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AN EXTENSION OF THE HERMITE-HADAMARD INEQUALITY FOR CONVEX SYMMETRIZED FUNCTIONS

Abstract

In this work, we extend the Hermite-Hadamard inequality to a new class of functions which do not satisfy the convex property. This result will be applied to both Haber and Fejér inequalities.

1 Introduction

In all what follows, we denote by I the closed real interval $[a, b]$.

Definition 1. *A real-valued function f is said to be convex on I if $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$ for all $x, y \in I$ and $0 \leqslant \lambda \leqslant 1$. Conversely, if the opposite inequality holds, the function is said to be concave on I.*

A function f that is continuous on I and twice differentiable on (a, b) is convex on I if and only if $f^{''}(x) \geqslant 0$ for all $x \in (a, b)$. (f is concave if and only if $f^{''}(x) \leqslant 0$ for all $x \in (a, b)$).

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Proposition 2. *Let $f : I \longrightarrow \mathbb{R}$, be a convex function, then the Hermite-Hadamard inequality [9]*

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)\,dx \leqslant \frac{f(a)+f(b)}{2} \tag{1}$$

holds.

It is obvious that the Hermite-Hadamard inequality gives us an estimate of the mean value of the convex function. Note that the first inequality in (1) was proved by Hadamard in 1893 [1]. The Hermite-Hadamard inequality is well-known but for more details on historical considerations, one can consult [3, 10, 11]. Generalizations, developments and refinements can be found in [2, 3, 5, 6, 7].

In [6], A.El Farissi, proved the following theorem for a convex function.

Theorem 3. *Assume that $f : I \to \mathbb{R}$ is a convex function on I. Then for all $\lambda \in [0,1]$, we have*

$$f\left(\frac{a+b}{2}\right) \leqslant l(\lambda) \leqslant \frac{1}{b-a}\int_a^b f(x)\,dx \leqslant L(\lambda) \leqslant \frac{f(a)+f(b)}{2},$$

where

$$l(\lambda) := \lambda f\left(\frac{\lambda b+(2-\lambda)a}{2}\right) + (1-\lambda) f\left(\frac{(1+\lambda)b+(1-\lambda)a}{2}\right)$$

and

$$L(\lambda) := \frac{1}{2}\left(f(\lambda b+(1-\lambda)a) + \lambda f(a) + (1-\lambda) f(b)\right).$$

Corollary 4. *Assume that $f : I \to \mathbb{R}$ is a convex function on I. Then we have the following inequality*

$$f\left(\frac{a+b}{2}\right) \leqslant \sup_{\lambda\in[0,1]} l(\lambda) \leqslant \frac{1}{b-a}\int_a^b f(x)\,dx \leqslant \inf_{\lambda\in[0,1]} L(\lambda) \leqslant \frac{f(a)+f(b)}{2},$$

where $l(\lambda), L(\lambda)$ are defined in Theorem (3).

2 Main results

The aim of our work is to extend these results to a new class of function, not necessarily convex. The following lemma will be used.

Let $f : I \longrightarrow \mathbb{R}$ be an arbitrary function, we define the new function:

$$\begin{aligned} F &: [a, b] \longrightarrow \mathbb{R} \\ x &\mapsto F(x) = f(a + b - x) + f(x). \end{aligned}$$

Definition 5. *A real-valued function f is said to be with convex symmetrization on I if F is convex.*

Theorem 6 (properties of F)**.** *Suppose that the function F is convex, then we have:*

1. *If f is a convex function then the function F is convex too. The converse is false.*
2. *The function F is symmetric to $\frac{a+b}{2}$ in the sense for all x on I, we have*
$$\forall x \in [a, b], F\,(a + b - x) = F(x).$$
3. $\forall x \in [a, b], F\left(\dfrac{a+b}{2}\right) \leq F(x) \leq F(a) = F(b) = f(a) + f(b).$
4. *The function F is increasing on $[\frac{a+b}{2}, b]$ and decreasing on $[a, \frac{a+b}{2}]$.*

Proof. The proof is left to the reader or one can consult [4] □

Example 7. *The function $f : [a, b] \longrightarrow \mathbb{R} : x \mapsto f(x) = \alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0$ such that $a < 0 < b$, $\alpha_{2,}\alpha_3 > 0$ and $a + b > 0$ is not necessarily convex on I, but $F(x) = f(a + b - x) + f(x)$ is convex. ($F'' > 0$).*

Example 8. *The function $f : [a, b] \longrightarrow \mathbb{R} : x \mapsto f(x) = shx = \dfrac{e^x - e^{-x}}{2}$ such that $a < 0 < b$ and $a + b > 0$ is not convex on I, but $F(x) = f(a+b-x)+f(x)$ is convex, $(F''(x) = 2sh\left(\dfrac{a+b}{2}\right) ch\left(\dfrac{a+b}{2} - x\right) > 0.$*

In Theorem 9, we establish the Hermite-Hadamard inequality for a class of functions, which are not necessarily convex.

