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A GENERALIZED MAXIMUM PRINCIPLE FOR CONVOLUTION OPERATORS IN BOUNDED REGIONS

Abstract

Dealing with the technically motivated concept of convolution operators in bounded regions of $\mathbb{R}^N$ with an underlying nearby boundary condition we extend a formerly proved result about the existence and uniqueness of suitable solutions for dimension $N \leq 2$ to arbitrary dimensions N. Thus, a first substantial result in a sufficiently generalized form, beyond the very specific case of rectangular regions, is established in this field. The result can also be seen as a generalized maximum principle for so called k-harmonic functions where k is the kernel of the given convolution operator.

1 Introduction

In [1] a specific type of Dirichlet problems for convolution operators in bounded regions was introduced. There, especially for kernels in certain Sobolev-spaces, which can be seen as kernels of generalized smoothing operators, substantial results for the very specific case of rectangular regions were proved. Moreover, due to [2], in [1] a generalization of these results for rectangular regions to general regions was mentioned, but only for dimension 1 and 2, and the proof in [2] *cannot* be extended to higher dimensions by slight modification. So, the result for arbitrary dimension was unproved and remained as an unsolved problem.

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In this paper we will give a proof for arbitrary dimensions $N \in \mathbb{N}$. Thus, a first substantial result in a sufficiently generalized form, beyond the very specific case of rectangulars, is established in this field.

There are two possible views on the presented theory. First, the concept is technically motivated by a measurement of a field f in the *interior* of some bounded region Ω with the aid of a sensor ω moving around in Ω such that the motion is completely *inside* Ω. More precisely, the measurement will be a weighted measurement, done with the aid of a weight-function k defined on ω. The task is to detect the original field f with the additional knowledge of the (maybe disturbed) values of the field near the boundary of the region Ω.

Another, more mathematical view to the theory, is dealing with the set of so called k-harmonic functions $f \in L^p(\Omega)$ which are defined by $T_k f = 0$ for a convolution operator T_k with convolution kernel k working *inside* Ω. The question is, if there exists some kind of maximum principle, which means that any such k-harmonic function can be controlled by their values near the boundary of Ω.

However, both views are essentially equivalent and the result of this paper gives an understanding of both topics.

2 Basics

In the following we use the notation of [1]. For a better understanding, we briefly introduce the relevant terms which are used throughout the paper.

All functions in the text have values in $\mathbb{K}$ where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Further, let $N \in \mathbb{N}$ and $\omega :=]0,1[^N$. (In [1] general, nonempty bounded regions ω were discussed, but since our main result in the present paper only deals with the case $\omega =]0,1[^N$ we restrict ourselves to this case.)

Let $\Omega \subseteq \mathbb{R}^N$ be a nonempty bounded region (i.e. open set) such that

$$\Omega_\omega := \{x \in \mathbb{R}^N |\ x + \overline{\omega} \subseteq \Omega\}$$

is *not empty.* Note that Ω_ω is open and $\Omega_\omega + \omega \subseteq \Omega$.

For $k \in L^1(\omega)$, $k \not\equiv 0$, and $p \in [1, \infty]$ define the convolution operator

$$T_k : L^p(\Omega) \to L^p(\Omega_\omega)$$

by

$$T_k f(x) := \int_{x+\omega} k(t-x) f(t) dt,\ x \in \Omega_\omega.$$

Note that T_k is a linear bounded operator with kernel k. Moreover, T_k is not injective (cf. [3, p. 140], [2, p. 14], [1, p. 177]).

