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THE HAKE'S THEOREM ON METRIC MEASURE SPACES

Abstract

In this paper, we extend the Hake's theorem over metric measure spaces. We provide its measure theoretic versions in terms of the Henstock variational measure V_F.

1 Introduction

A function $f : [0, 1] \to \mathbb{R}$ is said to be Henstock-Kurzweil integrable, with some $\lambda \in \mathbb{R}$ as its integral, if for every $\epsilon > 0$ there exists a positive function $\delta : [0, 1] \to (0, 1)$, such that the inequality

$$\left| \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) - \lambda \right| < \epsilon$$

is satisfied whenever $0 = x_0 < x_1 < x_2 < \cdots < x_n = 1$, $|x_i - x_{i-1}| < \delta(t_i)$ and the tags $t_i \in [x_{i-1}, x_i]$ for each $i = 1, \ldots, n$.

It is well known that the Henstock-Kurzweil integral, or simply the HK-integral, on real line generalizes the notions of Riemann, Lebesgue and improper integrals. In [19], Ng Wee Leng defined this integral over metric measure spaces. We further simplified that in [18] and proved some basic results of this integral over metric measure spaces.

The Hake's Theorem for real functions is as follows, see [8, Theorem 9.21].

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Theorem 1.1 (Hake). *A function $f : [0, 1] \to \mathbb{R}$ is Henstock-Kurzweil integrable if and only if f is Henstock-Kurzweil integrable over each subinterval $[c, 1]$ with $0 < c < 1$ and the following limit exists*

$$\lim_{c \to 0} \int_c^1 f.$$

Some generalizations of this theorem for functions on $\mathbb{R}^m$ were obtained by Faure, Muldowney and Skvortsov, see [6, 15]. But both of these use an abstract concept of integral convergence over a suitable increasing sequence of figures.

In [17], we proved some simplified measure theoretic extensions of the Hake's theorem on $\mathbb{R}^m$, in terms of the variational measures. Our proofs therein were dependent upon the Euclidean structure of $\mathbb{R}^m$ and thence not valid for functions over general metric measure spaces. We used the following measure theoretic characterization of the Henstock-Kurzweil integral, which was first proved by Bongiorno, Di Piazza, and Skvortsov for real valued functions on compact real intervals, see [1].

Theorem 1.2. *For an additive set function F, the following are equivalent:*

(i) There exists an HK-integrable function f with primitive F,

(ii) The corresponding variational measure V_F is absolutely continuous.

The proof of this theorem for functions with real compact domains was dependent upon the Fundamental Theorem of Calculus for the HK-integral. The similar result isn't available for functions on $\mathbb{R}^m$, as there are various regularity concerns for the derivative of interval functions on $\mathbb{R}^m$, see [23].

There are some extensions of the Fundamental Theorem of Calculus, for the HK-integral on $\mathbb{R}^m$, introduced by Lee Peng Yee, Lu Jitan and Emmanuel Cabral, see e.g. [10, 3, 4]. But those may not be used to extend the above theorem on $\mathbb{R}^m$.

In 2003, Lee Tuo-Yeong proved Theorem 1.2 for functions on compact intervals in $\mathbb{R}^m$ too, see [12]. He proposed a proof, independent of the Fundamental Theorem of Calculus, by using Kurzweil and Jarnik's results on the differentiability of interval functions in $\mathbb{R}^m$, see [11]. Lee also deduced a version of this theorem for the Lebesgue integral when the total variation is finite, see [12].

Two more alternative and simplified proofs of Theorem 1.2 on $\mathbb{R}^m$ were presented by Lee Tuo Yeong, see [13, 14]. But all of these proofs were intrinsically dependent upon the Euclidean structure of $\mathbb{R}^m$. Lee had also declared that Theorem 1.2 is unknown for functions on infinite dimensional domains, for more details see [12, 13, 14].

In this paper, we shall explore into this measure theoretic characterization of the HK-integral for functions over metric measure spaces. We deduce some partial results, raise some questions and prove some extensions of the Hake's theorem, alternatively.

2 Preliminaries

Throught this paper, we adopt our notations from [18]. Let (X, d) be a metric space with metric topology $\mathcal{T}$. An open ball in X, of radius r and center x, is denoted by $B(x, r)$, where $x \in X$ and $r \geq 0$.

