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# THE CLASS OF PURELY UNRECTIFIABLE SETS IN $\ell_2$ IS $\Pi^1_1$-COMPLETE

**Abstract**

The space $F(\ell_2)$ of all closed subsets of $\ell_2$ is a Polish space. We show that the subset $P \subset F(\ell_2)$ consisting of the purely 1-unrectifiable sets is $\Pi^1_1$-complete.

## 1 Introduction

The concepts of unrectifiable and purely unrectifiable sets are central in contemporary geometric measure theory; see e.g. [2]. In some sense, these are sets which are not capturable by smooth approximations: a set is *unrectifiable*, if it cannot be covered (up to a negligible set) by countably many $C^1$-curves, and 1-*purely unrectifiable* if its 1-dimensional Hausdorff measure restricted to any $C^1$-curve is zero. We only consider 1-purely unrectifiable sets in this article (as opposed to $m$-purely unrectifiable for $m > 1$), so we skip the "1" from the notation. There are several open questions concerning (partial) characterisations of purely unrectifiable sets, such as, for example, whether or not the two-dimensional Brownian motion is purely unrectifiable with probability 1 [3].

Another question asked by David Preiss (2013) is whether purely unrectifiable sets can be (in a certain sense) approximated by open sets; see Question 3.

Here we show that the notion of pure unrectifiability is subtle to the extent that any decision procedure for checking whether a given closed subset of $\ell_2$ is purely unrectifiable requires an exhaustive search through continuum many cases. That is to say, in the language of descriptive set theory, the set of all closed purely unrectifiable subsets of $\ell_2$ is $\Pi^1_1$-hard. On the other hand, there

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*is* a decision procedure of this sort, so the set is $\Pi^1_1$-complete (or *co-analytic complete*).

This might lead to a negative answer to Question 3; see discussion after the statement of the question.

**Acknowledgement** I would like to thank David Preiss for introducing me to this topic and pointing to this research direction.

## 2 Basic definitions

In order to define purely unrectifiable sets in $\ell_2$, let us review the definition of $C^1$-curve in $\ell_2$:

**Definition 1.** A *Fréchet derivative* of a function $f\colon [0,1] \to \ell_2$ at a point $x \in [0,1]$ is a linear operator $A_x\colon \mathbb{R} \to \ell_2$ such that

$$\lim_{h\to 0} \frac{\|f(x+h) - f(x) - A_x h\|_2}{|h|} = 0.$$

The function belongs to $C^1$ if the Fréchet derivative exists at every point and the map $x \mapsto A_x$ is continuous in the operator norm.

The linear operator $A_x$ is uniquely determined by the vector $A_x(1)$, so denote $f'(x) = A_x(1)$. Also denote the space of all $C^1$-curves by $C^1([0,1], \ell_2)$.

**Definition 2.** A subset $N$ of $\ell_2$ is *purely unrectifiable* if it is null on every $C^1$-curve. That is, given a $C^1$-map $f\colon [0,1] \to \ell_2$, the one-dimensional Hausdorff measure of $N \cap \operatorname{ran}(f)$, denoted $\mathcal{H}^1(N \cap \operatorname{ran}(f))$, equals 0. Denote the set of purely unrectifiable curves in $\ell_2$ by $P$.

**Question 3.** Let $e_0$ be the first basis vector of $\ell_2$. Let us call a closed set $N \subset \ell_2$ *weakly purely unrectifiable* if there exists $\tau > 0$ such that for every $\varepsilon > 0$ there exists open $G \subset \ell_2$ with $N \subset G$, such that for all $C^1$-curves $f$, if

$$\|f'(x) - e_0\|_2 < \tau$$

for all $x \in \operatorname{dom} f$, then the one-dimensional Hausdorff measure of $\operatorname{ran}(f) \cap G$ is less than $\varepsilon$. Denote the set of weakly purely unrectifiable curves by $P^*$. David Preiss asked the following question (2013): Is $P \subset P^*$?

Here I propose a possible strategy for the solution. We prove in this paper that the complexity of $P$ is exactly $\Pi^1_1$. What about $P^*$? The set of those $N$ that satisfy "$N \subset G$" in the formulation above is itself $\Pi^1_1$-complete for a fixed $G$, and additionally there is an existential quantifier over $G$. So $P^*$ is $\Sigma^1_2$, and here is a conjecture:

**Conjecture 4.** *$P^*$ is $\Sigma^1_2$-complete.*

Now, if the conjecture is correct and we could modify the definition of $P^*$ into $P^{**}$ such that the complexity is preserved and such that if $P \subset P^*$, then $P = P^{**}$, we would obtain a contradiction.