Furthermore, let

$$\partial_\omega\Omega := \Omega \cap \bigcup_{x \notin \Omega_\omega} (x+\omega) = \Omega \ \cap \bigcup_{x/x+\overline{\omega} \nsubseteq \Omega} (x+\omega).$$

This means that $\partial_\omega\Omega$ is the intersection of Ω with all translates of ω which closures are not completely inside Ω. For example, if Ω is a rectangular region (or cuboid) $\prod_{i=1}^{N}]a_i, b_i[$ with a,b $\in \mathbb{R}^N$, $|b_i - a_i| > 2$ then

$$\partial_\omega\Omega := \Omega \setminus \prod_{i=1}^{N} [a_i+1, b_i-1].$$

Note that $\partial\Omega \subseteq \overline{\partial_\omega\Omega}$ (cf. Lemma 4), thus $\partial_\omega\Omega$ can be seen as a specific kind of an extended N-dimensional boundary of Ω near the regular boundary $\partial\Omega$. Clearly $\partial_\omega\Omega$ is an open nonempty subset of Ω.

Note also that the "boundary" $\partial_\omega\Omega$ is a suitable set for fixing solutions of the convolution equation "$T_k f = g$", which means that for any given function $g \in L^p(\Omega_\omega)$ and $f_0 \in L^p(\partial_\omega\Omega)$ there exists at most one solution $f \in L^p(\Omega)$ such that $T_k f = g$ and $f_{|\partial_\omega\Omega} = f_0$ (cf. [1, p. 179]) due to Titchmarsh's convolution theorem (e.g. [4, p. 107]).

Nevertheless, in general there does not *exist* an exact solution $f \in L^p(\Omega)$ such that $T_k f = g$ and $f_{|\partial_\omega\Omega} = f_0$ and therefore, we are looking in such cases for best approximating solutions.

For this, and for more generality, let U be a measurable subset of Ω of positive N-dimensional measure. Assuming $f_0 \in L^p(U)$ and $g \in L^p(\Omega_\omega)$, we call $f \in L^p(\Omega)$ a *best approximation solution of ($T_k f = g$, $f_{|U} = f_0$)* if

$$T_k f = g$$

and

$$||f_{|U} - f_0|| \le ||h_{|U} - f_0||$$

for any $h \in L^p(\Omega)$ with $T_k h = g$.

Defining

$$\mathcal{N}_k := \{f \in L^p(\Omega) | T_k f = 0\},$$

$$\mathcal{N}_{k,U} := \{f_{|U} | f \in \mathcal{N}_k\}$$

and the trace operator

$$R_{k,U} : \mathcal{N}_k \to \mathcal{N}_{k,U}$$

by

$$R_{k,U} f := f_{|U}$$

and assuming $1 < p < \infty$, the existence of a unique best approximating solution of ($T_k f = g$, $f_{|U} = f_0$) for each $f_0 \in L^p(U)$ and each $g \in T_k(L^p(\Omega))$ is equivalent to the existence of a bounded inverse of the trace operator $R_{k,U}$ (cf. [1, p. 180]). Note that in this case the best approximating solution depends continuously on the boundary value $f_0 \in L^p(U)$ (cf. [1, p. 181]). In particular, if $T_k f = g$, $f_{|U} = f_0$ and $(h_n)_n \subseteq L^p(U)$ with $h_n \to f_0$ then the best aproximating solutions of ($T_k f = g$, $f_{|U} = h_n$) converges to f, which is the exact solution of ($T_k f = g$, $f_{|U} = f_0$) (and therefore is the best approximating solution of ($T_k f = g$, $f_{|U} = f_0$)).

Functions in $\mathcal{N}_k$ we call also *k−harmonic* functions. Obviously, the trace operator $R_{k,U}$ is bounded invertible if and only if there exists $C > 0$ such that for all k-harmonic functions $f \in L^p(\Omega)$ the relation $||f|| < C \cdot ||f_{|U}||$ holds. This relation we call also the *maximum principle for k-harmonic functions with respect to the boundary U*.