Theorem 9. *Let f be an integrable function defined on I with convex symmetrization F, then the function f satisfies Hermite-Hadamard inequality.*

Proof. Hermite-Hadamard inequality holds for F:

$$F\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b F(x)dx \leq \frac{F(a) + F(b)}{2}$$

substituting F

$$f\left(a+b-\frac{a+b}{2}\right)+f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b (f(a+b-x)+f(x))\,dx \leq \frac{2f(b)+2f(a)}{2}$$

using simple techniques of integration in particular $\int_a^b f(a+b-x)dx = \int_a^b f(x)dx$, we obtain

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(b)+f(a)}{2}.$$

□

Theorem 10. *Let f be an integrable function defined on I with convex symmetrization F, then for all $\lambda \in [0,1]$, we have*

$$f\left(\frac{a+b}{2}\right) \leqslant h(\lambda) \leqslant \frac{1}{b-a}\int_a^b f(x)\,dx \leqslant H(\lambda) \leqslant \frac{f(a)+f(b)}{2}, \qquad (2)$$

where

$$\begin{aligned} h(\lambda) := {} & \frac{\lambda}{2}\left[f\left(\frac{(2-\lambda)b+\lambda a}{2}\right)+f\left(\frac{\lambda b+(2-\lambda)a}{2}\right)\right] \\ & + \frac{(1-\lambda)}{2}\left[f\left(\frac{(1+\lambda)a+(1-\lambda)b}{2}\right)+f\left(\frac{(1-\lambda)a+(1+\lambda)b}{2}\right)\right] \end{aligned}$$

and

$$H(\lambda) := \frac{1}{4}\left[f(a)+f(b)+f(\lambda b+(1-\lambda)a)+f(\lambda a+(1-\lambda)b)\right].$$

Proof. Let F be a convex function on I. Applying (1) on the subinterval $[a, \lambda b+(1-\lambda)a]$, with $\lambda \neq 0$, we get

$$\begin{aligned} F\left(\frac{\lambda b+(2-\lambda)a}{2}\right) & \leqslant \frac{1}{\lambda(b-a)}\int_a^{\lambda b+(1-\lambda)a} F(x)\,dx \qquad (3) \\ & \leqslant \frac{F(a)+F(\lambda b+(1-\lambda)a)}{2}. \end{aligned}$$

Applying (1) again on $[\lambda b+(1-\lambda)a, b]$, with $\lambda \neq 1$ we get

$$\begin{aligned} F\left(\frac{(1+\lambda)b+(1-\lambda)a}{2}\right) & \leqslant \frac{1}{(1-\lambda)(b-a)}\int_{\lambda b+(1-\lambda)a}^{b} F(x)\,dx \qquad (4) \\ & \leqslant \frac{F(b)+F(\lambda b+(1-\lambda)a)}{2}. \end{aligned}$$

Multiplying (3) by λ, (4) by $(1-\lambda)$, and adding the resulting inequalities, we get

$$h(\lambda) \leqslant \frac{1}{b-a} \int_a^b F(x)\, dx \leqslant H(\lambda). \tag{5}$$

Using the fact that F is a convex function, we obtain

$$\begin{aligned} F\left(\frac{a+b}{2}\right) &= F\left(\lambda \frac{(\lambda b + (2-\lambda) a)}{2} + (1-\lambda) \frac{(1+\lambda) b + (1-\lambda) a}{2}\right) \\ &\leqslant \lambda F\left(\frac{\lambda b + (1-\lambda) a + a}{2}\right) + (1-\lambda) F\left(\frac{\lambda b + (1-\lambda) a + b}{2}\right) \\ &\leqslant \frac{1}{2}\left(F(\lambda b + (1-\lambda) a) + \lambda F(a) + (1-\lambda) F(b)\right) \leqslant \frac{F(a) + F(b)}{2}. \end{aligned} \tag{6}$$

Then by (5) and (6) we get (2). □

The following Theorem is a generalization of Theorem 3 to a large class of integrable functions with convex symmetrization. The calculus result is inspired by [6].

