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SOME NEW TYPES OF FILTER LIMIT THEOREMS FOR TOPOLOGICAL GROUP-VALUED MEASURES

Abstract

Some new types of limit theorems for topological group-valued measures are proved in the context of filter convergence for suitable classes of filters. We investigate (s)-boundedness, σ-additivity and regularity properties of topological group-valued measures. We consider also Schur-type theorems, using the sliding hump technique, and prove some convergence theorems in the particular case of positive measures. We deal with the notion of uniform filter exhaustiveness, by means of which we prove some theorems on existence of the limit measure, some other kinds of limit theorems and their equivalence, using known results on existence of countably additive restrictions of strongly bounded measures. Furthermore we pose some open problems.

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1 Introduction

The theory of filter convergence has been the object of several recent studies. The concept of filter convergence was introduced in [44]. An interesting case of filter convergence is the statistical convergence, introduced by H. Fast ([43]), H. Steinhaus ([52]) and I. Schoenberg ([48]). These topics have been several applications in the very recent literature (see also [26, 31, 39, 40]). Among them we recall, for instance, Functional Analysis (see for instance [2, 8, 9, 20, 21, 23, 24, 25]), Approximation Theory of positive operators, signal sampling, image and audio-video reconstruction (see also [4, 5, 12, 13, 16, 27]).

This paper is a free continuation of the research initiated in [14], where some aspects of filter convergence of sequences of topological group-valued measures are investigated. Here we deal with topological group-valued measures. Among the studies in the classical case we quote, for instance, [35, 41, 42]. In [35, 42] there are also some results about equivalence between classical versions of limit theorems. A survey on the literature about these topics can be found in [36] and in the bibliography therein. In [14] some Nikodým and Brooks-Jewett-type theorems are given with respect to filter convergence for topological group-valued measures. Here we give different types of limit theorems in this framework. In general, it is impossible to obtain results analogous to the classical ones, even for positive real-valued measures (see also [20, Example 3.4], [22, Remark 3.8]). Different versions of such kind of theorems are established in [2] for real-valued measures and [21, 22] for (ℓ)-group-valued measures. We deal with some basic properties of filter convergence and topological group-valued measures and some relations between them, and prove some Schur-type and limit theorems. As a particular case, we consider positive measures, and in this context we give some limit theorems by considering a larger class of filters. Some topics about topological groups can be found, for instance, in [33, 47].

Observe that, in the context of topological groups, it is sufficient to deal with a suitable basis of neighborhoods of zero, which allows us to give a direct approach to our theorems. Similar results are proved in [21] in the context of (ℓ)-groups and Riesz spaces, where one considers order sequences or regulators, playing a role similar to that of neighborhoods of zero. In lattice groups, among the more frequently used tools we recall the Fremlin lemma, which allows to replace countably many regulators by a single one and is useful in particular in matrix-diagonal processes (see also [1, 20]), and the Maeda-Ogasawara-Vulikh representation theorem, by means to which several properties of lattice group-valued measures can be studied, by investigating the corresponding ones of real-valued measures.

In the setting of topological groups it is possible to use different techniques, since we deal with a different kind of structure. To prove equivalence results between filter limit theorems, we apply some results about existence of suitable σ-additive restrictions of (s)-bounded measures, like in [41, 42], and without considering the Stone Isomorphism technique, though it is possible to get Stone-type extensions also for (s)-bounded topological group-valued measures (see also [49, 50]). In the lattice group setting (see [15]) it is dealt with the Stone extensions, since the nature of the convergence in such groups is not necessarily topological, and hence it is not advisable to argue with σ-additive restrictions. However, the Drewnowski-type technique here used is in general easier to handle than the Stone Isomorphism technique. Finally, we pose some open problems.

2 Preliminaries

We begin with recalling the basic properties of filters.

Let $Z \neq \emptyset$ be any set. A *filter* $\mathcal{F}$ of Z is a nonempty collection of subsets of Z with $\emptyset \notin \mathcal{F}$, $A \cap B \in \mathcal{F}$ whenever A, $B \in \mathcal{F}$, and such that for each $A \in \mathcal{F}$ and $B \supset A$ we get $B \in \mathcal{F}$. A filter of Z is said to be *free* iff it contains the Fréchet filter $\mathcal{F}_{\rm cofin}$ of all cofinite subsets of Z.

Let Q be a countable set and $\mathcal{F}$ be a filter of Q. A subset of Q is $\mathcal{F}$-*stationary* iff it has nonempty intersection with every element of $\mathcal{F}$. We denote by $\mathcal{F}^*$ the family of all $\mathcal{F}$-stationary subsets of Q. If $I \in \mathcal{F}^*$, then the *trace* $\mathcal{F}(I)$ of $\mathcal{F}$ on I is the family $\{F \cap I : F \in \mathcal{F}\}$.

Observe that $\mathcal{F}(I)$ is a filter of I. Indeed, if F_1, $F_2 \in \mathcal{F}(I)$, then $(F_1 \cap F_2) \cap I = (F_1 \cap I) \cap (F_2 \cap I) \in \mathcal{F}$, and hence $F_1 \cap F_2 \in \mathcal{F}(I)$.

Let now $F \in \mathcal{F}$ and $F \cap I \subset F' \subset I$, and set $F^* := F' \cup F$: then $F^* \in \mathcal{F}$ and $F^* \cap I \supset F \cap I$. It is easy to see that $F' \subset F^* \cap I$. To prove the converse inclusion, note that $F^* \cap I = (F' \cap I) \cup (F \cap I) \subset F'$. Hence, $F' = F^* \cap I \in \mathcal{F}(I)$, and thus we get the claim.

A free filter $\mathcal{F}$ of $\mathbb{N}$ is a P-*filter* iff for every sequence $(A_n)_n$ in $\mathcal{F}$ there is a sequence $(B_n)_n$ in $\mathcal{F}$, such that the symmetric difference $A_n \triangle B_n$ is finite for all $n \in \mathbb{N}$ and $\bigcap_{n=1}^{\infty} B_n \in \mathcal{F}$.

A filter $\mathcal{F}$ of Q is said to be *diagonal* iff for every sequence $(A_n)_n$ in $\mathcal{F}$ and for each $I \in \mathcal{F}^*$ there exists a set $J \subset I$, $J \in \mathcal{F}^*$ such that the set $J \setminus A_n$ is finite for all $n \in \mathbb{N}$ (see also [21, 22]).

Remark 2.1. Observe that every P-filter $\mathcal{F}$ is diagonal. Indeed, let $(A_n)_n$ be a sequence in $\mathcal{F}$ and $I \in \mathcal{F}^*$. As $\mathcal{F}$ is a P-filter, then by [3, Proposition 1] there exists $J_0 \in \mathcal{F}$, with the property that $J_0 \setminus A_n$ is finite for every $n \in \mathbb{N}$. We claim that $J := I \cap J_0 \in \mathcal{F}^*$. Indeed, if E is any element of $\mathcal{F}$, then $J_0 \cap E \in \mathcal{F}$. So, as $I \in \mathcal{F}^*$, we get $\emptyset \neq I \cap J_0 \cap E = J \cap E$. By arbitrariness of E, $J \in \mathcal{F}^*$, and thus we get the claim. Therefore, the set J satisfies the condition requested in the definition of diagonal filter.

From now on $\mathcal{F}$ denotes a free filter of $\mathbb{N}$, $R = (R, +)$ is a Hausdorff complete abelian topological group satisfying the first axiom of countability, with neutral element 0, and $\mathcal{J}(0)$ denotes a basis of closed and symmetric neighborhoods of 0 (see also [28, 29, 30]). Moreover, given $k \in \mathbb{N}$ and U, $U_1, \ldots, U_k \subset R$, put $U_1 + \cdots + U_k := \{u_1 + \ldots + u_k\colon\, u_1 \in U_1, \ldots, u_k \in U_k\}$, and $k\,U := U + \cdots + U$ (k times).

A sequence $(x_n)_n$ in R *$\mathcal{F}$-converges to* $x_0 \in R$ iff for every $U \in \mathcal{J}(0)$, $\{n \in \mathbb{N} : x_n - x_0 \in U\} \in \mathcal{F}$, and we write $(\mathcal{F}) \lim_n x_n = x_0$. Moreover, we say that a sequence $(B_n)_n$ of subsets of R *$\mathcal{F}$-converges to* 0 iff for each $U \in \mathcal{J}(0)$ the set $\{n \in \mathbb{N} : B_n \subset U\}$ belongs to $\mathcal{F}$, and we write $(\mathcal{F}) \lim_n B_n = 0$. We say that $\lim_n x_n = x_0$ (resp. $\lim_n B_n = 0$) iff $(\mathcal{F}_{\text{cofin}}) \lim_n x_n = x_0$ (resp. $(\mathcal{F}_{\text{cofin}}) \lim_n B_n = 0$). Furthermore, we denote by $\sum_{n=1}^{\infty} B_n$ the set of all elements $b \in R$ of the type $b = \sum_{k=1}^{\infty} b_k := \lim_n \sum_{k=1}^{n} b_k$ as b_k varies in B_k, $k \in \mathbb{N}$.

Note that the $\mathcal{F}$-limit is unique, since R is Hausdorff (see also [45]).

Observe that filter convergence satisfies the following property.

($\mathcal{U}$) If each subsequence of a given sequence $(x_n)_n$ has a sub-subsequence which $\mathcal{F}$-converges to x_0, then $(\mathcal{F}) \lim_n x_n = x_0$.

Otherwise, there exist $U \in \mathcal{J}(0)$ with $\mathcal{Z}(U) := \{n \in \mathbb{N} : x_n - x_0 \in U\} \notin \mathcal{F}$. Since $\mathcal{F}$ is free, the set $\mathcal{Y}(U) := \mathbb{N} \setminus \mathcal{Z}(U)$ is infinite, say $\mathcal{Y}(U) := \{n_1 < n_2 < \ldots < n_k < \ldots\}$. Thus the subsequence $(x_{n_k})_k$ does not have any sub-subsequence, $\mathcal{F}$-convergent to x_0.

Note that, in general, property ($\mathcal{U}$) is not true in the lattice context: for instance, this is the case of the space $L^0(X, \mathcal{B}, \mu)$ of all μ-measurable real-valued functions with identification up to μ-null sets, endowed with the almost everywhere convergence, where $\mu : \mathcal{B} \to [0, +\infty]$ is a σ-additive and σ-finite measure (see also [53]).

We now prove a Cauchy criterion for filter convergence. Similar results in the context of topological spaces can be found in [34] (see also [37], for the classical case).

Theorem 2.2. *Let R and $\mathcal{F}$ be as above, $x \in R$ and $(x_n)_n$ be a sequence in R. Then the following are equivalent:*

(j) $(\mathcal{F})\lim_n x_n = x$;

(jj) for every $U \in \mathcal{J}(0)$ there is $r \in \mathbb{N}$ with $\{n \in \mathbb{N} : x_n - x_r \in U\} \in \mathcal{F}$;

(jjj) for every $U \in \mathcal{J}(0)$ there is $F \in \mathcal{F}$ with $x_n - x_r \in U$ whenever n, $r \in F$.

Proof. $(jj) \Longrightarrow (j)$

Choose arbitrarily $U \in \mathcal{J}(0)$, let $U_0 \in \mathcal{J}(0)$ be with $3\,U_0 \subset U$, and $(U_p)_p$ be a decreasing countable basis of closed symmetric neighborhoods of 0. For each p, $q \in \mathbb{N}$ there are r_p, $r_q \in \mathbb{N}$ with

$$\{n \in \mathbb{N} : x_n - x_{r_p} \in U_p\} \cap \{n \in \mathbb{N} : x_n - x_{r_q} \in U_q\} \in \mathcal{F}.$$

So there exists $n_{p,q} \in \mathbb{N}$ with $x_{n_{p,q}} - x_{r_p} \in U_p$, $x_{n_{p,q}} - x_{r_q} \in U_q$, so that $x_{r_p} - x_{r_q} \in U_p + U_q$. Thus the sequence $(x_{r_p})_p$ is Cauchy in R in the classical sense and so it converges to an element $x \in R$, since R is complete. If $q \geq p$, we get $x_{r_p} - x_{r_q} \in 2\,U_p$. Taking the limit as q tends to $+\infty$, we obtain $x_{r_p} - x \in 2\,U_p$, since U_p is closed.