In the following we restrict ourselves to special kernel functions k in an appropriate Sobolev-space. For any arbitrary nonempty region G in $\mathbb{R}^N$ let

$$S_p^{\vec{1}}W(G) := \{f \in L^p(G) | D^\alpha f \in L^p(G), \alpha \in \{0,1\}^N\}$$

(endowed with the usual norm for Sobolev-spaces) and herewith

$$S_{1,1}^{\vec{1}}W(\omega) := \{f \in L^1(\omega) | f - 1 \in S_{1,0}^{\vec{1}}W(\omega)\}$$

where $S_{1,0}^{\vec{1}}W(\omega)$ is the closure of $C_0^\infty(\omega)$ in $S_1^{\vec{1}}W(\omega)$ (cf. [1, p. 181-182]).

Spaces of these types are called Nikol'skij-Sobolev spaces and were firstly introduced by Nikol'skij (cf. [5, 6]). By standard methods (e.g. [7] or [2, p. 28]) it can be seen that $S_1^{\vec{1}}W(\omega)$ is boundedly imbedded into $C(\overline{\omega})$. Thus, any $k \in S_{1,1}^{\vec{1}}W(\omega)$ coincides a.e. with a function $k' \in C(\overline{\omega})$ with value 1 on the boundary of ω. By this definition T_k can be seen as a generalized or disturbed version of the smoothing operator T_{k_0} with kernel $k_0 \equiv 1$.

Note that for $k \in S_{1,1}^{\vec{1}}W(\omega)$ the operator T_k maps $L^p(\Omega)$ boundedly into $S_p^{\vec{1}}W(\Omega_\omega)$ (cf. [1, p. 183]). Moreover, the range of T_k consists of all functions $g \in S_p^{\vec{1}}W(\Omega_\omega)$ where there exists an extension $\overline{g} \in S_p^{\vec{1}}W(\mathbb{R}^N)$ of g (cf. [1, p. 192]). Note also that for $1 < p < \infty$ Džabrailov constructed a class of domains G for which there exists a (linear and bounded) extension operator

$L : S_p^{\vec{1}}W(G) \to S_p^{\vec{1}}W(\mathbb{R}^N)$ (cf. [8, p. 192], for examples see also [2, p. 68]). Therefore, if Ω_ω is in this class, the range of T_k is even equal to $S_p^{\vec{1}}W(\Omega_\omega)$.

Finally, for the case of cuboids, another "boundary" plays an important role: If $Q = \prod_{i=1}^N]a_i, b_i[$ with a,b $\in \mathbb{R}^N$, $|b_i - a_i| > 1$, let

$$\partial_\omega^l Q := Q \setminus \prod_{i=1}^N [a_i + 1, b_i[.$$

This notation follows the idea as $\partial_\omega^l Q$ were the *left* part of $\partial_\omega Q$ in case of rectangular regions Q $(= \Omega)$.

The importance of $\partial_\omega^l Q$ is given by the result that in case of rectangular regions Q mentioned above for $k \in S_{1,1}^{\vec{1}}W(\omega)$ the trace operator $R_{k,\partial_\omega^l Q}$ is boundedly invertible (cf. [1, p. 192]).

3 Main result

The following theorem is a result of [1, p. 192] resp. [2]. But there, the result was only proved for dimension ≤ 2. The validity for any dimension remained as an unsolved problem. The proof in [2] uses a covering of $\Omega \setminus U$ by suitable subsets of Ω. But the used direct construction method for the covering is not applicable for higher dimensions because of the unmanageable complexity of Ω.

It is nearby to consider a technique by induction, but there, it is the difficulty to find a method to get a covering of $\Omega \setminus U$ in the next dimension on the basis of supposed coverings in lower dimensions at all. Moreover, it is necessary to ensure that the covering of $\Omega \setminus U$ fulfilled all required properties. In the proof in this paper we solve these problems by *sharpening* the required properties, but for which an approach by induction is possible. So, proving the stronger properties, we get the original properties which finally leads to the validity of the theorem.