## 3 Preliminaries in descriptive set theory

We follow the notation and presentation of the book "Classical Descriptive Set Theory" by A. Kechris [1] and refer frequently to it below when addressing well-known facts.

A *Polish space* is a separable topological space which is homeomorphic to a complete metric space. The Hilbert space $\ell_2$ is an example of a Polish space. A *standard Borel space* is a set $X$ endowed with a $\sigma$-algebra $S$ such that there exists a Polish topology on $X$ in which the Borel sets are precisely the sets in $S$.

Let $F(\ell_2)$ denote the set of all closed subsets of $\ell_2$. This is a standard Borel space where the $\sigma$-algebra is generated by the sets of the form

$$\{A \in F(\ell_2) \mid A \cap U \neq \varnothing\}, \tag{B}$$

where $U$ ranges over the basic open sets of $\ell_2$ [1, Thm. 12.6]. We need the following fact. Let $H$ be the Hilbert cube $H = [0,1]^{\mathbb{N}}$. By [1, Thm. 4.14], $\ell_2$ can be embedded into $H$ so that the image is a $G_\delta$ subset. Let $e$ be that embedding. Let $K(H)$ be the set of all compact non-empty subsets of $H$ equipped with the Hausdorff metric; $K(H)$ is a compact Polish space.

**Fact 5.** *The embedding $e\colon \ell_2 \to H$ induces an embedding of $F(\ell_2)$ into $K(H)$ such that the image of $F(\ell_2)$ is $G_\delta$ in $K(H)$, thus inducing a Polish topology on $F(\ell_2)$ [1, Thm. 3.17]. This topology gives rise to the same Borel sets as $(B)$ above.* □

By $\omega$ and by $\mathbb{N}$, we denote the set of natural numbers; by $\mathbb{N}_+$, the set of positive natural numbers. For $n \in \mathbb{N}$, $\omega^n$ is the set of all functions from $\{0, \ldots, n-1\}$ to $\omega$, $\omega^{<\omega} = \bigcup_{n\in\mathbb{N}} \omega^n$ and $\omega^\omega$ denotes the set of all functions from $\omega$ to $\omega$. Similarly, $2^\omega$ denotes the set of all functions from $\omega$ to $\{0,1\}$, and $2^{<\omega}$ the set of functions from $\{0, \ldots, n-1\}$ to $\{0,1\}$ for all $n$. The spaces $\omega^\omega$ and $2^\omega$ are Polish spaces in the product topology.

The set $\omega^{<\omega}$ can be ordered in a natural way: $p < q$ if $q \restriction \operatorname{dom} p = p$. This is an example of a *tree*. The set of all trees, Tr, is the set of all downward closed suborders of $\omega^{<\omega}$. The space Tr can be endowed naturally with a Polish topology as a closed subset of $2^{\omega^{<\omega}}$, which is in turn homeomorphic to $2^\omega$ via a bijection $\omega \to \omega^{<\omega}$. A *branch* of a tree $T \in \mathrm{Tr}$ is a sequence $(p_n)_{n<\omega}$ such that $p_n \in \omega^n$, $p_n < p_{n+1}$ and $p_n \in T$ for all $n$.

A subset of a Polish space $A \subset X$ is $\Sigma^1_1$ if there is a Polish space $Y$ and a Borel subset $B \subset X \times Y$ such that $A$ is the projection of $B$ to $X$. A set is $\Pi^1_1$ if it is the complement of a $\Sigma^1_1$ set.

**Definition 6.** A set $A \subset X$ is *Borel Wadge-reducible* to another $B \subset Y$ ($X$ and $Y$ are Polish) if there exists a Borel function $f\colon X \to Y$ such that for all $x \in X$, $x \in A \iff f(x) \in B$. We denote this by $A \leqslant_W B$.

