INROADS

Real Analysis Exchange
Vol. 38(2), 2012/2013, pp. 425–430

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A SHORT PROOF OF THE EXISTENCE OF UNIVERSAL FUNCTIONS

Abstract

We present a short proof of the existence of universal functions for period-doubling and critical golden-mean circle maps for all degrees of criticality $d > 1$. The method is based on H. Epstein's Herglotz-function technique.

During the past thirty years the theory of renormalisation has become an established part of dynamical systems analysis. The celebrated Feigenbaum universal function, which satisfies the functional equation

$$f(x) = -\lambda^{-1} f(f(-\lambda x))\,, \quad \lambda > 0\,, \tag{1}$$

(with f a unimodal map of an interval containing a dth order critical point at 0 ($d > 1$) and with normalization $f(0) = 1$), governs the universal metric properties of period-doubling cascades [6, 7]. (See also [1].)

Less well known are the universal functions related to critical circle mappings [12]. For golden circle mappings (i.e., those with average rotation per iteration equal to the the golden mean $(\sqrt{5}-1)/2$), the universal functions ξ, η satisfy the functional equations

$$\xi(x) = \beta^{-1}\eta(\beta x)\,, \quad \eta(x) = \beta^{-1}\eta(\xi(\beta x))\,, \quad \beta < 0\,. \tag{2}$$

Here ξ, η are increasing functions, defined and commuting on intervals containing 0 and each with a dth order critical point at 0 ($d > 1$), and with normallization $\xi(0) = 1$.

Mathematical Reviews subject classification: Primary: 37E20, 37E05, 37E10
Key words: Feigenbaum, functional equation, Herglotz function, universality
Received by the editors April 2, 2012
Communicated by: Zbigniew Nitecki

The existence of these universal functions, which underlie the theory, has been established by a variety of methods, ranging from computer-assisted proofs (see [9, 11] and, for a more recent example, see [8]) to topological [3, 4, 5] and complex analytic methods [13, 10, 14, 15]. All these proofs are somewhat lengthy, and involve some non-trivial analysis and estimates.

In this note we present a short proof of the existence of the universal functions in both the dissipative period-doubling and critical golden-mean circle mapping cases, using the Herglotz-function approach pioneered by Epstein. Our proof uses the Schauder-Tikhonov fixed-point theorem (and therefore comes into the topological class of proofs). It is based on Epstein's elegant paper [5], but is simpler in that it does not require a two-stage construction. Although Herglotz functions (also known as Pick functions) are defined on the upper and lower half-planes in $\mathbb{C}$, they have an integral representation in terms of a real-valued measure supported on $\mathbb{R}$ so that, as will be clear below, most of the analysis occurs on the real intervals on which the functions are real-analytic.

Let $\mathbb{C}_\pm$ denote the upper and lower complex plane respectively, and for $A < 0 < 1 < B$, let $\Omega(A, B)$ denote $\mathbb{C}_+ \cup \mathbb{C}_- \cup (A, B)$. We denote by $H(A, B)$ (resp. $AH(A, B)$) the space of Herglotz (resp. anti-Herglotz) functions on $\Omega(A, B)$, i.e., the space of analytic functions $f : \Omega(A, B) \to \mathbb{C}$ such that $f(\mathbb{C}_\pm) \subseteq \bar{\mathbb{C}}_\pm$ (resp. $f(\mathbb{C}_\pm) \subseteq \bar{\mathbb{C}}_\mp$). We refer to [2] for the general theory of Herglotz/Pick funtions and to [4] for the specific function spaces considered here.

Let $E(A, B)$ denote the space of normalised anti-Herglotz functions $\psi \in AH(A, B)$ satisfying $\psi(0) = 1$, $\psi(1) = 0$. Equiping these spaces with the topology of uniform convergence on compact subsets of $\Omega(A, B)$, we have, in particular, that $E(A, B)$ is compact and convex and that, by the Schauder-Tikhonov theorem, every continuous $T : E(A, B) \to E(A, B)$ has a fixed point. This result is our main topological tool.

Of great utility are the so-called *a priori* bounds, which are derived from the Herglotz representation theorem, which is the main analytic ingredient in the proof. Recall from [4], that a function $\psi \in E(A, B)$ may be written

$$\log \psi(x) = \int \sigma(t) \left[\frac{1}{t} - \frac{1}{t - x} \right] dt \,,$$

where σ is a density function on $\mathbb{R}$ satisfying $0 \le \sigma(t) \le 1$ for all $t \in \mathbb{R}$, $\sigma(t) = 0$ for $A < t < 1$ and $\sigma(t) = 1$ for $1 < t < B$. From this representation, we may derive the inequality, for the special case $A = -\lambda^{-1}$, $B = \lambda^{-2}$, $\lambda \in (0, 1)$,

$$\psi(x) \le \frac{1 - x}{1 - \lambda^2 x} \,, \quad x \in [0, 1], \quad \psi \in E(-\lambda^{-1}, \lambda^{-2}) \,.$$

One further result we shall need is that, if $f \in H(A, B)$ (resp. $f \in AH(A, B)$), then f is real and strictly increasing (resp. decreasing) on (A, B), and in both cases f has positive Schwarzian derivative. We refer the reader to Epstein's paper [4] for these and further results.