Theorem 1. *Let $k \in S_{1,1}^{\vec{1}}W(\omega)$ and U a measurable subset of Ω such that*

$$\overline{\Omega \setminus U} \cap \overline{\partial_\omega \Omega} = \emptyset.$$

Then there exists $C > 0$ such that for any k-harmonic function $f \in L^p(\Omega)$

$$||f|| < C \cdot ||f_{|U}||.$$

Thus, for k-harmonic functions $f \in L^p(\Omega)$ the maximum principle with respect to the boundary U does hold.

Theorem 2. *Let $k \in S^{\vec{1}}_{1,1}W(\omega)$ and U a measurable subset of Ω such that*

$$\overline{\Omega \setminus U} \cap \overline{\partial_\omega \Omega} = \emptyset$$

and let furthermore $1 < p < \infty$. Then there exists for each $f_0 \in L^p(U)$ and each $g \in T_k(L^p(\Omega))$ a best approximating solution of ($T_k f = g$, $f_{|U} = f_0$). The best approximating solution depends continuously on the boundary value $f_0 \in L^p(U)$.

Theorem 2 follows immediately by Theorem 1 since under the given conditions the trace operator $R_{k,U}$ is then bounded invertible. So, it is sufficient to prove Theorem 1.

Remark 1. *If U is an open subset of Ω then*

$$\overline{\Omega \setminus U} \cap \overline{\partial_\omega \Omega} = \emptyset$$

is equivalent to

$$\overline{\partial_\omega \Omega} \subseteq U$$

(relative closure in Ω).

Remark 1 was proved in [2, p. 48], for completeness we give the proof here.

Proof of Remark 1. Let $U \subseteq \Omega$ be open.

a) Let $\overline{\partial_\omega \Omega} \subseteq U$. We show that $(\Omega \setminus U)$ is closed. Herewith, we get

$$\begin{aligned} \overline{\partial_\omega \Omega} \cap \overline{\Omega \setminus U} &= \overline{\partial_\omega \Omega} \cap (\Omega \setminus U) = \overline{\partial_\omega \Omega} \cap (\Omega \cap \complement U) \\ &= (\overline{\partial_\omega \Omega} \cap \Omega) \cap \complement U \subseteq U \cap \complement U = \emptyset. \end{aligned}$$

The closedness of $(\Omega \setminus U)$ can be seen as follows. We have $\complement(\Omega \setminus U) = \complement\Omega \cup U = (\complement\overline{\Omega} \cup \partial\Omega) \cup U = (\complement\overline{\Omega} \cup U) \cup \partial\Omega$. First $(\complement\overline{\Omega} \cup U)$ is open. But let $x \in \partial\Omega$, w any point in ω and $z := x - w$. Then we have $x \in z + \omega$ and there exists $\epsilon > 0$ such that $B_\epsilon(x) \subseteq z + \omega$. Since $x \notin \Omega$ we have $(z + \overline{\omega}) \not\subseteq \Omega$, hence $z \notin \Omega_\omega$ and therefore

$$B_\epsilon(x) \subseteq z + \omega \subseteq \bigcup_{y \notin \Omega_\omega} (y + \omega).$$

Thus

$$\Omega \cap B_\epsilon(x) \subseteq \Omega \cap \bigcup_{y \notin \Omega_\omega} (y + \omega) = \partial_\omega \Omega.$$

This implies $\Omega \cap B_\epsilon(x) \subseteq \Omega \cap \overline{\partial_\omega \Omega} \subseteq U$. Finally, there holds

$$B_\epsilon(x) = (B_\epsilon(x) \cap \Omega) \cup (B_\epsilon(x) \cap \complement\Omega) \subseteq U \cup \complement\Omega = \complement(\Omega \setminus U).$$

Thus $\complement(\Omega \setminus U))$ is open.

b) $\overline{\Omega \setminus U} \cap \overline{\partial_\omega \Omega} = \emptyset \Rightarrow (\Omega \setminus U) \cap \overline{\partial_\omega \Omega} = \emptyset \Rightarrow (\Omega \cap \overline{\partial_\omega \Omega}) \cap \complement U = \emptyset \Rightarrow$

$$\Omega \cap \overline{\partial_\omega \Omega} \subseteq U.$$

□

As mentioned above Theorem 1 was proved in ([2, p. 46-55]) only for dimension $N \leq 2$ and there, moreover, with the additional restriction $\Omega_\omega + \omega = \Omega$. Since we work without this additional restriction we have to prove Theorem 1 also for dimension 1 (and then of course for all other dimensions).