Pick arbitrarily $p \in \mathbb{N}$. If $x_n - x_{r_p} \in U_p$, then

$$x_n - x = x_n - x_{r_p} + x_{r_p} - x \in 3\,U_p,$$

and thus

$$\{n \in \mathbb{N} : x_n - x \in 3\,U_p\} \supset \{n \in \mathbb{N} : x_n - x_{r_p} \in U_p\}.$$

Now, choose arbitrarily $U \in \mathcal{J}(0)$. There is $\overline{p} \in \mathbb{N}$ with $3\,U_{\overline{p}} \subset 3\,U_0 \subset U$, and so

$$\{n \in \mathbb{N} : x_n - x \in U\} \supset \{n \in \mathbb{N} : x_n - x \in 3\,U_{\overline{p}}\} \supset \{n \in \mathbb{N} : x_n - x_{r_{\overline{p}}} \in U_{\overline{p}}\}.$$

Since $\{n \in \mathbb{N} : x_n - x_{r_{\overline{p}}} \in U_{\overline{p}}\} \in \mathcal{F}$, then $\{n \in \mathbb{N} : x_n - x \in U\} \in \mathcal{F}$, and hence $(\mathcal{F})\lim_n x_n = x$, that is (j).

$(j) \Longrightarrow (jjj)$

Suppose that $(\mathcal{F})\lim_n x_n = x$, choose arbitrarily $U \in \mathcal{J}(0)$ and let $U^* \in \mathcal{J}(0)$ be such that $2\,U^* \subset U$. Then in correspondence with U^* there is $F \in \mathcal{F}$ with $x_n - x \in U^*$ for each $n \in F$, and so for every n, $r \in F$ we get $x_n - x_r \in U$.

$(jjj) \Longrightarrow (jj)$

Choose arbitrarily $U \in \mathcal{J}(0)$. Then there exists $F \in \mathcal{F}$ with $x_n - x_r \in U$ for all n, $r \in F$. If $r_0 = \min F$, then $\{n \in \mathbb{N} : x_n - x_{r_0} \in U\} \supset F$, and hence $\{n \in \mathbb{N} : x_n - x_{r_0} \in U\} \in \mathcal{F}$, since $F \in \mathcal{F}$. $\square$

We now consider some properties of filters.

Given an infinite set $I \subset Q$, a *blocking* of I is a countable partition $\{D_k : k \in \mathbb{N}\}$ of I into nonempty finite subsets.

A filter $\mathcal{F}$ of Q is said to be *block-respecting* iff for every $I \in \mathcal{F}^*$ and for each blocking $\{D_k : k \in \mathbb{N}\}$ of I there is a set $J \in \mathcal{F}^*$, $J \subset I$ with $\sharp(J \cap D_k) = 1$ for all $k \in \mathbb{N}$, where $\sharp$ denotes the number of elements of the set into brackets.

Some examples of filters satisfying these properties and of filters lacking them can be found in [2].

The following results will be useful in the sequel.

Proposition 2.3. (see [14, Proposition 2.1]) *If $\mathcal{F}$ is a block-respecting filter of $\mathbb{N}$, then $\mathcal{F}(I)$ is a block-respecting filter of I for every $I \in \mathcal{F}^*$.*

Proposition 2.4. *If $\mathcal{F}$ is any free filter, x_n, $n \in \mathbb{N}$, is a sequence in R, $\mathcal{F}$-convergent to $x \in R$, and $J \in \mathcal{F}^*$, then the sequence x_n, $n \in J$, $\mathcal{F}(J)$-converges to x.*

PROOF. Choose arbitrarily $U \in \mathcal{J}(0)$, and set $F := \{n \in \mathbb{N} : x_n \in U\}$. We get: $\{n \in J : x_n \in U\} = F \cap J \in \mathcal{F}(J)$, and so the assertion follows. $\square$

We recall the next technical lemma (see [14, Lemma 2.2 α)]); for similar results existing in the literature, see also [2, Lemma 3.3], [21, Lemma 2.2] and [22, Lemma 3.1]).

Lemma 2.5. *Let $(x_{j,n})_{j,n}$ be a double sequence in R, and $\mathcal{F}$ be a diagonal filter of $\mathbb{N}$.*

If $(\mathcal{F}) \lim\limits_{j \in \mathbb{N}} x_{j,n} = 0$ for each $n \in \mathbb{N}$, then for every $I \in \mathcal{F}^*$ there exists $J \in \mathcal{F}^*$, $J \subset I$, with $\lim\limits_{j \in J} x_{j,n} = 0$ for each $n \in \mathbb{N}$.

Also the following technical results hold (see [45, Theorem 8 (i)]).

Proposition 2.6. *Let $(x_n)_n$ be a sequence in R, $(\mathcal{F})$-convergent to $x \in R$. If $\mathcal{F}$ is a P-filter, then there exists an element $E \in \mathcal{F}$, with $\lim\limits_{n \in E} x_n = x$.*

A consequence of Lemma 2.5 is the following

Proposition 2.7. *Let $(x_{j,n})_{j,n}$ be a double sequence in R, $\mathcal{F}$ be any P-filter of $\mathbb{N}$, and suppose that $(\mathcal{F}) \lim\limits_{j} x_{j,n} = x_n$ for every $n \in \mathbb{N}$.*

Then there exists $B_0 \in \mathcal{F}$ such that $\lim\limits_{j \in B_0} x_{j,n} = x_n$ for all $n \in \mathbb{N}$.

PROOF. By hypothesis and Proposition 2.6 we get the existence of a sequence $(A_n)_n$ in $\mathcal{F}$, with $\lim_{j\in A_n} x_{j,n} = x_n$ for all $n \in \mathbb{N}$. As $\mathcal{F}$ is a P-filter, there is a sequence of sets $(B_n)_n$ in $\mathcal{F}$, such that $A_n \triangle B_n$ is finite for all $n \in \mathbb{N}$ and $B_0 := \bigcap_{n=1}^{\infty} B_n \in \mathcal{F}$. Thus, since $\lim_{j\in A_n} x_{j,n} = x_n$ for all $n \in \mathbb{N}$, we get also $\lim_{j\in B_n} x_{j,n} = x_n$, and a fortiori $\lim_{j\in B_0} x_{j,n} = x_n$, for all n. □

We now recall some main properties of topological group-valued measures, submeasures and Fréchet-Nikodým topologies.

Let Σ be a σ-algebra of parts of an abstract infinite set G. We say that a finitely additive measure $m : \Sigma \to R$ is (s)-*bounded* on Σ iff

$$\lim_k m(C_k) = 0 \quad \text{for each disjoint sequence } (C_k)_k \text{ in } \Sigma. \tag{1}$$

A finitely additive measure $m : \Sigma \to R$ is said to be σ-*additive* on Σ iff

$$m\Big(\bigcup_{k=1}^{\infty} C_k\Big) = \sum_{k=1}^{\infty} m(C_k) := \lim_i \Big(\sum_{k=1}^{i} m(C_k)\Big) \tag{2}$$

for every disjoint sequence $(C_k)_k$ in Σ.

A *submeasure* $\eta : \Sigma \to [0, +\infty]$ is a set function with $\eta(\emptyset) = 0$, $\eta(A) \le \eta(B)$ whenever A, $B \in \Sigma$, $A \subset B$, and $\eta(A \cup B) \le \eta(A) + \eta(B)$ whenever A, $B \in \Sigma$ and $A \cap B = \emptyset$.

A submeasure η is *order continuous* iff $\lim_k \eta(H_k) = 0$ for every decreasing sequence $(H_k)_k$ in Σ with $\bigcap_{k=1}^{\infty} H_k = \emptyset$.

For every σ-algebra $\mathcal{L} \subset \Sigma$, set $m^{\mathcal{L}}(A) := \bigcup\{m(B) : B \in \mathcal{L}, B \subset A\}$, $A \in \mathcal{L}$. Moreover, put

$$m^+(A) := m^{\Sigma}(A) = \bigcup\{m(B) : B \in \Sigma, B \subset A\}, \quad A \in \Sigma.$$

Given two finitely additive measures $m : \Sigma \to R$, $\lambda : \Sigma \to [0, +\infty]$, we say that m is λ-*absolutely continuous* or shortly λ-*continuous* on Σ, iff $\lim_k m^+(H_k) = 0$ for every decreasing sequence $(H_k)_k$ in Σ such that $\lim_k \lambda(H_k) = 0$.

We will see that m is λ-continuous if and only if $\lim_n m^+(A_n) = 0$ for any arbitrary sequence $(A_n)_n$ in Σ with $\lim_n \lambda(A_n) = 0$. Note that, in the lattice group setting, this property is in general not true (see [6, Remark 1.13.1]). We first extend [41, Lemma 4.6] to the topological group context.

Lemma 2.8. *Let $m : \Sigma \to R$ be an (s)-bounded measure and $(E_k)_k$ be any arbitrary sequence of elements of Σ.*

Then for every $U \in \mathcal{J}(0)$ there is $q \in \mathbb{N}$ with

$$m^+\Big(E_k \setminus \bigcup_{l=1}^{q} E_l\Big) \subset U \quad \text{for every } k \geq q.$$

Proof. If we deny the thesis, then it is possible to find a neighborhood $U \in \mathcal{J}(0)$ and to construct a strictly increasing sequence $(r_h)_h$ in $\mathbb{N}$, with $m^+(B_h) \not\subset U$ for every $h \in \mathbb{N}$, where $B_h := E_{r_{h+1}} \setminus \bigcup_{l=1}^{r_h} E_l$. It is not difficult to see that the B_h's are pairwise disjoint, so getting a contradiction with (s)-boundedness of m. □

We are in position to prove the following characterization of absolute continuity, using a technique similar to that of [41, Theorem 6.1 (a)].

Theorem 2.9. *Let $\lambda : \Sigma \to [0, +\infty]$ be a finitely additive measure. An (s)-bounded measure $m : \Sigma \to R$ is λ-absolutely continuous if and only if $\lim_n m^+(A_n) = 0$ for any sequence $(A_n)_n$ in Σ, such that $\lim_n \lambda(A_n) = 0$.*

Proof. The "if" part is straightforward.

We now turn to the "only if" part. If we deny the thesis, then there exist: a neighborhood $U \in \mathcal{J}(0)$, a decreasing sequence $(U_h)_h$ in $\mathcal{J}(0)$, a sequence $(A_n)_n$ in Σ, with $2\,U_h \subset U_{h-1}$ for every $h \in \mathbb{N}$, $2\,U_0 \subset U$, $\lim_n \lambda(A_n) = 0$ and $m^+(A_n) \not\subset U$ for each $n \in \mathbb{N}$. So, we can extract a subsequence $(A_{n_k})_k$ of $(A_n)_n$, with $\lambda(A_{n_k}) \leq 2^{-k}$ for all $k \in \mathbb{N}$.

Let $E_k := A_{n_k}$. At the first step, by Lemma 2.8 applied to the sequence E_k, $k \in \mathbb{N}$, in correspondence with U_1 there exists $k_1 \in \mathbb{N}$, with

$$\begin{aligned} m^+(E_k) \quad &\subset \quad m^+\Big(E_k \setminus \bigcup_{l=1}^{k_1} E_l\Big) + m^+\Big(E_k \cap \Big(\bigcup_{l=1}^{k_1} E_l\Big)\Big) \subset \\ &\subset \quad U_1 + m^+\Big(E_k \cap \Big(\bigcup_{l=1}^{k_1} E_l\Big)\Big) \quad \text{for every } k \geq k_1. \end{aligned} \tag{3}$$

Put $B_1 := \bigcup_{l=1}^{k_1} E_l$. From (3) we deduce

$$m^+(E_k \cap B_1) \not\subset U_0 + U_1, \tag{4}$$

otherwise we should get $m^+(E_k) \subset U_0 + 2\,U_1 \subset 2\,U_0 \subset U$, a contradiction. Hence, from (4) we obtain $m^+(B_1) \not\subset U_0$.