The result that we shall prove is the following, which is a restatement of the theorem in [5].

Theorem 1. *For all $d > 1$, $\nu \in [1, 2]$, there exists $b \in (0, 1)$ and $\psi \in E(-b^{-1}, b^{-2})$ such that*

$$\psi(x) = \lambda^{-\nu d}\psi(z_1\psi(-\lambda x)^{1/d}), \quad z_1 = \psi(-\lambda)^{-1/d}, \quad \lambda^{\nu d} = \psi(z_1)\,. \tag{3}$$

Proof. Let $d > 1$, $\nu \in [1, 2]$ be fixed. Let $b \in (0, 1)$ be the unique solution of the equation $b^{\nu d} = (1 - b^{\nu})/(1 - b^{2+\nu})$. It is straightforward to establish that such a solution exists, is unique, and that if $\lambda \in (0, 1)$ satisfies $\lambda^{\nu d} < (1 - \lambda^{\nu})/(1 - \lambda^{2+\nu})$, then $\lambda < b$.

Our method is to define a continuous map $T : E(-b^{-1}, b^{-2}) \to E(-b^{-1}, b^{-2})$ such that a fixed point of T is a solution of (3).

We define T by $T(\psi)(x) = \tau^{-1}\psi(\phi(x))$ where $\tau = \psi(\phi(0))$, $\phi(x) = z_1\psi(-\lambda x)^{1/d}$, $z_1 = \psi(-\lambda)^{-1/d}$, and $\lambda \in (0, 1)$ is defined by

$$\begin{aligned}\lambda = \sup\{x \mid 0 < x \leq b\,, x \leq \psi(\psi(-x)^{-1/d})^{1/(d\nu)}\,,\\ \text{and } \psi(-xb^{-2})^{1/d}/\psi(-x)^{1/d} \leq b^{-2}\}\,.\end{aligned} \tag{4}$$

We note that $f(x) = \psi(-xb^{-2})^{1/d}/\psi(-x)^{1/d}$ is an increasing function of x (see Appendix below), and that the graph of the function $\psi(\psi(-x)^{-1/d})^{1/(d\nu)}$ crosses the diagonal at a unique fixed point in $(0, 1)$. (For the right-hand side is Herglotz, and thus has positive Schwarzian derivative, $f(0) = 0, f'(0+) = \infty$, and $f(1) < 1$.) Hence we have that $\lambda(\psi)$ is a continuous function of ψ, and, by standard properties of composition operators, that T is also continuous. Moreover, we have $T(\psi)(0) = 1$, $T(\psi)(1) = 0$, so that T is a continuous map from $E = E(-b^{-1}, b^{-2})$ to itself. Applying the Schauder-Tikhonov fixed-point theorem, we obtain $\psi \in E$ with $T(\psi) = \psi$. The fixed-point ψ satisfies

$$\begin{aligned}&\psi(x) = \tau^{-1}\psi(\phi(x)) \text{ for } x \in (-b^{-1}, b^{-2})\,, \quad \phi(x) = z_1\psi(-\lambda x)^{1/d}\,,\\ &\tau = \psi(\psi(-\lambda)^{-1/d})\,.\end{aligned} \tag{5}$$

Our aim is now to show that, for this fixed-point ψ, we have the equation $\lambda = \psi(\psi(-\lambda)^{-1/d})^{1/(d\nu)}$, so that (3) is solved.

Our first observation is that, since $\psi \in E(-b^{-1}, b^{-2})$ and the function $\phi(x) = z_1\psi(-\lambda x)^{-1/d}$, we have that $\phi \in H(-\lambda^{-1}, \lambda^{-1}b^{-1})$ and $\phi((-\lambda^{-1}, 1)) \subseteq$

$(0,1)$ so that, using (5), ψ may be extended to $(-\lambda^{-1}, b^{-2})$ and, from the fixed-point equation (5), we have $\psi(x) < \tau^{-1}$ for $x \in (-\lambda^{-1}, 1)$.

Now, we know that the equation $x = f(x) = \psi(\psi(-x)^{-1/d})^{1/(d\nu)}$ has a unique solution in $(0,1)$. Hence, we need only show that λ, defined by (4), satisfies $\lambda < b$ and $\phi(b^{-2}) = \psi(-\lambda b^{-2})^{1/d}/\psi(-\lambda)^{1/d} < b^{-2}$.

