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72 + 42: CHARACTERIZATIONS OF THE COMPLETENESS AND ARCHIMEDEAN PROPERTIES OF ORDERED FIELDS

Abstract

This paper provides a list of statements of single-variable Real Analysis, including well-known theorems, that are equivalent to the completeness or Archimedean properties of totally ordered fields. There are 72 characterizations of completeness and 42 characterizations of the Archimedean property, among them many that appear to be new in the sense that they do not seem to have previously been mentioned in this context. An attempt is made to be as comprehensive as possible and to give a complete account of the current state of knowledge of the matter. Proofs are provided whenever they are not readily available in the literature.

1 Introduction

Completeness is a – in some ways *the* – defining property of the real numbers. While this statement may appear to be a truism, in the last decade or so it has taken on a new meaning. It has been recognized that large swaths of the landscape of single-variable Real Analysis are actually equivalent to completeness [12, 11, 18]. That is to say that many "marquee" theorems of Real Analysis could equivalently be used to *define* the completeness of the real numbers and could replace the standard definitions by means of the existence

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of suprema or the convergence of Cauchy sequences.[1] Or, put differently, if we were to imagine the Analysis landscape in a "mathematical world" (i.e. totally ordered field) without completeness, it would be barren one indeed: many of the most beloved landmarks of Analysis would simply not exist in such a world.

The purpose of this paper is to give an encyclopedic account of this phenomenon. Accordingly, the main part of this paper consists of a massive list of mostly familiar statements of Real Analysis, each of which turns out to be equivalent to completeness in totally ordered fields. We adopt Dedekind's original definition of completeness[2] by means of the "*Cut Axiom*", which stipulates the absence of "*gaps*"; see Section 2.3 below. In addition, we also provide a list of statements equivalent to the *Archimedean property* (AP) of ordered fields. About half of the items on the lists are new (to our knowledge) and have not been mentioned in this context before; the other half are, with few exceptions, drawn from the three references mentioned previously [12, 11, 18].

As Propp points out [11], the exercise of identifying Analysis statements equivalent to completeness has the flavour of *Reverse Mathematics* (RM), with second-order arithmetic used in RM [17] replaced with ZF(C);[3] if $\mathcal{O}$ denotes the collection of axioms for totally ordered fields, (CA) denotes the axiom of completeness, and T is a (possibly second-order) theorem of analysis, we want to show that, in ZF(C), $\mathrm{T} + \mathcal{O} \Leftrightarrow (\mathrm{CA}) + \mathcal{O}$, which is sometimes expressed as T being equivalent to (CA) "over $\mathcal{O}$". Since we only work in ordered fields, we simplify this notation by writing T $\Leftrightarrow$ (CA). Frequently, we need to enhance the axiom system $\mathcal{O}$ by (AP) or the weaker condition of the existence of unbounded sequences, called "*countable cofinality*", to obtain the equivalence of a given statement T to completeness. In these cases, we write "T+(AP) $\Leftrightarrow$ (CA)" or "T+($*$) $\Leftrightarrow$ (CA)", respectively. We emphasize, however, that this is *not* supposed to imply that the addition of the AP or countable cofinality is necessary. In a number of cases this is an open problem.

When proving a particular equivalence, the proof of the standard implication that completeness implies the property in question will often be omitted or reduced to a brief remark on how completeness enters the argument. The unusual part is the proof of the other implication ("reversal" in the parlance of RM), which is almost always accomplished by the construction of a counter example,[4] assuming that the underlying field is *not* complete.

[1] The latter definition requires the *Archimedean property* in addition, a fact that is sometimes overlooked.

[2] For a brief historical account, see [18].

[3] The full Axiom of Choice is never invoked.

[4] playing the rôle of a "*recursive counter example*" in RM

The Intermediate Value Theorem (IVT) may serve as a quintessential example of the basic reasoning applied in this paper. The fact that the IVT is critically based on completeness is well known; so the "reversal" part is the interesting one. Its proof amounts to the construction of a continuous function *not* satisfying the Intermediate Value Property. If the underlying field is incomplete, it possesses a gap, and the characteristic function associated with that gap provides a counterexample.

This construction illustrates a common feature of a large number of counterexamples to be exhibited in this paper: many continuous functions constructed to violate a particular property are piecewise constant and are more-or-less elaborate variations on the theme of step functions or "devil's staircases". This may be seen as a reflection of the fact that the order topology of an incomplete field is very fine[5] – one may say *too* fine, in the sense that simple continuity with respect to this topology allows for pathological functions which are piecewise constant. One possible solution to eliminating these pathological examples is to coarsen the topology, which is the idea behind the "weak topology" for the *Levi-Civita field* in [13, 15]. Another option is to strengthen the notion of continuity. And indeed, in many cases replacing the assumption of continuity by *uniform* continuity weakens a statement in such a way that it is no longer equivalent to completeness; instead, it may become equivalent to the AP. In fact, this may be turned into a systematic way of generating conjectures:[6] if a statement about continuous or differentiable functions is known to be equivalent to completeness, the same statement for *uniform* continuous or differentiable functions is a good candidate to be equivalent to the AP. While this scheme works in some instances such as (CA25) and (AP23), there are a number of cases where we don't know what happens. The most prominent – and most troubling to the authors – of those is again the IVT: while it is easy to see that the uniform version of the IVT implies the AP, it is unclear whether this version is still strong enough to imply completeness.

In some sense this is only the tip of the iceberg in that there are many open problems and conjectures that have arisen in the course of this project. A collection will be provided at the end of this paper.

Before outlining the structure of this paper, we briefly comment on how it might be read. Readers may want to look for their favourite fact of Analysis and see if it, too, happens to be equivalent to completeness; instructors of Real Analysis may want to consider regaling their students with defining $\mathbb{R}$ as the unique ordered field in which some obscure property is satisfied (for example, how about "every equicontinuous family of functions defined on a

[5] An incomplete field is totally disconnected; see (CA8) in Section 3.1.

[6] affectionately called the "conjecture machine" by the authors

closed and bounded interval is uniformly equicontinuous" or the Weierstraß Approximation Theorem?); or one may simply marvel at the sheer size of the lists: who would have thought that there are more than 70 ways of expressing completeness and another 40+ statements equivalent to the AP? The authors certainly had no idea when they started this project. Maybe readers will start feeling the thrill of the chase that the authors felt, and will start searching for their own characterizations of completeness or the AP. The authors certainly do not think that this is the end of the story, nor do they believe that the story will ever be complete...

This paper is organized as follows: we first present a quick overview of ordered fields and their properties, including a more formal description of some of the terms mentioned in this introduction. We then present the list of statements equivalent to completeness (Section 3.1). The proofs of equivalence appear in section following the list (3.2). We then repeat this with the list / proofs for statements equivalent to the Archimedean Property (Sections 4.1 and 4.2). Finally, we remark on some open problems and conjectures (Section 5).

2 Ordered fields: preliminaries

In this section we collect some basic definitions and properties of (totally) ordered fields, which we denote by $\mathbb{F}$ with order $<$.

2.1 Intervals, convex sets, and order topology

We use the familiar notations $[a,b] = \{x \in \mathbb{F} \mid a \leq x \leq b\}$, $(-\infty, b) = \{x \in \mathbb{F} \mid x < b\}$ etc., with the obvious meaning of $\leq$. Moreover, $\mathbb{F}^+ := (0, \infty) = \{x \in \mathbb{F} \mid x > 0\}$. A subset $I \subset \mathbb{F}$ is an *interval* if it is of one of the types $[a,b]$, $[a,b)$, $(a,b]$, (a,b), $(-\infty, b]$, $(-\infty, b)$, $[a, \infty)$, (a, ∞), or $(-\infty, \infty) = \mathbb{F}$ (with $a, b \in \mathbb{F}$, $a \leq b$). A subset $C \subset \mathbb{F}$ is *convex* if $[a,b] \subset C$ for all $a, b \in C$ with $a \leq b$. We consider $\mathbb{F}$ with the *order topology*, i.e. with the topology generated by the intervals (a,b).

It is easy to see that every interval is a convex set. However, the converse is not always true, but depends on the completeness of $\mathbb{F}$ (see (CA6) in Section 3.1). Similarly, every connected set is convex, but if $\mathbb{F}$ is not complete, there exist convex sets that are not connected such as $[0,1]$; see (CA5) and (CA7) in Section 3.1.

2.2 Archimedean property, countable cofinality, $\mathbb{Q}(X)$

The natural and rational numbers are contained in any ordered field $\mathbb{F}$; that is, there is a canonical order-preserving field homomorphism from $\mathbb{Q}$ into $\mathbb{F}$.

Definition 2.1 (Archimedean Property, Countable Cofinality).
A field $\mathbb{F}$ is said to be Archimedean *or to* have the Archimedean Property*, if, for every pair $a, b \in \mathbb{F}^+$, there is an $n \in \mathbb{N}$ such that $a < nb$.*
$\mathbb{F}$ is said to be countably cofinal *if there exists a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ that is unbounded in $\mathbb{F}$.*

Archimedean fields are countably cofinal (since $\{n\}_{n\in\mathbb{N}}$ is an unbounded sequence); the *rational fraction field* $\mathbb{Q}(X)$ provides an example of a non-Archimedean countably cofinal field. $\mathbb{Q}(X)$ is the field of fractions for the polynomial ring $\mathbb{Q}[X]$. The order for $\mathbb{Q}(X)$ is defined as follows: let x be an element of $\mathbb{Q}(X)$ with lowest-terms representation $x \equiv (\sum_{j=0}^{m} p_j X^j)/(\sum_{k=0}^{n} q_k X^k)$. Let p be the non-zero p_j with the smallest index (or 0 if all p_j are 0), and similarly let q be the non-zero q_k with the smallest index. Then x is positive iff p/q is positive. With this order, $\mathbb{Q}(X)$ is clearly not Archimedean, as, for example, $1/X > n$ for all $n \in \mathbb{N}$. However, $\mathbb{Q}(X)$ is countably cofinal; the sequence $\lambda_n = 1/X^n$ is easily seen to be unbounded.

In fact, elements like X and $1/X$ occur in every non-Archimedean field, which warrants the following definition.

Definition 2.2 (Infinitesimals and Infinitely Large Elements).
An element $\delta \in \mathbb{F}$ is called infinitesimal *if $|\delta| < \frac{1}{n}$ for all $n \in \mathbb{N}$. This is also denoted by $|\delta| \ll 1$. (Note that* 0 is *an infinitesimal.)*
An element $\lambda \in \mathbb{F}$ is called infinitely large *if $n < \lambda$ for all $n \in \mathbb{N}$. This is also denoted by $1 \ll \lambda$.*

2.3 Cuts, gaps, completeness

A pair of non–empty subsets $A, B \subset \mathbb{F}$ satisfying $A \cup B = \mathbb{F}$ and $a < b$ for all $a \in A$ and $b \in B$, is called a *cut* of $\mathbb{F}$. Cuts were introduced by Dedekind [3] for the purpose of defining a notion of completeness as well as constructing the real numbers.

If A, B is a cut of $\mathbb{F}$, a number $c \in \mathbb{F}$ such that $a \le c \le b$ for all $a \in A$ and $b \in B$ is called *a cutpoint of* A, B. It is easy to show that c is unique, so we might say it is *the* cutpoint of A, B.

A cut A, B without a cutpoint is called a *gap* (or *Dedekind cut* [16]) for $\mathbb{F}$.

Suppose A, B is a gap for $\mathbb{F}$. If we can find a $\delta \in \mathbb{F}^+$ such that for every $a \in A$ and $b \in B$, $b - a \ge \delta$, then we say that A, B is an *irregular gap* and that the *gap length* of A, B is (at least) δ. If no such δ exists, A, B is a *regular gap*. Note that if A, B is a gap, then both A and B are open (and closed).

Definition 2.3 (Cut Axiom for Completeness)**.**
A field is (Dedekind) complete *if there are no gaps; i.e., if every cut has a cutpoint.*

2.4 Integration

In general ordered fields — in particular non-Archimedean ones — integration is more multifaceted than in the reals, i.e. several definitions that are equivalent in $\mathbb{R}$ may result in different sets of integrable functions when considered in general fields $\mathbb{F}$.

Let $a, b \in \mathbb{F}$, $a \leq b$. The *set of step functions on* $[a, b]$, denoted $\mathcal{S}_{[a,b]}$, is the set of all functions that are of the form $\sum_{k=1}^{n} a_k \chi_{A_k}(x)$ where $\{A_n\}$ is an interval partition of $[a, b]$, $a_k \in \mathbb{F}$, and $\chi_{A_k}(x)$ denotes the *characteristic function for* A_k. Let $f : [a, b] \to \mathbb{F}$ be an arbitrary function. Then the sets of step functions above and below f are defined by $\mathcal{U}_f = \{s \in \mathcal{S}_{[a,b]} \mid s \geq f\}$ and $\mathcal{L}_f = \{s \in \mathcal{S}_{[a,b]} \mid s \leq f\}$, repectively.

Definition 2.4 (Darboux Integrability, Darboux Integral)**.**
A function $f : [a, b] \to \mathbb{F}$ is Darboux integrable *if, for any $\varepsilon \in \mathbb{F}^+$, there exist step functions $g \in \mathcal{L}_f$ and $h \in \mathcal{U}_f$ such that $\int_a^b h - \int_a^b g < \varepsilon$.*
We say that f has a Darboux integral *if $\mathcal{L}_f$ and $\mathcal{U}_f$ are non-empty and there is a unique number $J \in \mathbb{F}$ such that $\int_a^b g \leq J \leq \int_a^b h$ for all $g \in \mathcal{L}_f$ and for all $h \in \mathcal{U}_f$. If f has a Darboux integral, we set $\int_a^b f \equiv J$.*

It is easy to see that having a Darboux integral implies Darboux integrability; however, in an incomplete field, the converse is not true; see (CA34) below. This approach to integration is adapted from Olmsted [9]. However, Olmsted calls this "having a *Riemann* integral" and "*Riemann* integrable". We reserve the term "Riemann" for the standard integral defined, as in $\mathbb{R}$, by limits of Riemann sums. While Darboux integrability and the Darboux and Riemann integrals coincide on $\mathbb{R}$, this is generally not the case in incomplete and non-Archimedean fields.

The following definitions will prove useful later.

Definition 2.5 (Step-Regulated & Limit-Regulated Functions)**.**
A function $f : [a, b] \to \mathbb{F}$ is called step-regulated *if it can be written as the uniform limit of step functions. f is said to be* limit-regulated *if its one-sided limits exist at every point of $[a, b]$.*

The one-sided limits, if existent, are denoted by $f(x^-)$ and $f(x^+)$.

2.5 Uniform differentiability and uniform equicontinuity

Generally speaking, all "ε-δ-definitions" of Real Analysis such as (uniform) convergence, (uniform) continuity, or differentiability may be used verbatim in general ordered fields $\mathbb{F}$. In this section we explicitly state two lesser-familiar definitions, namely uniform differentiability and uniform equicontinuity.