A set $A \subset X$ is $\Pi^1_1$-*hard* if every $\Pi^1_1$ set $B$ is Wadge-reducible to it, $B \leqslant_W A$. We define $\Sigma^1_1$-hard similarly. A set is $\Pi^1_1$-*complete* ($\Sigma^1_1$-complete) if it is $\Pi^1_1$ and $\Pi^1_1$-hard ($\Sigma^1_1$ and $\Sigma^1_1$-hard).

Since the classes $\Sigma^1_1$ and $\Pi^1_1$ are closed under preimages in Borel maps [1, Thm. 14.4], it is clear that if $A$ is $\Sigma^1_1$ and $B \leqslant_W A$, then $B$ is also $\Sigma^1_1$. On the other hand, a simple diagonalisation argument together with Souslin's Theorem [1, Thm. 14.11] shows that there are $\Pi^1_1$ sets that are not $\Sigma^1_1$. Therefore, a $\Pi^1_1$-hard set cannot be $\Sigma^1_1$, because it Wadge reduces to some $\Pi^1_1$ set that is not $\Sigma^1_1$. In particular, it cannot be Borel.

An example of a $\Pi^1_1$-complete set is the set of those trees in Tr that do not have a branch [1, 27.1]. To sum up, the main conclusions in this paper are based on the following two facts:

**Fact 7.** *1. If $A$ is $\Pi^1_1$-hard and $A \leqslant_W B$, then $B$ is $\Pi^1_1$-hard.*

*2. The set $\{T \in \mathrm{Tr} \mid T$ has no branches$\}$ is $\Pi^1_1$-hard. [1, p. 209]* □

## 4 Main theorem

**Proposition 8.** *The set $P = \{A \in F(\ell_2) \mid A$ is purely unrectifiable$\}$ is $\Pi^1_1$.*

PROOF. The space $C^1(\mathbb{R}, \ell_2)$ is Polish in the topology given by the sup-norm. Let $A \subset F(\ell_2) \times C^1(\mathbb{R}, \ell_2)$ be the set of those pairs $(C, \gamma)$ such that

$$\mathcal{H}^1(C \cap \operatorname{ran} \gamma) > 0.$$

Then the projection of $A$ to the first coordinate is precisely the complement of $P$. It remains to show that $A$ is Borel.

Fix a dense countable subset $D$ of $\ell_2$ and define a basic open set of $\ell_2$ to be an open ball $B(x, r)$ where $r \in \mathbb{Q}$ and $x \in D$. Clearly, this is a countable basis.

Since $C \cap \operatorname{ran}\gamma$ is compact, the inequality $H^1(C \cap \operatorname{ran}\gamma) > 0$ is equivalent to the statement that there exists $n \in \mathbb{N}$ such that for all finite sequences $(B(x_1,r_1),\dots,B(x_k,r_k))$ of basic open sets of $\ell_2$, if $\sum_{i=1}^k r_i < 1/n$, then $C \cap \operatorname{ran}\gamma \not\subset \overline{\bigcup_{i=1}^k B(x_i,r_i)}$. Denoting

$$A^*(x_1,\dots,x_k,r_1,\dots,r_k) = \{(C,\gamma) \in F(\ell_2) \times C^1(\mathbb{R},\ell_2) \mid C \cap \operatorname{ran}\gamma \not\subset \overline{\bigcup_{i=1}^k B(x_i,r_i)}\},$$

we get

$$A = \bigcup_{n\in\mathbb{N}} \bigcap_{k\in\mathbb{N}} \bigcap_{\substack{\bar{x}\in D^k,\bar{r}\in\mathbb{Q}^k \\ r_1+\dots+r_k<1/n}} A^*(x_1,\dots,x_k,r_1,\dots,r_k).$$

Being a subset of a closed set is Borel, so $A^*(x_1,\dots,x_k,r_1,\dots,r_k)$ is Borel. Hence, $A$ is Borel. □

**Theorem 9** (Main Theorem)**.** *The set*

$$P = \{A \in F(\ell_2) \mid A \textit{ is purely unrectifiable}\}$$

*is* $\Pi^1_1$*-complete.*

*Proof of Theorem 9.* We have already shown (Proposition 8) that $P$ is $\Pi^1_1$, so we want to show that it is $\Pi^1_1$-hard. The proof is reminiscent of the proof of [1, Thm. 27.6, pp. 210–211].

