INROADS

Real Analysis Exchange
Vol. 38(2), 2012/2013, pp. 445–466

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FROM SCALAR MCSHANE INTEGRABILITY TO PETTIS INTEGRABILITY

Abstract

A new concept of McShane-tightness is introduced to pass from scalar (alias weak) McShane integrability to Pettis integrability. It is used also to derive a scalar McShane version of the Vitali theorem.

1 Introduction

In [17], weak (scalar) McShane integrability, a weakening of Pettis integrability, for functions defined on compact intervals in $\mathbb{R}^m$ into a Banach space X is introduced. There, it is shown that these two notions (scalar McShane integrability and Pettis integrability) are equivalent if and only if the Banach space X contains no copy of c_0. A similar result dealing with functions defined on compact intervals in $\mathbb{R}$ can already be found in [6].

It is the aim of this paper to describe more deeply the relationship between the scalar McShane integral and the Pettis integral. Our principal objective is to determine precisely when a scalar McShane integrable function is also Pettis integrable. For this purpose a necessary and sufficient tightness condition involving locally upper bounded McShane sums (namely $\mathcal{SLM}$-tightness) is introduced. The proof of our main result (Theorem 3.1) depends on an

Mathematical Reviews subject classification: Primary: 28B05; Secondary: 26A39
Key words: McShane integral, Pettis integral, scalar McShane integral, locally τ-upper McShane-bounded, locally McShane-tight
Received by the editors June 20, 2012
Communicated by: Luisa DiPiazza

exhaustion-type lemma (Lemma 3.1), which may be useful elsewhere.

As application of our methods, we provide a convergence result for the scalar McShane integral analogous to the one we have for the Pettis integral (see [5] and [10]), based on a sequential version of the $\mathcal{SLM}$-tightness condition used in Theorem 3.1. For various Kurzweil-Henstock versions, one can look at the contribution of L. Di Piazza ([13]).

2 Notations and preliminaries

Throughout this paper $[0,1]$ is the unit interval of the real line equipped with the usual topology and the Lebesgue measure λ. The family of Lebesgue measurable subsets of $[0,1]$ is denoted by $\mathcal{L}$. For $E \in \mathcal{L}$, we often consider the collection $\mathcal{L}^+(E)$ of all Lebesgue measurable subsets of E with (strictly) positive measure and denote $\mathcal{L}^+([0,1])$ by just $\mathcal{L}^+$. By $L^1_{\mathbb{R}}(\lambda)$ we denote the space (of classes) of Lebesgue-integrable functions defined on $[0,1]$. Recall that a subset $\mathcal{H}$ of $L^1_{\mathbb{R}}(\lambda)$ is uniformly integrable ([12]) if

$$\lim_{a\to\infty} \sup_{h\in\mathcal{H}} \int_{\{t\in[0,1]:|h(t)|\geq a\}} |h|\, d\lambda = 0.$$

It is well known ([12]) that $\mathcal{H}$ is uniformly integrable if and only if it is $L^1_{\mathbb{R}}(\lambda)$-*bounded* (i.e. $\sup_{h\in\mathcal{H}} \int_{[0,1]} |h|\, d\lambda$ is finite) and *equi-continuous*, i.e.

$$\lim_{\lambda(A)\to 0} \sup_{h\in\mathcal{H}} \int_A |h|\, d\lambda = 0.$$

Now let X be a Banach space, whose norm is denoted by $\|.\|$. We denote by X^* the topological dual of X and the closed unit ball of X^* by $\overline{B}_{X^*}$. A function $f : [0,1] \to X$ is said to be *scalarly measurable* (resp. *scalarly integrable*, alias *Dunford integrable*) if for every $x^* \in X^*$, the real-valued function $\langle x^*, f\rangle$ is measurable (resp. Lebesgue integrable). If $f : [0,1] \to X$ is a scalarly integrable function, then for each $E \in \mathcal{L}$, there is $x_E^{**} \in X^{**}$ such that

$$\langle x^*, x_E^{**}\rangle = \int_E \langle x^*, f\rangle\, d\lambda.$$

The vector x_E^{**} is called the *Dunford integral* of f over E, and is denoted by $(\mathcal{D})$- $\int_E f\, d\lambda$. In the case that $(\mathcal{D})$- $\int_E f\, d\lambda \in X$ for all $E \in \mathcal{L}$, then f is called *Pettis-integrable* and we write $(\mathcal{P}e)$- $\int_E f\, d\lambda$ instead of $(\mathcal{D})$- $\int_E f\, d\lambda$ to denote the *Pettis integral* of f over E. If $f : [0,1] \to X$ is a Pettis integrable function, then the set $\{\langle x^*, f\rangle : x^* \in \overline{B}_{X^*}\}$ is relatively weakly compact in $L^1_{\mathbb{R}}(\lambda)$ ([2], Theorem II. 3.8) (see also [11]); equivalently it is *uniformly integrable* ([2], Theorem III. 2.15). For an extensive study of Banach space-valued Pettis integral, the reader is referred to Musial ([11]).

A *partial McShane partition* (or simply a *McShane partition*) is a finite collection $\{(A_i, t_i) : 1 \leq i \leq m\}$, where $A_1, ..., A_m$ are pairwise disjoints measurable subsets of $[0,1]$ and t_i is a point of $[0,1]$ for each $i \leq m$. Let E be a member of $\mathcal{L}$. If the union of all the elements A_i of the partition equals (resp. is contained in) E, then it is a *McShane partition* of E (resp. in E). A *gauge* on $[0,1]$ is a function $\delta : [0,1] \to]0, +\infty[$. For a given δ on $[0,1]$, we say that a McShane partition $\{(A_i, t_i) : 1 \leq i \leq m\}$ is *subordinate* to δ if $A_i \subset (t_i - \delta(t_i), t_i + \delta(t_i))$ for every $1 \leq i \leq m$. Let $f : [0,1] \to X$ be a function. We set

$$\sigma(f, \mathcal{P}) := \sum_{i=1}^{i=m} \lambda(A_i) f(t_i),$$

for each McShane partition $\mathcal{P} := \{(A_i, t_i) : 1 \leq i \leq m\}$.

Definition 2.1. A function $f : [0,1] \to X$ is McShane integrable, with McShane integral ϖ, if for every $\varepsilon > 0$ there is a gauge δ on $[0,1]$ such that

$$\|\sigma(f, \mathcal{P}) - \varpi\| < \varepsilon,$$

for every δ-fine McShane partition $\mathcal{P}$ of $[0,1]$. We set $\varpi := (\mathcal{M})\text{-}\int_0^1 f \, d\lambda$.

Of course, the original definition of the McShane integrability involves partitions into non-overlapping subintervals of $[0,1]$ rather that McShane (measurable) partitions, but the translation from one to the other is possible ([9] and [14]). See also ([3]) and ([4]) for a more general setting.

It is known ([4]) that if a function $f : [a,b] \to X$ is McShane integrable on $[a,b]$, then it is Pettis integrable on $[a,b]$ and the integrals are equals. When X is separable, these two notions coincides ([7]).

Also, recall that a real-valued function is Lebesgue integrable on $[a,b]$ if and only if it is McShane integrable on $[a,b]$ and the integrals are equals in both cases (see [8]).