Definition 2.6 (Uniform Differentiability)**.**
A function $f : [a,b] \to \mathbb{F}$ *is* uniformly differentiable on $[a,b]$ *if there exists a function* $g : [a,b] \to \mathbb{F}$ *such that for every* $\varepsilon \in \mathbb{F}^+$, *there is a* $\delta \in \mathbb{F}^+$ *such that*

$$\left| \frac{f(x) - f(y)}{x - y} - g(y) \right| < \varepsilon$$

for every pair $x, y \in [a,b]$ *with* $0 < |x - y| < \delta$. *In this case, we write* $f' = g$.

A function from $\mathbb{R}$ to $\mathbb{R}$ is uniformly differentiable iff its derivative is uniformly continuous. It turns out that the "$\Leftarrow$"–direction of this statement is equivalent to the completeness of Archimedean fields; see (CA52).

Definition 2.7 (Uniform Equicontinuity)**.**
A family $\{f_n\}$ *of functions* $f_n : [a,b] \to \mathbb{F}$ *is called* uniformly equicontinuous on $[a,b]$ *if, for every* $\varepsilon \in \mathbb{F}^+$, *there is a* $\delta \in \mathbb{F}^+$ *such that* $|f_n(x) - f_n(y)| < \varepsilon$ *for every pair* $x, y \in [a,b]$ *with* $|x - y| < \delta$ *and* $n \in \mathbb{N}$.

In Real Analysis a family of functions is equicontinuous iff it is *uniformly* equicontinuous. While "$\Leftarrow$" is obvious, the "$\Rightarrow$"–implication is equivalent to completeness (see (CA62)).

3 Characterizations of Dedekind Completeness

The statements listed below are equivalent to the completeness of a totally ordered a (countably cofinal) field $\mathbb{F}$.

Many of these statements are collected from other authors and, therefore, a statement may be "tagged" with up to three references, each indicating an existing list and the position of the property on that list. **(R##)**, **(P##)** and **(T##)** are adapted from [12], [11] and [18], respectively, while any property that is unmarked is not included in those lists and will be proven equivalent to completeness in Section 3.2. If the author provides a proof, the tag will indicate as such with a '†'. If no proof is given by any of the authors, an additional citation will be given to a published – and to our knowledge, correct – proof, although this proof may not be the oldest or original version.

Additionally, a property may only be equivalent to the completeness of fields that are Archimedean. In that case, the property will include "**+(AP)**" and the additional condition that "$\mathbb{F}$ is Archimedean" will be added to the property's definition. If we make reference to a particular statement that is equivalent to the Archimedean Property, it will be listed as "(AP##)" and can be found in full in the second half of this document (Section 4.1). Statements whose proofs make use of the assumption of countable cofinality are marked with a '*'. Note however that we do not claim that the Archimedean Property (or countable cofinality) is necessary in all cases; it is possible that a statement is equivalent to completeness under some weaker assumption or perhaps with no additional assumptions at all.

To avoid any impression of circular reasoning, when proving a given result, we will only make use of results with lower indices on the list. The label "(CA)" means "any equivalent formulation of completeness"; more precisely, "(CA) $\Leftrightarrow$ (CAn)" is to be interpreted as "(CAk) $\Leftrightarrow$ (CAn) for all $k \leq n-1$."

3.1 The List

CA0. **(Cut Property)**: (R12), (P3), (T1)
If $A, B \subset \mathbb{F}$ are non–empty subsets such that $A \cup B = \mathbb{F}$ and $a < b$, for all $a \in A$ and $b \in B$, then there exists a $c \in \mathbb{F}$ such that $a \leq c \leq b$ for all $a \in A$ and $b \in B$.

CA1. **(Modified Cut Property / Tarksi's Axiom 3)**: (T2†)
If $A, B \subset \mathbb{F}$ are non–empty subsets such that $a < b$, for all $a \in A$ and $b \in B$, then there exists a $c \in \mathbb{F}$ such that $a \leq c \leq b$ for all $a \in A$ and $b \in B$.

CA2. **(Existence of Suprema)**: (R6†), (R6′), (P1), (T5†)
All bounded, non-empty subsets of $\mathbb{F}$ have a least upper bound.

CA3. **(Intervals are Connected)**: (R9†)
If $I \subset \mathbb{F}$ is an interval and $A, B \subset I$ are non-empty subsets of I such that $A \dot{\cup} B = I$, then one of A, B is not (relatively) open in I.

CA4. **($\mathbb{F}$ is Connected)**: (R11†), (P4†), (T3)
$\mathbb{F}$ is a connected topological space.

CA5. **(Unit Interval is Connected)**: (R7†)
The subset $[0, 1] \subset \mathbb{F}$ is connected.

CA6. **(Convex Sets Are Intervals)**:
Every convex subset of $\mathbb{F}$ is an interval.

CA7. **(Convex Sets Are Connected)**:
Every convex subset of $\mathbb{F}$ is connected.

CA8. **($\mathbb{F}$ is Not Totally Disconnected)**: (R8†)
There exists a connected subset $I \subset \mathbb{F}$ that contains more than one element.

CA9. **(Principle of Real Induction)**: (T4), [1]
If $S \subset [a, b]$ is a subset that satisfies (i) $a \in S$ (ii) $\forall x \in S \setminus \{b\}$ $\exists y > x$ such that $[x, y] \subset S$ (iii) $[a, x) \subset S \Rightarrow [a, x] \subset S$, then $S = [a, b]$.

CA10. **(Unit Interval is Compact)**: (R23†)
If $\{U_\alpha : \alpha \in A\}$ is a collection of open subsets of $\mathbb{F}$ such that $[0, 1] \subset \bigcup_{\alpha\in A} U_\alpha$, then there exists a finite subset J of A such that $[0, 1] \subset \bigcup_{j\in J} U_j$.

CA11. **(Heine-Borel)**:
Every closed and bounded subset of $\mathbb{F}$ is compact.

CA12. **(Lebesgue's Lemma)+(AP)**: (R24†)
$\mathbb{F}$ is Archimedean and for every open cover $\{U_\alpha : \alpha \in A\}$ of $[a, b]$ there exists a $\delta \in \mathbb{F}^+$ such that, if $x, y \in [a, b]$ with $|x - y| \leq \delta$, there exists $\beta \in A$ such that $x, y \in U_\beta$.

CA13. **(Bounded Monotone Convergence Property)**: (R2†), (P10†), (T6)
Every sequence $\{x_n\} \subset \mathbb{F}$ satisfying $x_n \leq x_{n+1} \leq B$, for all $n \in \mathbb{N}$ and some $B \in \mathbb{F}$, converges in $\mathbb{F}$.

CA14. **(Bolzano-Weierstrass)**: (R5†), (T7)
For every bounded sequence $\{x_n\}$ in $\mathbb{F}$, there exists a strictly increasing sequence $\{n_i\} \subset \mathbb{N}$ such that $\{x_{n_i}\}$ converges in $\mathbb{F}$.

CA15. **(Strong Nested Interval Property (NIP))+(AP)**: (R3†),(P18),(T8)
$\mathbb{F}$ is Archimedean and for every sequence of closed and bounded intervals $\{I_n\}$ satisfying $I_{n+1} \subset I_n$ for all $n \in \mathbb{N}$, there exists some $c \in \mathbb{F}$ such that $c \in I_n$ for all $n \in \mathbb{N}$.

CA16. **(Weak NIP)+(AP)**: (R4†)
$\mathbb{F}$ is Archimedean and for every sequence of intervals $\{[a_n, b_n]\}$ such that $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$ $(\forall n \in \mathbb{N})$ and the sequence $\{b_n - a_n\}$ converges to zero, there exists some $c \in \mathbb{F}$ such that $c \in [a_n, b_n]$ for all $n \in \mathbb{N}$.

CA17. **(Strong Base g NIP)**: (P(5.3))[7]
For all $g \in \mathbb{N}$ with $g > 1$ and $\{d_n\} \subset \mathbb{Z}_g$, and if the intervals I_n are given by

$$I_n = \left[\sum_{k=0}^{n} d_k g^{-k}, \sum_{k=0}^{n} d_k g^{-k} + g^{-n}\right]$$

for all $n \in \mathbb{N}$, then there exists a unique $c \in \mathbb{F}$ such that $c \in I_n$ for all $n \in \mathbb{N}$.

CA18. **(Arbitrary Base g Expansion Property)**: (R34†)
For every $g \in \mathbb{N}$ with $g > 1$, let the set G of formal power series be defined as

$$G \equiv \left\{ \sum_{n=0}^{\infty} a_n g^{k-n} \;\middle|\; \{a_n\} \subset \mathbb{Z}_g \text{ and } k \in \mathbb{Z} \right\}.$$

Then any element of G converges in $\mathbb{F}$ and every $x \in \mathbb{F}^+$ can be expressed as an element of G.[8]

CA19. **(Intermediate Value Property)**: (R10†), (P5†), (T9†)
For every continuous function $f : [a, b] \to \mathbb{F}$ and $y \in \mathbb{F}$ between $f(a)$ and $f(b)$, there is some $c \in [a, b]$ such that $f(c) = y$.

CA20. **(Strictly Monotonic Continuous Functions Have the IVP)**:
Every strictly monotonic, continuous function has the Intermediate Value Property.

CA21. **(Rolle's Property)**: (R15†)
For every continuous function $f : [a, b] \to \mathbb{F}$, differentiable on (a, b) with $f(a) = f(b)$, there is some $c \in [a, b]$ such that $f'(c) = 0$.

CA22. **(Mean Value Property)**: (R17†), (P8†), (T10†)
For every continuous function $f : [a, b] \to \mathbb{F}$, differentiable on (a, b), there is some $c \in [a, b]$ such that $f'(c) = (f(b) - f(a))/(b - a)$.

CA23. **(Cauchy's Mean Value Property)**: (R16†)
For every pair of continuous functions $f : [a, b] \to \mathbb{F}$ and $g : [a, b] \to \mathbb{F}$, both differentiable on (a, b), there is some $c \in [a, b]$ such that

$$(g(b) - g(a))f'(c) = (f(b) - f(a)g'(c).$$

[7] This property is described in Section 5.3 of [11], but is not formally listed as an axiom there, nor is a proof given.

[8] Putting these together gives a bijection between G and $\mathbb{F}^+$, but this formulation exposes the difference between this and (AP16).

CA24. **(Extreme Value Property)**: (R14†), (P7†), (T11†)
For every continuous function $f : [a, b] \to \mathbb{F}$ there is some $c \in [a, b]$ such that $f(x) \leq f(c)$ for any $x \in [a, b]$.

CA25. **(Bounded Value Property) + (*)**: (R27†), (P6)[9], (T14†)
For every continuous function $f : [a, b] \to \mathbb{F}$ there is some $B \in \mathbb{F}^+$ such that $|f(x)| \leq B$ for all $x \in [a, b]$.

CA26. **(Modified Bounded Value Property)**: (R28†)
For every continuous function $f : [a, b] \to \mathbb{F}$ there is some $B \in \mathbb{N}$ such that $|f(x)| \leq B$ for all $x \in [a, b]$.

CA27. **(Darboux's Property)**: (T12†)
For every differentiable function $f : [a, b] \to \mathbb{F}$, the derivative f' has the Intermediate Value Property on $[a, b]$.

CA28. **(Open Function Property)**: (T13), [8]
Every continuous and injective function maps open sets to open sets.

CA29. **(Uniform Continuity Property) + (*)**: (R25†), (T15†)
Every continuous function defined on a closed and bounded interval is uniformly continuous.

CA30. **(Extensibility to the Boundary) + (*)**:
Every uniformly continuous function $f : (a, b] \to \mathbb{F}$ has a continuous extension to $[a, b]$.

CA31. **(C^1 Functions Are Uniformly Continuous)+(AP)**:
$\mathbb{F}$ is Archimedean and every continuously differentiable function defined on a closed and bounded interval is uniformly continuous.

CA32. **(Darboux Integral Property) + (*)**: (T16†)
Every continuous function defined on a closed and bounded interval has a Darboux integral.

CA33. **(Uniform Darboux Integral Property)**:
Every uniformly continuous function defined on a closed and bounded interval has a Darboux integral.

CA34. **(Integral Equivalence Property)+(AP)**: (T18†)
$\mathbb{F}$ is Archimedean and every Darboux integrable function has a Darboux integral.

[9]Propp proves that the assumption of countable cofinality is necessary, unlike the other authors, but does not prove the result as a whole.

CA35. **(Cauchy Completeness)+(AP)**: (R1†), (P11)[10], (T17†)
$\mathbb{F}$ is Archimedean and all Cauchy sequences in $\mathbb{F}$ are convergent in $\mathbb{F}$.

CA36. **(Taylor's Expansion Property)**: (R18†)
For every function $f : [a, b] \to \mathbb{F}$ that is n times continuously differentiable on $[a, b]$ and $n+1$ times continuously differentiable on (a, b), there exists a number $c \in (a, x)$ for every $x \in (a, b]$ such that

$$f(x) = \left(\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \right) + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

CA37. **(Characterization of Polynomials)**: (R19†)
Every function $f : [a, b] \to \mathbb{F}$ that is n times continuously differentiable on $[a, b]$, $n+1$ times continuously differentiable on (a, b) and that satisfies $f^{(n+1)} \equiv 0$ is a polynomial of degree at most n.

CA38. **(Constant Value Property)**: (R20†), (P9†)
Every continuous function $f : [a, b] \to \mathbb{F}$, continuously differentiable on (a, b) with $f' \equiv 0$, is constant on $[a, b]$.

CA39. **(Monotonicity Property)**: (R21†)
Every function $f : [a, b] \to \mathbb{F}$, differentiable on (a, b) with $f'(x) \geq 0$ for all $x \in (a, b)$, is increasing.

CA40. **(Convexity Property)**: (R22†)
Every function $f : [a, b] \to \mathbb{F}$, twice differentiable on (a, b) with $f''(x) \geq 0$ for all $x \in (a, b)$, is convex.

CA41. **(Uniform Approximation by Step Functions)**: (R26†)
Every continuous function defined on a closed and bounded interval is step-regulated.

CA42. **(Increasing Antiderivative Property)**:
If $f : [a, b] \to \mathbb{F}$ is a continuous function satisfying $f(x) \geq 0$ for all $x \in [a, b]$, then any antiderivative of f is increasing.

CA43. **(Antiderivatives Differ by Constants)**:
For every continuous function $f : [a, b] \to \mathbb{F}$ and any pair of antiderivatives $F, G : [a, b] \to \mathbb{F}$ of f, the function $(F - G)$ is constant on $[a, b]$.

[10] Propp proves that (AP) is necessary but does not prove the result as a whole.

CA44. **(Maximal Archimedean Property)+(AP)**: (R33†)
$\mathbb{F}$ is Archimedean and any Archimedean totally ordered field can be embedded in $\mathbb{F}$, i.e. for every Archimedean totally ordered field $\mathbb{K}$ there exists an order-preserving field homomorphism from $\mathbb{K}$ to $\mathbb{F}$.

CA45. **(Fixed Point Property)**: (P12†)
For every continuous function $f : [a,b] \to [a,b]$, there exists a number $z \in [a,b]$ such that $f(z) = z$.