Corollary 11. *Assume that $f : I \to \mathbb{R}$ is an integrable function defined on I with convex symmetrization F, then we have the following inequality*

$$f\left(\frac{a+b}{2}\right) \leqslant \sup_{\lambda \in [0,1]} h(\lambda) \leqslant \frac{1}{b-a} \int_a^b f(x)\, dx \leqslant H\left(\frac{1}{2}\right) \leqslant \frac{f(a) + f(b)}{2},$$

where $h(\lambda), H\left(\frac{1}{2}\right)$ are defined in Theorem 10.

In the following theorem we will extend the Fejér inequality to the new class of functions. In what follows we assume that the function $f : I \longrightarrow \mathbb{R}$ is an integrable function defined on I with convex symmetrization F. Suppose that $g : I \longrightarrow [0, +\infty[$ is integrable and symmetric to $\dfrac{a+b}{2}$.

Theorem 12. *Let f, g be two functions defined on I as above. Then we have*

$$f\left(\frac{a+b}{2}\right) \int_a^b g(x)\, dx \leqslant \int_a^b g(x) f(x)\, dx \leqslant \frac{f(a) + f(b)}{2} \int_a^b g(x)\, dx. \tag{7}$$

Proof. The Fejér inequality was established for $f : I \longrightarrow \mathbb{R}$ convex and $g : I \longrightarrow [0, +\infty[$ integrable and symmetric to $\dfrac{a+b}{2}$. Here we have the same conditions with F and g, so we obtain

$$F\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx \leqslant \int_a^b g(x)F(x)\,dx \leqslant \frac{F(a)+F(b)}{2}\int_a^b g(x)\,dx.$$

Substituting F in the above formulae transforms the inequality into

$$2f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx \leqslant \int_a^b g(x)F(x)\,dx \leqslant \frac{2(f(a)+f(b))}{2}\int_a^b g(x)\,dx.$$

The change of variable $x = a+b-x$ transforms $\int_a^b g(x)f(a+b-x)\,dx$ into $\int_a^b g(a+b-x)f(x)\,dx$. The fact that g is symmetric to $\dfrac{a+b}{2}$, gives

$$\int_a^b g(a+b-x)f(x)\,dx = \int_a^b g(x)f(x)dx.$$

Using the last identity, we derive (7). □

3 Applications

1. Let a, b be two real numbers such that $a+b>0$. The function

   $$\begin{aligned} f_n : [a,b] &\longrightarrow \mathbb{R} \\ x &\mapsto x^n \end{aligned}$$

   is in general not convex for all integers, but the function F is convex

   $$\begin{aligned} F_n : [a,b] &\longrightarrow \mathbb{R} \\ x &\mapsto f_n(a+b-x)+f_n(x). \end{aligned}$$

   This can be proved by induction on n.

   According to the Theorem 9, we have.

   $$\left(\frac{a+b}{2}\right)^n \leq \frac{1}{b-a}\int_a^b x^n dx \leq \frac{a^n+b^n}{2}. \tag{8}$$

   We mentioned here that we can obtain this inequalities using Theorem 2.2 of [2].

2. We can verify easily the following identity

$$b^{n+1} - a^{n+1} = (b-a) \sum_{k=0}^{k=n} a^k b^{n-k}. \tag{9}$$

Replacing the identity (9) in inequality (8), we derive

$$\left(\frac{a+b}{2}\right)^n \leq \frac{1}{n+1} \sum_{k=0}^{k=n} a^k b^{n-k} \leq \frac{a^n + b^n}{2}$$

which is a generalization of Haber inequality [8]

$$\left(\frac{a+b}{2}\right)^n \leq \frac{1}{n+1} \sum_{k=0}^{k=n} a^k b^{n-k}$$

for $n \in \mathbb{N}$ and a, b two positive real numbers.

3. Let $a, b \in \mathbb{R}$ be such that $a + b > 0$. The function

$$\begin{aligned} f &: [a,b] \longrightarrow \mathbb{R} \\ x &\mapsto a_0 + a_1 x^1 + ... + a_n x^n \end{aligned}$$

where $a_k > 0$, for $k > 1$, is not necessarily convex, but the function

$$\begin{aligned} F &: [a,b] \longrightarrow \mathbb{R} \\ x &\mapsto f(a+b-x) + f(x) \end{aligned}$$

is convex.

According to Theorem 9 and in the case where all the coefficients are equal to 1 ($a_k = 1$), we have:

$$\sum_{k=0}^{k=n} \left(\frac{a+b}{2}\right)^k \leq \frac{1}{b-a} \sum_{k=0}^{k=n} \int_a^b x^k dx \leq \frac{1}{2} \sum_{k=0}^{k=n} (a^k + b^k).$$

Remark 13. *The particular case where $a < 0$, $n = 5$, is an example where the result of [2] does not apply.*

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