As in ([2]) we use the following definition.

Definition 1. *A subset U of Ω is called cuboid-regular if there exist finitly many cuboids $Q_i \subseteq \Omega,\ Q_i = \prod_{j=1}^{N}]a_{i,j}, b_{i,j}[$ with $|b_{i,j} - a_{i,j}| > 1$, $i \in \{1, \ldots, M\}$ such that*

$$\Omega \setminus U \subseteq \bigcup_{n=1}^{M} \overline{Q_n}$$

and for all $i \in \{1, \ldots, M\}$

$$\partial_\omega^l Q_i \cap (\Omega \setminus U) \subseteq \bigcup_{n=0}^{i-1} \overline{Q_n},\ (Q_0 := \emptyset).$$

Lemma 1. *Let $k \in S^{\vec{1}}_{1,1} W(\omega)$ and U a cuboid-regular measurable subset of Ω. Then there exists $C > 0$ such that for any k-harmonic function $f \in L^p(\Omega)$*

$$||f|| < C \cdot ||f_{|U}||.$$

The proof of Lemma 1 follows [2, p. 55], but we use here a technique that aims directly at the maximum principle.

PROOF OF LEMMA 1. Let $(Q_i)_{i \in \{1,\ldots,M\}}$ the finite family of cuboids mentioned in Definition 1. Thus, there exists $C_1, \ldots, C_M > 0$ such that

$$||f_{|Q_i}|| \leq C_i \cdot ||f_{|\partial_\omega^l Q_i}||$$

for all $f \in L^p(\Omega)$ with $T_k f = 0$.

Let

$$\Omega_i := U \cup \bigcup_{j=1}^{i} Q_i,\ i \in \{1, \ldots, M\}.$$

We prove by induction

$$||f_{|\Omega_i}|| \leq B_i \cdot ||f_{|U}||$$

for all $f \in L^p(\Omega)$ with $T_k f = 0$, with certain $B_i > 0$.

For $i = 1$ we have

$$\begin{aligned} ||f_{|U \cup Q_1}|| &\leq ||f_{|U}|| + ||f_{|Q_1}|| \\ &\leq ||f_{|U}|| + C_1 ||f_{|\partial^l_\omega Q_1}|| \leq B_1 ||f_{|U}|| \end{aligned}$$

where $B_1 := 1 + C_1$.

Now, suppose for $i_0, 1 \leq i_0 < M$

$$||f_{|\Omega_{i_0}}|| \leq B_{i_0} \cdot ||f_{|U}||$$

for all $f \in L^p(\Omega)$ with $T_k f = 0$. Then we get for any such f

$$||f_{|\Omega_{i_0+1}}|| \leq ||f_{|\Omega_{i_0}}|| + ||f_{|Q_{i_0+1}}|| \leq ||f_{|\Omega_{i_0}}|| + C_{i_0+1} ||f_{|\partial^l_\omega Q_{i_0+1}}||.$$

Since

$$\partial^l_\omega Q_{i_0+1} = (\partial^l_\omega Q_{i_0+1} \cap (\Omega \setminus U)) \cup (\partial^l_\omega Q_{i_0+1} \cap U) \subseteq \bigcup_{i=1}^{i_0} (\overline{Q_i} \cap \Omega) \cup U$$

we have

$$\begin{aligned} &||f_{|\Omega_{i_0+1}}|| \\ &\leq ||f_{|\Omega_{i_0}}|| + C_{i_0+1} ||f_{|\bigcup_{i=1}^{i_0} (\overline{Q_i} \cap \Omega) \cup U}|| \\ &= ||f_{|\Omega_{i_0}}|| + C_{i_0+1} ||f_{|\Omega_{i_0}}|| \\ &\leq B_{i_0} (1 + C_{i_0+1}) ||f_{|U}||. \end{aligned}$$