CA46. **(Contraction Mapping Property)**: (P13†)
For every function $f : \mathbb{F} \to \mathbb{F}$ satisfying $|f(x) - f(y)| < c|x - y|$ for all $x, y \in \mathbb{F}$ and some $c < 1$, there exists a number z such that $f(z) = z$.

CA47. **(Ratio Test)**: (P16†)
For every sequence $\{a_n\}$ in $\mathbb{F}$ such that $\lim_{n\to\infty} |a_{n+1}/a_n| < 1$, the series $\sum_{n=1}^{\infty} a_n$ converges in $\mathbb{F}$.

CA48. **(Comparison Test) + (AP)**: (R35)
$\mathbb{F}$ is Archimedean and if $\{a_n\}$, $\{b_n\}$ are non-negative sequences with $a_n \le b_n$ for all $n \in \mathbb{N}$, then $\sum_{n=0}^{\infty} a_n$ converges in $\mathbb{F}$ if $\sum_{n=0}^{\infty} b_n$ converges in $\mathbb{F}$.

CA49. **(Rearrangement Property) + (AP)**: (R36†)
$\mathbb{F}$ is Archimedean and if $\sum_{n=0}^{\infty} a_n$ is an absolutely convergent series, then any rearrangement series also converges absolutely.

CA50. **(Absolute Convergence Test) + (AP)**: (R37†)
$\mathbb{F}$ is Archimedean and if $\{a_n\}$ is a sequence such that $\sum_{n=0}^{\infty} |a_n|$ converges, then $\sum_{n=0}^{\infty} a_n$ also converges.

CA51. **(Weierstrass Approximation Property) + (*)**:
For every continuous function $f : [a,b] \to \mathbb{F}$ and $\varepsilon \in \mathbb{F}^+$, there exists a polynomial $p : [a,b] \to \mathbb{F}$ such that $|f(x) - p(x)| < \varepsilon$ for all $x \in [a,b]$.

CA52. **(f' Uniformly Continuous Implies f Uniformly Differentiable) +(AP)**:
$\mathbb{F}$ is Archimedean and every function $f : [a,b] \to \mathbb{F}$ whose derivative (exists and) is uniformly continuous is uniformly differentiable.

CA53. **(Modified FTC I)+(AP)**:
$\mathbb{F}$ is Archimedean and for every continuous function $f : [a,b] \to \mathbb{F}$ there exists a uniformly differentiable function $F : [a,b] \to \mathbb{F}$ such that $F'(x) = f(x)$ for every $x \in [a,b]$.

CA54. **(FTC II: Evaluation Property)**:
For every function $f : [a, b] \to \mathbb{F}$ possessing a Darboux intergral and an antiderivative F the identity $\int_a^b f(x)\,dx = F(b) - F(a)$ holds.

CA55. **(One-Sided Limits of Monotone Functions) + (*)**:
Every monotone function is limit-regulated.

CA56. **(Step-Regulated Implies Limit-Regulated) + (AP)**:
$\mathbb{F}$ is Archimedean and every step-regulated function is limit-regulated.[11]

CA57. **(Limit-Regulated Implies Step-Regulated)**:
Every limit-regulated function is step-regulated.

CA58. **(Limit-Regulated Implies Bounded) + (*)**:
Every limit-regulated function is bounded.

CA59. **(Jumps of Limit-Regulated Functions 1)**:
For every limit-regulated function $f : [a, b] \to \mathbb{F}$ and $\varepsilon \in \mathbb{F}^+$, the set $D_\varepsilon(f) = \{x \in [a, b] | \max\{|f(x^-) - f(x)|, |f(x^+) - f(x)|\} \geq \varepsilon\}$ is finite.

CA60. **(Jumps of Limit-Regulated Functions 2)**:
There exists an $\varepsilon \in \mathbb{F}^+$ such that the set $D_\varepsilon(f)$ is finite for every limit-regulated function f.

CA61. **(Convergence of Derivative & Uniform Convergence) + (*)**:
For every sequence $\{f_n\}$ of differentiable functions $f_n : [a, b] \to \mathbb{F}$ such that $\{f'_n\}$ converges uniformly in $\mathbb{F}$ and there is some $c \in [a, b]$ such that $\{f_n(c)\}$ converges in $\mathbb{F}$, $\{f_n\}$ converges uniformly in $\mathbb{F}$.

CA62. **(Equicontinuity Implies Uniform Equicontinuity) + (*)**:
Every equicontinuous family $\{f_n\}$ of functions $f_n : [a, b] \to \mathbb{F}$ is uniformly equicontinuous.

CA63. **(Arzelà–Ascoli 1)**:
Every uniformly bounded equicontinuous family of functions has a pointwise convergent subsequence.

CA64. **(Arzelà–Ascoli 2) + (*)**:
Every pointwise convergent equicontinuous family of functions converges uniformly.

[11] Note that limit– and step–regulated functions are defined on closed and bounded intervals; see Def. 2.5

CA65. **(C^1 Functions Are Lipschitz) + (*)**:
Every continuously differentiable function defined on a closed and bounded interval is Lipschitz.

CA66. **(Continuous Functions Preserve Cauchy Sequences) + (*)**:
If $f : [a, b] \to \mathbb{F}$ is a continuous function and $\{x_n\} \subset [a, b]$ is a Cauchy sequence, then $\{f(x_n)\}$ is also a Cauchy sequence.

CA67. **(L'Hôpital's Rule 1)+(*)**:
For every pair of differentiable functions $f, g : (a, b) \to \mathbb{F}$ with $g'(x) \neq 0$ anywhere, $\lim_{x\to a^+} f(x) = \lim_{x\to a^+} g(x) = 0$ and $\lim_{x\to a^+} f'(x)/g'(x) = L$ for some $L \in \mathbb{F}$, the limit $\lim_{x\to a^+} f(x)/g(x)$ exists and is equal to L.

CA68. **(L'Hôpital's Rule 2)+(*)**:
For every pair of differentiable functions $f, g : (a, \infty) \to \mathbb{F}$ with $g'(x) \neq 0$ anywhere, $\lim_{x\to\infty} g(x) = \infty$ and $\lim_{x\to\infty} f'(x)/g'(x) = L$ for some $L \in \mathbb{F}$, the limit $\lim_{x\to\infty} f(x)/g(x)$ exists and is equal to L.

CA69. **(Convergence Property for Darboux Integration)+(AP)**:
$\mathbb{F}$ is Archimedean and every function that is the uniform limit of functions possessing Darboux integrals has a Darboux integral as well.

CA70. **(Injective Functions are Monotonic)**:
Every continuous, injective function on $[a, b]$ is monotonic on $[a, b]$.

CA71. **(Approximate Intermediate Value Property)**:
For every continuous function $f : [a, b] \to \mathbb{F}$, $\varepsilon \in \mathbb{F}^+$ and $y \in \mathbb{F}$ between $f(a)$ and $f(b)$, there is some $c \in [a, b]$ such that $|f(c) - y| < \varepsilon$.

CA72. **(Weak Intermediate Value Property)**:
There exists an $\varepsilon \in \mathbb{F}^+$ such that, for every continuous function $f : [a, b] \to \mathbb{F}$ and $y \in \mathbb{F}$ between $f(a)$ and $f(b)$, there is some $c \in [a, b]$ such that $|f(c) - y| < \varepsilon$.

Remarks. (a) In (CA32), (CA33),(CA34),(CA54), and (CA69) "f has a Darboux integral" can be replaced with "f is Riemann-integrable."
(b) In (CA63) "equicontinuous" may be replaced by "*uniformly* equicontinuous".
(c) Technically, the list could be extended by another eight items, as there are also "approximate" and "weak" versions of Rolle's (CA21), the Mean Value (CA22), Cauchy's Mean Value (CA23) and Darboux's (CA27) Properties, which are all equivalent to completeness. However, we thought it sufficient to only list two representative cases, (CA71) and (CA72).

3.2 Assorted proofs

Proof of (CA) $\Leftrightarrow$ (CA6).
($\Rightarrow$): Standard Real Analysis (note that, if C is a bounded convex set, its endpoints are given by $\inf(C)$ and $\sup(C)$).
($\Leftarrow$): Let $\mathbb{F}$ be incomplete, A, B a gap, $a \in A$, and $C := [a, \infty) \cap A$. Then C is convex: to see this, let $\tilde{a}, \tilde{b} \in C$, $\tilde{a} < \tilde{b}$, and $\tilde{a} < x < \tilde{b}$. Then, clearly, $x \in (a, \infty)$. Moreover, if x were in B, then $\tilde{b} > x$ would be in B as well; contradiction. So $x \in A$ and $x \in C$. Now assume that there was an element $b \in \mathbb{F}$ such that $C = [a, b]$ or $C = [a, b)$ (the case $[a, \infty)$ does not occur, since C is bounded above by any element of B). Considering the first case, by the definition of C, $b \in A$ and $b < y$ for all $y \in B$. Since b cannot be a cut point for A, B, there must be an $x \in A$ such that $b < x$, which implies $x \in C \setminus [a, b]$, a contradiction. If $C = [a, b)$, $b \in A$ would lead to the same contradiction, so assume $b \in B$. As before, since b cannot be a cutpoint, there is an $y \in B$ such that $y < b$. But this means $y \in [a, b) = C \subset A$ and so $y \in A \cap B = \emptyset$, again a contradiction. $\square$

Proof of (CA) $\Leftrightarrow$ (CA7).
($\Rightarrow$): Follows from (CA3) and (CA6).
($\Leftarrow$): If $\mathbb{F}$ is incomplete, the set $[0, 1]$ is convex but not connected by (CA5). $\square$

Proof of (CA) $\Leftrightarrow$ (CA11).
($\Rightarrow$): Standard Real Analysis.
($\Leftarrow$): If we assume (CA11), it immediately follows that $[0, 1]$ is compact, which is the statement of (CA10). $\square$

Proof of (CA) $\Leftrightarrow$ (CA20).
This proof is straight-forward and omitted. $\square$

In the next lemma we list the properties of a particular function, which will serve as a counter example in a number of proofs. The idea of the construction goes back to an earlier version of [11]; a detailed proof is given in [4].

Lemma 3.1.
Let $\mathbb{F}$ be an incomplete Archimedean field (considered as a subfield of $\mathbb{R}$) and $c \in \mathbb{R} \setminus \mathbb{F}$ with $3/8 < c < 5/8$. There exists a (uniformly) continuous real function $f : [0, 1]_{\mathbb{R}} \to \mathbb{R}$ with $\int_0^1 f(x)\, dx = c$ so that $f|_{\mathbb{F}} : [0, 1] = [0, 1]_{\mathbb{R}} \cap \mathbb{F} \to \mathbb{F}$ is well-defined. Furthermore, the function takes on non-negative values everywhere in $[0, 1]_{\mathbb{R}}$ and hence its antiderivative function F is strictly increasing. Finally, there is a sequence of Riemann-integrable functions in $\mathbb{F}$ uniformly convergent to f.

Proof of (CA) $\Leftrightarrow$ (CA30).
($\Rightarrow$): This is a standard proof. However, we note that it only uses Cauchy completeness and makes no assumption about the field being Archimedean. Hence, the statement holds true in any Cauchy complete field and so the assumption of Archimedean is necessary.
($\Leftarrow$): If $\mathbb{F}$ is non-Archimedean, then the function constructed in the proof of [18, Prop.4 (ix)] provides a counterexample. So suppose $\mathbb{F}$ is an incomplete Archimedean ordered field. Let f be the function from Lemma 3.1 and define $F : (0,1]_{\mathbb{R}} \to \mathbb{R}$ by $F(x) = \int_x^1 f(t)\,dt$. F is uniformly continuous and hence $F|_{\mathbb{F}}$ is as well. However, if $F|_{\mathbb{F}}$ could be extended to $[0,1]$, it would agree with the extension of F to $[0,1]_{\mathbb{R}}$ at 0, i.e. $F|_{\mathbb{F}}(0) = F(0) = \int_0^1 f(t)\,dt = c$, which is impossible since $c \notin \mathbb{F}$. $\square$

Proof of (CA) $\Leftrightarrow$ (CA31).
($\Rightarrow$): This is a standard proof of Real Analysis based on (CA25) and (CA22).

($\Leftarrow$): Suppose $\mathbb{F}$ is not Dedekind complete and suppose $\mathbb{F}$ is Archimedean. Let A, B be a gap. Let $a \in A$ and $b \in B$ be arbitrary, and define $f : [a,b] \to \mathbb{F}$ as

$$f(x) = \begin{cases} 0, & \text{if } x \in A \cap [a,b] \\ 1, & \text{if } x \in B \cap [a,b] \end{cases}$$

Then $f' \equiv 0$, so f is continuously differentiable, but f is not uniformly continuous, since the gap A, B is regular by (AP9). $\square$

We now state a helpful lemma. We do not provide the proof; it is a standard exercise in undergraduate analysis.

Lemma 3.2 (Continuous Extension of Uniformly Continuous Functions)**.**
Let $\mathbb{F}$ be an ordered subfield of $\mathbb{R}$, $h_{\mathbb{F}} : [a,b] \to \mathbb{F}$ a uniformly continuous function, and $a, b \in \mathbb{F}$ with $[a,b] \subset \mathbb{F}$. Then there exists a unique continuous function $h : [a,b]_{\mathbb{R}} \to \mathbb{R}$ such that $h(x) = h_{\mathbb{F}}(x)$ for all $x \in [a,b]$. We term h the *continuous extension of $h_{\mathbb{F}}$ to $\mathbb{R}$. (Note also that h is even uniformly continuous.)*

Lemma 3.3.
Let $\mathbb{F}$ be a Archimedean field and let $f : [a,b]_{\mathbb{R}} \to \mathbb{R}$ be a function with a Darboux integral J. Let $F = f|_{\mathbb{F}}$ be the restriction of f to $[a,b] = [a,b]_{\mathbb{R}} \cap \mathbb{F}$ and suppose $F(\mathbb{F}) \subset \mathbb{F}$. Let $\mathcal{L}_F$ and $\mathcal{U}_F$ be the set of step functions with coefficients in $\mathbb{F}$ below and above F, respectively. If there is an $I \in \mathbb{F}$ such that $\forall g \in \mathcal{L}_F$, $\forall h \in \mathcal{U}_F$, $\int_a^b g \le I \le \int_a^b h$, then $I = J$ and hence F has a Darboux integral in $\mathbb{F}$ equal to J.

Proof.
Suppose $I \neq J$. We will consider the case $I < J$; the case where $I > J$ is similar. Since J is the Darboux integral for f and is hence unique, $J > (I+J)/2$ is not the Darboux integral, and so there is some step function $s \in \mathcal{L}_f$ so that $(I+J)/2 < \int_a^b s$.