Let $\mathcal{T}_0$ denotes the family of open balls in X. For $B \in \mathcal{T}_0$, $\overline{B}$ will denote its closure. Consider the following collections of sets:

$$\mathcal{I}_1 := \{\overline{B_1} \backslash \overline{B_2} : B_1, B_2 \in \mathcal{T}_0 \text{ where } B_1 \not\subset B_2 \text{ and } B_2 \not\subset B_1\},$$

$$\mathcal{I}_2 := \left\{ \bigcap_{i \in \wedge} X_i : \bigcap_{i \in \wedge} X_i \neq \emptyset \text{ where } X_i \in \mathcal{I}_1, \text{ for all } i \in \wedge \text{ and } \wedge \text{ is a finite set} \right\}.$$

Note that the sets in $\mathcal{I}_1$ are either closed balls or scalloped balls and any member of the collection of sets in $\mathcal{I}_2$ is a finite intersection of a combination of closed balls or scalloped balls.

Let $\mathcal{B}$ denote the σ-algebra of Borel subsets of X and $\mu : \mathcal{B} \to [0, \infty)$ be a measure satisfying $\mu(\{y \in X : d(x, y) = r\}) = 0$, for each $x \in X$ and $r \geq 0$. Let Ω denotes the μ-completion of the Borel σ-algebra $\mathcal{B}$, on subsets of X.

Sets in $\mathcal{I}_2$ are called *generalised intervals* or simply *intervals* whenever there is no ambiguity. Any finite (possibly just one) union of intervals in X will be called a *figure*. Note that, because of our choice of μ, we have $\mu(\overline{I}) = \mu(I)$ for each interval I in X.

Let E be compact figure in X and I be a subinterval of E. For any figure $J \subset E$, let $Sub(J)$ denotes the collection of compact subintervals of J and $\mathcal{F}(J)$ denotes the algebra generated by $Sub(J)$. Let $\mathcal{F} = \mathcal{F}(E)$. The following defines a Riemann-type integral on metric spaces.

Definition 2.1. (i) A finite collection $\{(x_i, I_i) : i = 1, \ldots, p\}$ of point-interval pairs is said to be a *partial division* in E if I_i's are mutually disjoint intervals and $x_i \in \overline{I_i}$, for each i. Further if $\cup_{i=1}^{p} I_i = E$, it is called a *division* of E.

(ii) A positive valued function $\delta : E \to (0, \infty)$ is called *a gauge on* E. A division $\{(x_i, I_i) : i = 1, \ldots, p\}$ of E is called *δ-fine* if $I_i \subset B(x_i, \delta(x_i))$ for each i.

(iii) A function $f : E \to \mathbb{R}$ is said to be *Henstock-Kurzweil integrable* (or simply HK-integrable), with some $A \in \mathbb{R}$ as its integral, if for every $\epsilon > 0$ there is a gauge $\delta : E \to (0, \infty)$ such that the inequality

$$\left|\sum_{i=1}^{p} f(x_i)\mu(I_i) - A\right| < \epsilon$$

is satisfied, for all δ-fine divisions $\{(x_i, I_i) : i = 1, \ldots, p\}$ of E.

We will denote the Henstock-Kurzweil integral of f over E by $(HK)\int_E f d\mu$. A function $F : \mathcal{F} \to \mathbb{R}$ is called the *primitive* of f if $F(J) = (HK)\int_J f d\mu$, for each $J \in \mathcal{F}$.

It is pertinent to mention that the generalized intervals in the HK-integral cannot be replaced with measurable sets or closed sets, as in that case the integral will be reduced to the McShane integral, see [16] for more details.

Remarks 2.2. Note that the integral is well-defined only if for each gauge δ on E there exists at least one δ-fine division of E.

A proof for the existence of such δ-fine divisions is given in [19]. But it assumes that a closed and bounded interval in a metric space is compact, which is not true in general. Since we have chosen E to be a compact set, the existence of a δ-fine division of E is assured.

Remarks 2.3. In [19], the authors state an additional regularity hypothesis on the measure. But that is redundant as a totally finite measure on a metric space is always regular, see [21, Proposition 19.13] for more details.

Given any finitely additive set function $F : \mathcal{F} \to \mathbb{R}$, the Henstock variational measure V_F on subsets of E is defined as follows:

Definition 2.4. (i) For $M \subset E$ and a gauge $\delta : M \to (0, \infty)$, define

$$V(F, M, \delta) := \sup_P \sum_{i=1}^{p} |F(I_i)|$$

where the supremum is taken over all δ-fine partial divisions $P = \{(x_i, I_i) : 1 \leq i \leq p\}$ in E, which are tagged in M.