Proceeding by induction, at the $h+1$-th step suppose that we have determined $k_1 < k_2 < \ldots < k_h \in \mathbb{N}$ and $B_1, \ldots B_h \in \Sigma$, with

$$B_0 := G,\, B_h = B_{h-1} \cap \Big(\bigcup_{l=k_{h-1}+1}^{k_h} E_l \Big),\, m^+(E_k \cap B_h) \not\subset U_0 + U_h \tag{5}$$

for all $h \in \mathbb{N}$ and $k \geq k_h$. By Lemma 2.8 applied to the sequence $E_k \cap B_h$, $k = k_h + 1,\, k_h + 2, \ldots$, in correspondence with U_{h+1} we find an integer $k_{h+1} > k_h$, with

$$\begin{aligned} & m^+(E_k \cap B_h) \subset m^+\Big((E_k \cap B_h) \setminus \bigcup_{l=k_h+1}^{k_{h+1}} E_l \Big) + \\ + \quad & m^+\Big((E_k \cap B_h) \cap \Big(\bigcup_{l=k_h+1}^{k_{h+1}} E_l \Big) \Big) \subset U_{h+1} + m^+(E_k \cap B_{h+1}) \end{aligned} \tag{6}$$

whenever $k \geq k_{h+1}$, where $B_{h+1} = B_h \cap \Big(\bigcup_{l=k_h+1}^{k_{h+1}} E_l \Big)$. From (6) we obtain that $m^+(E_k \cap B_{h+1}) \not\subset U_0 + U_{h+1}$, otherwise we should have $m^+(E_k \cap B_h) \subset U_0 + 2\,U_{h+1} \subset U_0 + U_h$, which contradicts (5). Hence, $m^+(B_{h+1}) \not\subset U_0$.

By construction, $(B_h)_h$ is a decreasing sequence in Σ, $\lim_h \lambda(B_h) = 0$ and $m^+(B_h) \not\subset U_0$ for every $h \in \mathbb{N}$, which contradicts λ-absolute continuity of m. $\square$

A topology τ on Σ is a *Fréchet-Nikodým topology* iff the functions $(A, B) \mapsto A \triangle B$ and $(A, B) \mapsto A \cap B$ from $\Sigma \times \Sigma$ (endowed with the product topology) to Σ are continuous, and for every τ-neighborhood V of $\emptyset$ in Σ there is a τ-neighborhood U of $\emptyset$ in Σ with the property that, if $E \in \Sigma$ is contained in some suitable element of U, then $E \in V$ (see also [41, §1]).

Observe that a topology τ on Σ is a Fréchet-Nikodým topology if and only if there is a family of submeasures $\Xi := \{\eta_i : i \in \Lambda\}$, with the property that a base of τ-neighborhoods of $\emptyset$ in Σ is given by

$$\mathcal{D} := \{U_{\varepsilon,J} := \{A \in \Sigma : \eta_i(A) < \varepsilon \text{ for all } i \in J\} : \varepsilon > 0,\, J \subset \Lambda \text{ is finite}\}$$

(see also [11, 15, 41, 42]).

Let τ be a Fréchet-Nikodým topology on Σ. A finitely additive measure $m : \Sigma \to R$ is *τ-continuous* on Σ, iff $\lim_k m^+(H_k) = 0$ for each decreasing sequence $(H_k)_k$ in Σ, with $\tau\text{-}\lim_k H_k = \emptyset$.

Note that, when λ is a finitely additive non-negative real-valued measure defined on Σ and τ is the topology generated by the pseudo-λ-distance defined by $d_\lambda(A, B) := \lambda(A \triangle B)$, A, $B \in \Sigma$, then τ-continuity is equivalent to λ-absolute continuity (see also [9, 38]).

A finitely additive measure $m : \Sigma \to R$ is said to be *positive* iff

$$m^+(A) = \{m(A)\} \quad \text{for every } A \in \Sigma. \tag{7}$$

It is readily seen that, in the classical case $R = \mathbb{R}$, every positive measure in the sense of the usual order of $\mathbb{R}$ is positive according to (7).

It is not difficult to see that a finitely additive measure $m : \Sigma \to R$ is (s)-bounded on Σ if and only if $\lim_k m^+(C_k) = 0$ for all disjoint sequences $(C_k)_k$ in Σ. Otherwise, there exist a disjoint sequence $(C_k)_k$, a neighborhood $U \in \mathcal{J}(0)$ and two sequences $(n_k)_k$, $(B_k)_k$ in $\mathbb{N}$ and Σ respectively, with $\lim_k n_k = +\infty$, $B_k \subset C_k$ and $m(B_k) \notin U$ for each $k \in \mathbb{N}$, getting a contradiction with (1), since the B_k's are pairwise disjoint.

We now give the following property of (s)-bounded topological group-valued measures (see also [28, 49, 50]).

Proposition 2.10. *Let $m : \Sigma \to R$ be an (s)-bounded measure. Then*

$$\lim_k m^+(H_k) = 0 \tag{8}$$

for each decreasing sequence $(H_k)_k$ in Σ, satisfying

$$m\Big(B \cap \Big(\bigcap_{k=1}^{\infty} H_k\Big)\Big) = 0 \quad \textit{for every } B \in \Sigma. \tag{9}$$

Proof. Let m and $(H_k)_k$ be as in the hypothesis. First of all we prove that

$$\lim_k \Big(\bigcup_{p \geq q \geq k} m^+(H_q \setminus H_p) \Big) = 0,$$

that is for every $U \in \mathcal{J}(0)$ there is $\overline{k} \in \mathbb{N}$ with the property that

$$m(E) \in U \quad \text{for any } p \geq q \geq \overline{k} \text{ and for each } E \in \Sigma \text{ with } E \subset H_q \setminus H_p. \tag{10}$$

If (10) is not true, then there are: a neighborhood $U \in \mathcal{J}(0)$, two sequences $(k_h)_h$, $(p_h)_h$ in $\mathbb{N}$, with $\lim_h k_h = +\infty$, a sequence $(B_h)_h$ in Σ, with $B_h \subset$

$H_{k_h} \setminus H_{k_h+p_h}$ and $m(B_h) \notin U$ for every $h \in \mathbb{N}$. Without loss of generality, we can choose the integers k_h in such a way that $k_{h+1} > k_h + p_h$ for every h.

So, the B_h's are pairwise disjoint, and hence we obtain a contradiction with (s)-boundedness of m.

We now prove (8). To this aim, we claim that for each $U \in \mathcal{J}(0)$ there exists $\overline{k} \in \mathbb{N}$ with $m(E) \in U$ whenever $E \subset H_k$ and $k \geq \overline{k}$. Set $H_\infty := \bigcap_{k=1}^{\infty} H_k$, $E' := E \setminus H_\infty$ and $E_p := E \setminus H_p$, $p \in \mathbb{N}$. Note that $(E_p)_p$ is an increasing sequence in Σ, and that $\bigcup_{p=1}^{\infty} E_p = E \setminus H_\infty = E'$. Let $\overline{k}$ be as in (10), and $p \geq k \geq \overline{k}$. Since $E_p \subset H_k \setminus H_p$, from (10) it follows that $m(E_p) \in U$. Since U is closed, by (9) we get that $m(E) = m(E') = \lim_p m(E_p) \in U$. This proves the claim, and hence (8). $\square$

A consequence of Proposition 2.10 is the following characterization of σ-additivity.

Theorem 2.11. *A finitely additive measure $m : \Sigma \to R$ is σ-additive on Σ if and only if $\lim_k m^+(H_k) = 0$ for each decreasing sequence $(H_k)_k$ in Σ, with $\bigcap_{k=1}^{\infty} H_k = \emptyset$.*

We now prove the following property of σ-additive topological group-valued measures.

Theorem 2.12. *Let $m : \Sigma \to R$ be a σ-additive measure, and $(E_k)_k$ be any sequence in Σ. Then we get*

$$m^+\Big(\bigcup_{k=1}^{\infty} E_k\Big) \subset \sum_{k=1}^{\infty} m^+(E_k). \tag{11}$$

Proof. Set $C_1 := E_1$, $C_k := E_k \setminus \Big(\bigcup_{i=1}^{k-1} E_i\Big)$, $k \geq 2$. Note that the C_k's are pairwise disjoint and $\bigcup_{k=1}^{\infty} C_k = \bigcup_{k=1}^{\infty} E_k$. Choose arbitrarily $B \in \Sigma$, $B \subset \bigcup_{k=1}^{\infty} E_k$, and set $B_k := B \cap C_k$, $k \in \mathbb{N}$. Taking into account σ-additivity of m, we get

$$m(B) = \sum_{k=1}^{\infty} m(B_k) \in \sum_{k=1}^{\infty} m^+(C_k) \subset \sum_{k=1}^{\infty} m^+(E_k). \tag{12}$$

By (12) and arbitrariness of B we obtain (11). This ends the proof. $\square$

We now turn to a Drewnowski-type theorem on existence of σ-additive restrictions of (s)-bounded topological group-valued measures. This will be useful in the sequel in order to prove some equivalence results between the filter limit theorems involved. We first recall the following

Theorem 2.13. ([32, Lemma 2.3]) *Let $m : \Sigma \to R$ be an (s)-bounded measure. Then for each disjoint sequence $(C_k)_k$ in Σ there exists an infinite subset $P_0 \subset \mathbb{N}$, with*

$$\lim_h \Big(\bigcup \Big\{ m\Big(\bigcup_{k \in Y, k \geq h} C_k \Big) : Y \subset P_0 \Big\} \Big) = 0,$$

and m is σ-additive on the σ-algebra generated by the sets C_k, $k \in P_0$.

Theorem 2.14. *Let $m_j : \Sigma \to R$, $j \in \mathbb{N}$, be a sequence of finitely additive measures. Then for any disjoint sequence $(C_k)_k$ in Σ there exists an infinite subset $P \subset \mathbb{N}$, with*

$$\lim_h \Big(\bigcup \Big\{ m_j\Big(\bigcup_{k \in Y, k \geq h} C_k \Big) : Y \subset P \Big\} \Big) = 0$$

for every $j \in \mathbb{N}$, and each m_j is σ-additive on the σ-algebra generated by the sets C_k, $k \in P$.

Proof. By Theorem 2.13 there is an infinite subset $P_1 \subset \mathbb{N}$ with

$$\lim_h \Big(\bigcup \Big\{ m_1\Big(\bigcup_{k \in Y, k \geq h} C_k \Big) : Y \subset P_1 \Big\} \Big) = 0.$$

At the second step, an infinite subset $P_2 \subset P_1$ can be found, such that

$$\lim_h \Big(\bigcup \Big\{ m_2\Big(\bigcup_{k \in Y, k \geq h} C_k \Big) : Y \subset P_2 \Big\} \Big) = 0.$$

Proceeding by induction, we find a strictly increasing $(p_j)_j$ in $\mathbb{N}$ and a decreasing sequence of infinite subsets $P_j \subset \mathbb{N}$, with $p_j = \min P_j$ and

$$\lim_h \Big(\bigcup \Big\{ m_j\Big(\bigcup_{k \in Y, k \geq h} C_k \Big) : Y \subset P_j \Big\} \Big) = 0 \quad \text{for every } j \in \mathbb{N}. \tag{13}$$

Let $P := \{p_j : j \in \mathbb{N}\}$. For every $j \in \mathbb{N}$ there is $h' \in \mathbb{N}$ large enough (depending on j), such that for each $h \geq h'$ we get $\{k \in P : k \geq h\} \subset \{k \in P_j : k \geq h\}$, and hence

$$\bigcup\Big\{m_j\Big(\bigcup_{k\in Y, k\geq h} C_k\Big) : Y \subset P\Big\} \subset \bigcup\Big\{m_j\Big(\bigcup_{k\in Y, k\geq h} C_k\Big) : Y \subset P_j\Big\}. \tag{14}$$

From (13) and (14) it follows that

$$\lim_h\Big(\bigcup\Big\{m_j\Big(\bigcup_{k\in Y, k\geq h} C_k\Big) : Y \subset P\Big\}\Big) = 0 \quad \text{for all } j \in \mathbb{N}. \tag{15}$$