Let the representation of s be given by $s(x) \equiv \sum_{k=1}^n \alpha_k \chi_{A_k}(x)$ where $\{\alpha_k\} \subset \mathbb{R}$ and the A_ks are disjoint intervals and have union $[a,b]$. By (AP3), for each k, there exists a $\beta_k \in \mathbb{Q}$ such that $\alpha_k - (J-I)/(2(b-a)) < \beta_k < \alpha_k$. Define the step function t by $t(x) \equiv \sum_{k=1}^n \beta_k \chi_{A_k}(x)$ and note that $\beta_k < \alpha_k$ for all k and so $t < s < f$. Since t takes on only rational values, $t|_{\mathbb{F}}$ maps only into $\mathbb{F}$. $t|_{\mathbb{F}} < f|_{\mathbb{F}}$ and therefore $t|_{\mathbb{F}} \in \mathcal{L}_F$. Then,

$$\int_a^b t|_{\mathbb{F}} = \int_a^b t > \int_a^b \left(s - \frac{J-I}{2(b-a)} \right) = \left(\int_a^b s \right) - \frac{J-I}{2} > \frac{I+J}{2} - \frac{J-I}{2} = I.$$

But this contradicts the definition of I, since $t|_{\mathbb{F}} \in \mathcal{L}_F$ and yet $\int_a^b t|_{\mathbb{F}} > I$. So $I = J$, from which it is obvious that I is the unique value with this property and so F has a Darboux integral in $\mathbb{F}$ equal to $I = J$. □

Corollary 3.4.
Let $\mathbb{F}$ be an Archimedean field and let $f : [a,b]_{\mathbb{R}} \to \mathbb{R}$ be a function with a Darboux integral $J \in \mathbb{R}$. Suppose $F = f|_{\mathbb{F}}$ satisfies $F(\mathbb{F}) \subset \mathbb{F}$. Then F has a Darboux integral in $\mathbb{F}$ if and only if $J \in \mathbb{F}$.

Proof.
($\Rightarrow$): Let $I \in \mathbb{F}$ be the value of the Darboux integral of F. Then by definition, $\forall g \in \mathcal{L}_F$, $\forall h \in \mathcal{U}_F$, $\int_a^b g \leq I \leq \int_a^b h$ and hence by Lemma 3.3, $I = J$ and so $J \in \mathbb{F}$.
($\Leftarrow$): All step functions in $\mathcal{L}_F$ and $\mathcal{U}_F$ appear in $\mathcal{L}_f$ and $\mathcal{U}_f$, respectively, and thus J satisfies the conditions of I in Lemma 3.3. Therefore, F has a Darboux integral with value J, as desired. □

This corollary provides a criterion for a restricted function having a Darboux integral, based on the value of the Darboux integral of the unrestricted function.

Proof of (CA) $\Leftrightarrow$ (CA33).
($\Rightarrow$): In $\mathbb{R}$, all continuous functions have Darboux integrals, so in particular, the uniformly continuous ones.
($\Leftarrow$): If $\mathbb{F}$ is non-Archimedean, then as proven in [18] (Lemma D), there exists a function (which is uniformly continuous) that does not have a Darboux integral.

So suppose $\mathbb{F}$ is Archimedean and let $f : [0,1]_{\mathbb{R}} \to \mathbb{R}$ be the function in Lemma 3.1, and $f_{\mathbb{F}} : [0,1] \to \mathbb{F}$ its restriction to $\mathbb{F}$. Note that $f_{\mathbb{F}}$ is uniformly continuous. Since f has (Darboux) integral c but $c \notin \mathbb{F}$, by Cor. 3.4, $f_{\mathbb{F}}$ does not have a Darboux integral in $\mathbb{F}$. □

PROOF OF (CA) ⇔ (CA42) ⇔ (CA43).

(CA) ⇒ (CA42): This is a simple argument based on (CA22).

(CA42) ⇒ (CA43): Let F, G be antiderivatives for a function f. Define $H = G - F$. Then $H' \equiv 0$ and hence by assumption both H and $(-H)$ are increasing, i.e. H is constant.

(CA43) ⇒ (CA) : Suppose $f : [a,b] \to \mathbb{F}$ is continuous on $[a,b]$ with $f' \equiv 0$. Then both f and the zero function are antiderivatives of the zero function and so they differ by a constant, i.e. f is constant. This proves (CA38), which is equivalent to completeness. □

Remark. These two axioms were originally proven to be equivalent to completeness in [12]. Those proofs used countable cofinality, however, while the proofs presented here do not.

PROOF OF (CA) ⇔ (CA48).
(⇒): This is a standard proof, using (CA13) to establish convergence of partial sums.
(⇐): Let $\mathbb{F}$ be an incomplete Archimedean ordered field. Then there is some value $c \in \mathbb{R} \setminus \mathbb{F}$. Without loss of generality, we may assume $0 < c < 1$. Then c has a binary expansion with bit sequence $\{d_n\}$ so that $\sum_{n=1}^{\infty} d_n/2^n = c$ with each d_n either 0 or 1. Let $a_n = d_n/2^n$ and $b_n = 1/2^n$. Then $a_n \leq b_n$ for all n, as required, and $\sum_{n=1}^{\infty} b_n = 1$. But $\sum_{n=1}^{\infty} a_n = c$ by definition and hence cannot converge in $\mathbb{F}$ since $c \notin \mathbb{F}$. □

We now state two standard results without proof. It is easy to verify that the proofs are valid in any totally ordered field.

Lemma 3.5.
If a sequence of bounded functions converges uniformly on a closed interval $[a,b]$, then the limit function is bounded on $[a,b]$.

Lemma 3.6.
Let $p : [a,b] \to \mathbb{F}$ be a polynomial with coefficients in $\mathbb{F}$. Then p is bounded on $[a,b]$.

PROOF OF (CA) $\Leftrightarrow$ (CA51).
($\Rightarrow$): Weierstrass Approximation Theorem.
($\Leftarrow$): Suppose $\mathbb{F}$ is not Dedekind complete, and yet (CA51) still holds. We know by $\neg$(CA25) there exists a continuous function on a closed interval that is unbounded, call it f. By assumption, there exists a sequence of polynomials uniformly convergent to f. But, by Lemma 3.6, polynomials are bounded on closed intervals and so by Lemma 3.5, their uniform limit, f, must be bounded as well. Contradiction. $\square$

PROOF OF (CA) $\Leftrightarrow$ (CA52).
($\Rightarrow$): It is relatively simple to show that this follows directly from (CA22).
($\Leftarrow$): Suppose $\mathbb{F}$ is not Dedekind complete and Archimedean. The the function in the proof of "(CA31) $\Rightarrow$ (CA)" above provides a counter example here as well — for a similar reason. $\square$

PROOF OF (CA) $\Leftrightarrow$ (CA53). See [4] $\square$

PROOF OF (CA) $\Leftrightarrow$ (CA54).
($\Rightarrow$): This is a standard proof, using (CA22).
($\Leftarrow$): Let $\mathbb{F}$ be incomplete and A, B a gap in $\mathbb{F}$. Choose some $a \in A$ and $b \in B$ and define

$$F(x) = \begin{cases} 0, & \text{if } x \in [a,b] \cap A \\ 1, & \text{if } x \in [a,b] \cap B. \end{cases}$$

Then $F' \equiv 0$ on $[a,b]$. If we now consider $f = F'$, we obviously get that f has a Darboux integral with $\int_a^b f(x)\,dx = 0$, however $F(b) - F(a) = 1$, proving the result. $\square$

PROOF OF (CA) $\Leftrightarrow$ (CA55).
($\Rightarrow$): This follows easily from (CA13).
($\Leftarrow$): Let $\mathbb{F}$ be a countably cofinal ordered field and assume first that $\mathbb{F}$ is non-Archimedean, i.e. contains an infinitely large element λ (cf. (AP2) in Section 4.1). Let $\{\varepsilon_n\}$ be a positive, decreasing null sequence in $\mathbb{F}$, and define $x_n = 1 - \varepsilon_n$. Define $f : \mathbb{F} \to \mathbb{F}$ by

$$f(x) \equiv \begin{cases} 0, & \text{if } x \leq x_0 \\ n, & \text{if } x_{n-1} < x \leq x_n \\ \lambda, & \text{if } x \geq 1. \end{cases} \quad .$$

Certainly, f is increasing. We claim that f has no left-hand limit at 1 and hence (CA55) fails on, say, the interval $[0,2]$. But this is obvious, as $f(x_n) = n$ is clearly not a Cauchy sequence, and hence not convergent.

So $\mathbb{F}$ must be Archimedean. If $\mathbb{F}$ is incomplete, we know by Lemma 3.1 that there is some function $f : [0,1]_{\mathbb{R}} \to \mathbb{R}$ with integral $c \notin \mathbb{F}$. Define, similar to the proof of (CA30), $F : [0,1]_{\mathbb{R}} \to \mathbb{R}$ by $F(x) = \int_x^1 f(t)\,dt$ and let $g : [-1,1] \to \mathbb{F}$ be defined by

$$g(x) = \begin{cases} -F|_{\mathbb{F}}(-x) + 5/8 \text{ if } x \in [-1,0) \\ 0, \text{ if } x = 0 \\ F|_{\mathbb{F}}(x) - 5/8, \text{ if } x \in (0,1]. \end{cases}$$

Now let $\{q_n\}$ be a positive null sequence in $\mathbb{Q}$. Then $g(0^+) = \lim_{n\to\infty} g(q_n) = \lim_{n\to\infty} F|_{\mathbb{F}}(q_n) - 5/8 = \lim_{n\to\infty} F(q_n) - 5/8 = F(0) - 5/8 = c - 5/8 \notin \mathbb{F}$ and hence $g(0^+)$ does not exist in $\mathbb{F}$. Similarly, $g(0^-)$ does not exist in $\mathbb{F}$. However, g is clearly (strictly) monotonic on $[-1,1]$ and hence (CA55) fails, proving the result. □

Proof of (CA) ⇔ (CA56).
(⇒): The proof is given in [5]. We note that it only uses completeness in the sense of Cauchy convergence, and hence our additional stipulation of the Archimedean Property is necessary for equivalence.
(⇐): Let $\mathbb{F}$ be Archimedean and incomplete. Let g be the function used in the previous proof. We know from that proof that g is not limit-regulated, so it remains to show that g is step-regulated. We only show that g is step-regulated on $(0,1]$, since it is then easy to show that it is step-regulated on $[-1,1]$.

Let $\varepsilon \in \mathbb{F}^+$ be given. We will construct an ε-close step function for g on $(0,1]$. Since g is continuous on $(0,1]$, for every $x \in (0,1]$, there is some $\delta_x \in \mathbb{F}^+$ so that $|x-y| < \delta_x$ implies $|g(x)-g(y)| < \varepsilon$. Let $I_x = (x-\delta_x, x+\delta_x)$. The set of open intervals $\{I_q \,|\, q \in (0,1] \cap \mathbb{Q}\}$ is an open cover of $[0,1]_{\mathbb{R}}$ by the density of $\mathbb{Q}$ in $\mathbb{R}$ and hence by (CA10) has a finite subcover $\{I_{q_0}, I_{q_1}, ..., I_{q_n}\}$. Then the function $s(x) = g(q_0)\chi_{(0,q_0)}(x) + \sum_{i=1}^{n-1} g(q_i)\chi_{[q_i,q_{i+1}]}(x) + g(q_n)\chi_{[q_n,1]}(x)$ is a ε-close step function for g on $(0,1]$ defined using only values in $\mathbb{F}$. □

Proof of (CA) ⇔ (CA57).
(⇒): The proof is given in [5] and uses (CA11) to construct the partition used for the desired step function.
(⇐): Clearly, all continuous functions are limit-regulated. Hence, by assumption, they are step-regulated, proving (CA41), which is equivalent to completeness. □

Proof of (CA) ⇔ (CA58).
(⇒): Standard proof by contradiction using (CA14).

($\Leftarrow$): Suppose $\mathbb{F}$ is countably cofinal and incomplete. Then $\neg$(CA25) gives a continuous function that is unbounded. Since continuous functions are obviously limit-regulated, this proves the result. $\square$

PROOF OF (CA) $\Leftrightarrow$ (CA59) $\Leftrightarrow$ (CA60).

(CA) $\Rightarrow$ (CA59): This may be shown by contradiction. Although the proof does not always seem to be part of the real-analysis syllabus, we nevertheless omit it here.

(CA59) $\Rightarrow$ (CA60): Trivial.

(CA60) $\Rightarrow$ (CA): Let $\mathbb{F}$ be an incomplete field. If $\mathbb{F}$ is Archimedean, choose some $c \in (0,2)_{\mathbb{R}} \setminus \mathbb{F}$ and some strictly increasing sequence $\{\varepsilon_n\}$ drawn from $\mathbb{F}$ convergent to c in $\mathbb{R}$. If $\mathbb{F}$ is non-Archimedean, let $\varepsilon_n = n\delta$, where δ is a positive infinitesimal in $\mathbb{F}$ (cf. (AP1) in Section 4.1).

Note that in either case, all ε_n are contained in $[0,2]$. Let $\varepsilon \in \mathbb{F}^+$ be given. Then we define the function $f : [0,2] \to \mathbb{F}$ as

$$f(x) \equiv \frac{\varepsilon}{2} \cdot \begin{cases} (-1)^n, & \text{if } x \in [\varepsilon_n, \varepsilon_{n+1}) \\ 1, & \text{otherwise} \end{cases}$$

and note that since f is piecewise-constant, it must be limit-regulated. Then at every ε_n, we have a "jump" of size ε and so $\{\varepsilon_n\} \subset D_\varepsilon(f)$, which means $D_\varepsilon(f)$ is not finite. $\square$

We now present a lemma that allows us to shift gaps. The result is quite intuitive, so we omit the proof.

Lemma 3.7.
Let $\mathbb{F}$ be an incomplete field, and let A, B be a gap. Suppose $f : \mathbb{F} \to \mathbb{F}$ is surjective and strictly increasing. Define $f(A) = \{f(x) \mid x \in A\}$ and similarly define $f(B) = \{f(x) \mid x \in B\}$. Then $f(A), f(B)$ is also a gap in $\mathbb{F}$.

Corollary 3.8.
Let $\mathbb{F}$ be an incomplete field, and let A, B be a gap. Let $m \in \mathbb{F}^+$ and define $mA = \{mx \mid x \in A\}$ and similarly define $mB = \{mx \mid x \in B\}$. Then mA, mB is also a gap in $\mathbb{F}$.

PROOF.
The result follows directly from Lemma 3.7, using $f(x) = mx$. $\square$

Definition 3.9.
Let $\mathbb{F}$ be an incomplete field, with gap $\tilde{A}, \tilde{B}$ so that $\tilde{A} \cap \mathbb{F}^+$ is non-empty, and let $\{x_n\}_{n=0}^{\infty} \subset \mathbb{F}^+$ be a strictly decreasing sequence in $\mathbb{F}$ convergent to 1. Define $\tilde{A}_n = x_n \tilde{A}$ and $\tilde{B}_n = x_n \tilde{B}$. We can define a partition $A, B_0, B_1, \ldots$ of $\mathbb{F}$ by

$$B_0 = \tilde{B}_0 \qquad B_k = \tilde{B}_k \cap \tilde{A}_{k-1} \ (k \geq 1) \qquad A = \bigcap_{i=0}^{\infty} \tilde{A}_i$$

We say that this partition is the gap sequence *generated by $\{x_n\}$ using the gap $\tilde{A}, \tilde{B}$.*

Although the definition is somewhat technical, the idea of a gap sequence is relatively simple. A pictorial example of the construction is shown in Figure 1.

