INROADS

Real Analysis Exchange
Vol. 38(2), 2012/2013, pp. 487–492

Marian Jakszto, Faculty of Mathematics and Computer Sciences, University of Łódź, ul. Stefana Banacha 22, 90-238 Łódź, Poland. email: jakszto@math.uni.lodz.pl

ON SOME MODES OF CONVERGENCE IN SPACES WITH THE WEAK BANACH-SAKS PROPERTY

Abstract

The paper proves a general theorem that relates some modes of convergence, such as pointwise a.e. convergence, to weak convergence in any space with the weak Banach-Saks property. Some results that follow immediately from the main theorem are given for specific functional spaces.

1 Introduction

The starting point for stating the main result of the paper is

Theorem 1. *Let $f^i \in L^p(\Omega)$, $1 < p < +\infty$, $\Omega \subseteq \mathbb{R}^N$. Suppose that the sequence $\{f^i\}$ is bounded in norm. If $\{f^i\}$ converges pointwise a.e. to f, then $f \in L^p(\Omega)$, and the sequence $\{f^i\}$ converges weakly to f in $L^p(\Omega)$. In other words, L^p-bounded pointwise a.e. convergence implies weak convergence.*

This theorem connects pointwise a.e. convergence of Lebesgue measurable functions (real analysis) with weak convergence in the space L^p (functional analysis).

The classical proof of this theorem, given in [7, Theorem 13.44], is real-analytic in nature: It uses Fatou's Lemma, the absolute continuity of the Lebesgue integral, and Egorov's Theorem. Another proof, proposed in [9],

Mathematical Reviews subject classification: Primary: 28A20, 46B20; Secondary: 46E30, 46E35
Key words: weak Banach-Saks property, weak convergence, pointwise a.e. convergence, convergence in measure
Received by the editors October 26, 2012
Communicated by: Alexander Olevskii

leans basically on functional analysis, namely on Riesz's Theorem and the Banach-Saks Theorem.

The objective of the present paper is to apply the functional-analytic method of [9] in a more general setting. This leads to the main result of the paper, Theorem 3, and permits to state many consequent theorems that relate some modes of convergence, such as pointwise a.e. convergence, to weak convergence in specific functional spaces with the weak Banach-Saks property. Two such theorems are provided in Section 3. The last section of the paper, Section 4, discusses a variant of the Main Theorem.

2 Main result

Recall that a Banach space F is said to have the weak Banach-Saks property iff every weakly null sequence $\left\{f^k\right\} \subseteq F$ contains a subsequence $\left\{f^{k_l}\right\}$ such that

$$\frac{1}{n}\sum_{l=1}^{n} f^{k_l} \longrightarrow 0 \qquad \text{in the norm of } F.$$

Every uniformly convex space and the space $L^1[0,1]$ are examples of a Banach space that has the weak Banach-Saks property. See [5, Theorem 1 on p. 78] and [11].

In the Assumptions that follow, a *convergence* in a set $\widehat{F}$ means any operation of assigning limits to countable sequences of elements of $\widehat{F}$, such that, firstly, any sequence has one limit at most and, secondly, any subsequence of each convergent sequence converges to the same limit as the initial sequence.

The terms of the sequence $\left\{\frac{1}{n}\sum\limits_{l=1}^{n} x_l\right\}$ will be, traditionally, called the Cesàro means of a sequence $\{x_n\}$.

We make the following three Assumptions.

1. $\widehat{F}$ is a vector space over $\mathbb{R}$. This space is endowed additionally with some convergence τ in such a way that the Cesàro means of each convergent sequence tend to the limit of this sequence.

2. F is a linear subspace of $\widehat{F}$. The space F is equipped with a norm which makes it a Banach space with the weak Banach-Saks property.

3. Every sequence $\{\phi^n\} \subseteq F$ that converges in the norm of F contains a subsequence $\{\phi^{n_q}\}$ that τ-converges to the same limit.

Assumption 1 is satisfied, for instance, by any locally convex Hausdorff topological real vector space. In such a space, if $x_n \longrightarrow x$, then $\frac{1}{n}\sum\limits_{l=1}^{n} x_l \longrightarrow x$, which has been easily proven at the beginning of [3]. Note however that Assumption 1 requires no topology, in general.

For a typical example of a space that satisfies Assumptions 1–3 consider F being $L^p[0,1]$, $p \in [1,+\infty)$, and $\widehat{F}$ being the vector space of all equivalence classes of Lebesgue measurable functions defined over the unit interval, with pointwise a.e. convergence τ. This convergence is not topological; see [10].

To prove the Main Theorem, we will need a well-known proposition.

