a) there exists a subset of $\mathbf{R}$ which is simultaneously a Vitali set and a Bernstein set;

(b) there exists a subset of $\mathbf{R}$ which is simultaneously a Hamel basis and a Bernstein set;

(c) there exists no subset of $\mathbf{R}$ which is simultaneously a Hamel basis and a Vitali set.

Remark 6. Let μ be an arbitrary measure from the class $\mathcal{M}$. By using Lemma 1, it is not difficult to prove within **ZF** & **DC** theory that if there exists a μ-nonmeasurable subset of $\mathbf{R}$, then there exists a partition of $\mathbf{R}$ into two μ-thick subsets. So, taking into account Lemma 1 and Theorem 3, we may conclude that the following four assertions are equivalent in **ZF** & **DC** theory:

(a) there exists a μ-nonmeasurable subset of $\mathbf{R}$;

(b) there exists a partition of $\mathbf{R}$ into two μ-thick subsets;

(c) there exists a partition of $\mathbf{R}$ into continuum many μ-thick subsets;

(d) there exists a function $g : \mathbf{R} \to \mathbf{R}$ such that $\mathrm{ran}(g|X) = \mathbf{R}$ for each μ-measurable set $X \subset \mathbf{R}$ with $\mu(X) > 0$.

In this context, the transfinite construction given in [11] becomes superfluous. At the same time, it seems that an analogue of Theorem 2 cannot be deduced within **ZF** & **DC** theory by assuming that there exists a λ-nonmeasurable subset of $\mathbf{R}$.

Remark 7. Consider the theory **ZF** & **DC** & $(\omega_1 \leq \mathbf{c})$, where ω_1 denotes, as usual, the least uncountable cardinal. It was proved in this theory that there exists a λ-nonmeasurable subset of $\mathbf{R}$ (see [14] and [15]). Consequently, within the same theory, there exists a partition of $\mathbf{R}$ into continuum many λ-thick subsets.

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