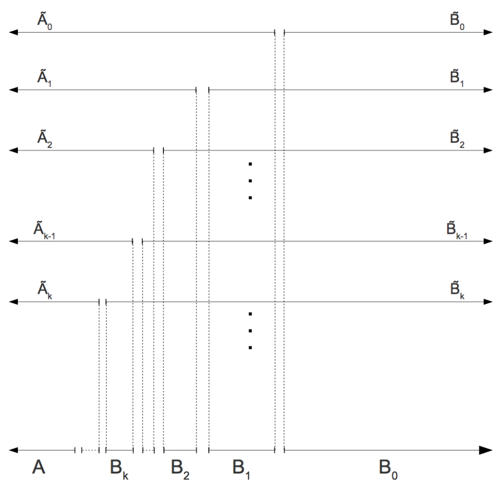

Figure 1: Example construction of a gap sequence.

A gap sequence, such as the one shown in the figure, has a number of useful properties, which we present as a lemma but do not prove.

Lemma 3.10.
Let $A, \{B_k\}$ be the gap sequence generated by some sequence $\{x_n\}$ using some gap $\tilde{A}, \tilde{B}$. Then

- *$A = \tilde{A}$ and $\bigcup_{k=0}^{\infty} B_k = \tilde{B}$.*

- *$A < B_k$ for all k, i.e. if $x \in A$ and $y \in B_k$ for some k, then $x < y$.*
- *$B_i < B_j$ whenever $i > j$.*
- *A and all B_k are open and convex sets.*

PROOF OF (CA) $\Leftrightarrow$ (CA61).
($\Rightarrow$): This is a standard proof, which establishes the sequence of functions is uniformly Cauchy and then uses (CA35) to show uniform convergence.
($\Leftarrow$): Let $\{\lambda_n\}$ be an unbounded sequence in $\mathbb{F}$. W.l.o.g. we may assume that the sequence is strictly increasing and that $\lambda_0 = 1$. Define $x_n = 1 + 1/\lambda_n$ and note that $\{x_n\}$ is a sequence in $\mathbb{F}$ convergent to 1. Suppose $\mathbb{F}$ is not complete and let A, B be a gap in $\mathbb{F}$. W.l.o.g. assume that $0 \in A$ and $1/2 \in B$. Let $A_0, \{B_n\}$ be the gap sequence generated by A, B using $\{x_n\}$.

We define the sequence of functions $\{f_n\}$ on $[-1, 1]$ as follows:

$$f_n(x) = \begin{cases} \lambda_k, & \text{if } x \in B_k \cap [-1,1] \text{ for some } k \leq n, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Since each f_n is locally constant, we have $f_n' \equiv 0$ and $\{f_n'\}$ trivially converges uniformly. Furthermore, since $1 \in B_0$, $f_n(1) = \lambda_0 = 1$ for all $n \in \mathbb{N}$, and so $\{f_n(1)\}$ converges in $\mathbb{F}$.

However, each f_n is bounded on $[-1, 1]$ by λ_n and so if $\{f_n\}$ converged uniformly, then by Lemma 3.5, the limit function f would be bounded. But it is obvious from the definition of $\{f_n\}$ that the limit cannot be bounded. Contradiction. $\square$

PROOF OF (CA) $\Leftrightarrow$ (CA62).
($\Rightarrow$): This is a standard proof and is almost exactly the same as the proof that (CA29) holds in $\mathbb{R}$.
($\Leftarrow$): Let $\mathbb{F}$ be a countably cofinal incomplete field. Then $\neg$(CA29) gives us some function f that is continuous but not uniformly continuous on some interval $[a, b]$. Define $f_n = f$ for all $n \in \mathbb{N}$. Then $\{f_n\}$ is obviously equicontinuous but not uniformly equicontinuous (note that each function of a uniformly equicontinuous family is uniformly continuous). $\square$

PROOF OF (CA) $\Leftrightarrow$ (CA63).
($\Rightarrow$): For this argument, usually a step in proof of the Arzelà-Ascoli Theorem, the density of $\mathbb{Q}$ may be used.
($\Leftarrow$): If $\mathbb{F}$ is not complete, there exists a bounded sequence $(y_n) \subset \mathbb{F}$ that does not have a convergent subsequence (by $\neg$(CA14)). Define $f_n \equiv y_n$, which is equicontinuous (even uniformly equicontinuous), as each function is

constant. However, $\{f_n\}$ cannot have a pointwise convergent subsequence, since $y_n = f_n(x)$ does not have one. $\square$

Proof of (CA) $\Leftrightarrow$ (CA64).
($\Rightarrow$): Standard argument in the proof of the Arzelà-Ascoli Theorem.
($\Leftarrow$): Suppose $\mathbb{F}$ is incomplete and let $\{f_n\}$ be the family of functions defined in (1). It is obvious that the sequence is pointwise convergent on $[-1,1]$ and that the family is equicontinuous on that interval. But, as shown above, it is impossible for $\{f_n\}$ to converge uniformly. $\square$

Proof of (CA) $\Leftrightarrow$ (CA65).
($\Rightarrow$): This standard proof relies on (CA22).
($\Leftarrow$): Let f be the pointwise limit of the sequence $\{f_n\}$ defined in (1). f exists since at each point in $[-1,1]$ the sequence of function values associated with that point is eventually constant. Since $f' \equiv 0$, f is obviously continuously differentiable on $[-1,1]$. However, f is not Lipschitz since it is unbounded by construction. $\square$

Proof of (CA) $\Leftrightarrow$ (CA66).
($\Rightarrow$): This statement does not require completeness if the function is *unifomly* continuous. The crux of the proof, then, is appealing to (CA29).
($\Leftarrow$): Let f be the function used in the previous proof. Let $\{y_n\}$ be any sequence so that $y_n \in B_n$ for all n, i.e. $f(y_n) = \lambda_n$ and so forms an unbounded – hence not Cauchy – sequence. It remains to show that $\{y_n\}$ is Cauchy.

Let $\varepsilon \in \mathbb{F}^+$ be given. Since $\{x_n\}$ monotonically tends to 1 from above, there is some $N \in \mathbb{N}$ so that $x_N - 1 < \varepsilon$. Suppose $m > n > N$. A simple argument shows that $y_n \in \tilde{A}_N$. Since $m > n$, $y_m < y_n$, and since $y_m \in B_m \subset B$, $x_N y_m \in \tilde{B}_N$ and so $x_N y_m > y_n$.

So $|y_n - y_m| = y_n - y_m < x_N y_m - y_m = (x_N - 1)y_m < \varepsilon y_m < \varepsilon$, where the last inequality follows from $1 \in B_0$ and $m > N \geq 0$. $\square$

We now present two alternate types of gap sequences. These are natural modifications to make and hence we leave it to the reader to verify that they are well-defined in the same way as the previous gap sequence.

Definition 3.11.
Let $\mathbb{F}$ be a (countably cofinal) incomplete field, with gap $\tilde{A}, \tilde{B}$ so that $\tilde{A} \cap \mathbb{F}^+$ is non-empty, and let $\{x_n\}_{n=0}^{\infty} \subset \mathbb{F}^+$ be a strictly decreasing sequence in $\mathbb{F}$ convergent to 0. Then we define the gap sequence A, B_k in the same way as the standard gap sequence. This definition yields the following properties:

- $A = (-\infty, 0]$ *and* $\bigcup_{k=0}^{\infty} B_k = (0, \infty)$.

- *$B_i < B_j$ whenever $i > j$.*
- *All B_k are open and convex sets.*

We might refer to this modification as a null gap sequence*.*

Definition 3.12.
Let $\mathbb{F}$ be a (countably cofinal) incomplete field, with gap $\tilde{A}, \tilde{B}$ so that $\tilde{A} \cap \mathbb{F}^+$ is non-empty, and let $\{x_n\}_{n=0}^{\infty} \subset \mathbb{F}^+$ be a strictly increasing unbounded sequence in $\mathbb{F}^+$. Then we define the gap sequence A_k in the following way, which bears much similarity to the standard gap sequence:

$$A_0 = \tilde{A}_0 \qquad\qquad A_k = \tilde{A}_k \cap \tilde{B}_{k-1} \ (k \geq 1)$$

This definition yields the following properties:

- *$\bigcup_{k=0}^{\infty} A_k = \mathbb{F}$.*
- *$A_i < A_j$ whenever $i < j$.*
- *All A_k are open and convex sets.*

We might refer to this modification as a unbounded gap sequence*.*

We now use these two new gap sequences to prove similar results, namely equivalence of completeness and the two versions of l'Hôpital's Rule.

Proof of (CA) $\Leftrightarrow$ (CA67).
($\Rightarrow$): Standard analysis proof, typically based on (CA23).
($\Leftarrow$): Let $\{\lambda_n\}$ be an unbounded sequence in $\mathbb{F}$. W.l.o.g. we may assume that the sequence is strictly increasing, $\lambda_{n+1} \geq 2\lambda_n$ for all $n \in \mathbb{N}$ and that $\lambda_0 = 1$ and $\lambda_1 = 2$. Denote $1/\lambda_n$ by ε_n and note that $\{\varepsilon_n\}$ is a null sequence in $\mathbb{F}$. Suppose $\mathbb{F}$ is not Dedekind complete and let $\tilde{A}, \tilde{B}$ be a gap in $\mathbb{F}$. W.l.o.g. assume that $1 \in \tilde{A}$ and $2 \in \tilde{B}$. Let $A, \{B_n\}$ be the null gap sequence generated by $\tilde{A}, \tilde{B}$ using $\{\varepsilon_n\}$. We define $g : (0, 1) \to \mathbb{F}$ by $g(x) = x$, and define f as $f(x) = -x + 2\varepsilon_{n-1}$ if $x \in B_n$. Since a gap sequence is a partition of $\mathbb{F}$ and $1 \in B_1$, this is well-defined.

We must show that f and g satisfy the conditions of l'Hôpital's Rule. Certainly g is differentiable on $(0, 1)$ and has derivative 1 everywhere. Furthermore, $\lim_{x\to 0^+} g(x) = 0$. Additionally, it is easy to see that f is differentiable with derivative -1 on $(0, 1)$, since it is linear on each B_n.

We claim that $\lim_{x\to 0^+} f(x) = 0$. To this end, let $\varepsilon \in \mathbb{F}^+$ be arbitrary and fixed. Since $\{\varepsilon_n\}$ is a null sequence, there exists some $N \in \mathbb{N}$ such that $\varepsilon_N < \varepsilon/2$. Take $\delta = \varepsilon_N$, and let $x \in (0, \delta)$ be arbitrary. Then it is easily seen that $\varepsilon_N \in B_{N+1}$ and hence, since $x > 0$, there is some $n \geq N+1$ such that $x \in B_n$. We know $2\varepsilon_{n-1} > f(x)$ by definition of f on B_n. Furthermore, by definition of the gap sequence, if $x \in B_n$, then $x \in \varepsilon_{n-1}\tilde{A}$ and so $x < 2\varepsilon_{n-1}$, since $2 \in \tilde{B}$. Thus, $f(x) = -x + 2\varepsilon_{n-1} > 0$ and so $|f(x)| = f(x) < 2\varepsilon_{n-1} \leq 2\varepsilon_N < \varepsilon$, which implies $\lim_{x\to 0^+} f(x) = 0$, as desired.

Now, since $f'(x) = -1$ and $g'(x) = 1$ on $(0,1)$, it must be the case that

$$\lim_{x\to 0^+} \frac{f'(x)}{g'(x)} = \frac{-1}{1} = -1$$

But both f and g are positive on $(0,1)$ and so $f(x)/g(x)$ is positive on $(0,1)$. Thus $\lim_{x\to 0^+} f(x)/g(x) \geq 0 \neq -1$ or $\lim_{x\to 0^+} f(x)/g(x)$ does not exist, both of which violate the conclusion of l'Hôpital's Rule. □

We now modify this proof for the second version of l'Hôpital's Rule, namely the "infinite limit" case.

Proof of (CA) $\Leftrightarrow$ (CA68).
($\Rightarrow$): This is proven in a manner similar to (CA67).
($\Leftarrow$): The proof is similar to the previous one. Let $\{\lambda_n\} \subset (2,\infty)$ again be an unbounded sequence, w.l.o.g. strictly increasing. Let $\tilde{A}, \tilde{B}$ be a gap with $2 \in \tilde{B}$ and let $\{A_n\}$ be the unbounded gap sequence generated by $\tilde{A}, \tilde{B}$ using $\{\lambda_n\}$. We define $g : (1,\infty) \to \mathbb{F}$ as $g(x) = x$, and define f as $f(x) = -x + 2\lambda_n^2$ if $x \in A_n$.

We must show that f and g satisfy the conditions of l'Hôpital's Rule. Certainly g is differentiable on $(1,\infty)$ and has derivative 1 everywhere. It is obvious that $\lim_{x\to\infty} g(x) = \infty$. Additionally, it is easy to see that f is differentiable with derivative -1 on $(0,1)$. It remains to be shown that $\lim_{x\to\infty} f(x) = \infty$. Let $x \in (1,\infty)$ be fixed. We know that there is some $n \in \mathbb{N}$ so that $x \in A_n$. Hence $x \in \tilde{A}_n$ and since $2 \in \tilde{B}$, $2\lambda_n \in \tilde{B}_n$, i.e. $x < 2\lambda_n$. Hence $f(x) = -x + 2\lambda_n^2 > 2\lambda_n(-1+\lambda_n) > \lambda_n$, which is a unbounded sequence by assumption.

Now the proof may be concluded as above. □

Proof of (CA) $\Leftrightarrow$ (CA69).
($\Rightarrow$): This is a standard result of Real Analysis.
($\Leftarrow$): From Lemma 3.1 we know that in a incomplete Archimedean field, we have a function f that is the uniform limit of functions with Darboux integrals, but cannot itself have a Darboux integral, for its value would be $\int_a^b f(x)\,dx = c \notin \mathbb{F}$, which is impossible. □

Proof of (CA) $\Leftrightarrow$ (CA70).
This proof is quite easy and omitted. □

PROOF OF (CA) $\Leftrightarrow$ (CA71) $\Leftrightarrow$ (CA72).
(CA) $\Rightarrow$ (CA71): Immediately follows from (CA19).
(CA71) $\Rightarrow$ (CA72): Trivial.
(CA72) $\Rightarrow$ (CA) : Suppose $\mathbb{F}$ is incomplete and let A, B be a gap in $\mathbb{F}$, $a \in A$ and $b \in B$. Let $\varepsilon \in \mathbb{F}^+$ be given. Define $f : [a, b] \to \mathbb{F}$ by

$$f(x) \equiv \begin{cases} 0, & \text{if } x \in [a, b] \cap A \\ 2\varepsilon, & \text{if } x \in [a, b] \cap B \end{cases}$$

and note that f is continuous on $[a, b]$. Then between a and b, f takes on only values of 0 and 2ε and hence there is no value c in $[a, b]$ so that $|f(c) - \varepsilon| < \varepsilon$. $\square$

4 Characterizations of the Archimedean Property

The following statements are equivalent to the Archimedean Property of a totally ordered field $\mathbb{F}$ (assumed to be countably cofinal).