We now turn to the last assertion. Let $C_* := \bigcup_{q\in P} C_q = \bigcup_{l=1}^{\infty} C_{p_l}$ and pick any decreasing sequence $(H_s)_s$ in the σ-algebra $\mathcal{L}$ generated in C_* by the sets C_{n_l}, with $\bigcap_{s=1}^{\infty} H_s = \emptyset$. For every $s \in \mathbb{N}$ there exists $h(s) \in \mathbb{N}$ with $H_s \subset \bigcup_{l\geq h(s)} H_{p_l}$. Note that $\lim_s h(s) = +\infty$. From this and (15) it follows that $\lim_s m_j^{\mathcal{L}}(H_s) = 0$ for every $j \in \mathbb{N}$, that is σ-additivity of every m_j on $\mathcal{L}$. This ends the proof. $\square$

We say that the finitely additive measures $m_j : \Sigma \to R$, $j \in \mathbb{N}$, are *uniformly (s)-bounded* on Σ iff $\lim_k\Big(\bigcup_{j=1}^{\infty} m_j^+(C_k)\Big) = 0$ for each disjoint sequence $(C_k)_k$ in Σ. The m_j's are *uniformly σ-additive* on Σ iff $\lim_k\Big(\bigcup_{j=1}^{\infty} m_j^+(H_k)\Big) = 0$ for each decreasing sequence $(H_k)_k$ in Σ with $\bigcap_{k=1}^{\infty} H_k = \emptyset$. If λ is a finitely additive measure on Σ, then the m_j's are said to be *uniformly λ-absolutely continuous* or shortly *uniformly λ-continuous* on Σ iff $\lim_k\Big(\bigcup_{j=1}^{\infty} m_j^+(H_k)\Big) = 0$ for each decreasing sequence $(H_k)_k$ in Σ with $\lim_k \lambda(H_k) = 0$. If τ is a Fréchet-Nikodým topology on Σ, then the m_j's are *uniformly τ-continuous* on Σ iff $\lim_k\Big(\bigcup_{j=1}^{\infty} m_j^+(H_k)\Big) = 0$ for each decreasing sequence $(H_k)_k$ in Σ with τ-$\lim_k H_k = \emptyset$.

We now recall the following property, which will be useful in the sequel in order to prove our limit theorems in the topological group setting.

Theorem 2.15. ([32, Corollary 3.15]) *Let G be any infinite set, Σ be a σ-algebra of subsets of G, $m_j : \Sigma \to R$, $j \in \mathbb{N}$, be a sequence of uniformly (s)-bounded measures, and $(H_k)_k$ be any decreasing sequence in Σ, with $\lim_k m_j^+(H_k) = 0$ for every $j \in \mathbb{N}$.*

Then $\lim_k \Big(\bigcup_{j=1}^{\infty} m_j^+(H_k) \Big) = 0$.

Note that, by arguing analogously as in Theorem 2.9, it is possible to prove the following

Theorem 2.16. *Let $\lambda : \Sigma \to [0, +\infty]$ be a finitely additive measure. A sequence $m_j : \Sigma \to R$, $j \in \mathbb{N}$, of uniformly (s)-bounded measures is uniformly λ-absolutely continuous if and only if $\lim_n \Big(\bigcup_j m_j^+(A_n) \Big) = 0$ for any sequence $(A_n)_n$ in Σ, with $\lim_n \lambda(A_n) = 0$.*

Indeed, it will be enough to consider the quantity $\bigcup_j m_j^+$ instead of m^+.

Let now $\mathcal{G}$, $\mathcal{H} \subset \Sigma$ be two lattices, such that $\mathcal{G}$ is closed with respect to countable disjoint unions, and the complement of every element of $\mathcal{H}$ belongs to $\mathcal{G}$. Some cases investigated in the literature are when G is a normal topological space (resp. a locally compact Hausdorff space), $\mathcal{G}$ is the class of all open subsets of G, $\mathcal{H}$ is the family of all closed (resp. compact) subsets of G, Σ is the σ-algebra of all Borel subsets of G (see also [32]). We say that $m : \Sigma \to R$ is *regular* on Σ iff for every $A \in \Sigma$ there exist two sequences $(G_k)_k$ in $\mathcal{G}$, $(F_k)_k$ in $\mathcal{H}$, with $F_k \subset F_{k+1} \subset A \subset G_{k+1} \subset G_k$ for every k and $\lim_k m^+(G_k \setminus F_k) = 0$. Observe that, if $m_j : \Sigma \to R$, $j \in \mathbb{N}$, are regular measures, then the sequences $(G_k)_k$, $(F_k)_k$ can be taken independently of j (see [32, Remark 3.5]). The measures $m_j : \Sigma \to R$, $j \in \mathbb{N}$, are said to be *uniformly regular* on Σ iff to every $A \in \Sigma$ there correspond two sequences $(G_k)_k$ in $\mathcal{G}$, $(F_k)_k$ in $\mathcal{H}$, with $F_k \subset F_{k+1} \subset A \subset G_{k+1} \subset G_k$ for every k and $\lim_k \Big(\bigcup_{j=1}^{\infty} m_j^+(G_k \setminus F_k) \Big) = 0$.

We now prove the following relation between σ-additivity and regularity of measures.

Theorem 2.17. *Let (G, d) be a compact metric space, Σ be the σ-algebra of all Borel sets of G, $\mathcal{G}$ and $\mathcal{H}$ be the lattices of all open and all closed subsets of G respectively. Then a measure $m : \Sigma \to R$ is regular if and only if it is σ-additive.*

PROOF. We begin with the "if" part.

Let $\mathcal{T} := \{A \in \Sigma :$ for every $U \in \mathcal{J}(0)$ there are $D \in \mathcal{G}$, $F \in \mathcal{H}$ with $F \subset A \subset D$ and $m^+(D \setminus F) \subset U\}$. Observe that $\mathcal{H} \subset \mathcal{T}$. Indeed, pick arbitrarily $W \in \mathcal{H}$ and for each $k \in \mathbb{N}$ set $D_k := \{x \in G : d(x, W) < 1/k\}$, $W_k := D_k \setminus W$. Note that the sequence $(W_k)_k$ is decreasing, and $\bigcap_{k=1}^{\infty} W_k = \emptyset$. By σ-additivity of m, for every $U \in \mathcal{J}(0)$ there is $k_0 \in \mathbb{N}$, with $m^+(D_{k_0} \setminus W) \subset U$. Since $D_{k_0} \in \mathcal{G}$, $W \in \mathcal{H}$ and $W \subset D_{k_0}$, it follows that $W \in \mathcal{T}$.

We now prove that $\mathcal{T}$ is a σ-algebra. It is easy to see that, if $A \in \mathcal{T}$, then $G \setminus A \in \mathcal{T}$. Let now $(A_k)_k$ be a disjoint sequence of elements of $\mathcal{T}$, with $A := \bigcup_{k=1}^{\infty} A_k$. We claim that $A \in \mathcal{T}$.

Choose arbitrarily $U \in \mathcal{J}(0)$, let $(U_k)_k$ be a sequence in $\mathcal{J}(0)$, such that $2\,U_k \subset U_{k-1} \subset U_0 \subset U$ for every k and $2\,U_0 \subset U$. Note that $\sum_{k=1}^{n} U_k \subset U_0$ for all $n \in \mathbb{N}$, and hence, since U_0 is closed, we get also $\sum_{k=1}^{\infty} U_k \subset U_0$.

By hypothesis there are two sequences $(D_k)_k$ and $(F_k)_k$ in $\mathcal{G}$ and $\mathcal{H}$ respectively, with $F_k \subset A_k \subset D_k$ and $m^+(D_k \setminus F_k) \subset U_k$ for every k. Since $(F_k)_k$ is disjoint, by σ-additivity of m there exists $k_0 \in \mathbb{N}$ with

$$m^+\Big(\Big(\bigcup_{k=1}^{\infty} F_k\Big) \setminus \Big(\bigcup_{k=1}^{k_0} F_k\Big)\Big) = m^+\Big(\bigcup_{k=k_0+1}^{\infty} F_k\Big) \subset U_1.$$

Set $D := \bigcup_{k=1}^{\infty} D_k$, $F := \bigcup_{k=1}^{k_0} F_k$. Note that $F \subset A \subset D$, $D \in \mathcal{G}$, $F \in \mathcal{H}$, and taking into account Theorem 2.12 we get:

$$\begin{aligned} m^+(D \setminus F) &\subset m^+\Big(D \setminus \Big(\bigcup_{k=1}^{\infty} F_k\Big)\Big) + m^+\Big(\Big(\bigcup_{k=1}^{\infty} F_k\Big) \setminus F\Big) \\ &\subset m^+\Big(\bigcup_{k=1}^{\infty} (D_k \setminus F_k)\Big) + U_0 \subset \sum_{k=1}^{\infty} m^+(D_k \setminus F_k) + U_0 \subset 2U_0 \subset U. \end{aligned}$$

From this it follows that $A \in \mathcal{T}$, that is the claim. Therefore, $\mathcal{T}$ is a σ-algebra. Since $\mathcal{T} \supset \mathcal{H}$, then $\mathcal{T} = \Sigma$. Since R satisfies the first axiom of countability, there is a family $(U_k)_k$, which is a basis of neighborhoods of 0.

In correspondence with U_k and every $A \in \Sigma$, there are $D_k^* \in \mathcal{G}$, $F_k^* \in \mathcal{H}$, with $F_k^* \subset A \subset D_k^*$ and $m^+(D_k^* \setminus F_k^*) \subset U_k$. For every $k \in \mathbb{N}$, let $D_k := \bigcap_{i=1}^{k} D_i^*$, $F_k := \bigcup_{i=1}^{k} F_i^*$. We get: $F_k \subset F_{k+1} \subset A \subset G_{k+1} \subset G_k$, $D_k \in \mathcal{G}$, $F_k \in \mathcal{H}$,

$$m^+(D_k \setminus F_k) \subset m^+(D_k^* \setminus F_k^*) \subset U_k,$$

and hence $\lim_k m^+(D_k \setminus F_k) = 0$. Thus, m is regular on Σ. This proves the "if" part.

We now turn to the "only if" part. Let $(C_k)_k$ be a disjoint sequence in Σ, and set $C := \bigcup_{k=1}^{\infty} C_k$. Fix arbitrarily $U \in \mathcal{J}(0)$, and let $(U_k)_k$ be a sequence in $\mathcal{J}(0)$, with $2\,U_k \subset U_{k-1}$ for every k and $2\,U_0 \subset U$. By hypothesis, m is regular, and so in correspondence with C_k and U_k there are $D_k \in \mathcal{G}$, $F_k \in \mathcal{H}$, with $F_k \subset C_k \subset D_k$ and $m^+(D_k \setminus C_k) \subset m^+(D_k \setminus F_k) \subset U_k$. Moreover a set $K \in \mathcal{H}$, $K \subset C$ can be found, with $m^+(C \setminus K) \subset U_0$. Note that, since G is compact, K is also compact, and hence, since $K \subset \bigcup_{k=1}^{\infty} D_k \in \mathcal{G}$, there exists $N \in \mathbb{N}$ with $K \subset \bigcup_{k=1}^{N} D_k$.

Choose arbitrarily $B \subset \bigcup_{k=N+1}^{\infty} C_k = C \setminus \Big(\bigcup_{k=1}^{N} C_k\Big)$. Since

$$B \cap K \subset \Big(\bigcup_{k=1}^{N} D_k\Big) \setminus \Big(\bigcup_{k=1}^{N} C_k\Big),$$

taking into account Theorem 2.12 we get:

$$\begin{aligned} m(B) &= m(B \setminus K) + m(B \cap K) \in m^+(C \setminus K) + \\ &+ m^+\Big(\Big(\bigcup_{k=1}^{N} D_k \Big) \setminus \Big(\bigcup_{k=1}^{N} C_k \Big) \Big) \subset m^+(C \setminus K) + \\ &+ m^+\Big(\bigcup_{k=1}^{N} (D_k \setminus C_k) \Big) \subset m^+(C \setminus K) + \\ &+ \sum_{k=1}^{N} m^+(D_k \setminus C_k) \subset \\ &\subset m^+(C \setminus K) + \sum_{k=1}^{N} U_k \subset 2\, U_0 \subset U, \end{aligned}$$

and hence $m^+\Big(\bigcup_{k=N+1}^{\infty} C_k \Big) \subset U$. This proves σ-additivity of m and ends the proof. $\square$

Remarks 2.18. (a) Note that, arguing similarly as above, it is possible to prove that, under the same hypotheses as in Theorem 2.17, given a sequence $m_j : \Sigma \to R$, $j \in \mathbb{N}$ of measures, the m_j's are uniformly regular if and only if they are uniformly σ-additive.