Lemma 2.1. *(Saks-Henstock) Let $f : [0,1] \to X$ be a McShane integrable function and let $\varepsilon > 0$. Suppose that δ is a gauge on $[0,1]$ such that*

$$\|\sigma(f, \mathcal{P}) - (\mathcal{M})\text{-}\int_{[0,1]} f \, d\lambda\| < \varepsilon$$

whenever $\mathcal{P}$ is a McShane partition of $[0,1]$ that is subordinate to δ. If $\{(A_i, t_i) : i = 1, ..., m\}$ is an arbitrary (partial) McShane partition subordinate to δ, then

$$\|\sum_{i=1}^{i=m}[\lambda(A_i)f(t_i) - (\mathcal{M})\text{-}\int_{A_i} f\,d\lambda]\| \leq \varepsilon.$$

Proof. It is a consequence of Lemma 2B in [3] or Lemma 2 in [15]. □

Definition 2.2. A function $f : [0,1] \to X$ is said to be *scalarly McShane integrable* (alias *weakly McShane integrable* [17]) on $[0,1]$ ($\mathcal{SM}$-integrable for short), if for each $[a,b] \subset [0,1]$ there exists $\varpi_{[a,b]} \in X$ with the following property: for every $\varepsilon > 0$ and for every $x^* \in X^*$ there exists a gauge δ on $[a,b]$ such that

$$|\langle x^*, \sigma(f,\mathcal{P})\rangle - \langle x^*, \varpi_{[a,b]}\rangle| < \varepsilon$$

for every McShane partition $\mathcal{P}$ of $[a,b]$ subordinate to δ.

We set

$$\varpi_{[a,b]} = (\mathcal{SM})\text{-}\int_{[a,b]} f\,d\lambda.$$

The function f is $\mathcal{SM}$-integrable on a set $E \in \mathcal{L}$ if $1_E f$ is $\mathcal{SM}$-integrable on $[0,1]$.

In [17], the following equivalent formulation of scalar McShane integrability is given.

Definition 2.3. A function $f : [0,1] \to X$ is said to be $\mathcal{SM}$-integrable on $[0,1]$, if it is scalarly integrable and for each subinterval $[a,b]$ of $[0,1]$, there exists $\varpi_{[a,b]} \in X$ such that

$$\langle x^*, \varpi_{[a,b]}\rangle = \int_{[a,b]} \langle x^*, f\rangle\,d\lambda \text{ for all } x^* \in X^*.$$

Before going further, we list below the properties of the Scalar McShane integral ([17], Theorem 24). Taking into account Definition 2.3, the proofs are virtually identical to the McShane integral of real valued functions (see [8]).

Theorem 2.1. *Let f, g: $[0,1] \to X$ be two functions.*

(1) If f is $\mathcal{SM}$-integrable on $[0,1]$, then it is $\mathcal{SM}$-integrable on every subinterval of $[0,1]$.

(2) Let I_1 and I_2 be two non-overlapping subintervals of $[0,1]$. If f is $\mathcal{SM}$-integrable on each of the intervals I_1 and I_2, then it is $\mathcal{SM}$-integrable on $I_1 \cup I_2$ and

$$(\mathcal{SM})\text{-}\int_{I_1 \cup I_2} f\, d\lambda = (\mathcal{SM})\text{-}\int_{I_1} f\, d\lambda + (\mathcal{SM})\text{-}\int_{I_2} f\, d\lambda.$$

(3) If f, g are $\mathcal{SM}$-integrable and if α is a real number, then $\alpha f + g$ is $\mathcal{SM}$-integrable and

$$(\mathcal{SM})\text{-}\int_{[0,1]} \alpha f + g\, d\lambda = \alpha\,(\mathcal{SM})\text{-}\int_{[0,1]} f\, d\lambda + (\mathcal{SM})\text{-}\int_{[0,1]} g\, d\lambda.$$

(4) If f is $\mathcal{SM}$-integrable and if $f = g$ λ-a.e., then the function g is $\mathcal{SM}$-integrable and

$$(\mathcal{SM})\text{-}\int_{[0,1]} g\, d\lambda = (\mathcal{SM})\text{-}\int_{[0,1]} f\, d\lambda.$$

The next proposition is an immediate consequence of the previous definitions. See also ([17] and [18]).

Proposition 2.1. *Let $f : [0,1] \to X$ be a function.*

(1) If f is $\mathcal{SM}$-integrable, then it is Dunford integrable and

$$(\mathcal{D})\text{-}\int_{[a,b]} f\, d\lambda = (\mathcal{SM})\text{-}\int_{[a,b]} f\, d\lambda \textit{ for every } [a,b] \subset [0,1].$$

(2) If f is Pettis integrable, then it is $\mathcal{SM}$-integrable and

$$(\mathcal{SM})\text{-}\int_{[a,b]} f\, d\lambda = (\mathcal{P}_e)\text{-}\int_{[a,b]} f\, d\lambda \textit{ for every } [a,b] \subset [0,1].$$

Clearly, these notions coincides when X is reflexive. When X is not reflexive, this may not be the case.

Example 1. Define $f : [0,1] \to c_0$ by

$$f(t) := (1_{[0,1]}, 2.1_{[0,\frac{1}{2}]}, ..., n.1_{[0,\frac{1}{n}]}),$$

([2], p. 53). This function is scalarly integrable, but not $\mathcal{SM}$-integrable, because $(\mathcal{D})\text{-}\int_{[0,1]} f\, d\lambda \in \ell_\infty \setminus c_0$.

Example 2. For each $n \geq 1$ let

$$I_n^1 := (\frac{1}{n+1}, \frac{n+\frac{1}{2}}{n(n+1)}) \text{ and } I_n^2 := (\frac{n+\frac{1}{2}}{n(n+1)}, \frac{1}{n})$$

and define the function $f : [0,1] \to c_0$ by

$$f(t) := (2n(n+1)(1_{I_n^1}(t) - 1_{I_n^2}(t))).$$

Then f is $\mathcal{SM}$-integrable, but not Pettis integrable since if $E := \cup_n I_n^1$ we have $\int_E f\, d\lambda = 1 \in \ell_\infty \setminus c_0$. See ([6] and [17]) for more details.

Remark 2.1. Thanks to Example 2 if each $\mathcal{SM}$-integrable function is Pettis integrable, then it is easy to see that X cannot contain a copy of c_0. The converse implication also holds as was shown by Gordon ([7], Theorem 18). See ([17]) for the case of functions defined on compact intervals in $\mathbb{R}^m$.

In order to pass from scalar McSchane integrability to Pettis integrability, we introduce the following new concepts of *local upper McShane-boundedness* and *local McShane-tightness*.

Definition 2.4. Let $\tau > 0$. A function $f : [0,1] \to \mathbb{R}$ is said to be *locally τ-upper McShane-bounded* if for each gauge δ on $[0,1]$ and for each set E in $\mathcal{L}^+$, there is an F in $\mathcal{L}^+(E)$ and a McShane partition $\mathcal{P}$ of F subordinate to δ such that

$$m_F(f, \mathcal{P}) := \frac{1}{\lambda(F)} \sigma(f, \mathcal{P}) \leq \tau.$$

A function $f : [0,1] \to X$ is said to be *scalarly locally τ-upper McShane-bounded* if, for each $x^* \in \overline{B}_{X^*}$, $\langle x^*, f \rangle$ is locally τ-upper McShane-bounded.