Again, many of these statements are collected from other authors and a statement may be tagged with up to two references, each indicating an existing list and the position of the property on that list. **(R#)** and **(T#)** are adapted from [12] and [18], respectively, while any property that is unmarked is not included on either list and will be proven equivalent to the Archimedean Property in the following section.

Again, we mark those statements as in the previous list with '*' and '†' or a reference.

4.1 The List

AP0. **(Archimedean Property)**: (Riii)
For every pair of elements a, b in $\mathbb{F}^+$, there is some $n \in \mathbb{N}$ such that $a < nb$.

AP1. **(No Infinitesimals)**: (Tii), [7]
$\mathbb{F}$ has no non-zero infinitesimals.

AP2. **(All Elements are Finite)**: (Tiii), [7]
$\mathbb{F}$ has no infinitely large elements.

AP3. **(Density of $\mathbb{Q}$)**: (Rvi†), (Rvii†), (Tiv)
$\mathbb{Q}$ is dense in $\mathbb{F}$, that is for every pair $a, b \in \mathbb{F}$ with $a < b$, there is some $q \in \mathbb{Q}$ such that $q \in (a, b)$.

AP4. **(Subfield of $\mathbb{R}$)**: (Tv), [10]
$\mathbb{F}$ can be embedded in $\mathbb{R}$, i.e. there exists an order-preserving homomorphism from $\mathbb{F}$ into $\mathbb{R}$.

AP5. **(Harmonic Sequence 1)**: (Rii†), (Tvi)
The sequence $\{1/n \mid n \in \mathbb{N},\, n > 1\}$ converges to zero.

AP6. **(Harmonic Sequence 2)**:
The sequence $\{1/n \mid n \in \mathbb{N},\, n > 1\}$ converges in $\mathbb{F}$.

AP7. **($\mathbb{N}$ is Unbounded)**: (Ri†)
The sequence $\{n : n \in \mathbb{N}\}$ is unbounded in $\mathbb{F}$.

AP8. **(Rational Cauchy Sequence)**:
There is some rational sequence $\{q_n\} \subset \mathbb{Q}$ that is Cauchy in $\mathbb{F}$ but not eventually constant.

AP9. **(Cuts Are Regular)**: (Tvii†)
For every cut A, B and $\varepsilon \in \mathbb{F}^+$ here exist $x \in A, y \in B$ such that $y - x < \varepsilon$.

AP10. **(Bounded Monotone Cauchy Property)**: (Tviii†)
Every bounded monotone sequence is a Cauchy sequence.

AP11. **(Bolzano-Weierstrass-Cauchy Property)**:
Every bounded sequence has a subsequence that is a Cauchy sequence.

AP12. **(Monotone Functions Are Darboux Integrable)**: (Tx†)
Every monotone function defined on $[a, b]$ is Darboux integrable.

AP13. **(Floor Function)**:
For every x in $\mathbb{F}^+$, there is some $n \in \mathbb{N}$ such that $n \leq x < n + 1$.

AP14. **(Analytically Nilpotent Elements)**: (Riv†)
For every $x \in \mathbb{F}$ such that $|x| < 1$, the sequence $\{x^n\}$ converges to zero.

AP15. **(Analytically Nilpotent Rational)**:
There is some $q \in \mathbb{Q}^+$ so that $q^n \to 0$ in $\mathbb{F}$.

AP16. **(Base g Expansion)**: (Rv†)
For every $g \in \mathbb{N}$ with $g > 1$, let the set G of formal power series be defined as

$$G \equiv \left\{ \sum_{n=0}^{\infty} a_n g^{k-n} : \{a_n\} \subset \mathbb{Z}_g \text{ and } k \in \mathbb{Z} \right\}.$$

For any $x \in \mathbb{F}^+$, there exists an element of G convergent to x.

AP17. **(Step Function Approximation 1)**:
$f(x) = x$, defined on $[0, 1]$, is step-regulated.

AP18. **(Step Function Approximation 2)**:
Every uniformly continuous function defined on $[a, b]$ is step-regulated.

AP19. **(Polynomial Approximation)**:
Every polynomial of degree n defined on $[a, b]$ can be uniformly approximated by piecewise polynomial functions of degree at most $n - 1$.

AP20. **(Identity Has a Darboux Integral)**:
$f(x) = x$, defined on $[0, 1]$, has a Darboux integral.

AP21. **(Identity is Darboux Integrable)**:
$f(x) = x$, defined on $[0, 1]$, is Darboux integrable.

AP22. **(Uniformly Continuous Functions Are Darboux Integrable)**:
Every uniformly continuous function on $[a, b]$ is Darboux integrable.

AP23. **(Uniform Bounded Value Property) + (*)**: (Tix†)
For every uniformly continuous function $f : [a, b] \to \mathbb{F}$ there is some $B \in \mathbb{F}^+$ such that $|f(x)| \leq B$ for all $x \in [a, b]$.

AP24. **(Modified Uniform Bounded Value Property)**:
For every uniformly continuous function $f : [a, b] \to \mathbb{F}$ there is some $B \in \mathbb{N}$ such that $|f(x)| \leq B$ for all $x \in [a, b]$.

AP25. **(Uniformly Differentiable Functions Are Lipschitz) + (*)**:
For every uniformly differentiable function $f : [a, b] \to \mathbb{F}$ is Lipschitz.

AP26. **(Uniformly Differentiable Functions Are Bounded) + (*)**:
For every uniformly differentiable function $f : [a, b] \to \mathbb{F}$ is bounded.

AP27. **(Modified Weierstrass Approximation Property) + (*)**:
For every uniformly continuous function $f : [a, b] \to \mathbb{F}$ and $\varepsilon \in \mathbb{F}^+$ there exists a polynomial $p : [a, b] \to \mathbb{F}$ such that $|f(x) - p(x)| < \varepsilon$ for all $x \in [a, b]$.

AP28. **(Modified** (CA61)**) + (*)**:
For every sequence $\{f_n\}$ of uniformly differentiable functions
$f_n : [a, b] \to \mathbb{F}$ such that $\{f_n'\}$ is uniformly Cauchy and there is some $c \in [a, b]$ such that $\{f_n(c)\}$ is Cauchy, $\{f_n\}$ is uniformly Cauchy.

AP29. **(Modified Arzelà–Ascoli 1)**:
Every uniformly bounded uniformly equicontinuous family of functions has a subsequence that is pointwise Cauchy.

AP30. **(Modified Arzelà–Ascoli 2) + (*)**:
Every uniformly equicontinuous family of functions that is pointwise Cauchy is uniformly Cauchy.

AP31. **(Null Sequences 1)**:
There exists a null sequence $\{x_n\} \subset \mathbb{F}$ such that $\{nx_n\}$ does not converge in $\mathbb{F}$.

AP32. **(Null Sequences 2)**:
There exists a null sequence $\{x_n\} \subset \mathbb{F}$ and a sequence $\{r_n\} \subset \mathbb{Q}$ such that $\{r_n x_n\}$ does not converge in $\mathbb{F}$.

AP33. **(Non-Trivial Quasi-Cauchy Sequences)**:
There exists a sequence $\{a_n\}$ such that for all $\varepsilon \in \mathbb{F}^+$, there is some $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_{n+1} - a_n| < \varepsilon$ and yet $\{x_n\}$ is not a Cauchy sequence. Such a sequence is called a *quasi-Cauchy sequence*.

AP34. **($\mathbb{Q}(X)$ Not a Subfield)**:
There does not exist an order-preserving field homomorphism from $\mathbb{Q}(X)$ into $\mathbb{F}$.

AP35. **(Series Convergence Test)**:
There is a null sequence $\{a_n\} \subset \mathbb{F}$ such that $\{\sum_{k=1}^n a_k\}$ is not a Cauchy sequence.

AP36. **(Jumps of Monotone Functions 1) + (*)**:
For every monotone function $f : [a, b] \to \mathbb{F}$ and $\varepsilon \in \mathbb{F}^+$, the set $D_\varepsilon(f) \equiv \{x \in [a, b] \mid \forall \delta \in \mathbb{F}^+, \exists y \in (x - \delta, x + \delta)\colon |f(x) - f(y)| \geq \varepsilon\}$ is finite.

AP37. **(Jumps of Monotone Functions 2) + (*)**:
There exists an $\varepsilon \in \mathbb{F}^+$ such that the set $D_\varepsilon(f)$ is finite for every monotone function $f : [a, b] \to \mathbb{F}$.

AP38. **(Rational Geometric Series)**:
There exists an $x \in \mathbb{Q}^+$ such that the series $\sum_{n=1}^\infty x^n$ converges in $\mathbb{F}$.

AP39. **(Rational Power Series 1)**:
For every sequence $\{c_n\}$ drawn from $\mathbb{Q}$ with $\limsup_{n\to\infty} \sqrt[n]{|c_n|} < \infty$ (where the limit superior is taken in $\mathbb{R}$), there exists an $x \in \mathbb{Q}^+$ such that the series $\sum_{n=1}^\infty c_n x^n$ is Cauchy in $\mathbb{F}$ (i.e. its sequence of partial sums is Cauchy in $\mathbb{F}$).

AP40. **(Rational Power Series 2)**:
There is some sequence $\{c_n\}$ drawn from $\mathbb{Q}$ with $\limsup_{n\to\infty} \sqrt[n]{|c_n|} < \infty$ and $x \in \mathbb{Q}^+$ so that the series $\sum_{n=1}^\infty c_n x^n$ is Cauchy in $\mathbb{F}$.

AP41. **(Approximate Uniform Intermediate Value Property)**:
For every uniformly continuous function $f : [a, b] \to \mathbb{F}$, $y \in \mathbb{F}$ and $\varepsilon \in \mathbb{F}^+$ between $f(a)$ and $f(b)$, there is some $c \in [a, b]$ such that $|f(c) - y| < \varepsilon$.

AP42. **(Weak Uniform Intermediate Value Property)**:
There exists an $\varepsilon \in \mathbb{F}^+$ such that, for every uniformly continuous function $f : [a, b] \to \mathbb{F}$ and $y \in \mathbb{F}$ between $f(a)$ and $f(b)$, there is some $c \in [a, b]$ such that $|f(c) - y| < \varepsilon$.

Remark. As with completeness, the list could be extended by another eight items, as there are also "approximate" and "weak" versions of Rolle's, the Mean Value, Cauchy's Mean Value, and Darboux's properties for *uniformly differentiable* functions, which are all equivalent to the AP.

4.2 Assorted proofs

Proof of (AP) $\Leftrightarrow$ (AP6).
($\Rightarrow$): Clear by (AP5).
($\Leftarrow$): Suppose $\mathbb{F}$ is non-Archimedean. Then by $\neg$(AP1), there exists a positive infinitesimal δ. Suppose the harmonic sequence converges in $\mathbb{F}$ to some $L \in \mathbb{F}$. It is obvious that L cannot be negative. Furthermore, if $L = 0$, then (AP5) would hold and $\mathbb{F}$ would be Archimedean, so $L > 0$.

In fact, L must be an infinitesimal, for, if not, there would be an $N \in \mathbb{N}$ such that $1/N < L$, and so for all $n > N$, $\left|L - \frac{1}{n}\right| = L - \frac{1}{n} > \frac{1}{N} - \frac{1}{n} > \delta$ and so $\{1/n\}$ would not converge to L.

So there exists an $N' \in \mathbb{N}$ such that $1/n - L < \delta$ for all $n > N'$. But then $L + \delta > 1/N'$, and so δ is not infinitesimal, a contradiction. So the harmonic sequence does not converge in $\mathbb{F}$. $\square$

Proof of (AP) $\Leftrightarrow$ (AP8).
($\Rightarrow$): It is easy to see that the sequence $\{(1/2)^n\}$ is a Cauchy sequence in any Archimedean ordered field.
($\Leftarrow$): Suppose $\mathbb{F}$ is non-Archimedean. Let $\{q_n\}$ be a rational sequence that is not eventually constant. Then for all n, $|q_{n+1} - q_n| \in \mathbb{Q}^+$ and hence, if $\delta \in \mathbb{F}^+$ is infinitesimal, $|q_{n+1} - q_n| > \delta$. So $\{q_n\}$ is not Cauchy. $\square$

Proof of (AP) $\Leftrightarrow$ (AP13).
($\Rightarrow$): Let $\mathbb{F}$ be an Archimedean field, and x be an arbitrary element of $\mathbb{F}^+$. Then using $a = x$, $b = 1$ in (AP0) gives us a $k \in \mathbb{N}$ such that $x < k$. Then the set $A = \{n \in \mathbb{N} : x < n\}$ contains k and is thus non-empty. By well-ordering, A contains a least element which we denote by m. It is easy to see that defining n as $m - 1$ will work and is unique.
($\Leftarrow$): Assuming $\neg$(AP13) implies $\neg$(AP2) through a simple argument. $\square$

PROOF OF (AP) ⇔ (AP15).
(⇒): Follows from (AP14) using any $q \in \mathbb{Q}^+$ such that $q < 1$.
(⇐): Suppose $\mathbb{F}$ is a non-Archimedean ordered field and let $\delta \in \mathbb{F}^+$ be infinitesimal. Suppose $q = a/b$ for some $a, b \in \mathbb{N}$. Then $1/b^n > \delta$ by definition of infinitesimals, and hence $q^n = a^n/b^n \geq 1/b^n > \delta$ and so q^n does not converge to 0. □

PROOF OF (AP) ⇔ (AP17).
(⇒): For an arbitrary $\varepsilon \in \mathbb{F}^+$, define $s : [0,1] \to \mathbb{F}$ by $s(x) = \varepsilon\lfloor x/\varepsilon \rfloor$, which is well-defined by (AP13). One readily checks that $|s(x) - f(x)| < \varepsilon$, as desired.