(b) Observe that, even when $R = \mathbb{R}$, in general the concepts of regularity and σ-additivity are different. Indeed, with the above notations, if $\mathcal{G} = \mathcal{H} = \Sigma$, every finitely additive measure is obviously regular, but not necessarily σ-additive. Conversely, if $\mathcal{G} = \mathcal{H} = \{\emptyset, G\}$, and $m : \Sigma \to \mathbb{R}$ is any σ-additive measure such that $m(\emptyset) = 0$, $m(G) = 1$ and there exist $E \in \Sigma$ and $\alpha \in (0, 1)$ with $m(E) = \alpha$, then it is not difficult to see that m is not regular.

3 The Schur-type and convergence theorems

In [14] we proved some Brooks-Jewett and Nikodým-type theorems for topological-group valued measures. Here we continue this investigation, and we prove some other versions of filter limit theorems in this setting. We begin with a Schur-type theorem (for related results existing in the recent literature see also [2, Theorems 2.6 and 3.5] in the Banach space setting and [21, Lemma 3.1 and Theorems 3.1, 4.1 and 4.2] for lattice group-valued measures). Note that the hypothesis that the involved filter is block-respecting is essential, even when $R = \mathbb{R}$ (see also [2, Remark 3.4]).

Theorem 3.1. *Let $\mathcal{F}$ be a block-respecting filter of $\mathbb{N}$, $m_j : \mathcal{P}(\mathbb{N}) \to R$, $j \in \mathbb{N}$, be a sequence of σ-additive measures, and assume that*

(i) $\lim\limits_j m_j(\{n\}) = 0$ *for any* $n \in \mathbb{N}$, *and*

(ii) $(\mathcal{F}) \lim\limits_j m_j(A) = 0$ *for every* $A \subset \mathbb{N}$.

Then we have:

$\beta)$ $(\mathcal{F}) \lim\limits_j m_j^+(\mathbb{N}) = 0$;

$\beta\beta)$ *if $\mathcal{F}$ is also diagonal, then the only condition (ii) is sufficient to get* $\beta)$.

Proof. We begin with proving $\beta)$. If $\beta)$ is not true, then there exists $U \in \mathcal{J}(0)$ such that

$$I^* := \Big\{ j \in \mathbb{N} : m_j^+(\mathbb{N}) \subset U \Big\} \notin \mathcal{F}. \tag{16}$$

From this it follows that every element F of $\mathcal{F}$ is not contained in I^*, that is F has nonempty intersection with $\mathbb{N} \setminus I^*$: otherwise, if $F \in \mathcal{F}$ and $F \subset I^*$, then we should have $I^* \in \mathcal{F}$. Thus the set $I := \mathbb{N} \setminus I^*$ is $\mathcal{F}$-stationary. Note that I is an infinite set, since $\mathcal{F}$ is a free filter.

Let now $(U_k)_k$ be a decreasing sequence in $\mathcal{J}(0)$, with $2\,U_0 \subset U$, and $2\,U_k \subset U_{k-1}$ for every $k \in \mathbb{N}$ (such a sequence does exist, see also [29]).

Put $n_0 := 1$. By σ-additivity of m_1, there exists an integer $l(1) > 1$ such that $m_1^+(]l(1), +\infty[) \subset U_1$ (here and in the sequel, the intervals and halflines involved are meant in $\mathbb{N}$). Moreover, by (i), there is $n_1 > l(1)$ with $m_s(L) \in U_1$ for all $s \geq n_1$ and for each finite subset $L \subset [1, l(1)]$, and hence $m_s^+([1, l(1)]) \subset U_1$ for any $s \geq n_1$.

Subsequently, by σ-additivity of $m_1, \ldots, m_{n_1}$, we find a natural number $l(n_1) > n_1$, with $m_r^+(]l(n_1), +\infty[) \subset U_2$ for every $r \leq n_1$, and also an integer $n_2 > l(n_1)$ for which $m_s^+([1, l(n_1)]) \subset U_2$ whenever $s \geq n_2$.

By induction, we construct two strictly increasing sequences $(n_h)_h$ and $(l(n_h))_h$, such that, for any $h \in \mathbb{N}$, $n_{h-1} < l(h) < n_h$, $m_r^+(]l(n_h), +\infty[) \subset U_{h+1}$ for each $r \leq n_h$, and $m_s^+([1, l(n_h)]) \subset U_{h+1}$ whenever $s \geq n_{h+1}$. Observe that the n_h's can be chosen in such a way that the sets $I \cap [n_{h-1}, n_h[$, $h \in \mathbb{N}$, are nonempty, so forming a blocking of I. Therefore there exists an $\mathcal{F}$-stationary set $J \subset I$, such that J intersects each interval $[n_h, n_{h+1}[$ in exactly one point. So we can write $J = \{j_0, j_1, j_2, \ldots\}$. Since $J \in \mathcal{F}^*$, then at least one of the two sets $J_1 := \{j_1, j_3, j_5, \ldots\}$ and $J_2 := \{j_0, j_2, j_4, \ldots\}$ is $\mathcal{F}$-stationary. Without loss of generality, suppose that $J_1 \in \mathcal{F}^*$. Now, for each fixed natural number h, we have

$$m_{j_{2h-1}}^+([l(n_{2h}, +\infty[) \subset U_{2h} \subset U_2, \quad m_{j_h}^+([1, l(n_{2h-2}]) \subset U_{2h-1} \subset U_1. \tag{17}$$

From this and since $m^+_{j_{2h-1}}(\mathbb{N}) \not\subset U$, for each index h we get

$$m^+_{j_{2h-1}}(]l(n_{2h-2}), l(n_{2h})]) \not\subset U_0 : \tag{18}$$

otherwise, from (17) and (18) we should have $m^+_{j_{2h-1}}(\mathbb{N}) \subset U_0+U_1+U_2 \subset U_0+2\,U_1 \subset 2\,U_0 \subset U$, a contradiction. By (18) there is a set $Q_h \subset]l(n_{2h-2}), l(n_{2h})]$ with

$$m_{j_{2h-1}}(Q_h) \notin U_0. \tag{19}$$

Note that the Q_h's are pairwise disjoint. Set now $H := \bigcup_{h=1}^{\infty} Q_h$. For each index h we have

$$\begin{aligned} m_{j_{2h-1}}(H) &= m_{j_{2h-1}}(H \cap [1, l(n_{2h-2})]) + m_{j_{2h-1}}(H \cap]l(n_{2h}), +\infty[) + \\ &+ m_{j_{2h-1}}(H \cap]l(n_{2h-2}), l(n_{2h})]) = \\ &= m_{j_{2h-1}}(H \cap [1, l(n_{2h-2})]) + m_{j_{2h-1}}(H \cap]l(n_{2h}), +\infty[) + \\ &+ m_{j_{2h-1}}(Q_h), \end{aligned}$$

and so we see that

$$m_{j_{2h-1}}(H) - m_{j_{2h-1}}(Q_h) \in U_1 + U_2. \tag{20}$$

Thanks to (20), we obtain $m_{j_{2h-1}}(H) \notin U_2$ for all h, otherwise $m_{j_{2h-1}}(Q_h) \in U_1 + U_2 + U_2 \subset U_1 + U_1 \subset U_0$, which contradicts (19). But by (ii), in correspondence with U_2 there exists an element $F \in \mathcal{F}$ with $m_j(H) \in U_2$ for all $j \in F$, and, since J_1 is $\mathcal{F}$-stationary, we get that F has at least an element j_* in common with J_1. So we have contemporarily $m_{j_*}(H) \notin U_2$ and $m_{j_*}(H) \in U_2$, a contradiction. This proves β).

$\beta\beta$) Let $\mathcal{F}$ be a diagonal and block-respecting filter of $\mathbb{N}$. If the thesis is not true, then, proceeding analogously as in β), we get the existence of a neighborhood $U \in \mathcal{J}(0)$ and of an infinite $\mathcal{F}$-stationary set $I \subset \mathbb{N}$, with

$$m^+_j(\mathbb{N}) \not\subset U \quad \text{for every } j \in I. \tag{21}$$

Since R satisfies the first axiom of countability, by (ii) and Lemma 2.5, in correspondence with I there is $J \in \mathcal{F}^*$, $J \subset I$, with $\lim_{j \in J} m_j(\{n\}) = 0$ for every $n \in \mathbb{N}$. Moreover, from (ii) and Proposition 2.4 it follows also that $(\mathcal{F}(J)) \lim_{j \in J} m_j(A) = 0$ for every $A \subset \mathbb{N}$. Furthermore observe that, since $J \in \mathcal{F}^*$ and $\mathcal{F}$ is block-respecting, then $\mathcal{F}(J)$ is block-respecting too. By β) applied to the sequence $m_j : \mathcal{P}(\mathbb{N}) \to R$, $j \in J$, and to the filter $\mathcal{F}(J)$ of J, we get $(\mathcal{F}(J)) \lim_{j \in J} m^+_j(\mathbb{N}) = 0$, contradicting (21). This ends the proof of $\beta\beta$). $\square$

The following result will be useful in the sequel.

Theorem 3.2. *Let $\mathcal{F}$ be a diagonal filter of $\mathbb{N}$, $m_j : \mathcal{P}(\mathbb{N}) \to R$, $j \in \mathbb{N}$, be a sequence of σ-additive measures, and suppose that $(\mathcal{F})\lim_j m_j^+(\mathbb{N}) = 0$.*

Then for every $I \in \mathcal{F}^*$ there is $J \subset I$, $J \in \mathcal{F}^*$, with

$$\lim_k \Big(\bigcup_{j \in J} m_j^+([k, +\infty[) \Big) = 0.$$

PROOF. For every j, $k \in \mathbb{N}$, let $x_{j,k} := m_j^+(\mathbb{N})$. Since R satisfies the first axiom of countability, by Lemma 2.5 it follows that for every $I \in \mathcal{F}^*$ there exists $J \subset I$, $J \in \mathcal{F}^*$, with $\lim_{j \in J} m_j^+(\mathbb{N}) = 0$. So, if $U \in \mathcal{J}(0)$ is chosen arbitrarily and $U_0 \in \mathcal{J}(0)$ is such that $2\,U_0 \subset U$, there is a natural number $\overline{j}$, without loss of generality $\overline{j} \in J$, with $m_j(A) \in U_0$ for every $j \geq \overline{j}$, $j \in J$, and $A \subset \mathbb{N}$. By σ-additivity of the m_j's, in correspondence with $j \in \mathbb{N}$ there exists $\overline{k}_j \in \mathbb{N}$ with $m_j(A) \in U_0$ for every $A \subset [\overline{k}, +\infty[$. If $k^* := \max\{\overline{k}_1, \dots, \overline{k}_{\overline{j}-1}\}$, then we get

$$m_j(A) \in U_0 \subset U \quad \text{for each } A \subset [k^*, +\infty[\text{ and } j \in [1, \overline{j} - 1]. \tag{22}$$

Moreover, we have

$$m_j(A) \in 2\,U_0 \subset U \quad \text{for every } A \subset [k^*, +\infty[\text{ and } j \geq \overline{j},\ j \in J. \tag{23}$$

The assertion follows from (22) and (23). □

We now prove a Vitali-Hahn-Saks-type theorem, as a consequence of Theorems 3.1 and 3.2.