Proposition 2. *Given a sequence $\{\alpha^i\} \subseteq \mathbb{R}$, if each subsequence $\{\alpha^{i_j}\}$ contains a subsequence $\{\alpha^{i_{j_k}}\}$ that converges to α, then $\alpha^i \longrightarrow \alpha$.*

Theorem 3 (Main Theorem)**.** *Let spaces $\widehat{F}$ and $F \subseteq \widehat{F}$ satisfy Assumptions 1–3. Suppose that a sequence $\{f^i\} \subseteq F$ is relatively compact in the weak topology of F and τ-converges to $f \in \widehat{F}$. Then $f \in F$, and $\{f^i\}$ converges weakly to f in F.*

Proof. Take any subsequence $\{f^{i_j}\}$. By assumption, $\{f^{i_j}\}$ is relatively weakly compact, therefore it is possible to extract a subsequence $\{f^{i_{j_k}}\}$ that converges weakly to some $f^0 \in F$. To finish the proof, it is enough to show that $f^0 = f$; after this, put $\alpha^i := \langle f^i, \lambda\rangle$ and $\alpha := \langle f, \lambda\rangle$ where λ is an arbitrary element of F^*, and use Proposition 2.

For convenience, the elements of the subsequence $\{f^{i_{j_k}}\}$ will be denoted by f^k. According to this notation, $\{f^k\}$ tends weakly to f^0 in F. By Assumption 2, there is a subsequence $\{f^{k_l}\}$ such that

$$\frac{1}{n}\sum_{l=1}^{n} f^{k_l} \longrightarrow f^0 \qquad \text{in the norm of } F.$$

By Assumption 3 there exists a subsequence $\{n_q\}$ for which

$$\frac{1}{n_q}\sum_{l=1}^{n_q} f^{k_l} \xrightarrow{\tau} f^0.$$

On the other hand, by the last assumption of the theorem and by Assumption 1

$$\frac{1}{n_q}\sum_{l=1}^{n_q} f^{k_l} \xrightarrow{\tau} f,$$

which proves that $f^0 = f$, since the τ-limit is unique by the definition of the convergence τ. $\square$

3 Examples using the main result

3.1 Banach Function Spaces.

Let (Ω, Σ, μ) be a complete σ-finite measure space and let $\mathcal{M} = \mathcal{M}(\Omega, \Sigma, \mu)$ be the space of all equivalence classes of measurable real functions defined over the set Ω. A Banach space $F \subseteq \mathcal{M}$ is called a Banach function space on (Ω, Σ, μ) iff

1) F satisfies the property: if $x \in \mathcal{M}$, $y \in F$, and $|x| \leq |y|$ almost everywhere, then $x \in F$ and $\|x\| \leq \|y\|$;

2) there exists a function $u \in F$ such that $u > 0$ almost everywhere.

The primary example of Banach function spaces is the Lebesgue spaces L^p, $1 \leq p \leq +\infty$. Other examples are the classes of Marcinkiewicz, Lorentz, and Musielak-Orlicz spaces. Some of them have the weak Banach-Saks property. See [2], [6], and [8].

By Lemma 5.21 in [1], norm convergence in any Banach function space implies pointwise a.e. convergence, up to a subsequence. As a result, using the Main Theorem with $\widehat{F} = \mathcal{M}$ endowed with pointwise a.e. convergence τ gives

Theorem 4. *Let F be a reflexive Banach function space on (Ω, Σ, μ) with the weak Banach-Saks property. Suppose that a sequence of functions $\{f^i\} \subseteq F$ is bounded in norm. If $\{f^i\}$ converges pointwise μ-a.e. to a function f, then $f \in F$ and $\{f^i\}$ converges weakly to f in F.*

If $F = L^p$, $1 < p < +\infty$, then Theorem 4 is the classical Theorem 1.

3.2 Sobolev Spaces.

Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $p \in (1, +\infty)$. Consider the Sobolev space

$$W^{1,p}(\Omega) = \{u \in L^p(\Omega) : \partial_\nu u \in L^p(\Omega) \text{ for } \nu = 1, 2, \ldots, n\}$$

endowed with the norm

$$\|u\| := \left(\int_\Omega |u|^p + \sum_{\nu=1}^{n} \int_\Omega |\partial_\nu u|^p \right)^{\frac{1}{p}},$$

where $\partial_\nu u$ is the distributional partial derivative. This space is uniformly convex and therefore has the weak Banach-Saks property.