(ii) The Henstock variational measure V_F on a set $M \subset E$ is defined as

$$V_F(M) := \inf_\delta V(F, M, \delta)$$

where the infimum is taken over all the gauges $\delta : M \to (0, \infty)$.

It can be easily seen that when M is a compact real interval then $V_F(M)$ is equal to the standard total variation of F over M, see [22, Lemma 2.2].

In [7, Proposition 3.3], it is proved that on finite dimensional Euclidean spaces, V_F is a metric outer measure. The same proof holds true in case of metric measure spaces too. Further, an application of [2, Theorem 3.7] shows that V_F is a Borel measure. Finally, if V_F is absolutely continuous with respect to μ then V_F is a measure on Ω, see [12, Theorem 3.7].

3 The Main Results

First we restate the Hake's theorem as follows:

Theorem 3.1. *Let f and F be real valued functions over $[0,1]$ such that for each interval $[c,1]$ with $0 < c < 1$, f is HK-integrable over $[c,1]$ with $(\mathcal{HK})\int_c^1 f = F(1) - F(c)$.*

Then f is HK-integrable over $[0,1]$ if and only if F is continuous at 0. Moreover, in that case we have, $\int_0^1 f = F(1) - F(0)$.

We generalize this version of the Hake's theorem over metric spaces which also extends our previous results on Hake-type theorems, see [17, Theorem 5.2, Theorem 5.4]. We observe that the following partial result, as a particular case of [5, Proposition 2], is valid even on metric measure spaces.

Theorem 3.2. *Let $f : E \to \mathbb{R}$ be an HK-integrable function with primitive F. Then V_F is absolutely continuous with respect to μ.*

Next we present some extensions of Theorem 3.1 in our setting.

Theorem 3.3. *Let $I = \overline{B}(x,r)$ be a closed ball in E and its boundary be $\partial I := \{y \in X : d(x,y) = r\}$. Assume that for each compact interval $J \subset I$ with $J \cap \partial I = \emptyset$, f is HK-integrable over J, with $(\mathcal{HK})\int_J f d\mu = F(J)$.*

Then f is HK-integrable over I if and only if $V_F(\partial I) = 0$. Moreover, in that case we have, $(\mathcal{HK})\int_I f d\mu = F(I)$.

Proof. If f is HK-integrable over I then by Theorem 3.2, V_F is absolutely continuous with respect to μ. Thus $V_F(\partial I) = 0$.

For the converse, assume that $V_F(\partial I) = 0$ and let $\epsilon > 0$ be given. We choose an increasing sequence of closed balls $A_n = \overline{B}\big(x, r - \frac{1}{n}\big)$ inside I such that $(\cup_{n=1}^{\infty} A_n) \cup \partial I = I$.

By our hypothesis, f is HK-integrable over A_n, for each $n \in \mathbb{N}$. Using Saks-Henstock Lemma, we choose a gauge $\delta_n : A_n \to (0,\infty)$ so that the inequality

$$\sum_{i=1}^{p} |f(t_i)\mu(J_i) - F(J_i)| \leq \frac{\epsilon}{2^{n+1}}$$

is satisfied for any δ_n−fine partial division $\{(t_i, J_i) : 1 \leq i \leq p\}$ of A_n.

Now we divide the proof in two cases. First we consider the case when $f(t) = 0$, for all $t \in \partial I \cup (\cup_n \partial A_n)$. Set $B := \partial I \cup (\cup_n \partial A_n)$. Since f is HK-integrable over each A_n, Theorem 3.2 implies that $V_F(\partial A_n) = 0$, for all $n \in \mathbb{N}$. Now since V_F is a metric outer measure, we have

$$V_F(B) = V_F(\partial I \cup (\cup_n \partial A_n)) \leq V_F(\partial I) + \sum_n V_F(\partial A_n) = 0.$$

Since $V_F(B) = 0$, we can choose a gauge $\delta_0 : B \to (0, \infty)$ such that for every δ_0-fine partial division $\{(t_i, J_i) : 1 \leq i \leq p\}$ anchored in B, the following inequality is satisfied

$$\sum_{i=1}^{p} |F(J_i)| < \frac{\epsilon}{2}.$$

Now, we define a gauge $\delta : I \to (0, \infty)$ as follows:

$$\delta(t) = \begin{cases} \delta_0(t) & \text{for } t \in B, \\ \min\{\delta_n(t), \frac{1}{2}\text{dist}(t, \partial A_n \cup \partial A_{n-1})\} & \text{for } t \in (A_n \setminus A_{n-1})^o. \end{cases}$$