Definition 2.5. A function $f : [0,1] \to X$ is said to be *scalarly locally McShane-tight* (*$\mathcal{SLM}$-tight* for short) if for each $\varepsilon > 0$ there exist $\tau_\varepsilon > 0$ and $\eta_\varepsilon > 0$ such that for every $x^* \in \overline{B}_{X^*}$, there exists a measurable set L_ε such that
(a) $\langle x^*, 1_{L_\varepsilon} f \rangle$ is locally τ_ε-upper McShane-bounded.
(b) Given any gauge δ on $[0,1]$ and any finite collection $\{I_i : i = 1, ..., p\}$ of non-overlapping sub-intervals of $[0,1]$ with $\sum_{i=1}^{i=p} \lambda(I_i) \leq \eta_\varepsilon$ we have

$$\langle x^*, \sigma(f, \mathcal{P}) \rangle \leq \varepsilon,$$

for some McShane partition $\mathcal{P}$ of $\cup_{i=1}^{i=p} I_i \setminus L_\varepsilon$ subordinate to δ.

Note that every scalarly locally τ-upper McShane-bounded function $f : [0,1] \to X$ is $\mathcal{SLM}$-tight (take $\tau_\varepsilon = \tau$ and $L_\varepsilon = [0,1]$ in Definition 2.5).

It is also interesting to introduce the following formulation of tightness for scalarly integrable functions. It involves integrals instead of McShane sums.

Definition 2.6. A function $f : [0,1] \to X$ is said to be *scalarly locally tight* ($\mathcal{SL}$*-tight* for short) if for each $\varepsilon > 0$ there exist $\tau_\varepsilon > 0$ and $\eta_\varepsilon > 0$ such that for every $x^* \in \overline{B}_{X^*}$, there exists a measurable set L_ε such that

(a)$'$ For each set E in $\mathcal{L}^+$, there is an F in $\mathcal{L}^+(E)$ such that

$$m_F(\langle x^*, 1_{L_\varepsilon} f\rangle) := \frac{1}{\lambda(F)} \int_{F \cap L_\varepsilon} \langle x^*, f\rangle \, d\lambda \le \tau_\varepsilon.$$

(b)$'$ $$\int_{E \setminus L_\varepsilon} \langle x^*, f\rangle \, d\lambda \le \varepsilon \text{ whenever } E \in \mathcal{L} \text{ and } \lambda(E) \le \eta_\varepsilon.$$

Proposition 2.2. *Let $f : [0,1] \to X$ be a scalarly integrable function. If it is $\mathcal{SL}$-tight, then it is also $\mathcal{SLM}$-tight.*

Proof. Let τ_ε and η_ε be the positive real numbers corresponding to ε in Definition 2.6, and take any gauge δ on $[0,1]$ and any x^* in $\overline{B}_{X^*}$. Then there exists a measurable set L_ε such that (a)$'$ and (b)$'$ hold. To see that (a) holds, fix $E \in \mathcal{L}^+$ and, with the help of (a)$'$, choose an F in $\mathcal{L}^+(E)$ such that

$$m_F(\langle x^*, 1_{L_\varepsilon} f\rangle) \le \tau_\varepsilon.$$

Next, since $\langle x^*, f\rangle$ is Lebesgue integrable, $\langle x^*, 1_{L_\varepsilon} f\rangle$ is also Lebesgue integrable and so there exists a gauge δ_0 on $[0,1]$ with $\delta_0 \le \delta$ such that

$$|\langle x^*, \sigma(1_{L_\varepsilon} f, \mathcal{P})\rangle - \int_{[0,1]} \langle x^*, 1_{L_\varepsilon} f\rangle \, d\lambda| < \varepsilon\lambda(F),$$

for every McShane partition $\mathcal{P}$ of $[0,1]$ subordinate to δ_0. Now fix a McShane partition $\mathcal{P}_0$ of F subordinate to δ_0. Then, by the Henstock Lemma, one has

$$|\langle x^*, \sigma(1_{L_\varepsilon} f, \mathcal{P}_0)\rangle - \int_F \langle x^*, 1_{L_\varepsilon} f\rangle \, d\lambda| \le \varepsilon\lambda(F),$$

whence

$$\langle x^*, \sigma(1_{L_\varepsilon} f, \mathcal{P}_0)\rangle \le \int_F \langle x^*, 1_{L_\varepsilon} f\rangle \, d\lambda + \varepsilon\lambda(F).$$

Thus

$$m_F(\langle x^*, 1_{L_\varepsilon} f\rangle, \mathcal{P}_0) \le m_F(\langle x^*, 1_{L_\varepsilon} f\rangle) + \varepsilon \le \tau_\varepsilon + \varepsilon.$$

Thus (a) holds. Let us prove (b). Invoking again the Lebesgue integrability of $\langle x^*, f\rangle$, we find a gauge δ_0' of $[0,1]$ with $\delta_0' \le \delta$ such that

$$|\langle x^*, \sigma(f, \mathcal{Q})\rangle - \int_{[0,1]} \langle x^*, f\rangle \, d\lambda| < \varepsilon,$$

for every McShane partition $\mathcal{Q}$ of $[0,1]$ subordinate to δ_0'. Now let $\{I_i : i = 1, ..., p\}$ be any finite collection of non-overlapping sub-intervals of $[0,1]$ such that $\lambda(E) \le \eta_\varepsilon$, where $E := \cup_{i=1}^{i=p} I_i$, and select a McShane partition $\mathcal{Q}_0$ of $E \setminus L_\varepsilon$ subordinate to δ_0'. Then, by (b)$'$, we have

$$\int_{E \setminus L_\varepsilon} \langle x^*, f\rangle \, d\lambda \le \varepsilon \text{ whenever } E \in \mathcal{L} \text{ and } \lambda(E) \le \eta_\varepsilon.$$

Further, applying once again the Henstock Lemma we get

$$|\langle x^*, \sigma(f, \mathcal{Q}_0)\rangle - \int_{E \setminus L_\varepsilon} \langle x^*, f\rangle \, d\lambda| \le \varepsilon,$$

whence

$$\langle x^*, \sigma(f, \mathcal{Q}_0)\rangle \le \int_{E \setminus L_\varepsilon} \langle x^*, f\rangle \, d\lambda + \varepsilon \le 2\varepsilon.$$

This gives (b). Thus f is $\mathcal{SLM}$-tight. □

Remark 2.2. Actually the converse of Proposition 2.2 also holds, using Lemma 3.2 to come in Section 3 via the next corollary.