We will show that the set $NB$ of those trees $T \in \mathrm{Tr}$ which do not have a branch is Wadge-reducible to $P$. That is, we will find a Borel function $H\colon \mathrm{Tr} \to F(\ell_2)$ such that $H(T)$ is not purely unrectifiable if and only if $T$ has a branch. The result follows then from Fact 7.

A Cantor set $C \subset \mathbb{R}$ with a positive Lebesgue measure can be constructed by removing an open interval of length $1/4$ from the middle of the closed unit interval $[0,1]$, and then removing open intervals of length $1/16$ from the middle of each of the remaining intervals and so on. At the $n^{\text{th}}$ step we have a disjoint union of $2^n$ closed intervals. From left to right, label these intervals by $C_n^1,\dots,C_n^{2^n}$ and set $C = \bigcap_{n=0}^\infty \bigcup_{k=1}^{2^n} C_n^k$.

Let $\{e_{n,m} \mid n,m \in \mathbb{N}\}$ be a basis for $\ell_2$. For each $s \in \omega^{<\omega}$, let us define a finite subset $v_s$ of $\ell_2$ as follows:

$$v_s = \Big\{ \sum_{n=0}^{\operatorname{dom}(s)-1} \frac{1+p(n)}{\sqrt{2^n}} e_{n,s(n)} \mid p \in 2^{\operatorname{dom}(s)} \Big\}.$$

Then for every tree $T \in \mathrm{Tr}$, let

$$H(T) = \overline{\bigcup_{s \in T} v_s}.$$

**Claim 9.1.** If $T \in \mathrm{Tr}$ has a branch, then there is a $C^1$-function $f\colon [0,1] \to \ell_2$ such that the one-dimensional Hausdorff measure of $H(T) \cap \operatorname{ran} f$ is positive.

*Proof of Claim 9.1.* Suppose that $T$ has a branch and that $b \in \omega^\omega$ is such that $b \restriction n \in T$ for all $n$. Let us construct a $C^1$-function $f\colon [0,1] \to \ell_2$ as follows. For $n \in \mathbb{N}$ define $f_n\colon [0,1] \to \mathbb{R}$ to be a smooth function such that

- $f_n(x) = \frac{1}{\sqrt{2^n}}$ for $x \in C_n^k$ when $k$ is odd, and $f_n(x) = \frac{2}{\sqrt{2^n}}$ for $x \in C_n^k$ when $k$ is even,
- range of $f_n$ is $\left[\frac{1}{\sqrt{2^n}}, \frac{2}{\sqrt{2^n}}\right]$, and
- if $I$ is an open interval which is removed at the $k^{\text{th}}$ stage in the construction of $C$ and $x \in I$, then

$$0 < |f_n'(x)| \leqslant \frac{4^{k+1}}{\sqrt{2^n}}.$$

The derivative can be bounded in this way because if $I$ is an open interval that is removed at the $k^{\text{th}}$ stage, then $|I| = 4^{-k}$, and in this interval, the function is only required to either raise from $1/\sqrt{2^n}$ to $2/\sqrt{2^n}$ or decrease the same amount in the opposite direction. On the other hand, if $x \in C$, then the derivative of $f_k$ is 0 for all $k$.

Now let $f(x) = \sum_{n=0}^{\infty} f_n(x) e_{n,b(n)}$. Clearly, $f(x) \in \ell_2$ for all $x$:

$$\begin{aligned}
\|f(x)\|_2^2 &= \sum_{n=0}^{\infty} |f_n(x)|^2 \\
&\leqslant \sum_{n=0}^{\infty} \left|\frac{2}{\sqrt{2^n}}\right|^2 \\
&= \sum_{n=0}^{\infty} \frac{2}{2^n} \\
&= 4.
\end{aligned}$$

**Subclaim 9.1.1.** The function $f$ has a Fréchet derivative at each $x \in [0,1]$.

*Proof of Subclaim 9.1.1.* The vector $A_x = \sum_{n=0}^{\infty} f'_n(x) e_{n,b(n)}$ is in $\ell_2$, because the absolute value of $f'_n(x)$ is bounded by $\frac{4^{k+1}}{\sqrt{2^n}}$, where $k$ is a constant natural number that depends on $x$. Thus, $A_x$ defines a bounded linear operator $h \mapsto A_x h$. We claim that $A_x$ is the Fréchet derivative of $f$ at $x$. For that we need to show that