(⇐): Suppose $\mathbb{F}$ is non-Archimedean. Then by ¬(AP9), $\mathbb{F}$ has an irregular gap, A, B, with gap length $\delta \in \mathbb{F}^+$, w.l.o.g. $\delta \in (0,1)$. If (AP17) were to hold, we could find a $s(x)$ for $\varepsilon = \delta/2$. Let $S = \{x_k\}_{k=1}^n$ be the partition points of $s(x)$. Let $a = \max\{x : x \in S \cap A\}$ and $b = \min\{x : x \in S \cap B\}$, i.e. (a,b) contains the gap and $s|_{(a,b)}(x) = c$, where c is a constant. Choose $x \in A \cap (a,b)$ and $y \in B \cap (a,b)$. Then $y - x = |y - x| = |y - c + c - x| \leq |y - s(y)| + |s(x) - x| < \delta/2 + \delta/2 = \delta$ and so $y - x < \delta$. But $x \in A$ and $y \in B$ and so $y - x \geq \delta$, a contradiction. So (AP17) cannot hold. □

PROOF OF (AP) ⇔ (AP18).
(⇒): This is a typical proof when dealing with the regulated integral.[12] Those proofs require the uniform continuity of the function – although in $\mathbb{R}$, this is easily achieved by continuous functions on a closed and bounded interval by (CA29) – and uses the Archimedean Property implicitly when splitting the interval into finitely many subintervals of an arbitrary width, since if the width happened to be infinitesimal, this would be impossible on any interval of finite length.
(⇐): If (AP18) holds, we can easily verify (AP17). □

PROOF OF (AP) ⇔ (AP19).
(⇒): Let $\mathbb{F}$ be Archimedean and let $P(n)$ be the statement "Every polynomial on $[0,1]$ of degree n can be uniformly approximated by piecewise polynomial functions of degree at most $n-1$". We proceed by induction on n. $P(1)$ is equivalent to (AP17) and so holds, since our field is Archimedean by assumption. Now we assume $P(n-1)$ holds. Let $p : [0,1] \to \mathbb{F}$ be a polynomial of degree n. Approximating p allows us to approximate the entire class of functions $\{p(x) + c : c \in \mathbb{F}\}$ and so without loss of generality we may assume $p(0) = 0$, i.e. $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x$. Now define $q(x)$ to be $q(x) \equiv a_n x^{n-1} + a_{n-1}x^{n-2} + \cdots + a_1$, a polynomial of degree $n-1$ such

[12]See, e.g., Theorem 1.5.5 in [6].

that $q(x) = xp(x)$. By $P(n-1)$, q can be uniformly approximated on $[0,1]$ by piecewise polynomial functions of degree at most $n-2$. That is, for any $\varepsilon \in \mathbb{F}^+$, there is a function $\bar{q}(x)$ defined by $\bar{q}(x) \equiv \sum_{n=1}^{N} q_k(x)\chi_{I_k}(x)$ where $\{q_k\}_{k=1}^{N}$ is a sequence of polynomials of degree at most $n-2$ and $\{I_k\}_{k=1}^{N}$ is a sequence of disjoint intervals whose union is $[0,1]$, such that for any $x \in [0,1]$, $|q(x) - \bar{q}(x)| < \varepsilon$.
Define $\bar{p}(x)$ to be $\bar{p}(x) \equiv \sum_{n=1}^{N} (xq_k(x))\,\chi_{I_k}(x)$ and note that $\bar{p}$ is a piecewise polynomial of degree at most $n-1$. It is now not difficult to show that $\bar{p}$ approximates p uniformly ε-closely.
($\Leftarrow$): Since this holds for any polynomial on $[0,1]$, it in particular holds for $f(x) = x$. Thus, $f(x) = x$ can be uniformly approximated by piecewise constant functions, i.e. step functions. This is exactly the statement of (AP17), which we know by the above to be equivalent to the AP. $\square$

Proof of (AP) $\Leftrightarrow$ (AP20) $\Leftrightarrow$ (AP21).

(AP) $\Rightarrow$ (AP20): Assume $\mathbb{F}$ is Archimedean. We know $g : [0,1]_{\mathbb{R}} \to \mathbb{R}$ with $g(x) = x$ has a Darboux integral in $\mathbb{R}$ with value $1/2$. It is also easy to see that $g|_{\mathbb{F}} = f$ has $f(\mathbb{F}) = \mathbb{F}$, and so by Lemma 3.4, f has a Darboux integral as well (with value $1/2$).

(AP20) $\Rightarrow$ (AP21): If $f(x) = x$ has a Darboux integral, then is it Darboux integrable as well.

(AP21) $\Rightarrow$ (AP) : Let $\mathbb{F}$ be non-Archimedean and A, B a gap of length $\delta \in (0,1)$. Suppose $f(x) = x$ is Darboux integrable. Then there exist $g \in \mathcal{L}_f$ and $h \in \mathcal{U}_f$ (w.l.o.g. based on a common partition) such that $\int_0^1 h - \int_0^1 g < \delta^2$. Let $[a,b] \subset [0,1]$ be an interval containing the gap with g and h constant on (a,b). Let $x \in A \cap (a,b)$ and $y \in B \cap (a,b)$ be arbitrary. Note that $y - x \geq \delta$. Since $g \in \mathcal{L}_f$, $g(x) \leq f(x) = x$ and since $h \in \mathcal{U}_f$, $h(y) \geq f(y) = y$. So $\int_0^1 h - \int_0^1 g \geq \int_x^y h - \int_x^y g \geq \int_x^y y - \int_x^y x = (y-x)^2 \geq \delta^2$. This contradicts the choice of h and g and so f cannot be Darboux integrable. $\square$

Proof of (AP) $\Leftrightarrow$ (AP22).
($\Rightarrow$): Suppose $\mathbb{F}$ is Archimedean and let $f : [a,b] \to \mathbb{F}$ be uniformly continuous. Let $\varepsilon \in \mathbb{F}^+$ be given. Then by (AP18) we can find an $\varepsilon/(4(b-a))$-close step function for f, call it s. So $|s(x) - f(x)| < \varepsilon/(4(b-a))$ and thus $s(x) + \varepsilon/(4(b-a)) > f(x)$ and $s(x) - \varepsilon/4(b-a)) < f(x)$. Define $h(x) = s(x) + \varepsilon/(4(b-a))$ and

$g(x) = s(x) - \varepsilon/(4(b-a))$. Clearly, h and g are step functions. Furthermore, $h \in \mathcal{U}_f$ and $g \in \mathcal{L}_f$. Thus,

$$\int_a^b h - \int_a^b g = \int_a^b (h-g) = \int_a^b \varepsilon/(2(b-a)) = \varepsilon/2 < \varepsilon,$$

so f is Darboux integrable, as desired.
($\Leftarrow$): (AP22) obviously implies (AP21). $\square$

A proof of "(AP) $\Leftrightarrow$ (AP23)" is given in [18]; however the presentation has the slightly unsatisfactory feature that the counterexample constructed for the "$\Leftarrow$" direction is defined on $[0, 1)$ rather than $[0, 1]$.[13] We repeat the construction to show how this restriction can be avoided. The argument is based on the following refinement of Lemma A in [18].

Lemma 4.1.
Let $\mathbb{F}$ be a non–Archimedean ordered field. Then, for any $x_0 \in \mathbb{F}$, positive infinitesimal δ_0, and $n_0 \in \mathbb{N} \setminus \{0, 1\}$, there exists an irregular gap A, B satisfying

$$x_0 \in A, \tag{2a}$$
$$\forall a \in A, b \in B \qquad b - a \geq \delta_0^{n_0}, \qquad \textit{and} \tag{2b}$$
$$\forall y \in [x_0, \infty) \qquad \Big(\exists n \in \mathbb{N}, n < n_0 \quad y - x_0 \geq \delta_0^n/2\Big) \Rightarrow y \in B. \tag{2c}$$

Proof. Let $\delta := \delta_0^{n_0}$ and define A, B as in Lemma A of [18]. This gap will satisfy (2a) and (2b) according to that lemma. So it remains to be shown that (2c) holds. To this end, assume that $y - x_0 \geq \delta_0^n/2$ for some $n < n_0$, and that $y \in A$. Then there exists $m \in \mathbb{N}$ such that $y < x_0 + m\delta_0^{n_0}$. Thus, by assumption, $\delta_0^n/2 \leq y - x_0 < m\delta_0^{n_0} \Rightarrow \frac{1}{2m} < \delta_0^{n_0-n}$, which is impossible, since $\delta_0^{n_0-n}$ is an infinitesimal. $\square$

Proof of (AP) $\Leftrightarrow$ (AP23).
($\Rightarrow$): Simple consequence of (AP4) and Lemma 3.2.
($\Leftarrow$): Let $\mathbb{F}$ be non–Archimedean (with countable cofinality), let $x_0 = 0$, and $x_n = \frac{1}{2} + n\delta_0$ ($n \geq 1$). Define $\tilde{C} :=$ (downward closure of $\{x_n\}$) and $\tilde{D} := \mathbb{F} \setminus \tilde{C}$. Then $\tilde{C}, \tilde{D}$ is an irregular gap with $\tilde{D} - \tilde{C} \geq \delta_0$. Finally, let $C := \tilde{C} \cap [0, 1]$ and $D := \tilde{D} \cap [0, 1]$ and $y_n = \frac{x_{n+1} + x_n}{2}$. Note that $\bigcup_{n \geq 0} [x_n, x_{n+1}] = C$. Now, for each $n \in \mathbb{N}$, let $\tilde{A}_n, \tilde{B}_n$ be an irregular gap satisfying

$$y_n \in \tilde{A}_n, \tag{3a}$$
$$\forall a \in \tilde{A}_n, b \in \tilde{B}_n \qquad b - a \geq \delta_0^2, \qquad \text{and} \tag{3b}$$
$$\forall y \in [y_n, \infty) \qquad y - y_n \geq \delta_0/2 \text{ implies that } y \in \tilde{B}_n \tag{3c}$$

[13] Accordingly, the statement was phrased "Every uniformly continuous function maps bounded sets to bounded sets."

(exist by Lemma 4.1 above). Note that (3a) and (3c) imply $A_n := \tilde{A}_n \cap [x_n, x_{n+1}] \neq \emptyset$ and $B_n := \tilde{B}_n \cap [x_n, x_{n+1}] \neq \emptyset$, respectively, since $y_n \in (x_n, x_{n+1})$ and $x_{n+1} - y_n = \frac{\delta_0}{2}$. Finally, let $(\lambda_n)_{n\in\mathbb{N}} \subset \mathbb{F}$ be an unbounded sequence. Now define $f : [0,1] \to \mathbb{F}$ by

$$\forall x \in [x_n, x_{n+1}], \quad f(x) = \begin{cases} \lambda_n, & x \in A_n \\ \lambda_{n+1}, & x \in B_n \end{cases} \quad \text{and } \forall x \in D, \quad f(x) = 0. \tag{4}$$

Clearly, f is well–defined and unbounded. It remains to be shown that it is uniformly continuous. This follows as in the proof of [18, Prop.4 (ix)], since $f(x) = f(y)$ for all $x, y \in [0,1]$ such that $|x - y| < \delta_0^2$. □

Remark. It is not difficult to verify that the function defined in (4) is also uniformly differentiable.

PROOF OF (AP) $\Leftrightarrow$ (AP24).
($\Rightarrow$): Suppose $\mathbb{F}$ is Archimedean, and let $f : [a,b] \to \mathbb{F}$ be a uniformly continuous function. Then by (AP4), Lemma 3.2 and (CA26), f can be extended to a continuous function in $\mathbb{R}$, which is bounded by the completeness of $\mathbb{R}$. Since $\mathbb{R}$ is Archimedean, the bound may be chosen as a natural number. Hence f is bounded by a natural number as well.
($\Leftarrow$): Suppose $\mathbb{F}$ is non-Archimedean. Then the function $f(x) = \lambda$ for some infinite λ provides a trivial counterexample. □

Next we quote a critical result on uniformly differentiable functions; see [4, Thrm 4] for a proof.

Lemma 4.2 (Extensions of Uniformly Differentiable Functions)**.**
Let $\mathbb{F}$ be Archimedean and $f : [a,b] \to \mathbb{F}$ a uniformly differentiable function. Then f is uniformly continuous and its unique (uniformly) continuous extension $\bar{f} : [a,b]_{\mathbb{R}} \to \mathbb{R}$ (where $[a,b] = [a,b]_{\mathbb{R}} \cap \mathbb{F}$) is uniformly differentiable with $\bar{f}'(x) = f'(x)$ for all $x \in [a,b]$. Furthermore, the extension of f' exists (uniquely) and coincides with $\bar{f}'$ on $[a,b]_{\mathbb{R}}$.

PROOF OF (AP) $\Leftrightarrow$ (AP25) $\Leftrightarrow$ (AP26).

(AP) $\Rightarrow$ (AP25): Suppose $\mathbb{F}$ is Archimedean and $f : [a,b] \to \mathbb{F}$ is a uniformly differentiable function. Let $\bar{f} : [a,b]_{\mathbb{R}} \to \mathbb{R}$ be the extension of f to $[a,b]_{\mathbb{R}}$ according to Lemma 4.2, where $[a,b] = [a,b]_{\mathbb{R}} \cap \mathbb{F}$. $\bar{f}$ is uniformly differentiable, in particular C^1 and thus, by (CA65), Lipschitz. Clearly, its restriction $f = \bar{f}|_{[a,b]}$ is also Lipschitz.

(AP25) $\Rightarrow$ (AP26): Clear.

(AP32) $\Rightarrow$ (AP): Assume that $\mathbb{F}$ is not Archimedean. Then the function constructed in the proof of the implication "(AP23) $\Rightarrow$ (AP)" can be shown to be uniformly differentiable and unbounded. $\square$

PROOF OF (AP) $\Leftrightarrow$ (AP27).
($\Rightarrow$): Suppose $\mathbb{F}$ is Archimedean, and let $f : [0,1] \to \mathbb{F}$ be uniformly continuous and $\varepsilon \in \mathbb{F}^+$ arbitrary. Then, by Lemma 3.2, f has a continuous extension to $\mathbb{R}$, which we denote $f_{\mathbb{R}}$. Since $\mathbb{R}$ is complete, (CA51) holds and so there exists a polynomial p with coefficients in $\mathbb{R}$ such that $|f_{\mathbb{R}}(x) - p(x)| < \varepsilon/2$ for all $x \in [0,1]$.

Let $p(x) = \sum_{k=0}^{n} a_k x^k$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, for each a_k, there exists a $b_k \in \mathbb{Q}$ such that $|a_k - b_k| < \varepsilon/(2(n+1))$. Define the polynomial $q \equiv \sum_{k=0}^{n} b_k x^k$. A simple calculation shows that q is a polynomial with coefficients in $\mathbb{Q} \subset \mathbb{F}$ which is ε-close to f.
($\Leftarrow$): Suppose $\mathbb{F}$ is non-Archimedean. Then by $\neg$(AP23) there exists a uniformly continuous and unbounded function $f : [0,1] \to \mathbb{F}$. If (AP27) were true, we could find a sequence of polynomials that converge uniformly to f, which is impossible in light of Lemmas 3.5 and 3.6. $\square$

PROOF OF (AP) $\Leftrightarrow$ (AP28).
($\Rightarrow$): Let $\mathbb{F}$ be Archimedean and $\bar{f}_n : [a,b]_{\mathbb{R}} \to \mathbb{R}$ (where $[a,b] = [a,b]_{\mathbb{R}} \cap \mathbb{F}$) the uniformly differentiable extension of $f_n : [a,b] \to \mathbb{F}$ according to Lemma 4.2. Then the assertion follows by applying (CA61) to the sequence $\{\bar{f}_n\}$.
($\Leftarrow$): Let $f : [0,1] \to \mathbb{F}$ be the function constructed in (4) and f_n defined by

$$f_n|_{[0,x_{n+1}] \cup D} = f \quad \text{and} \quad f_n|_{C \setminus [0,x_{n+1}]} \equiv \lambda_{n+1}. \tag{5}$$

Then $f_n' \equiv 0$ for all n, so $\{f_n'\}$ converges uniformly and $\{f_n(\frac{1}{2})\}$ converges pointwise. However, $\{f_n\}$ cannot be uniformly Cauchy, since the set $\{f_n(x) \mid x \in [0,1],\ n \in \mathbb{N}\}$ is unbounded. $\square$

PROOF OF (AP) $\Leftrightarrow$ (AP29).
($\Rightarrow$): Since $\{f_n\}$ is *uniformly* equicontinuous, every function $f_n : [a,b] \to \mathbb{F}$ is in particular uniformly continuous. So, by Lemma 3.2 (and the assumption that $\mathbb{F}$ is Archimedean), each function f_n may be extended to a (uniformly continuous) function $\bar{f}_n : [a,b]_{\mathbb{R}} \to \mathbb{R}$, where $[a,b] = [a,b]_{\mathbb{R}} \cap \mathbb{F}$. Now the assertion follows by applying (CA63) to the sequence $\{\bar{f}_n\}$.
($\Leftarrow$): Assume $\neg$(AP) and let δ be a positive infinitesimal. Then $y_n = n\delta$ is a bounded sequence that does not possess a subsequence that is Cauchy. Now define $f_n \equiv y_n$ ($n \in \mathbb{N}$), which is obviously uniformly bounded and uniformly equicontinuous. However, $\{f_n\}$ cannot possess a pointwise Cauchy subsequence, since $\{f_n(x)\} = \{y_n\}$ does not have one. $\square$

PROOF OF (AP) $\Leftrightarrow$ (AP30).
($\Rightarrow$): Identical to "($\Rightarrow$)" of the previous proof with (CA63) replaced by (CA64).
($\Leftarrow$): Assume $\neg$(AP) and let $\{f_n\}$ the sequence constructed in (5). Then $\{f_n\}$ is uniformly equicontinuous and pointwise convergent, but $\{f_n\}$ cannot be uniformly Cauchy, since $\{f_n(x) \mid x \in [0,1],\ n \in \mathbb{N}\}$ is unbounded. $\square$

PROOF OF (AP) $\Leftrightarrow$ (AP31) $\Leftrightarrow$ (AP32).