Theorem 3.3. *Let $\mathcal{F}$ be a diagonal and block-respecting filter of $\mathbb{N}$, τ be a Fréchet-Nikodým topology on Σ, $m_j : \Sigma \to R$, $j \in \mathbb{N}$, be a sequence of τ-continuous measures, with*

$$(\mathcal{F})\lim_j m_j(A) = 0 \quad \textit{for every } A \in \Sigma. \tag{24}$$

Then for each decreasing sequence $(H_k)_k$ in Σ with τ-$\lim_k H_k = \emptyset$ and for every $\mathcal{F}$-stationary set $I \subset \mathbb{N}$ there is an $\mathcal{F}$-stationary set $J \subset I$, with

$$\lim_k \Big(\bigcup_{j \in J} m_j^{\mathcal{L}}(H_k) \Big) = 0,$$

where $\mathcal{L}$ is the σ-algebra generated by the H_k's in H_1.

PROOF. Let I and $(H_k)_k$ be as in the hypotheses, set $C_k := H_k \setminus H_{k+1}$ for every $k \in \mathbb{N}$ and put $H_\infty := \bigcap_{k=1}^{\infty} H_k$. Since the m_j's are τ-continuous, we get

$$\lim_k m_j^+(H_k) = 0 \quad \text{for all } j \in \mathbb{N}. \tag{25}$$

For all $A \in \mathcal{P}(\mathbb{N})$ and $j \in \mathbb{N}$, set

$$\nu_j(A) = m_j\Big(\bigcup_{k \in A} C_k\Big).$$

We claim that the ν_j's are σ-additive. We get:

$$\begin{aligned} m_j^+(H_k) &= \bigcup\{m_j(B) : B \in \Sigma, B \subset H_k\} = \\ &= \bigcup\{m_j(B \setminus H_\infty) : B \in \Sigma, B \subset H_k\} = \\ &= \bigcup\{m_j(C) : C \in \Sigma, C \subset H_k \setminus H_\infty\} = \\ &= m_j^+(H_k \setminus H_\infty) = m_j^+\Big(\bigcup_{l=k}^{\infty} C_l\Big) \end{aligned} \tag{26}$$

for every j, $k \in \mathbb{N}$. By arguing analogously as in (26), it is possible to prove also that

$$m_j^{\mathcal{L}}(H_k) = m_j^{\mathcal{K}}\Big(\bigcup_{l=k}^{\infty} C_l\Big), \tag{27}$$

where $\mathcal{K}$ is the σ-algebra generated by the C_k's in H_1 (see also [19, Theorem 3.2], [18, Lemma 2.4]). From (25) and (26) it follows that $\lim_k m_j^+\Big(\bigcup_{l=k}^{\infty} C_l\Big) = 0$ for every decreasing sequence $(H_k)_k$ in Σ with $\tau\text{-}\lim_k H_k = \emptyset$. From this, since

$$\nu_j^+([k, +\infty[) := \bigcup\{\nu_j(D) : D \subset [k, +\infty[\} \subset m_j^+\Big(\bigcup_{l=k}^{\infty} C_l\Big) \quad \text{for every } j, k \in \mathbb{N},$$

we get

$$\lim_k \nu_j^+([k, +\infty[) = 0, \quad j \in \mathbb{N}. \tag{28}$$

We now are in position to prove σ-additivity of the ν_j's. Let $(A_k)_k$ be a decreasing sequence in $\mathcal{P}(\mathbb{N})$ with $\bigcap_{k=1}^{\infty} A_k = \emptyset$. Without loss of generality, we can and do assume that $A_k \supsetneq A_{k+1}$ for every k. Hence, $A_k \subset [k, +\infty[$ and so $\nu_j^+(A_k) \subset \nu_j^+([k, +\infty[)$ for all j, $k \in \mathbb{N}$. From this and (28) we have $\lim\limits_k \nu_j^+(A_k) = 0$, getting σ-additivity of ν_j, for every $j \in \mathbb{N}$.

Moreover, observe that, since the m_j's satisfy (24), then the ν_j's fulfil condition (ii) of Theorem 3.1. Since $\mathcal{F}$ is diagonal and block-respecting, by $\beta\beta)$ of Theorem 3.1 we get $(\mathcal{F}) \lim\limits_j m_j^+(\mathbb{N}) = 0$ for every $A \subset \mathbb{N}$. From this and Theorem 3.2, taking into account (27), it follows that for every $I \in \mathcal{F}^*$ there is $J \subset I$, $J \in \mathcal{F}^*$, with

$$\begin{aligned} 0 &= \lim_k \Big(\bigcup_{j \in J} \nu_j^+([k, +\infty[) \Big) = \\ &= \lim_k \Big(\bigcup_{j \in J} m_j^{\mathcal{K}} \Big(\bigcup_{l=k}^{\infty} C_l \Big) \Big) = \lim_k \Big(\bigcup_{j \in J} m_j^{\mathcal{L}}(H_k) \Big). \end{aligned}$$

This concludes the proof. □

Similarly as Theorem 3.3, it is possible to prove the following Nikodým convergence-type theorem (note that in this case σ-additivity of the ν_j's is a direct consequence of σ-additivity of the m_j's and (2)).

Theorem 3.4. *Let $\mathcal{F}$ be as in Theorem 3.3, $m_j : \Sigma \to R$, $j \in \mathbb{N}$, be a sequence of σ-additive measures, satisfying condition (24). Then for each decreasing sequence $(H_k)_k$ in Σ with $\bigcap\limits_{k=1}^{\infty} H_k = \emptyset$ and for every $I \in \mathcal{F}^*$ there exists $J \in \mathcal{F}^*$, $J \subset I$, with*

$$\lim_k \Big(\bigcup_{j \in J} m_j^{\mathcal{L}}(H_k) \Big) = 0.$$

In the following theorems, which are formulated for positive topological group-valued measures, the involved filter is required to be only diagonal, and not necessarily block-respecting. A meaningful example of such a filter is the class of all subsets of $\mathbb{N}$ having asymptotic density 1, which is also a P-filter (see also [2]).

The next theorem extends [17, Theorem 2.5] to the setting of topological group-valued measures.

Theorem 3.5. *Let G be any infinite set, $\Sigma \subset \mathcal{P}(G)$ be a σ-algebra, $m_j : \Sigma \to R$, $j \in \mathbb{N}$, be a sequence of positive (s)-bounded measures, $\mathcal{F}$ be a diagonal filter of $\mathbb{N}$. Assume that $m_0(E) := (\mathcal{F}) \lim\limits_j m_j(E)$ exists in R for every $E \in \Sigma$, and that m_0 is σ-additive and positive on Σ.*

Then for every set $I \in \mathcal{F}^*$ and for every disjoint sequence $(C_k)_k$ in Σ there exists $J \in \mathcal{F}^*$, $J \subset I$, with

$$\lim_k \Big(\bigcup_{j \in J} m_j^+(C_k) \Big) = \lim_k \Big(\bigcup_{j \in J} m_j(C_k) \Big) = 0.$$

Proof. Let $I \in \mathcal{F}^*$, $(C_k)_k$ be any disjoint sequence in Σ, and $\mathcal{K}$ be the σ-algebra generated by the C_k's in $\bigcup_{k=1}^{\infty} C_k$. For every $B \in \mathcal{K}$ there exists $P \subset \mathbb{N}$ with $B = \bigcup_{k \in P} C_k$. Since $\mathcal{F}$ is diagonal, by Lemma 2.5 there is $J \in \mathcal{F}^*$, $J \subset I$, with

$$m_0 \Big(\bigcup_{k \in E} C_k \Big) = \lim_j m_j \Big(\bigcup_{k \in E} C_k \Big) \tag{29}$$

for every $E \in \mathcal{I}_{\text{fin}} \cup \{\mathbb{N}\}$, where $\mathcal{I}_{\text{fin}}$ is the (countable) class of all finite subsets of $\mathbb{N}$. Moreover, by σ-additivity of m_0, we get

$$\lim_k \Big(m_0^+ \Big(\bigcup_{l=k}^{\infty} C_l \Big) \Big) = 0. \tag{30}$$

Choose arbitrarily $U \in \mathcal{J}(0)$, and let $U_0 \in \mathcal{J}(0)$ be such that $5\,U_0 \subset U$: such a neighborhood does exist (see also [28]). In correspondence with U_0 there exists $k_0 \in \mathbb{N}$ with $m_0 \Big(\bigcup_{k > k_0} C_k \Big) \in U_0$ and therefore, by positivity of m_0, $m_0 \Big(\bigcup_{k > k_0, k \in P} C_k \Big) \in U_0$. Moreover there is $j_0 \in J$, $j_0 = j_0(U, k_0)$ such that for every $j \in J$ with $j \ge j_0$ we have:

$$m_j \Big(\bigcup_{k \le k_0, k \in P} C_k \Big) - m_0 \Big(\bigcup_{k \le k_0, k \in P} C_k \Big) \in U_0, m_j \Big(\bigcup_{k \le k_0} C_k \Big) - m_0 \Big(\bigcup_{k \le k_0} C_k \Big) \in U_0,$$

$$m_j \Big(\bigcup_{k=1}^{\infty} C_k \Big) - m_0 \Big(\bigcup_{k=1}^{\infty} C_k \Big) \in U_0,$$

and hence

$$m_j \Big(\bigcup_{k > k_0} C_k \Big) - m_0 \Big(\bigcup_{k > k_0} C_k \Big) \in 2\,U_0.$$

Choose arbitrarily $B \in \mathcal{K}$. Taking into account positivity of the m_j's and of m_0, for every $j \in J$, $j \geq j_0$, we have:

$$\begin{aligned}
& m_j(B) - m_0(B) = m_j\Big(\bigcup_{k \in P} C_k\Big) - m_0\Big(\bigcup_{k \in P} C_k\Big) \in \\
\in \quad & \Big\{ m_j\Big(\bigcup_{k \leq k_0, k \in P} C_k\Big) - m_0\Big(\bigcup_{k \leq k_0, k \in P} C_k\Big)\Big\} \\
+ \quad & m_0^+\Big(\bigcup_{k > k_0, k \in P} C_k\Big) + m_j^+\Big(\bigcup_{k > k_0, k \in P} C_k\Big) \subset \\
\subset \quad & \Big\{ m_j\Big(\bigcup_{k \leq k_0, k \in P} C_k\Big) - m_0\Big(\bigcup_{k \leq k_0, k \in P} C_k\Big)\Big\} \\
+ \quad & m_0^+\Big(\bigcup_{k > k_0} C_k\Big) + m_j^+\Big(\bigcup_{k > k_0} C_k\Big) \subset \\
\subset \quad & \Big\{ m_j\Big(\bigcup_{k \leq k_0, k \in P} C_k\Big) - m_0\Big(\bigcup_{k \leq k_0, k \in P} C_k\Big)\Big\} \\
+ \quad & \Big\{ m_j\Big(\bigcup_{k > k_0} C_k\Big) - m_0\Big(\bigcup_{k > k_0} C_k\Big)\Big\} + 2\, m_0^+\Big(\bigcup_{k > k_0} C_k\Big) \subset 5\, U_0 \subset U.
\end{aligned}$$

Thus, $\lim_{j \in J} m_j(B) = m_0(B)$ for all $B \in \mathcal{K}$. Therefore, the finitely additive R-valued measures m_j, $j \in J$, satisfy the hypotheses of the classical version of the Brooks-Jewett theorem on $\mathcal{K}$ for topological group-valued measures (see [29, Theorem 2.6], [32, Theorem 2.4]). In particular, we get

$$\lim_k \Big(\bigcup_{j \in J} m_j^+(C_k)\Big) = \lim_k \Big(\bigcup_{j \in J} m_j(C_k)\Big) = 0.$$

This ends the proof. □

We now turn to a Vitali-Hahn-Saks-type theorem, extending [17, Theorem 2.6].