Suppose that $\lambda^{\nu d} < \psi(\psi(-\lambda)^{-1/d}) = \tau$. Then since $-\lambda \in (-\lambda^{-1}, 1)$, we have $\psi(-\lambda) < \tau^{-1} < \lambda^{-\nu d}$, so that $z_1 > \lambda^{\nu}$. Suppose now that $\lambda = b$. Then $\psi \in E(-\lambda^{-1}, \lambda^{-2})$ so, by the *a priori* bounds, we have

$$\lambda^{\nu d} = \psi(z_1) < \psi(\lambda^{\nu}) \leq \frac{1-\lambda^{\nu}}{1-\lambda^{2+\nu}}\,,$$

so that $\lambda < b$, a contradiction. We therefore conclude that $\lambda < b$. Now since $\psi \in E(-\lambda^{-1}, \lambda^{-1}b^{-1})$, we may extend ϕ so that $\phi \in H(-\lambda^{-1}, \lambda^{-2})$. Moreover, we have $\phi(\lambda^{-2}) = z_1\psi(-\lambda^{-1})^{1/d} < \tau^{-1} < \lambda^{-\nu} \leq \lambda^{-2}$. Furthermore, ϕ is Herglotz (and thus has positive Schwarzian derivative), $\phi(-\lambda^{-1}) = 0$, $\phi(1) = 1$, and $1 < b^{-2} < \lambda^{-2}$ so it follows that $\phi(b^{-2}) < b^{-2}$. The proof is complete. □

Following [5], we now give an outline of the construction of the universal functions satisfying equations (1) and (2) from a solution of (3) given by Theorem 1. Let $\psi \in E(-\lambda^{-1}, \lambda^{-2})$ satisfy (3) for a fixed $d > 1$ and $\nu \in [1,2]$. For the construction of the universal functions we shall take $\nu = 1$ for equation (1) and $\nu = 2$ for equation (2).

We now define $U \in AH(-\lambda^{-1}, \lambda^{-2})$ by $\lambda^{(1-\nu)d} z_1^d \psi$. Then U satisfies the functional equation

$$U(x) = \lambda^{-\nu d} U\left(\lambda^{\nu-1} U\left(-\lambda x\right)^{1/d}\right)\,,$$

valid for $x \in (-\lambda^{-1}, \lambda^{-2})$. Then, taking limits where necessary, we have $U(-\lambda^{-1}) = \lambda^{-\nu d}U(0)$ and $U(\lambda^{-2}) = \lambda^{-\nu d}U(z_1\lambda^{-\nu}) < 0$ since $z_1 > \lambda^{\nu}$ and $U(x) < 0$ for $1 < x < \lambda^{-2}$. Since U is strictly decreasing on $(-\lambda^{-1}, \lambda^{-2})$, with $U(1) = 0$, we may invert U to give a strictly decreasing, negative schwarzian, real-analytic function F defined on the interval $(U(\lambda^{-2}), U(-\lambda^{-1}))$ containing 0, and satisfying the functional equation

$$F(x) = -\lambda^{-1} F\left(\lambda^{(1-\nu)d} F\left(\lambda^{\nu d} x\right)^{d}\right)\,,$$

with normalization $F(0) = 1$.

In the case $\nu = 1$ (corresponding to period-doubling), we write $f(x) = F(|x|^d)$, which gives an even unimodal function with a degree-d critical point at 0 satisfying equation (1). This function is real-analytic when d is an even integer. In the case $\nu = 2$, we define $\xi(x) = F(-x|x|^{d-1})$, $\beta = -\lambda$, so that ξ satisfies the functional equation

$$\xi(x) = \beta^{-1}\xi\left(\beta^{-1}\xi\left(\beta^2 x\right)\right) .$$

Writing $\eta(x) = \beta\xi(\beta^{-1}x)$, it follows that ξ and η are both increasing functions defined on intervals containing 0 and satisfying (2), each with a degree-d critical point at 0. That ξ and η commute on an interval around 0 may be established by a more detailed analysis of the functional equations.

A Appendix.

To show that $\psi(-xb^{-2})^{1/d}/\psi(-x)^{1/d}$ is increasing it is clearly sufficient to show that $(d/dx)(\log\psi(-xb^{-2}) - \log\psi(-x)) > 0$ for $0 < x < b$. From the integral representation, we have, for $0 < x < b$,

$$\begin{aligned}
&\frac{d}{dx}(\log\psi(-xb^{-2}) - \log\psi(-x)) = \int \sigma(t)\left[\frac{b^{-2}}{(t+xb^{-2})^2} - \frac{1}{(t+x)^2}\right] dt \\
&= \int \frac{\sigma(t)}{(t+x)^2(t+xb^{-2})^2}\left[(b^{-2}-1)(t^2 - x^2b^{-2})\right] dt \\
&> 0\,,
\end{aligned}$$

since $b^{-2} > 1$, $x^2b^{-2} < 1$, and $\sigma(t) = 0$ for $|t| < 1$.

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