(AP) $\Rightarrow$ (AP31): Suppose $\mathbb{F}$ is Archimedean. Then by (AP5), $\{(-1)^n/n\}$ is a null sequence and $\{n \cdot (-1)^n/n\}$ does not converge.

(AP31) $\Rightarrow$ (AP32): If (AP31) holds, then using $r_n = n$ gives a non-convergent sequence for (AP32) as well.

(AP32) $\Rightarrow$ (AP): Suppose $\mathbb{F}$ is non-Archimedean and let $\{x_n\}$ be a null sequence in $\mathbb{F}$. Since $\mathbb{F}$ is non-Archimedean, by $\neg$(AP1), there exists a positive infinitesimal δ. Let $\varepsilon \in \mathbb{F}^+$ be given. Since $\{x_n\}$ is a null sequence, there exists $N \in \mathbb{N}$ such that $|x_n| < \delta\varepsilon$ for all $n \geq N$. Then for all such n, $|r_n x_n| = |r_n||x_n| < |r_n|\delta\varepsilon < \varepsilon$ and so $\{r_n x_n\}$ is a null sequence as well, and thus converges. $\square$

PROOF OF (AP) $\Leftrightarrow$ (AP33).
($\Leftarrow$): Suppose $\mathbb{F}$ is non-Archimedean and let $\{a_n\}$ be a quasi-Cauchy sequence. Let δ be a positive infinitesimal and $\varepsilon \in \mathbb{F}^+$ be given. Then we can find an $N \in \mathbb{N}$ such that $|a_{n+1} - a_n| < \delta\varepsilon$ for all $n \geq N$. Let $m, n \geq N$ be arbitrary and assume $m > n$. Then $|a_m - a_n| = \left|\sum_{k=n}^{m-1} a_{k+1} - a_k\right| \leq \sum_{k=n}^{m-1} |a_{k+1} - a_k| < \sum_{k=n}^{m-1} \delta\varepsilon = (m-n)\delta\varepsilon < \varepsilon$ and so $\{a_n\}$ is a Cauchy sequence since the above holds for arbitrarily small ε.
($\Rightarrow$): Suppose $\mathbb{F}$ is Archimedean. Consider the sequence $s_n \equiv \sum_{k=1}^{n} 1/n$. Then $\{s_n\}$ is certainly a quasi-Cauchy sequence by (AP5). However, we know that the harmonic series does not converge in $\mathbb{R}$, and hence not in $\mathbb{F}$ either, so $\{s_n\}$ cannot be Cauchy. $\square$

PROOF OF (AP) $\Leftrightarrow$ (AP34).
($\Rightarrow$): Let $\mathbb{F}$ be an Archimedean field, and suppose for a contradiction that there exists an order-preserving field homomorphism ϕ from $\mathbb{Q}(X)$ into $\mathbb{F}$. Let λ be infinitely large in $\mathbb{Q}(X)$, for example $1/X$. Let $n \in \mathbb{N}$ be fixed and arbitrary. Since n is a finite element in $\mathbb{Q}(X)$, $\lambda > n$ and so $\phi(\lambda) > \phi(n) = n$. This means that $\phi(\lambda)$ is infinitely large in $\mathbb{F}$, which violates (AP2). So no such ϕ exists.
($\Leftarrow$): Suppose $\mathbb{F}$ is non-Archimedean and let $\delta \in \mathbb{F}^+$ be an infinitesimal. We

claim taking $\phi : \mathbb{Q}(X) \hookrightarrow \mathbb{F}$ as the evaluation of $\mathbb{Q}(X)$ at δ will work. The evaluation is automatically a field homomorphism because of the properties of rational fractions, so it remains to show that it preserves the positive elements of $\mathbb{Q}(X)$. To this end, let x be an arbitrary positive element of $\mathbb{Q}(X)$; x can be written as $x \equiv X^i(\sum_{k=0}^n p_k X^k)/(\sum_{j=0}^m q_j X^j)$ in lowest terms, $\{p_k\}_{k=0}^n$, $\{q_j\}_{j=0}^m \subset \mathbb{Q}$, $n, m \in \mathbb{N}$, $i \in \mathbb{Z}$, and since $x > 0$, we know $p_0/q_0 > 0$. Now, $\phi(x)$ is given by $\phi(x) = \delta^i(\sum_{k=0}^n p_k\delta^k)/(\sum_{j=0}^m q_j\delta^j)$. Since $\delta \in \mathbb{F}^+$, it suffices to show that $\left(\sum_{k=0}^n p_k\delta^k\right) / \left(\sum_{j=0}^m q_j\delta^j\right) > 0$. We prove that the numerator is positive; the proof for the denominator is the same.

We know δ^k is infinitesimal for $k > 0$ and so $|p_k\delta^k|$ is also infinitesimal for $k > 0$ since $p_k \in \mathbb{Q}$. Now, since $p_0/n \in \mathbb{Q}$ is finite, $|p_k\delta^k| < p_0/n$. Thus, $\sum_{k=0}^n p_k\delta^k = p_0 + \sum_{k=1}^n p_k\delta^k \geq p_0 - \left|\sum_{k=1}^n p_k\delta^k\right| \geq p_0 - \sum_{k=1}^n |p_k\delta^k| > p_0 - \sum_{k=1}^n \frac{p_0}{n} = 0$ and so the numerator is positive, as desired. Therefore, $\phi(x) > 0$ and so ϕ preserves the positive elements of $\mathbb{Q}(X)$. $\square$

Proof of (AP) $\Leftrightarrow$ (AP35).
($\Rightarrow$): The harmonic sequence is an example.
($\Leftarrow$): Suppose $\mathbb{F}$ is non-Archimedean, let $\{a_n\}$ be a null sequence and define $s_n \equiv \sum_{k=0}^n a_k$. Then $s_{n+1} - s_n = a_n$, so $\{s_n\}$ is quasi–Cauchy. By $\neg$(AP33) this means that $\{s_n\}$ is in fact Cauchy. $\square$

Proof of (AP) $\Leftrightarrow$ (AP36) $\Leftrightarrow$ (AP37).

(AP) $\Rightarrow$ (AP36): Assume without loss of generality that f is increasing and let $\varepsilon \in \mathbb{F}^+$ be given. Then since f is Archimedean, there is some $N \in \mathbb{N}$ so that $N\varepsilon > f(b) - f(a)$. We will show that $|D_\varepsilon(f)| < N < \infty$. Suppose not. Then there are at least N points $\{x_1, x_2, \ldots, x_N\}$ in $D_\varepsilon(f)$. We may assume without loss of generality that these points are in increasing order. By definition, there are corresponding points y_n such that $|f(x_n) - f(y_n)| \geq \varepsilon$. Let I_n be the interval with endpoints $f(x_n)$ and $f(y_n)$. Since f is monotonic and we can find these y_n arbitrarily close to each x_n, we can enforce that the collection of I_n are pairwise disjoint. Then $f(b) - f(a) = \text{length}([f(a), f(b)]) \geq \sum_{n=1}^N \text{length}(I_n) = \sum_{n=1}^N |f(x_n) - f(y_n)| \geq N\varepsilon$ which contradicts the definition of N.

(AP36) $\Rightarrow$ (AP37): Trivial.

(AP37) $\Rightarrow$ (AP) : Let $\mathbb{F}$ be non-Archimedean, and let $\delta \in \mathbb{F}^+$ be infinitesimal. Let $\{\varepsilon_n\}$ be a strictly decreasing null sequence with $\varepsilon_1 = 1$. Let $\varepsilon \in \mathbb{F}^+$ be given. We will construct a function $f : [0, 1] \to \mathbb{F}$ so that $D_\varepsilon(f)$ is infinite.

Define f by

$$f(x) \equiv \begin{cases} n\varepsilon, & \text{if } x \in (\varepsilon_{n+1}, \varepsilon_n] \\ \varepsilon/\delta, & \text{otherwise, i.e. when } x = 0. \end{cases}$$

Then each ε_n is in $D_\varepsilon(f)$ and hence $D_\varepsilon(f)$ is infinite. Note that f is monotonically decreasing on $[0, 1]$ since $\varepsilon/\delta > n\varepsilon$ for all $n \in \mathbb{N}$. □

PROOF OF (AP) ⇔ (AP38).
(⇒): It is easy to verify that taking $x = 1/2$ will yield a geometric series convergent to 2 in an Archimedean field.
(⇐): In a non-Archimedean field, ¬(AP35) yields the result that if $\sum_{n=0}^{\infty} x^n$ is convergent in $\mathbb{F}$, then $x_n \to 0$. But this cannot happen for any $q \in \mathbb{Q}$ by ¬(AP15). □

PROOF OF (AP) ⇔ (AP39) ⇔ (AP40).

(AP) ⇒ (AP39): If we interpret this series as a power series in $\mathbb{R}$, the condition yields a non-zero radius of convergence $R = 1/\limsup_{n\to\infty} \sqrt[n]{|c_n|}$. Hence for any $q \in \mathbb{Q}^+$ with $q < R$ the series will converge in $\mathbb{R}$, i.e. the sequence of partial sums is Cauchy. Note that, while the series may not converge in $\mathbb{F}$, the sequence of partial sums is still Cauchy by the density of $\mathbb{F}$ in $\mathbb{R}$, which is all we require.

(AP39) ⇒ (AP40): Clear.

(AP40) ⇒ (AP) : Suppose $\mathbb{F}$ is non-Archimedean. Obviously, each term in the sequence of partial sum is rational, since all $c_n \in Q$ and $x \in \mathbb{Q}^+$. Since x is non-zero and $\limsup_{n\to\infty} \sqrt[n]{|c_n|} < \infty$, the sequence of partial sums is not eventually constant. But then by ¬(AP8), it is impossible for such a sequence to be Cauchy. □

PROOF OF (AP) ⇔ (AP41) ⇔ (AP42).
(AP) ⇒ (AP41): Suppose $\mathbb{F}$ is an Archimedean field and $f : [a, b] \to \mathbb{F}$ is uniformly continuous. Let y between $f(a)$ and $f(b)$ be given. Then, by Lemma 3.2, we can extend f to a continuous function $\bar{f} : [a, b]_\mathbb{R} \to \mathbb{R}$. Since $\mathbb{R}$ is complete, we can use (CA19) to find some $c \in [a, b]_\mathbb{R}$ so that $\bar{f}(c) = y$. Now let $\varepsilon \in \mathbb{F}^+$ be given. By the continuity of $\bar{f}$, we can find some $\delta > 0$ so that $x \in [a, b]_\mathbb{R}$ being δ-close to c implies $\bar{f}(x)$ is ε-close to y. In particular, we can choose some $q \in \mathbb{Q} \cap [a, b]_\mathbb{R}$ by the density of $\mathbb{Q}$ in $\mathbb{R}$. Hence taking $q \in \mathbb{F}$ will yield $f(q) \in \mathbb{F}$ so that $f(q)$ is ε-close to y.
(AP41) ⇒ (AP42): Trivial.

(AP42) $\Rightarrow$ (AP) : Suppose $\mathbb{F}$ is non-Archimedean and let A, B be an irregular gap in $\mathbb{F}$, $a \in A$ and $b \in B$. Let $\varepsilon \in \mathbb{F}^+$ be given. Define $f : [a, b] \to \mathbb{F}$ by

$$f(x) \equiv \begin{cases} 0, & \text{if } x \in [a,b] \cap A \\ 2\varepsilon, & \text{if } x \in [a,b] \cap B \end{cases}$$

and note that f is uniformly continuous on $[a, b]$. Then between a and b, f takes on only values of 0 and 2ε and hence there is no value c in $[a, b]$ so that $|f(c) - \varepsilon| < \varepsilon$. □

5 Some open problems and conjectures

As mentioned in the introduction, in the course of this project several questions have arisen that remain unsolved. In this final section we list a few of them.

Intermediate Value Theorems. One of the most intriguing problems has already been described in the introduction: is the IVT when restricted to *uniformly* continuous functions still equivalent to completeness? Note that this is *not* the case for the "approximate" version of the IVT (CA71): its uniform version is no longer equivalent to completeness; instead, it is equivalent to the AP, see (AP41). In fact, a whole scale of IVTs can be defined by replacing "continuous" in the IVT with "uniformly continuous", "Lipschitz", "convergent power series" etc. For each of these versions of the IVT, one can ask which property (if any) of an ordered field $\mathbb{F}$ it characterizes. For instance, the "polynomial" version is equivalent to $\mathbb{F}$ being "real-closed." Pressed for conjectures, we would probably venture to say that the "uniform" version *is* equivalent to completeness, whereas the "power-series" version *might* be in Archimedean fields.

Uniform differentiability. Similar questions as described for the IVT can be asked for the Mean Value, Rolle's and Darboux's Theorems for *uniformly* differentiable (instead of just differentiable) functions.

Characterizations of countable cofinality. Curiously, very few of those seem to be known to date. While this is not the subject of this paper, the interested reader can find a short list in [2].

Power series. We think that there should be additional characterizations of completeness or the AP that make reference to power series, such as the existence of a well-defined radius of convergence.

Integration. (a) Riemann integrability and the existence of Darboux integrals

can be shown to be equivalent in Archimedean fields. We conjecture that this property is *equivalent* to the AP.
(b) More generally, Darboux and Riemann integration in non-Archimedean fields is of limited use as not even the identity function is integrable; see (AP20). This raises the question of an elementary definition of a more suitable integration process.[14]
(c) Related to this is the question of finding an analog of (CA53) in general ordered fields; i.e. of finding a formulation of the first part of the Fundamental Theorem of Calculus equivalent to completeness *without* assuming the AP.

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