Theorem 3.6. *Let G, Σ, $\mathcal{F}$ be as in Theorem 3.5, τ be a Fréchet-Nikodým topology on Σ, $m_j : \Sigma \to R$, $j \in \mathbb{N}$, be a sequence of positive finitely additive (s)-bounded and τ-continuous measures. Assume that $m_0(E) := (\mathcal{F}) \lim_j m_j(E)$ exists in R for each $E \in \Sigma$, and that m_0 is σ-additive and positive on Σ.*

Then for every set $I \in \mathcal{F}^*$ and for each decreasing sequence $(H_k)_k$ in Σ with τ-$\lim_k H_k = \emptyset$ there exists a set $J \in \mathcal{F}^*$, $J \subset I$, with

$$\lim_k \Big(\bigcup_{j \in J} m_j^+(H_k)\Big) = \lim_k \Big(\bigcup_{j \in J} m_j(H_k)\Big) = 0.$$

Proof. Let τ, I, $(H_k)_k$ be as in the hypotheses, put $H_\infty := \bigcap_{k=1}^{\infty} H_k$, $C_k := H_k \setminus H_{k+1}$, $k \in \mathbb{N}$, and let $\mathcal{L}$ be the σ-algebra generated by the C_k's and H_∞ in H_1. Proceeding analogously as in Theorem 3.5, by virtue of [29, Theorem 2.6] and [32, Theorem 2.4], we get the existence of a set $J \subset I$, $J \in \mathcal{F}^*$, with the property that the m_j's, $j \in J$, are uniformly (s)-bounded on $\mathcal{L}$. Moreover, by τ-continuity and positivity of the m_j's, we get $\lim_k m_j^+(H_k) = 0$ for every $j \in \mathbb{N}$. By Theorem 2.15 applied to the sequence of measures $m_j : \Sigma \to R$, $j \in J$, we obtain that

$$0 = \lim_k \Big(\bigcup_{j \in J} m_j^+(H_k) \Big) = \lim_k \Big(\bigcup_{j \in J} m_j(H_k) \Big),$$

that is the assertion. □

Analogously as in Theorem 3.6 it is possible to prove the following Nikodým-type theorem.

Theorem 3.7. *Let G, Σ, $\mathcal{F}$ be as in Theorem 3.6, $m_j : \Sigma \to R$, $j \in \mathbb{N}$, be a sequence of positive σ-additive measures. If $m_0(A) := (\mathcal{F}) \lim_j m_j(A)$ exists in R for each $A \in \Sigma$, and m_0 is σ-additive and positive on Σ, then for each $I \in \mathcal{F}^*$ and for every decreasing sequence $(H_k)_k$ in Σ with $\bigcap_{k=1}^{\infty} H_k = \emptyset$ there exists $J \in \mathcal{F}^*$, $J \subset I$, with*

$$\lim_k \Big(\bigcup_{j \in J} m_j^+(H_k) \Big) = \lim_k \Big(\bigcup_{j \in J} m_j(H_k) \Big) = 0.$$

We now turn to a Dieudonné-type theorem, extending [18, Theorems 3.8, 3.10] to the context of topological groups.

Theorem 3.8. *Let G, Σ, $\mathcal{F}$ be as in Theorem 3.6, $\mathcal{G}$, $\mathcal{H} \subset \Sigma$ be as above, $m_j : \Sigma \to R$, $j \in \mathbb{N}$, be a sequence of positive regular measures, such that $m_0(E) := (\mathcal{F}) \lim_j m_j(E)$ exists in R for every $E \in \Sigma$, and m_0 is σ-additive and positive.*

Furthermore, let $A \in \Sigma$ and $(G_k)_k$, $(F_k)_k$ be two sequences in $\mathcal{G}$, $\mathcal{H}$ respectively, with $F_k \subset F_{k+1} \subset A \subset G_{k+1} \subset G_k$ for every $k \in \mathbb{N}$, and

$$\lim_k m_j(G_k \setminus F_k) = 0 \quad \textit{for every } j \in \mathbb{N}. \tag{31}$$

Then for each $I \in \mathcal{F}^*$ there is $J \in \mathcal{F}^*$, with

$$\lim_k \Big(\bigcup_{j \in J} m_j^+(G_k \setminus F_k) \Big) = 0. \tag{32}$$

Proof. Let A, $(G_k)_k$, $(F_k)_k$ be as in the hypothesis, $\mathcal{L}$ be the σ-algebra generated by the sets $G_k \setminus F_k$, $k \in \mathbb{N}$, and $I \in \mathcal{F}^*$. Since the m_j's are (s)-bounded, then, arguing analogously as in the proof of Theorem 3.5, by [29, Theorem 2.6] and [32, Theorem 2.4] we find a set $J \subset I$, $J \in \mathcal{F}^*$, such that the m_j's, $j \in J$, are uniformly (s)-bounded on $\mathcal{L}$. Moreover, by hypothesis and taking into account positivity of the m_j's, we have $\lim_k m_j^+(G_k \setminus F_k) = 0$ for every $j \in \mathbb{N}$. From this and Theorem 2.15 applied to the m_j's, $j \in J$, we get (32). □

4 Uniform filter exhaustiveness and equivalence between filter limit theorems

In this section we deal with the tool of uniform filter exhaustiveness for sequences of measures, by means of which it is possible to prove some results of existence of limit measures and some versions of convergence theorems, by considering a subsequence, indexed by a suitable element of the filter involved, on which it is possible to apply some classical versions of limit theorems. We prove also equivalence between filter Brooks-Jewett, Vitali-Hahn-Saks, Nikodým and Dieudonné-type theorems, extending results of [42].

Let $\mathcal{F}$ be a free filter of $\mathbb{N}$, Σ be a σ-algebra of parts of an infinite set G, and $\lambda : \Sigma \to [0, +\infty]$ be a finitely additive measure, such that Σ is separable with respect to the Fréchet-Nikodým topology generated by λ (shortly, *λ-separable*). Let $\mathcal{B} := \{F_i : i \in \mathbb{N}\}$ be a countable λ-dense subset of Σ. Assume that $m_j : \Sigma \to R$, $j \in \mathbb{N}$, is a sequence of finitely additive measures.

We say that the m_j's are *λ-uniformly $\mathcal{F}$-exhaustive on* Σ iff for every $U \in \mathcal{J}(0)$ there exist $\delta > 0$ and $V \in \mathcal{F}$ with $m_j(E) - m_j(F) \in U$ whenever $E, F \in \Sigma$ with $|\lambda(E) - \lambda(F)| \le \delta$ and for any $j \in V$.

We now prove the following result about extensions of filter limit measures in the topological group setting (for similar results existing in the (ℓ)-group context see also [10, Theorem 3.3], [11, Theorem 3.8, Lemma 3.9, Theorem 3.10], [15, Lemma 3.1]).

Theorem 4.1. *Let $(m_j)_j$ be a sequence of finitely additive measures, λ-uniformly $\mathcal{F}$-exhaustive on Σ, such that $m(F_i) := (\mathcal{F})\lim_j m_j(F_i)$ exists in R for every $i \in \mathbb{N}$. Then,*

(γ) *there is a finitely additive extension $m_0 : \Sigma \to R$ of m, with*

$$(\mathcal{F})\lim_j m_j(E) = m_0(E) \quad \textit{for all } E \in \Sigma.$$

($\gamma\gamma$) *Moreover, if $\mathcal{F}$ is a P-filter, then there is a set $M_0 \in \mathcal{F}$ such that*

$$\lim_{j \in M_0} m_j(E) = m_0(E) \quad \textit{for every } E \in \Sigma.$$

Proof. Choose arbitrarily $E \in \Sigma$ and $U \in \mathcal{J}(0)$, and let $U_0 \in \mathcal{J}(0)$ be with $3\,U_0 \subset U$. By hypothesis, there exist $\delta > 0$ and $V \in \mathcal{F}$ such that, if $|\lambda(E) - \lambda(F)| \le \delta$ and $j \in V$, then $m_j(E) - m_j(F) \in U_0$. By λ-separability of Σ, there is $\overline{i} \in \mathbb{N}$ with $|\lambda(E) - \lambda(F_{\overline{i}})| \le \delta$. By Theorem 2.2, there is a set $W^{(\overline{i})} \in \mathcal{F}$ with $m_j(F_{\overline{i}}) - m_l(F_{\overline{i}}) \in U$ whenever $j,\, l \in W^{(\overline{i})}$. In particular we get

$$\begin{aligned} m_j(E) - m_l(E) &= m_j(E) - m_j(F_{\overline{i}}) + m_j(F_{\overline{i}}) - m_l(F_{\overline{i}}) + \\ + \quad & m_l(F_{\overline{i}}) - m_l(E) \in 3\,U_0 \subset U \end{aligned}$$

for every $j,\, l \in V \cap W^{(\overline{i})}$. By Theorem 2.2, there is a set function $m_0 : \Sigma \to R$, extending m, with $(\mathcal{F}) \lim\limits_j m_j(E) = m_0(E)$. It is not difficult to see that m_0 is finitely additive on Σ. This proves (γ).

($\gamma\gamma$) Let $(U_p)_p$ be a base of neighborhoods of 0. By λ-uniform $\mathcal{F}$-exhaustiveness, for every $p \in \mathbb{N}$ there are a $\delta > 0$ and a set $M'_p \in \mathcal{F}$, with $m_j(E) - m_j(F) \in U$ whenever $E,\, F \in \Sigma$ with $|\lambda(E) - \lambda(F)| \le \delta$ and $j \in M'_p$. Since $\mathcal{F}$ is a P-filter, in correspondence with M'_p there exists $M_p \in \mathcal{F}$ such that $M_p \triangle M'_p$ is finite for each $p \in \mathbb{N}$ and $M := \bigcap\limits_{p=1}^{\infty} M_p \in \mathcal{F}$. Let $Z_p := M \setminus M'_p$, $p \in \mathbb{N}$. Note that Z_p is finite for every $p \in \mathbb{N}$, and so we get $m_j(E) - m_j(F) \in U_p$ whenever $E,\, F \in \Sigma$ with $|\lambda(E) - \lambda(F)| \le \delta$ and $j \in M \setminus Z_p$. Moreover, thanks to Proposition 2.7, there is a set $B_0 \in \mathcal{F}$ such that for every $j,\, p \in \mathbb{N}$ there exists $\overline{j} \in B_0$ with $m_j(F_i) - m(F_i) \in U_p$ whenever $j \ge \overline{j}$, $j \in B_0$. Without loss of generality, we can take $\overline{j} \in B_0 \cap M$. Set $M_0 := B_0 \cap M$: we get $M_0 \in \mathcal{F}$. The sequence m_j, $j \in M_0$, is λ-uniformly $\mathcal{F}_{\text{cofin}}$-exhaustive, and $\lim\limits_{j \in M_0} m_j(F_i) = m(F_i)$ for every $i \in \mathbb{N}$. From this and (γ) applied to m_j, $j \in M_0$ and $\mathcal{F}_{\text{cofin}}$, we find a finitely additive extension m_0 of m, defined on Σ, with $\lim\limits_{j \in M_0} m_j(E) = m_0(E)$ for each $E \in \Sigma$. Thus M_0 is the requested set. $\square$

The next step is to give some sufficient conditions on an $\mathcal{F}$-convergent sequence m_j, $j \in \mathbb{N}$, of topological group-valued measures, to get the existence of a set $M_0 \in \mathcal{F}$ such that the subsequence m_j, $j \in M_0$, is uniformly (s)-bounded (resp. uniformly σ-additive, uniformly τ-continuous, uniformly regular). These results yield also sufficient conditions for (s)-boundedness (resp. σ-additivity, τ-continuity, regularity) of the limit measure.

Observe that in this framework, even when $R = \mathbb{R}$, the hypothesis of λ-uniform $\mathcal{F}$-exhaustiveness in general cannot be dropped (see also [15, Remark 3.8 (c)]). However, without requiring filter exhaustiveness, it is possible to prove the following theorem on the existence of the filter limit measure, which extends [9, Theorem 4.12] to the topological group context.

Theorem 4.2. *Let $\Sigma \subset \mathcal{P}(G)$ be a σ-algebra, $\mathcal{L}$ be an algebra of sets generating Σ, and suppose that $m_j : \Sigma \to R$, $j \in \mathbb{N}$, is a sequence of uniformly σ-additive measures, such that $(\mathcal{F}) \lim_j m_j(E)$ exists in R for each $E \in \mathcal{L}$. Then $(\mathcal{F}) \lim_j m_j(E)$ exists in R for all $E \in \Sigma$.*

Proof. Let $\Pi := \{E \in \Sigma : (\mathcal{F}) \lim_j m_j(E) \text{ exists in } R\}$. By hypothesis, $\mathcal{L} \subset \Pi$. If we show that Π is a monotone class, then $\Pi = \Sigma$, and so the result will be proved.