Corollary 2.1. *Let $f : [0,1] \to X$ be a scalarly integrable function. If the set $\{\langle x^*, f\rangle : x^* \in \overline{B}_{X^*}\}$ is equi-continuous, then f is $\mathcal{SL}$-tight, and hence $\mathcal{SLM}$-tight.*

Proof. Let $\varepsilon > 0$. By hypothesis, there exists $\eta_\varepsilon > 0$ such that

$$\int_A |\langle x^*, f\rangle| \, d\lambda \le \varepsilon, \text{ for all } x^* \in \overline{B}_{X^*},$$

whenever $A \in \mathcal{L}$ and $\lambda(A) \le \eta_\varepsilon$. This shows that f is $\mathcal{SL}$-tight by taking in Definition 2.6 $\tau_\varepsilon = 1$ and $L_\varepsilon = \emptyset$. The $\mathcal{SLM}$-tightness then follows from Proposition 2.2. □

Corollary 2.2. *Let $f : [0,1] \to X$ be a function. If it is Pettis integrable, then it is also $\mathcal{SL}$-tight, and hence $\mathcal{SLM}$-tight.*

PROOF. If f is Pettis integrable then $\{\langle x^*, f\rangle : x^* \in \overline{B}_{X^*}\}$ is uniformly integrable (see Section 2), and hence f is $\mathcal{SL}$-tight, in view of Corollary 2.1. $\square$

Example 3. *A* $(\mathcal{SM})$*-integrable function that is not* $\mathcal{SL}$*-tight.*

PROOF. Let $f : [0,1] \to c_0$ be the function defined in **Example 2**. We have already seen that f is $\mathcal{SM}$-integrable, and now we want to prove that it is not $\mathcal{SL}$-tight. Proceeding by contradiction, assume that f is $\mathcal{SL}$-tight. Let $0 < \varepsilon < 1$ be fixed and let τ_ε and η_ε be the positive real numbers corresponding to ε in Definition 2.6. Let (e_n^*) be the nth unit vector in ℓ^1. Then to each $n \geq 1$ corresponds a measurable set $L_{n,\varepsilon}$ such that

(2.1) For each $n \geq 1$ and each set E in $\mathcal{L}^+$, there is an F in $\mathcal{L}^+(E)$ (which may depends on n) such that

$$\int_{F \cap L_{n,\varepsilon}} \langle e_n^*, f\rangle \, d\lambda \leq \tau_\varepsilon \lambda(F).$$

$$\int_{E \setminus L_{n,\varepsilon}} \langle e_n^*, f\rangle \, d\lambda \leq \varepsilon \text{ whenever } E \in \mathcal{L} \text{ and } \lambda(E) \leq \eta_\varepsilon. \tag{2.2}$$

We distinguish two cases.

Case 1). Suppose there exists $n_0 \geq 1$ *such that* $\lambda(I_n^1 \cap L_{n,\varepsilon}) > 0$ *for all* $n \geq n_0$. Then applying (2.1) for each $E_n := I_n^1 \cap L_{n,\varepsilon}$ we find $F_n \subset I_n^1 \cap L_{n,\varepsilon}$ with $\lambda(F_n) > 0$ such that

$$\int_{F_n \cap L_{n,\varepsilon}} \langle e_n^*, f\rangle \, d\lambda = \int_{F_n} \langle e_n^*, f\rangle \, d\lambda = 2n(n+1)\lambda(F_n) \leq \tau_\varepsilon \lambda(F_n),$$

for every $n \geq n_0$, a contradiction. We turn now to the second case.

Case 2). Suppose now there exists a strictly increasing sequence (k_n) *of positive integers such that* $\lambda(I_{k_n}^1 \cap L_{k_n,\varepsilon}) = 0$, *for all* $n \geq 1$. We then have

$$\int_{I_{k_n}^1 \setminus L_{k_n,\varepsilon}} \langle e_{k_n}^*, f\rangle \, d\lambda = \int_{I_{k_n}^1} \langle e_{k_n}^*, f\rangle \, d\lambda = 2k_n(k_n+1)\frac{1}{2k_n(k_n+1)} = 1,$$

for every $n \geq 1$. On the other hand, since $\lambda(I_{k_n}^1)$ tends to zero as n tends to infinity, there exists an index m such that $\lambda(I_{k_m}^1) \leq \eta_\varepsilon$. Thus, by (2.2), we have

$$\int_{I_{k_m}^1 \setminus L_{k_m,\varepsilon}} \langle e_{k_m}^*, f\rangle \, d\lambda \leq \varepsilon < 1,$$

which contradicts the last calculation. □

Proposition 2.3. *If $f : [0,1] \to X$ is a scalarly integrable function, then for every $\varepsilon > 0$, there exist $\tau_\varepsilon > 0$, and a measurable set L_ε with $\lambda([0,1] \setminus L_\varepsilon) \le \varepsilon$ such that*

$$(a)'' \qquad \sup_{x^* \in \overline{B}_{X^*}} m_E(|\langle x^*, 1_{L_\varepsilon} f\rangle|) \le \tau_\varepsilon \text{ for all } E \in \mathcal{L}^+.$$

Proof. Let $\varepsilon > 0$. For each $\ell \ge 1$, set

$$A_\ell := \{t \in [0,1] : \|f(t)\| \le \ell\}.$$

Then $\lambda([0,1] \setminus \cup_{\ell \ge 1} A_\ell) = 0$ so there is an $\ell_\varepsilon \ge 1$ such that $\lambda^*(A_{\ell_\varepsilon}) \ge 1 - \varepsilon$, where λ^* stands for the outer measure induced by λ. Let $L_\varepsilon \in \mathcal{L}$ be such that $A_{\ell_\varepsilon} \subset L_\varepsilon$ and $\lambda(L_\varepsilon) = \lambda^*(A_{\ell_\varepsilon})$ (equivalently L_ε is a measurable envelope of A_{ℓ_ε}, that is, $A_{\ell_\varepsilon} \subset L_\varepsilon$ and $\lambda(E \cap L_\varepsilon) = \lambda^*(E \cap A_{\ell_\varepsilon})$ for every $E \in \mathcal{L}$). Then, as f is scalarly integrable and for $t \in A_{\ell_\varepsilon}$, $|\langle x^*, f(t)\rangle| \le \ell_\varepsilon$ for all $x^* \in \overline{B}_{X^*}$, we have

$$\int_{E \cap L_\varepsilon} |\langle x^*, f\rangle| \, d\lambda \le \ell_\varepsilon \lambda(E \cap L_\varepsilon) \le \ell_\varepsilon \lambda(E) := \tau_\varepsilon \lambda(E)$$

for every $x^* \in \overline{B}_{X^*}$ and for every $E \in \mathcal{L}^+$, thereby proving (a)$''$. □

With the help of Proposition 2.3, Corollary 2.1 can be improved as follows.

Proposition 2.4. *Let $f : [0,1] \to X$ be a scalarly integrable function. If the set $\{\langle x^*, f\rangle : x^* \in \overline{B}_{X^*}\}$ is equi-continuous, then the following property (which is stronger than the $\mathcal{SL}$-tightness condition) is satisfied.*

For each $\varepsilon > 0$ there exist $\tau_\varepsilon > 0$ and a measurable set L_ε with $\lambda([0,1] \setminus L_\varepsilon) \le \varepsilon$ such that

$$(a)'' \qquad \sup_{x^* \in \overline{B}_{X^*}} m_E(|\langle x^*, 1_{L_\varepsilon} f\rangle|) \le \tau_\varepsilon \text{ for all } E \in \mathcal{L}^+.$$

$$(b)'' \qquad \sup_{x^* \in \overline{B}_{X^*}} \int_{[0,1] \setminus L_\varepsilon} |\langle x^*, f\rangle| \, d\lambda \le \varepsilon.$$

Consequently, the set $\{\langle x^*, f\rangle : x^* \in \overline{B}_{X^*}\}$ is uniformly integrable.