$$\lim_{h \to 0} \frac{\|f(x+h) - f(x) - A_x h\|_2}{|h|} = 0.$$

So assume that $\varepsilon > 0$. The numerator can be rewritten as

$$\sqrt{\sum_{n=0}^{\infty} |f_n(x+h) - f_n(x) - f'_n(x)h|^2}.$$

Let us show first that there exists $k \in \mathbb{N}$ such that for all $h$

$$\sum_{n=k}^{\infty} |f_n(x+h) - f_n(x) - f'_n(x)h|^2 \leqslant \varepsilon^2 h^2 :$$

$$\begin{aligned}
|f_n(x+h) - f_n(x) - f'_n(x)h|^2 &\leqslant (|f_n(x+h) - f_n(x)| + |f'_n(x)h|)^2 \\
\text{(mean value theorem)} \quad &= (|f'_n(\xi)||h| + |f'_n(x)||h|)^2 \\
&= (|f'_n(\xi)| + |f'_n(x)|)^2 h^2 \\
\text{(for some constant } K) \quad &\leqslant \left(\frac{K}{2^n}\right)^2 h^2.
\end{aligned}$$

The last inequality follows from the definition of $f$. Therefore, for each $i \in \mathbb{N}$ we have

$$\sum_{n=i}^{\infty} |f_n(x+h) - f_n(x) - f'_n(x)h|^2 \leqslant \sum_{n=i}^{\infty} \left(\frac{K}{2^n}\right)^2 h^2.$$

Now, by choosing $k$ big enough, we can make sure that $\sum_{n=k}^{\infty} \left(\frac{K}{2^n}\right)^2 < \varepsilon^2$, so pick this $k$. Then, for each $n < k$, let $h_n > 0$ be a small enough real number such that $|f_n(x+h_n) - f_n(x) - f'_n(x)h_n| \leqslant \frac{\varepsilon}{2^n} h_n$, and let $h = h_\varepsilon = \min_{n<k} h_n$. Then we have

$$\begin{aligned}
\frac{\|f(x+h) - f(x) - A_x h\|_2}{|h|} &= \frac{\sqrt{\sum_{n=0}^{\infty} |f_n(x+h) - f_n(x) - f'_n(x)h|^2}}{|h|} \\
&\leqslant \frac{\sqrt{\left(\sum_{n=0}^{k-1} |f_n(x+h) - f_n(x) - f'_n(x)h|^2\right) + \varepsilon^2 h^2}}{|h|}
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \frac{\sqrt{\Big(\sum_{n=0}^{k-1}(\frac{\varepsilon}{2^n}h)^2\Big)+\varepsilon^2h^2}}{|h|}\\
&<\frac{\sqrt{4\varepsilon^2h^2+\varepsilon^2h^2}}{|h|}\\
&=\sqrt{5}\varepsilon.
\end{aligned}
$$

$\square_{\text{Subclaim 9.1.1}}$

**Subclaim 9.1.2.** The Fréchet derivative of $f$ is continuous.
Thus $f \in C^1([0,1], \ell_2)$.

*Proof of Subclaim 9.1.2.* Let $x \in [0,1]$ and $\varepsilon > 0$. Denote by $A_x$ the Fréchet derivative of $f$ at $x$, which has the following form by the previous proof:

$$A_x = \sum_{n=0}^{\infty} f_n'(x) e_{n,b(n)}.$$

The norm of a linear operator from $\mathbb{R}$ to $\ell_2$ (such as $A_x$) is determined by the norm of the value at 1; thus for example,

$$\|A_x\| = \|A_x(1)\|_2 = \sum_{n=0}^{\infty} |f_n'(x)|^2.$$

So for every $y \in [0,1]$, we have

$$
\begin{aligned}
\|A_x - A_y\| &= \Big\| \sum_{n=0}^{\infty} (f_n'(x) - f_n'(y)) e_{n,b(n)} \Big\|_2\\
&= \sqrt{\sum_{n=0}^{\infty} |f_n'(x) - f_n'(y)|^2}.
\end{aligned}
$$