For any given δ-fine division $P = \{(t_i, I_i) : 1 \leq i \leq p\}$ of I, we have

$$\begin{aligned} \left| \sum_{i=1}^{p} f(t_i)\mu(I_i) - F(I) \right| &\leq \sum_{t_i \in B} |f(t_i)\mu(I_i) - F(I_i)| + \sum_{t_i \notin B} |f(t_i)\mu(I_i) - F(I_i)| \\ &\leq \sum_{t_i \in B} |F(I_i)| + \sum_n \sum_{t_i \in (A_n \setminus A_{n-1})^o} |f(t_i)\mu(I_i) - F(I_i)| \\ &< \tfrac{\epsilon}{2} + \sum_n \frac{\epsilon}{2^{n+1}} < \epsilon. \end{aligned}$$

This proves that f is HK-integrable over I with primitive F, when $f(t) = 0$ for all $t \in B$.

For the general case we define a function $g : I \to \mathbb{R}$ as $g = f - f.\chi_B$, where χ_B denotes the characteristic function of the set B. Then $g(t) = 0$ for all $t \in B$. Note that for a compact interval $J \subset I \setminus \partial I$, g is HK-integrable over J with integral $F(J)$, as $g(t) = f(t)$ for almost all $t \in I$. As earlier, we get $V_F(B) = 0$.

Thence, using the previous case, we conclude that g is HK-integrable over I with integral $F(I)$. Since $f(t) = g(t)$ for almost all $t \in I$, we see that f is HK-integrable over I with $(\mathcal{HK}) \int_I f d\mu = F(I)$, as the desired conclusion.

□

On the similar lines one can also prove the above theorem when I is a generalized interval. Now we present an alternative proof of [17, Theorem 5.4] for V_F. We observe that the following version of [22, Lemma 3.7], is true for V_F, even in our setting.

Lemma 3.4. *Let $f : E \to \mathbb{R}$ be HK-integrable with $(HK)\int_J f = F(J)$, for every interval $J \subset I$. Then for every $M \subset E$*

$$V_F(M) \leq \mu(E). \sup\{|f(t)| : t \in M\}.$$

Theorem 3.5. *Let $A \subset E$ be a closed set such that*
(a) f is HK-integrable over A.
(b) For each compact interval $J \subset E \setminus A$, f is HK-integrable over J, with integral $F(J)$.

Then $V_F(A) = 0$ if and only if f is HK-integrable over E with

$$(\mathcal{HK})\int_E f d\mu = F(E) + (\mathcal{HK})\int_A f d\mu. \tag{1}$$

PROOF. Since A is a closed subset of E, the set $E \setminus A$ can be written as a union of balls, open in the metric space (E, d). Being a compact metric space, (E, d) is Lindeloff and thus there exists a countable subfamily of those balls, say $\{B_n : n \in \mathbb{N}\}$, which covers $E \setminus A$. For each $n \in \mathbb{N}$, define a figure U_n as $U_n := B_n \setminus \cup_{m<n} B_m$.

As in the previous theorem, we first take the case when $f(t) = 0$, for all $t \in A \cup (\cup_n \partial U_n)$. Set $B := A \cup (\cup_n \partial U_n)$. If f is HK-integrable over E, Lemma 3.4 gives us $V_F(A) = 0$. For the converse, we assume that $V_F(A) = 0$. For any $n \in \mathbb{N}$, we write

$$\partial U_n = (\partial U_n \cap A) \cup (\partial U_n \cap (E \setminus A)).$$

We find a compact figure $J \subset (E \setminus A)$ such that $\partial U_n \cap (E \setminus A) \subset J$. Using our hypothesis, f is HK-integrable over J. Now by Theorem 3.2, we have $V_F \ll \mu$ on J and thence

$$V_F(\partial U_n \cap (E \setminus A)) = 0.$$

Since V_F is an outer measure, we have

$$V_F(\partial U_n) \leq V_F(\partial U_n \cap A) + V_F(\partial U_n \cap (E \setminus A)) \leq V_F(A) = 0.$$