Let $(E_r)_r$ be a monotone sequence of elements of Π with $\lim_r E_r = E \in \Sigma$ in the set-theoretic sense, choose arbitrarily $U \in \mathcal{J}(0)$, and let $(U_r)_r$ be a family of elements of $\mathcal{J}(0)$, with $2\,U_r \subset U_{r-1}$ for each r and $2\,U_0 \subset U$. For every $r \in \mathbb{N}$, since $E_r \in \Pi$, the sequence $(m_j(E_r))_j$ is $\mathcal{F}$-convergent, and so by Theorem 2.2 there exists $W_r \in \mathcal{F}$ with $m_p(E_r) - m_q(E_r) \in U_r$ whenever p, $q \in W_r$. Moreover, since the m_j's are uniformly σ-additive, there is $\overline{r} \in \mathbb{N}$ with $m_j(E_{\overline{r}}) - m_j(E) \in U_1$ for all $j \in \mathbb{N}$. Thus for every p, $q \in W_{\overline{r}}$ we get:

$$\begin{aligned} & m_p(E) - m_q(E) = [m_p(E) - m_p(E_{\overline{r}})] + [m_p(E_{\overline{r}}) - m_q(E_{\overline{r}})] + \\ + \quad & [m_q(E_{\overline{r}}) - m_q(E)] \in 2\,U_1 + U_{\overline{r}} \subset 2U_1 + U_0 \subset 2\,U_0 \subset U. \end{aligned}$$

By Theorem 2.2, the limit $(\mathcal{F}) \lim_j m_j(E)$ exists in R. The assertion follows from arbitrariness of $E \in \Sigma$. $\square$

We now introduce the following condition, which will be useful in the sequel.

A sequence of finitely additive measures $m_j : \Sigma \to R$, $j \geq 0$, together with λ, *satisfies property* $(*)$ *with respect to* R *and* $\mathcal{F}$ iff it is λ-uniformly $\mathcal{F}$-exhaustive on Σ and $(\mathcal{F}) \lim_j m_j(E) = m_0(E)$ for any $E \in \Sigma$.

The next result is an immediate consequence of Lemma 4.1 $(\gamma\gamma)$.

Lemma 4.3. *Let $m_j : \Sigma \to R$, $j \in \mathbb{N}$, satisfy together with λ property $(*)$ with respect to R and $\mathcal{F}$.*

Then there exists a set $M_0 \in \mathcal{F}$ such that the measures m_j, $j \in M_0$, and m_0 satisfy together with λ property $(*)$ with respect to R and $\mathcal{F}_{cofin}$.

We now deal with equivalence between filter limit theorems in the (ℓ)-group setting. We begin with recalling the following Brooks-Jewett-type theorem in the topological group context (for similar versions in the lattice group setting, see also [7, Theorem 3.1], [11, Theorem 3.4]).

Theorem 4.4. (see [29, Theorem 2.6], [32, Theorem 2.4]) *Let $m_j : \Sigma \to R$, $j \in \mathbb{N}$, be a sequence of (s)-bounded measures, convergent pointwise on Σ to a measure m_0.*

Then the measures m_j, $j \in \mathbb{N}$, are uniformly (s)-bounded and m_0 is (s)-bounded on Σ.

We now prove the following filter limit theorems for topological group-valued measures and their equivalence (for similar results in the (ℓ)-group setting, see [15, §3]). Note that in our context, since we deal with topological group-valued measures, we can use a Drewnowski-type approach, considering suitable σ-additive restrictions of (s)-bounded measures. In the lattice group setting, since the convergence does not have always a topological nature, it is not advisable to apply such an argument, and the tool of the Stone Isomorphism technique is used (see [15]), though it is possible to construct Stone-type extensions even for topological group-valued measures (see for instance [49, 50]).

In what follows, let us assume that:
H) $\lambda : \Sigma \to [0, +\infty]$ is a finitely additive measure, Σ is a λ-separable σ-algebra, $\mathcal{F}$ is a P-filter of $\mathbb{N}$, m_0, $m_j : \Sigma \to R$, $j \in \mathbb{N}$, are finitely additive measures, satisfying together with λ property $(*)$ with respect to R and $\mathcal{F}$ on Σ, and Σ_0 is a sub-σ-algebra on Σ.

Theorem 4.5. (Brooks-Jewett (BJ)) *If the m_j's are (s)-bounded on Σ_0, then there exists a set $M_0 \in \mathcal{F}$, such that the measures m_j, $j \in M_0$, are uniformly (s)-bounded on Σ_0.*

Theorem 4.6. (Vitali-Hahn-Saks (VHS)) *If every m_j is (s)-bounded and τ-continuous on Σ_0, then there exists a set $M_0 \in \mathcal{F}$, such that the measures m_j, $j \in M_0$, are uniformly (s)-bounded and uniformly τ-continuous on Σ_0.*

Theorem 4.7. (Nikodým (N)) *If each m_j is σ-additive on Σ_0, then there is $M_0 \in \mathcal{F}$, such that the measures m_j, $j \in M_0$, are uniformly σ-additive on Σ_0.*

Theorem 4.8. (Dieudonné (D)) *If each m_j is (s)-bounded and regular on Σ_0, then there is $M_0 \in \mathcal{F}$ with the property that the measures m_j, $j \in M_0$, are uniformly (s)-bounded and uniformly regular on Σ_0.*

To prove Theorem 4.5 (BJ), observe that there exists $M_0 \in \mathcal{F}$, satisfying the thesis of Lemma 4.3. The assertion of (BJ) follows by applying Theorem 4.4 to the sequence m_j, $j \in M_0$.

We now prove equivalence between (BJ), (VHS), (N) and (D).

We begin with the implication (BJ) $\Longrightarrow$ (VHS). Let $m_j : \Sigma \to R$, $j \in \mathbb{N}$, be a sequence of measures, fulfilling together with λ property $(*)$ with respect to R and $\mathcal{F}$, (s)-bounded and τ-continuous on Σ_0. By (BJ), there is $M_0 \in \mathcal{F}$ such that the measures m_j, $j \in M_0$, are uniformly (s)-bounded on Σ_0. So, $\lim_k \Big(\bigcup_{j \in M_0} m_j^+(C_k) \Big) = 0$ for every disjoint sequence $(C_k)_k$ in Σ_0.

Fix arbitrarily any decreasing sequence $(H_k)_k$ in Σ_0, with τ-$\lim_k H_k = \emptyset$. By τ-continuity of each m_j, $j \in \mathbb{N}$, on Σ_0, we get $\lim_k m_j^+(H_k) = 0$ for every $j \in \mathbb{N}$.

By Theorem 2.15, we obtain

$$\lim_k \left(\bigcup_{j \in M_0} m_j^+(H_k) \right) = 0, \tag{33}$$

so getting uniform τ-continuity of the m_j's, $j \in M_0$, on Σ_0. Thus, (BJ) implies (VHS).

The proof of (BJ) $\Longrightarrow$ (D) is similar to that of (BJ) $\Longrightarrow$ (VHS).

We now prove (VHS) $\Longrightarrow$ (N). Let τ be the Fréchet-Nikodým topology generated by the class of all order continuous submeasures defined on Σ_0. If $(H_k)_k$ is any decreasing sequence in Σ_0 with τ-$\lim_k H_k = \emptyset$ and $H_\infty = \bigcap_{k=1}^{\infty} H_k$, then we have $\eta(H_\infty) = 0$ for each order continuous submeasure η defined on Σ_0, and so it follows that $H_\infty = \emptyset$. Thus we obtain that, if $(m_j)_j$ is a sequence of measures, σ-additive on Σ_0, then they are τ-continuous on Σ_0. Since every m_j is also (s)-bounded on Σ_0, then by (VHS) they are uniformly τ-continuous on Σ_0, and so also uniformly σ-additive. Thus, (VHS) implies (N).

We prove (N) $\Longrightarrow$ (BJ). Let $m_j : \Sigma \to R$, $j \in \mathbb{N}$, be a sequence of (s)-bounded measures, satisfying together with λ property $(*)$ with respect to $\mathcal{F}$ and R.

Pick arbitrarily a disjoint sequence $(C_k)_k$ in Σ_0, and choose any subsequence $(C_{k_r})_r$ of $(C_k)_k$. By Theorem 2.14, there is a sub-subsequence $(C_{k_{r_s}})_s$, such that every m_j is σ-additive on the σ-algebra $\mathcal{L}$ generated by $(C_{k_{r_s}})_s$.

By (N) used with $\mathcal{F}$ and the sub-σ-algebra $\mathcal{L}$, where $\mathcal{L} \subset \Sigma_0 \subset \Sigma$, we find a set $M^* \in \mathcal{F}$, such that the measures $m_{j|\mathcal{L}}$, $j \in M^*$, are uniformly σ-additive, and hence also uniformly (s)-bounded, on $\mathcal{L}$. So we get that

$$\lim_s \Big(\bigcup_{j \in M^*} m_j(C_{k_{r_s}}) \Big) = 0. \tag{34}$$

By arbitrariness of the subsequence $(C_{k_r})_r$ and property $(\mathcal{U})$ used with $\mathcal{F} = \mathcal{F}_{\text{cofin}}$, from (34) it follows that

$$\lim_k \Big(\bigcup_{j \in M^*} m_j(C_k) \Big) = 0, \tag{35}$$

and hence (N) implies (BJ).

We now prove (D) $\Longrightarrow$ (BJ). Let $m_j : \Sigma \to R$, $j \geq 0$ satisfy, together with λ, property $(*)$ with respect to R and $\mathcal{F}$, and (s)-bounded on Σ_0, Of course, if we take $\mathcal{G} = \mathcal{H} = \Sigma_0$, then we get that the m_j's are regular on Σ_0 (with respect to this choice of $\mathcal{G}$ and $\mathcal{H}$). By (D), there exists a set $M_0 \in \mathcal{F}$, such that the measures m_j, $j \in M_0$, are uniformly (s)-bounded and uniformly regular on Σ_0. This proves that (D) implies (BJ).

Open problems:

(a) Prove similar Schur and limit theorems using m^+ instead of $m^{\mathcal{L}}$ and/or without assuming either "good" properties for the limit measure or filter uniform exhaustiveness.

(b) Investigate similar results by considering weaker notions of (s)-boundedness and σ-additivity.

(c) Study similar theorems by considering some other classes of filters.

(d) Investigate some other properties of filter exhaustiveness, weak filter convergence and filter (α)-convergence in the topological group setting (see also [9]).

(e) Investigate similar topics by dropping the hypothesis that the topological group involved satisfies the first axiom of countability (in this context some like classical properties do not hold, see also [46, 51]).

5 Conclusions

We have seen that several versions of limit theorems, which were proved with respect of filter convergence in the lattice group setting in [10, 11, 15, 17, 18, 21], hold even for filter convergence in topological group-valued measures. After having investigated some classes of filters and their properties, and examined some features of filter convergence, we have studied some properties of topological group-valued measures, and in particular some relations between regularity and σ-additivity, some aspects of absolute continuity and some Drewnowski-type theorems on existence of countably additive restrictions of (s)-bounded measures.

We have investigated three kinds of limit theorems. First, we have considered some particular classes of filters and Schur-type theorems for measures defined on the class of all subsets of $\mathbb{N}$ and we have deduced, as consequences, some Vitali-Hahn-Saks, Nikodým and Dieudonné-type theorems.

We have examined in particular positive measures, showing that in this case it is possible to prove some versions of these kinds of theorems under weaker assumptions on the filter involved.

Finally we have dealt with the powerful tool of filter exhaustiveness, which has allowed us to find a sub-sequence of the original sequence of measures, indexed by a suitable element of the filter involved, to which it is possible to apply the classical theorems in [29] and [32], obtaining some results about the existence of limit measures and further convergence theorems. Their equivalence has been proved, using a Drewnowski-type result about the existence of σ-additive restrictions of (s)-bounded measures. This is possible, because topological convergence satisfies property $(\mathcal{U})$, which in general is not fulfilled in lattice groups (see also [15, 26, 53]).

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