Now, similar to the previous proof, let us find $k \in \mathbb{N}$ such that

$$\sum_{n=k}^{\infty} |f_n'(x) - f_n'(y)|^2 < \varepsilon^2.$$

But

$$|f_n'(x) - f_n'(y)|^2 \leqslant (|f_n'(x)| + |f_n'(y)|)^2 \leqslant \left(\frac{K}{2^n}\right)^2,$$

where $K$ is some constant (this follows again from the definition of $f$). So we can find a big enough $k$ as required. Now, for every $i < k$ pick $\delta_i$ such that for every $y$ in the $\delta_i$-neighbourhood of $x$ we have $|f'_n(x) - f'_n(y)| < \varepsilon/2^n$. This is possible since $f_n$ are smooth by definition. Then let $\delta = \min_{i<k} \delta_i$. Now, if $y$ is the $\delta$-neighbourhood of $x$, then by applying the above, we have

$$\begin{aligned}
\|A_x - A_y\| &= \sqrt{\sum_{n=0}^{\infty} |f'_n(x) - f'_n(y)|^2} \\
&= \sqrt{\Big(\sum_{n=0}^{k-1} |f'_n(x) - f'_n(y)|^2\Big) + \sum_{n=k}^{\infty} |f'_n(x) - f'_n(y)|^2} \\
&\leqslant \sqrt{\Big(\sum_{n=0}^{k-1} |f'_n(x) - f'_n(y)|^2\Big) + \varepsilon^2} \\
&\leqslant \sqrt{\Big(\sum_{n=0}^{k-1} (\varepsilon/2^n)^2\Big) + \varepsilon^2} \\
&< \sqrt{2\varepsilon^2 + \varepsilon^2} \\
&= \sqrt{3}\varepsilon.
\end{aligned}$$

$\square_{\text{Subclaim 9.1.2}}$

**Subclaim 9.1.3.** $f$ is a homeomorphism onto its image.

*Proof of Subclaim 9.1.3.* Since $\operatorname{dom} f$ is compact, it is sufficient to show that it is injective. Let $x, y \in [0,1]$. If there is an interval $I$ which is removed at some stage $n$ in the construction of $C$ such that $x, y \in I$, then $f_n(x) \neq f_n(y)$, because $f'_n(z) > 0$ for all $z \in I$ by the definition of $f_n$. If not, find the least $m$ and an interval $I$ such that $I$ is removed at the $m^{\text{th}}$ stage and $I$ is between $x$ and $y$ or $x \in I \iff y \notin C$. Then clearly again, $f_m(x) \neq f_m(y)$. $\square_{\text{Subclaim 9.1.3}}$

**Subclaim 9.1.4.** $(f \restriction C)^{-1}$ is Lipschitz.

*Proof of Subclaim 9.1.4.* If $\eta \in 2^\omega$, denote by $g(\eta)$ the unique point in $C$ which is obtained by going "left" at stage $n$ if $\eta(n) = 0$ and "right" if $\eta(n) = 1$. That is, $g$ is the canonical homeomorphism of $2^\omega$ onto $C$. It is not hard to see that

$$g(\eta) = \sum_{n=1}^{\infty} \eta(n) \frac{2^{n+1} + 6}{4^{n+1}}.$$

Now, $f_n(g(\eta))$ is the image of $g(\eta)$ under $f_n$, and by the definition of $f_n$, we have $f_n(g(\eta)) = 1/\sqrt{2^n}$ if $\eta(n) = 0$ and $f_n(g(\eta)) = 2/\sqrt{2^n}$ if $\eta(n) = 1$; that is, $f_n(g(\eta)) = (1 + \eta(n))/\sqrt{2^n}$. Let $\eta$ and $\xi$ be two arbitrary elements of $2^\omega$, thus corresponding to the two (arbitrary) elements $g(\eta)$ and $g(\xi)$ of $C$. Denote $c_n = |\eta(n) - \xi(n)|$. Note that for all $n \in \mathbb{N}$, $c_n^2 = c_n$. Then