Again the outer measurability of V_F implies

$$V_F(B) \leq V_F(A) + \sum_n V_F(\partial U_n) = 0.$$

Let $\epsilon > 0$ be given. We can choose a gauge $\delta_0 : B \to (0, \infty)$ such that for each δ_0-fine partial division $P := \{(t_i, J_i) : 1 \leq i \leq p\}$ anchored in B, the following inequality is satisfied

$$\sum_{i=1}^{p} |F(J_i)| < \frac{\epsilon}{2}.$$

Since $V_F(\partial(U_n)) = 0$, f is HK-integrable over each $\overline{U}_n$. Using Saks-Henstock Lemma, we choose a gauge $\delta_n : \overline{U}_n \to (0, \infty)$ such that the inequality

$$\sum_{i=1}^{p} |f(t_i)\mu(J_i) - F(J_i)| \leq \frac{\epsilon}{2^{n+1}}$$

is satisfied for any δ_n-fine partial division $\{(t_i, J_i) : 1 \leq i \leq p\}$ of $\overline{U}_n$. Next, we define a gauge $\delta : E \to (0, \infty)$ as follows:

$$\delta(t) = \begin{cases} \delta_0(t) & \text{for } t \in B, \\ \min\{\delta_n(t), \frac{1}{2}\text{dist}(t, \partial U_n)\} & \text{for } t \in (U_n)^o. \end{cases}$$

Now for any δ-fine division $P := \{(t_i, I_i) : 1 \leq i \leq p\}$ of E, the following assertions hold true, due to our choice of δ.

$$\begin{aligned} \left| \sum_{i=1}^{p} f(t_i)\mu(I_i) - F(E) \right| &\leq \sum_{t_i \in B} |f(t_i)\mu(I_i) - F(I_i)| + \sum_{t_i \notin B} |f(t_i)\mu(I_i) - F(I_i)| \\ &\leq \sum\nolimits_{t_i \in B} |F(I_i)| + \sum_{n} \sum_{t_i \in (U_n)^o} |f(t_i)\mu(I_i) - F(I_i)| \\ &< \tfrac{\epsilon}{2} + \sum_{n} \frac{\epsilon}{2^{n+1}} = \epsilon. \end{aligned}$$

Thus f is HK-integrable over E with $(\mathcal{HK}) \int_E f d\mu = F(E)$. Hence we have proved our result for the case when $f(t) = 0$ for all $t \in B$.

For the general case, define a function $g : E \to \mathbb{R}$ as $g = f - f.\chi_B$, where χ_B is the characteristic function of B. Then $g(t) = 0$ for all $t \in B$ and $g(t) = f(t)$ for all $t \in E \setminus B$.

Note that for any compact interval $J \subset (E \setminus B) \subset (E \setminus A)$, since f is HK-integrable over J with integral $F(J)$ and $f(t) = g(t)$ for almost all $t \in J$, g is HK-integrable over J with integral $F(J)$.

Now as above, we have $V_F(A) = 0$ if and only if g is HK-integrable over E with $(\mathcal{HK}) \int_E g d\mu = F(E)$, that is, if and only if $f - f.\chi_A$ is HK-integrable over E with $(\mathcal{HK}) \int_E (f - f.\chi_A) d\mu = F(E)$.

Since f is given to be integrable over A we observe that $V_F(A) = 0$ if and only if f is HK-integrable over E with $(\mathcal{HK}) \int_E (f - f.\chi_A) d\mu = F(E)$, that is,

$$(\mathcal{HK}) \int_E f d\mu = F(E) + (\mathcal{HK}) \int_A f d\mu.$$

□

4 Notes and Remarks

We remark that equation (1) in Theorem 3.5 may appear a bit unintuitive, as one would naturally expect $(\mathcal{HK}) \int_E f d\mu = F(E)$, as the conclusion. This happens since we are not given any information about the relationship between f and F, on A. The set function F is given to be the primitive of f, only on the compact intervals inside $E \setminus A$.

It should be noted that the one way implications of Theorem 3.3 and Theorem 3.5 are true even for the full variational measure W_F, as $V_F(M) \leq W_F(M)$ for each $M \subset E$. But we are not certain about the other way implications, see [22] for more details about W_F.

In [9], Henstock presented a generalization of the HK-integral over uncountable copies of $\mathbb{R}$. He considered the integration of point-interval functions. The properties of this integral are not much unexplored.

Since Theorem 3.2 was the main tool behind our versions of the Hake's property, the following questions remain open:

Q 1. Let E be a compact subset of a metric measure space and $F : \mathcal{F} \to \mathbb{R}$ be a finitely additive set function satisfying $V_F \ll \mu$. Does there exist an HK-integrable function $f : E \to \mathbb{R}$ having primitive F?

Q 2. Does there exist an analogue of Theorem 3.2 for point-interval functions, as considered by Henstock in [9]?

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