With the Main Theorem in mind, take F to be the space $W^{1,p}(\Omega)$ and $\widehat{F}$ to be the space of all equivalence classes of Lebesgue measurable functions defined over Ω, with pointwise a.e. convergence τ. This leads to

Theorem 5. *Suppose that a sequence of functions $\left\{f^i\right\} \subseteq W^{1,p}(\Omega)$ is bounded in norm. If $\left\{f^i\right\}$ converges pointwise a.e. to a measurable function f, then $f \in W^{1,p}(\Omega)$, and $\left\{f^i\right\}$ converges weakly to f in $W^{1,p}(\Omega)$.*

Under the assumptions of this theorem, pointwise a.e. convergence of $W^{1,p}$-functions affects the behavior of their derivatives and implies the existence of the derivative of the limit function; notably $f \in L^1_{loc}(\Omega)$, the distributional derivative ∇f exists as a vector function, and for every $g \in L^{\frac{p}{p-1}}(\Omega)$

$$\int_\Omega \nabla f^i \cdot g \longrightarrow \int_\Omega \nabla f \cdot g.$$

Remark. *Using the Main Theorem, a theorem analogous to Theorem 5 may be stated and immediately proven for the variable exponent Sobolev space $W^{1,p(\cdot)}$. See Theorem 2.3.13, Lemma 2.3.15, and Theorem 8.1.6 in [4].*

4 A variant of the main theorem

Convergence in (Lebesgue) measure does not fit into the framework of Section 2, since this mode of convergence does not necessarily preserve the limit of a convergent sequence after passing to the sequence of its Cesàro means. Even so, some alteration to the Assumptions of Section 2 permits to extend the Main Theorem to cover the case involving convergence in measure.

Recall that every sequence that converges in measure contains a subsequence that converges pointwise a.e. Taking account of this fact, we make an extra Assumption using the same meaning of 'convergence' as in Assumption 1:

1a. The space $\widehat{F}$ is endowed with another convergence τ_s such that every sequence $\left\{\widehat{\phi}^n\right\} \subseteq \widehat{F}$ that τ_s-converges to some point in $\widehat{F}$ contains a subsequence which τ-converges to the same limit.

Theorem 6. *Let $\widehat{F}$ and $F \subseteq \widehat{F}$ satisfy Assumptions 1–3 and 1a. Suppose that a sequence $\left\{f^i\right\} \subseteq F$ is relatively compact in the weak topology of F and τ_s-converges to $f \in \widehat{F}$. Then $f \in F$, and $\left\{f^i\right\}$ converges weakly to f in F.*

If $\tau_s = \tau$, then Theorem 6 becomes the Main Theorem.

The proof of this theorem is much the same as that of the Main Theorem. A slight modification of the proof for this case consists in extracting, at the very beginning, a τ-convergent subsequence out of $\{f^{i_j}\}$ and only then choosing a weakly convergent subsequence. Such a τ-convergent subsequence exists by Assumption 1a.

Theorem 6 permits to replace pointwise a.e. convergence in Theorems 4 and 5 by convergence in measure.

References

[1] Y. A. Abramovich and C. D. Aliprantis, *An Invitation to Operator Theory*, American Mathematical Society, Providence, RI, 2002.

[2] S. V. Astashkin and F. A. Sukochev, *Banach-Saks property in Marcinkiewicz spaces*, J. Math. Anal. Appl., **336(2)** (2007), 1231–1258.

[3] H. Çakalli and B. Thorpe, *Cesàro means in topological groups*, J. Anal., **4** (1996), 9–15.

[4] L. Diening, P. Harjulehto, P. Hästö and M. Růžička, *Lebesgue and Sobolev Spaces With Variable Exponents*, Springer, Heidelberg, 2011.

[5] J. Diestel, *Geometry of Banach Spaces—Selected Topics*, Springer-Verlag, Berlin–New York, 1975.

[6] P. G. Dodds, E. M. Semenov and F. A. Sukochev, *The Banach-Saks property in rearrangement invariant spaces*, Studia Math., **162(3)** (2004), 263–294.

[7] E. Hewitt and K. Stromberg, *Real and Abstract Analysis*, Springer-Verlag, New York–Heidelberg, 1975.

[8] H. Hudzik and A. Kamińska, *On uniformly convexifiable and B-convex Musielak-Orlicz spaces*, Comment. Math. Prace Mat., **25(1)** (1985), 59–75.

[9] M. Jakszto, *Another proof that L^p-bounded pointwise convergence implies weak convergence*, Real Anal. Exchange, **36(2)** (2010/2011), 479–482.

[10] E. T. Ordman, *Classroom notes: Convergence almost everywhere is not topological*, Amer. Math. Monthly, **73(2)** (1966), 182–183.

[11] W. Szlenk, *Sur les suites faiblement convergentes dans l'espace L*, Studia Math., **25** (1965), 337–341.