$$\begin{aligned}
d(g(\eta), g(\xi)) &= \Big|\sum_{n=1}^{\infty} \eta(n)\frac{2^{n+1}+6}{4^{n+1}} - \sum_{n=1}^{\infty} \xi(n)\frac{2^{n+1}+6}{4^{n+1}}\Big| \\
&= \Big|\sum_{n=1}^{\infty} (\eta(n) - \xi(n))\frac{2^{n+1}+6}{4^{n+1}}\Big| \\
&\leqslant \Big|\sum_{n=1}^{\infty} |\eta(n) - \xi(n)|\frac{2^{n+1}+6}{4^{n+1}}\Big| \\
&= \sum_{n=1}^{\infty} c_n \frac{2^{n+1}+6}{4^{n+1}} \\
&= \sum_{n=1}^{\infty} \frac{c_n}{\sqrt{2^n}} \cdot \frac{2^{n+1}+6}{2^{n+1}\sqrt{2^{n+2}}} \\
\text{(Cauchy-Schwarz)} \quad &\leqslant \sqrt{\sum_{n=0}^{\infty} \frac{c_n^2}{2^n}} \cdot \underbrace{\sqrt{\sum_{n=1}^{\infty} \left(\frac{2^{n+1}+6}{2^{n+1}\sqrt{2^{n+2}}}\right)^2}}_{=:L} \\
&= L \cdot \sqrt{\sum_{n=1}^{\infty} \frac{c_n^2}{2^n}} \\
&= L \cdot \sqrt{\sum_{n=1}^{\infty} \Big|\frac{\eta(n)}{\sqrt{2^n}} - \frac{\xi(n)}{\sqrt{2^n}}\Big|^2} \\
&= L \cdot \sqrt{\sum_{n=1}^{\infty} \Big|\frac{1+\eta(n)}{\sqrt{2^n}} - \frac{1+\xi(n)}{\sqrt{2^n}}\Big|^2} \\
&= L \cdot \sqrt{\sum_{n=1}^{\infty} \Big|f_n(g(\eta)) - f_n(g(\xi))\Big|^2} \\
&= L \cdot \|f(g(\eta)) - f(g(\xi))\|_2.
\end{aligned}$$

This verifies that the function $(f \restriction C)^{-1}$ is Lipschitz. $\square_{\text{Subclaim 9.1.4}}$

Since $C$ has positive measure, this implies that the one-dimensional Hausdorff measure of $f[C] = ((f \restriction C)^{-1})^{-1}C$ must also have positive measure. So it remains to show that $f[C] \subset H(T)$, and then the proof of Claim 9.1 is done.

**Subclaim 9.1.5.** $f[C] \subset H(T)$.

*Proof of Subclaim 9.1.5.* Suppose $\eta \in 2^\omega$ and let $g(\eta)$ be as in the previous proof, the canonical image of $\eta$ in $C$. Then, as above,

$$f_n(g(\eta)) = (1 + \eta(n))/\sqrt{2^n},$$

so

$$f(g(\eta)) = \sum_{n=0}^{\infty} \frac{1 + \eta(n)}{\sqrt{2^n}} e_{n,b(n)}.$$

Now, by looking at the definition of $v_s$, one can see that the approximations of $f(g(\eta))$ of the form

$$\sum_{n=0}^{k-1} \frac{1 + \eta(n)}{\sqrt{2^n}} e_{n,b(n)}$$

appear in $v_{b\restriction k}$, so $f(g(\eta)) \in \overline{\bigcup_{s\in T} v_s} = H(T)$. $\square_{\text{Subclaim 9.1.5}}$

$\square_{\text{Claim 9.1}}$

**Claim 9.2.** If $T$ does not have a branch, then $H(T)$ is countable.

*Proof of Claim 9.2.* If $H(T)$ is uncountable, then, because $\bigcup_{s\in T} v_s$ is countable, there is a point $x$ in $\overline{\bigcup_{s\in T} v_s} \setminus \bigcup_{s\in T} v_s$. Let $(p_i)_{i\in\mathbb{N}}$ be a Cauchy sequence of elements of $\bigcup_{s\in T} v_s$ converging to $x$. By going to a subsequence, we can assume that for all $i \in \mathbb{N}$, $d(p_{i+1}, p_i) < 2^{-i}$. The latter inequality implies, by the definition of the sets $v_s$, that if $\operatorname{dom} s \leqslant i$, then

$$p_i \restriction \operatorname{dom} s \in v_s \iff p_{i+1} \restriction \operatorname{dom} s \in v_s.$$

So, we can find $b \in \omega^\omega$ such that $p_i \in v_{b\restriction i}$ for all $i$, and so $(b \restriction n)_{n\in\mathbb{N}}$ must be a branch in $T$. $\square_{\text{Claim 9.2}}$

By Claims 9.1 and 9.2, $T$ has no branch if and only if $H(T)$ is purely unrectifiable which concludes the proof. $\square_{\text{Theorem 9}}$

## References

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