Proof. Fix $\varepsilon > 0$. By the equi-continuity of f, there exists $0 < \eta_\varepsilon \leq \varepsilon$ such that for each $E \in \mathcal{L}$ with $\lambda(E) \leq \eta_\varepsilon$ we have

$$\sup_{x^* \in \overline{B}_{X^*}} \int_E |\langle x^*, f \rangle| \, d\lambda \leq \varepsilon.$$

On the other hand, by Proposition 2.3, there exist $\tau_\varepsilon > 0$, and a measurable set L_ε with $\lambda([0,1] \setminus L_\varepsilon) \leq \eta_\varepsilon$ for which (a)$''$ is satisfied. Moreover, it is plain that property (b)$''$ is also satisfied, in virtue of the inequality above. At last, since (a)$''$ and (b)$''$ imply

$$\sup_{x^* \in \overline{B}_{X^*}} \int_{[0,1]} |\langle x^*, f \rangle| \, d\lambda < \infty,$$

we conclude that the set $\{\langle x^*, f \rangle : x^* \in \overline{B}_{X^*}\}$ is uniformly integrable, since it is equi-continuous by hypothesis. $\square$

3 From scalar McShane integrability to Pettis integrability

The concept of $\mathcal{SLM}$-tightness, introduced above, allows us to pass from scalar McShane integrability to Pettis integrability as the following theorem shows.

Theorem 3.1. *A function $f : [0,1] \to X$ is Pettis integrable if and only if*
(i) f is $\mathcal{SM}$-integrable
(ii) f is $\mathcal{SLM}$-tight.

The proof of Theorem 3.1 involves the two following lemmas. The first one is an exhaustion lemma. It is the key step in the proof that (i) and (ii) imply the Pettis integrability of f. We need some extra definitions ([3]).

Given a measurable subset E of $[0,1]$, a *generalized McShane partition* of E is a countable collection $\{(A_i, t_i) : i \geq 1\}$ such that $\{(A_i, t_i) : 1 \leq i \leq m\}$ is a partial McShane partition in E for each $m \geq 1$, and $\lambda(E \setminus \cup_{i=1}^{\infty} A_i) = 0$. A generalized McShane partition $\{(A_i, t_i) : i \geq 1\}$ is subordinate to a gauge δ on $[0,1]$ if $A_i \subset (t_i - \delta(t_i), t_i + \delta(t_i))$ for every $i \geq 1$.

Lemma 3.1. *Let $f : [0,1] \to \mathbb{R}$ be a locally τ-upper McShane-bounded function for some $\tau > 0$ and L be a member of $\mathcal{L}^+$. Then, given any gauge δ on $[0,1]$, there exists a generalized McShane partition $\{(A_i, t_i) : i \geq 1\}$ of L subordinate to δ such that*

$$f(t_i) \leq \tau \text{ for all } i \geq 1.$$

Proof. The proof is an exhaustion-type argument in the spirit of [16]. Fix a gauge δ on $[0,1]$.

Let $\mathcal{A}_1$ denote the collection of subsets $B \in \mathcal{L}^+(L)$ such that there is $t \in [0,1]$ for which

$$B \subset (t-\delta(t), t+\delta(t)) \text{ and } f(t) \leq \tau.$$

Since f is locally τ-upper McShane-bounded, there is an F in $\mathcal{L}^+(L)$ and a McShane partition $\mathcal{P} := \{(B_i, u_i) : 1 \leq i \leq m\}$ of F subordinate to δ such that

$$\frac{1}{\lambda(F)} \sum_{i=1}^{i=m} \lambda(B_i) f(u_i) \leq \tau.$$

So there exists $i \in \{1, ..., m\}$ such that

$$B_i \subset (u_i - \delta(u_i), u_i + \delta(u_i)) \text{ and } f(u_i) \leq \tau.$$

Thus the collection $\mathcal{A}_1$ is not empty. If there is a set $B \in \mathcal{A}_1$ with $\lambda(L \setminus B) = 0$, then we are finished. Otherwise, let ℓ_1 be the smallest positive integer for which there is a set $A_1 \in \mathcal{A}_1$ with $\frac{1}{\ell_1} \leq \lambda(A_1) < \lambda(L)$. Accordingly, there is $t_1 \in [0,1]$ such that

$$A_1 \subset (t_1 - \delta(t_1), t_1 + \delta(t_1)) \text{ and } f(t_1) \leq \tau. \tag{†}$$

Observe that necessarily $\ell_1 > 1$, because $\frac{1}{\ell_1} < \lambda(L) \leq 1$.
Let $\mathcal{A}_2$ denote the collection of subsets $B \in \mathcal{L}^+(L \setminus A_1)$ such that there is $t \in [0,1]$ for which

$$B \subset (t-\delta(t), t+\delta(t)) \text{ and } f(t) \leq \tau. \tag{††}$$

Since f is locally τ-upper McShane-bounded, we see that $\mathcal{A}_2$ is not empty. If there is a set $B \in \mathcal{A}_2$ with $\lambda(L \setminus (A_1 \cup B)) = 0$, then we are finished. For in this case, the conclusion of the Lemma holds in view of (†) and (††). Otherwise, let ℓ_2 be the smallest positive integer strictly greater than 1 for which there is a set $A_2 \in \mathcal{A}_2$ with $\frac{1}{\ell_2} \leq \lambda(A_2) < \lambda(L \setminus A_1)$. Accordingly, there is $t_2 \in [0,1]$ such that

$$A_2 \subset (t_2 - \delta(t_2), t_2 + \delta(t_2)) \text{ and } f(t_2) \leq \tau.$$

Continue in this way. If the process stops in a finite numbers of steps then we are finished. If the process does not stop, then we obtain a countable family (A_i) of pairwise disjoint measurable subsets of L and a sequence (t_i) in $[0,1]$ such that

$$A_i \subset (t_i - \delta(t_i), t_i + \delta(t_i)), \ \frac{1}{\ell_i} \leq \lambda(A_i) < 1 \text{ and}$$

$$f(t_i) \le \tau \text{ for all } i \ge 1.$$

(ℓ_i being the smallest positive integer strictly greater than 1 for which there is a set $B \in \mathcal{A}_i$ with $\frac{1}{\ell_i} \le \lambda(B) < \lambda(L)$).

Set $A_\infty := \cup_{i=1}^{\infty} A_i$. We claim that $\lambda(L \setminus A_\infty) = 0$. Indeed, if $\lambda(L \setminus A_\infty) > 0$, then the local τ-upper McShane-boundedness insures the existence of $B \in \mathcal{L}^+$ contained in $L \setminus A_\infty$ and $t \in [0,1]$ such that

$$B \subset (t - \delta(t), t + \delta(t)) \text{ and } f(t) \le \tau.$$

Since for each positive integer n

$$\sum_{i=1}^{i=n} \frac{1}{\ell_i} \le \lambda(\cup_{i=1}^{i=n} A_i) \le 1$$

and $\ell_n > 1$, we can choose an integer $n \ge 1$ such that

$$\frac{1}{\ell_n - 1} \le \lambda(B).$$

As

$$B \subset L \setminus A_\infty \subset L \setminus \cup_{i=1}^{i=n-1} A_i,$$

we conclude that B is a member of $\mathcal{A}_n$. This contradicts the definition of ℓ_n. Thus $\lambda(L \setminus A_\infty) = 0$ as claimed. □

Lemma 3.2. *Let $f : [0,1] \to X$ be a function. If it is scalarly integrable and $\mathcal{SLM}$-tight, then $\{\langle x^*, f\rangle : x^* \in \overline{B}_{X^*}\}$ is uniformly integrable.*

PROOF. Let τ_ε and η_ε be the positive real numbers corresponding to ε in Definition 2.5, and $x^* \in \overline{B}_{X^*}$ be arbitrary fixed. Then there exists a measurable set L_ε such that
(3.1) $\langle x^*, 1_{L_\varepsilon} f\rangle$ is locally τ_ε-upper McShane-bounded.

(3.2) Given any gauge δ on $[0,1]$ and any finite collection $\{I_i : i = 1, ..., p\}$ of non-overlapping sub-intervals of $[0,1]$ with $\sum_{i=1}^{i=p} \lambda(I_i) \le \eta_\varepsilon$ we have

$$\langle x^*, \sigma(f, \mathcal{P})\rangle \le \varepsilon,$$

for some McShane partition $\mathcal{P}$ of $\cup_{i=1}^{i=p} I_i \setminus L_\varepsilon$ subordinate to δ.

Next, as $\langle x^*, 1_{L_\varepsilon} f\rangle$ and $\langle x^*, f\rangle$ are Lebesgue integrable on $[0,1]$, we may select a gauge δ_0 on $[0,1]$ such that

$$|\langle x^*, \sigma(1_{L_\varepsilon} f, \mathcal{P})\rangle - \int_{[0,1]} \langle x^*, 1_{L_\varepsilon} f\rangle \, d\lambda| < \frac{\varepsilon}{16} \text{ and} \tag{3.3}$$

$$|\langle x^*, \sigma(f, \mathcal{P})\rangle - \int_{[0,1]} \langle x^*, f\rangle \, d\lambda| < \frac{\varepsilon}{32}, \tag{3.4}$$

for every McShane partition $\mathcal{P}$ of $[0,1]$ subordinate to δ_0. Now, in view of (3.1), Lemma 3.1 insures the existence of a generalized McShane partition $\{(A_i, t_i) : i \geq 1\}$ of L_ε subordinate to δ_0 such that

$$\langle x^*, 1_{L_\varepsilon} f(t_i)\rangle \leq \tau_\varepsilon \text{ for all } i \geq 1. \tag{3.5}$$

Next, put $\zeta_\varepsilon := \min(\frac{\varepsilon}{8\tau_\varepsilon}, \eta_\varepsilon)$ and select a measurable set E of the form $E := \cup_{i=1}^{i=p} I_i$ such that $\lambda(E) \leq \zeta_\varepsilon$, where $\{I_i : i = 1, ..., p\}$ is a finite collection of non-overlapping sub-intervals of $[0,1]$. Then by (3.2), there exists a McShane partition $\mathcal{P}'$ of $E \setminus L_\varepsilon$ adapted to δ_0 such that

$$\langle x^*, \sigma(f, \mathcal{P}')\rangle \leq \frac{\varepsilon}{32}. \tag{3.6}$$

Now taking into account (3.3) and (3.4), the Henstock-Saks Lemma applied to the McShane partitions $\mathcal{P}_m := \{(E \cap A_i, t_i) : 1 \leq i \leq m\}$, $(m \geq 1)$ and $\mathcal{P}'$ entails

$$|\langle x^*, \sigma(1_{L_\varepsilon} f, \mathcal{P}_m)\rangle - \int_{E \cap \cup_{i=1}^{i=m} A_i} \langle x^*, 1_{L_\varepsilon} f\rangle \, d\lambda| \leq \frac{\varepsilon}{16} \text{ for all } m \geq 1 \text{ and}$$

$$|\langle x^*, \sigma(f, \mathcal{P}')\rangle - \int_{E \setminus L_\varepsilon} \langle x^*, f\rangle \, d\lambda| \leq \frac{\varepsilon}{32}.$$

Whence

$$\begin{aligned}
\int_{E \cap \cup_{i=1}^{i=m} A_i} \langle x^*, 1_{L_\varepsilon} f\rangle \, d\lambda &\leq \langle x^*, \sigma(1_{L_\varepsilon} f, \mathcal{P}_m)\rangle + \frac{\varepsilon}{16} \\
&\leq \sup_{i \geq 1} \langle x^*, 1_{L_\varepsilon} f(t_i)\rangle \sum_{i=1}^{i=m} \lambda(E \cap A_i) + \frac{\varepsilon}{16} \\
&\leq \tau_\varepsilon \lambda(E \cap \cup_{i=1}^{i=m} A_i) + \frac{\varepsilon}{16} \quad \text{(by (3.5))} \\
&\leq \tau_\varepsilon \lambda(E) + \frac{\varepsilon}{16} \leq \frac{3\varepsilon}{16},
\end{aligned}$$

(because $\lambda(E) \leq \frac{\varepsilon}{8\tau_\varepsilon}$) for every $m \geq 1$ and

$$\int_{E \setminus L_\varepsilon} \langle x^*, f \rangle \, d\lambda \leq \langle x^*, \sigma(f, \mathcal{P}') \rangle + \frac{\varepsilon}{32} \leq \frac{\varepsilon}{16},$$

by (3.6). Adding the two previous inequalities, we get

$$\int_{E \cap \cup_{i=1}^{i=m} A_i} \langle x^*, 1_{L_\varepsilon} f \rangle \, d\lambda + \int_{E \setminus L_\varepsilon} \langle x^*, f \rangle \, d\lambda \leq \frac{\varepsilon}{4}.$$

This yields

$$\int_E \langle x^*, f \rangle \, d\lambda = \lim_{m \to \infty} \int_{E \cap \cup_{i=1}^{i=m} A_i} \langle x^*, 1_{L_\varepsilon} f \rangle \, d\lambda + \int_{E \setminus L_\varepsilon} \langle x^*, f \rangle \, d\lambda \leq \frac{\varepsilon}{4},$$

since $(\cup_{i=1}^{i=m} A_i)$ is an increasing sequence with union L_ε λ-a.e. At last, recalling that a measurable set is an G_σ set and that an open set is expressible as a countable union of subintervals, it is easy to check that

$$|\int_E \langle x^*, f \rangle \, d\lambda| \leq \frac{\varepsilon}{4} \text{ whenever } E \in \mathcal{L} \text{ and } \lambda(E) \leq \zeta_\varepsilon,$$

and thus $\int_E |\langle x^*, f \rangle| \, d\lambda \leq 4.\frac{\varepsilon}{4} = \varepsilon$, by a standard inequality for scalar measures (see for instance [1], p.97). Since this holds for all $x^* \in B_{X^*}$, we conclude that $\{\langle x^*, f \rangle : x^* \in \overline{B}_{X^*}\}$ is equi-continuous. In turn, by Proposition 2.4, the desired conclusion follows. $\square$

Proof of Theorem 3.1. The implication in one direction follows from Proposition 2.1 (2) and Corollary 2.2. As for the other direction, suppose that f satisfies conditions (i) and (ii) of Theorem 3.1. Condition (i) and Proposition 2.1 give that f is Dunford integrable and $(\mathcal{D})$-$\int_{[a,b]} f \, d\lambda \in X$ for every interval $[a, b] \subset [0, 1]$. Further, since f is also $\mathcal{SLM}$-tight (by (ii)), Lemma 3.2 shows that $\{\langle x^*, f \rangle : x^* \in \overline{B}_{X^*}\}$ is uniformly integrable, therefore equi-continuous. It follows that for every $\varepsilon > 0$, there is an $\eta > 0$ such that

$$\|(\mathcal{D})\text{-}\int_E f \, d\lambda\| \leq \int_E |\langle x^*, f \rangle| \, d\lambda \leq \varepsilon \text{ for all } x^* \in X^*,$$

whenever $\lambda(E) \leq \eta$. Consequently, it is possible to invoke Proposition 2B of [4], which shows that f is Pettis integrable.

Every scalarly locally τ-upper McShane-bounded function is $\mathcal{SLM}$-tight. So we have the following.

Corollary 3.1. *Let $f : [0,1] \to X$ be a function. If the following two conditions holds*
(i) f *is* $\mathcal{SM}$*-integrable*
(ii)$^+$ f *is scalarly locally* τ*-upper McShane-bounded for some* $\tau > 0$,
then f *is Pettis integrable.*

4 A convergence theorem

In this section we provide a convergence result for the scalar McShane integral analogous to the one we have for the Pettis integral (see [5] and [10]), based on the following sequential version of the $\mathcal{SLM}$-tightness.

Definition 4.1. A sequence (f_n) of functions from $[0,1]$ into X is said to be *uniformly* $\mathcal{SLM}$*-tight* if for each $\varepsilon > 0$ there exist $\tau_\varepsilon > 0$ and $\eta_\varepsilon > 0$ such that for every $x^* \in \overline{B}_{X^*}$, there exists a measurable set L_ε such that

(1) $(\langle x^*, 1_{L_\varepsilon} f_n\rangle)$ is locally τ_ε-upper McShane-bounded for each $n \geq 1$.
(2) Given any sequence (δ_n) of gauges on $[0,1]$ and any finite collection $\{I_i : 1 \leq i \leq p\}$ of non-overlapping sub-intervals of $[0,1]$ with $\sum_{i=1}^{i=p} \lambda(I_i) \leq \eta_\varepsilon$ we have

$$\liminf_{n\to\infty} \langle x^*, \sigma(f_n, \mathcal{P}_n)\rangle \leq \varepsilon,$$

for some sequence of McShane partitions $(\mathcal{P}_n)$ of $\cup_{i=1}^{i=p} I_i \setminus L_\varepsilon$ adapted to (δ_n), that is, $\mathcal{P}_n$ is subordinate to δ_n for each $n \geq 1$.

Theorem 4.1. *Let $f : [0,1] \to X$ be a scalarly integrable function. If there is a sequence (f_n) of* $\mathcal{SM}$*-integrable functions from $[0,1]$ into X such that*

(j) (f_n) *is uniformly* $\mathcal{SLM}$*-tight, and*
(jj) $\lim_{n\to\infty} \int_I \langle x^*, f_n\rangle\, d\lambda = \int_I \langle x^*, f\rangle\, d\lambda$ *for each* $x^* \in \overline{B}_{X^*}$ *and each interval* $I \subset [0,1]$,

then f *is Pettis integrable.*

Corollary 4.1. *Let $\tau > 0$ and let $f : [0,1] \to X$ be a scalarly integrable function. If there is a sequence (f_n) of* $\mathcal{SM}$*-integrable functions from $[0,1]$ into X such that*

(j)$^+$ f_n *is scalarly locally* τ*-upper McShane-bounded for each* $n \geq 1$, *and*
(jj) $\lim_{n\to\infty} \int_I \langle x^*, f_n\rangle\, d\lambda = \int_I \langle x^*, f\rangle\, d\lambda$ *for each* $x^* \in \overline{B}_{X^*}$ *and each interval* $I \subset [0,1]$,

then f is $\mathcal{SM}$-integrable.

For the proof of Theorem 4.1 we need the following lemma.

Lemma 4.1. *Let $f : [0,1] \to X$ be a scalarly integrable function. Suppose there exists a uniformly $\mathcal{SLM}$-tight sequence (f_n) of scalarly integrable functions from $[0,1]$ to X such that*

$$\lim_{n\to\infty} \int_I \langle x^*, f_n\rangle \, d\lambda = \int_I \langle x^*, f\rangle \, d\lambda$$

for each $x^* \in \overline{B}_{X^*}$ and each interval $I \subset [0,1]$. Then the set $\{\langle x^*, f\rangle : x^* \in \overline{B}_{X^*}\}$ is uniformly integrable.

Proof. Let τ_ε and η_ε be the positive real numbers corresponding to ε in Definition 4.1 and $x^* \in \overline{B}_{X^*}$ be arbitrary fixed. Then there exists a measurable set L_ε such that
(4.1) $(\langle x^*, 1_{L_\varepsilon} f_n\rangle)$ is locally τ_ε-upper McShane-bounded for each $n \geq 1$.

(4.2) Given any sequence (δ_n) of gauges on $[0,1]$ and any finite collection $\{I_i : 1 \leq i \leq p\}$ of non-overlapping sub-intervals of $[0,1]$ with $\sum_{i=1}^{i=p} \lambda(I_i) \leq \eta_\varepsilon$ we have

$$\liminf_{n\to\infty} \langle x^*, \sigma(f_n, \mathcal{P}_n)\rangle \leq \varepsilon$$

for some sequence of McShane partitions $(\mathcal{P}_n)$ of $\cup_{i=1}^{i=p} I_i \setminus L_\varepsilon$ adapted to (δ_n).

Next, as for each $n \geq 1$ the functions $\langle x^*, 1_{L_\varepsilon} f_n\rangle$ and $\langle x^*, f_n\rangle$ are Lebesgue integrable on $[0,1]$, one can find a gauge δ_n on $[0,1]$ such that

$$|\langle x^*, \sigma(1_{L_\varepsilon} f_n, \mathcal{P})\rangle - \int_{[0,1]} \langle x^*, 1_{L_\varepsilon} f_n\rangle \, d\lambda| < \frac{\varepsilon}{16} \text{ and} \tag{4.3}$$

$$|\langle x^*, \sigma(f_n, \mathcal{P})\rangle - \int_{[0,1]} \langle x^*, f_n\rangle \, d\lambda| < \frac{\varepsilon}{32} \tag{4.4}$$

for every McShane partition $\mathcal{P}$ of $[0,1]$ subordinate to δ_n. Now, according to (4.1), Lemma 3.1 provides, for each $n \geq 1$, a generalized McShane partition $\{(A_{n,i}, t_{n,i}) : i \geq 1\}$ of L_ε subordinate to δ_n such that

$$\langle x^*, 1_{L_\varepsilon} f(t_{n,i})\rangle < \tau_\varepsilon, \text{ for all } i \geq 1. \tag{4.5}$$

Next, put $\zeta_\varepsilon := \min(\frac{\varepsilon}{8\tau_\varepsilon}, \eta_\varepsilon)$ and select a measurable set E of the form $E := \cup_{i=1}^{i=p} I_i$ such that $\lambda(E) \leq \zeta_\varepsilon$, where $\{I_i : i = 1, ..., p\}$ is a finite collection of non-overlapping sub-intervals of $[0,1]$. Then, by (4.2), there exists a sequence of McShane partitions $(\mathcal{P}'_n)$ of $E \setminus L_\varepsilon$ adapted to (δ_n) such that

$$\liminf_{n\to\infty} \langle x^*, \sigma(f_n, \mathcal{P}'_n)\rangle \leq \frac{\varepsilon}{32}. \tag{4.6}$$

Now, in view of (4.3) and (4.4), the classical Henstock-Saks Lemma applied to the partial McShane partitions $\mathcal{P}_{n,m} := \{(E \cap A_{n,i}, t_{n,i}) : 1 \leq i \leq m\}$ and $\mathcal{P}'_n$ $(m,n \geq 1)$ gives

$$|\langle x^*, \sigma(1_{L_\varepsilon} f_n, \mathcal{P}_{n,m})\rangle - \int_{E\cap\cup_{i=1}^{i=m} A_{n,i}} \langle x^*, 1_{L_\varepsilon} f_n\rangle \, d\lambda| \leq \frac{\varepsilon}{16}; \text{ and}$$

$$|\langle x^*, \sigma(f_n, \mathcal{P}'_n)\rangle - \int_{E\setminus L_\varepsilon} \langle x^*, f_n\rangle \, d\lambda| \leq \frac{\varepsilon}{32}$$

for every $m, n \geq 1$. Whence,

$$\begin{aligned}
\int_{E\cap\cup_{i=1}^{i=m} A_{n,i}} \langle x^*, 1_{L_\varepsilon} f_n\rangle \, d\lambda &\leq \langle x^*, \sigma(1_{L_\varepsilon} f_n, \mathcal{P}_n^m)\rangle + \frac{\varepsilon}{16} \\
&\leq \sup_{i\geq 1} \langle x^*, 1_{L_\varepsilon} f_n(t_i)\rangle \sum_{i=1}^{i=m} \lambda(E \cap A_{n,i}) + \frac{\varepsilon}{16} \\
&\leq \tau_\varepsilon \lambda(E \cap \cup_{i=1}^{i=m} A_{n,i}) + \frac{\varepsilon}{16} \quad \text{(by (4.5))} \\
&\leq \tau_\varepsilon \lambda(E) + \frac{\varepsilon}{16} \leq \frac{3\varepsilon}{16},
\end{aligned}$$

(because $\lambda(E) \leq \frac{\varepsilon}{8\tau_\varepsilon}$) and

$$\int_{E\setminus L_\varepsilon} \langle x^*, f_n\rangle \, d\lambda \leq \langle x^*, \sigma(f_n, \mathcal{P}'_n)\rangle + \frac{\varepsilon}{32}$$

for every $m, n \geq 1$. Adding the two previous inequalities, we get

$$\int_{E\cap\cup_{i=1}^{i=m} A_{n,i}} \langle x^*, f_n\rangle \, d\lambda + \int_{E\setminus L_\varepsilon} \langle x^*, f_n\rangle \, d\lambda \leq \langle x^*, \sigma(f_n, \mathcal{P}'_n)\rangle + \frac{7\varepsilon}{32}$$

for every $m, n \geq 1$. This yields

$$\begin{aligned}
\int_E \langle x^*, f_n\rangle \, d\lambda &= \int_{E\cap L_\varepsilon} \langle x^*, f_n\rangle \, d\lambda + \int_{E\setminus L_\varepsilon} \langle x^*, f_n\rangle \, d\lambda \\
&= \lim_{m\to\infty} \int_{E\cap\cup_{i=1}^{i=m} A_{n,i}} \langle x^*, f_n\rangle \, d\lambda + \int_{E\setminus L_\varepsilon} \langle x^*, f_n\rangle \, d\lambda \\
&\leq \langle x^*, \sigma(f_n, \mathcal{P}'_n)\rangle + \frac{7\varepsilon}{32}
\end{aligned}$$

for every $n \geq 1$. Since, by hypothesis, $\lim_{n\to\infty} \int_E \langle x^*, f_n \rangle \, d\lambda = \int_E \langle x^*, f \rangle \, d\lambda$, it follows from the previous inequality and (4.6) that

$$\int_E \langle x^*, f \rangle \, d\lambda \leq \liminf_{n\to\infty} \langle x^*, \sigma(f_n, \mathcal{P}'_n) \rangle + \frac{7\varepsilon}{32} \leq \frac{\varepsilon}{4}.$$

Finally, to obtain the desired conclusion, it suffices to repeat the arguments used at the end of the proof of Lemma 3.2. □

Proof of Theorem 4.1. It is a direct consequence of Lemma 4.1 and the following version for the $\mathcal{SM}$-integral of a well known result of Geitz ([5], Theorem 3). See also ([10], Theorem 1).

Theorem 4.2. *Let $f : [0,1] \to X$ be a function satisfying the following two conditions*

(j) $\{\langle x^*, f \rangle : x^* \in \overline{B}_{X^*}\}$ is uniformly integrable.
(jj) There exists a sequence (f_n) of $\mathcal{SM}$- integrable functions from $[0,1]$ into X such that $\lim_{n\to\infty} \int_I \langle x^*, f_n \rangle \, d\lambda = \int_I \langle x^*, f \rangle \, d\lambda$ for each subinterval I of $[0,1]$.

Then f is $\mathcal{SM}$-integrable. Consequently, f is Pettis integrable.

Proof. Repeating mutatis mutandis the arguments of the proof of Theorem 3 in [5] (or Theorem 1 in [10]), we see that f is $\mathcal{SM}$-integrable, therefore Dunford integrable and $(\mathcal{D})$-$\int_{[a,b]} f \, d\lambda \in X$ for every interval $[a,b] \subset [0,1]$. Further, the set $\{\langle x^*, f \rangle : x^* \in \overline{B}_{X^*}\}$ is equi-continuous, since it is uniformly integrable (by (j)). It follows that for every $\varepsilon > 0$, there is an $\eta > 0$ such that

$$\|(\mathcal{D})\text{-}\int_E f \, d\lambda\| \leq \int_E |\langle x^*, f \rangle| \, d\lambda \leq \varepsilon \text{ for all } x^* \in X^*,$$

whenever $\lambda(E) \leq \eta$. Consequently, according to Proposition 2B of [4], f is Pettis integrable. □

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