Hence $B_{i_0+1} := B_{i_0}(1 + C_{i_0+1})$ satisfies the required condition. By this way there exists $B_M > 0$ such that for all $f \in L^p(\Omega)$ with $T_k f = 0$

$$||f_{|\Omega_M}|| \leq B_M \cdot ||f_{|U}||.$$

By assumption we have $\Omega \subseteq \bigcup_{n=1}^M \overline{Q_n} \cup U$ which finally implies

$$||f_{|\Omega}|| = ||f_{|\Omega_M}|| \leq B_M \cdot ||f_{|U}||$$

for all $f \in L^p(\Omega)$ with $T_k f = 0$.

$\square$

Theorem 1 is now proved by

Lemma 2. *Every subset U of Ω with*

$$\overline{\Omega \setminus U} \cap \overline{\partial_\omega \Omega} = \emptyset$$

is cuboid-regular.

The main result of this paper is the proof of this Lemma for all dimensions N. It remains in [2] as an open problem. The used technique in [2] cannot be extended to arbitrary dimensions. In the present paper we use a newly developed induction method. Instead of proving the cuboid-regularity of U directly, we prove an even *stronger property* of U for which this induction method is applicable. Herewith, we get the validity of Lemma 2.

So, we will prove the following result.

Lemma 3. *Let $U \subseteq \Omega, U \neq \Omega$ with*

$$\overline{\Omega \setminus U} \cap \overline{\partial_\omega \Omega} = \emptyset.$$

Then for any $\epsilon > 0$ there exist finitly many cuboids $Q_i \subseteq \Omega,\ Q_i = \prod_{j=1}^{N}]a_{i,j}, b_{i,j}[$ with $|b_{i,j} - a_{i,j}| > 1$, $i \in \{1, \ldots, M\}$ such that

$$\Omega \setminus U \subseteq \bigcup_{n=1}^{M} \overline{Q_n}$$

and for all $i \in \{1, \ldots, M\}$

$$\partial_\omega^l Q_i \cap (\Omega \setminus U) \subseteq \bigcup_{n=0}^{i-1} \overline{Q_n},\ \ (Q_0 := \emptyset)$$

and with $R_i := \prod_{j=1}^{N} [a_{i,j} + 1, b_{i,j}[$

$$\overline{R_i} \cap (\overline{\Omega \setminus U}) \neq \emptyset$$

and

$$\text{length}(R_i) < \epsilon$$

where $\text{length}(R_i) := \max\{|b_{i,j} - (a_{i,j} + 1)| :\ j \in \{1, \ldots, N\}\}$.

Clearly, Lemma 2 holds if Lemma 3 is true, at least for the case $U \neq \Omega$. But for $U = \Omega$ Lemma 2 is trivially true: Because of $\Omega_\omega \neq \emptyset$ there exists some $x \in \mathbb{R}^N$ with $x + \overline{\omega} \subseteq \Omega$. Since Ω is open, there exists also a cuboid $Q = \prod_{i=1}^{N}]a_i, b_i[$ with $|b_i - a_i| > 1$ and $x + \overline{\omega} \subseteq Q \subseteq \Omega$. Q satisfies the required conditions in this case.

PROOF OF LEMMA 3. (Hint: The proof is quite elementary and very technically already for dimension $N = 1$. Nevertheless, particularly for dimension $N > 1$, the proof is of complex nature and nontrivial.)

Case $N = 1$: Let $U \subseteq \Omega$, $U \neq \Omega$, $\overline{\Omega \setminus U} \cap \overline{\partial_\omega \Omega} = \emptyset$ and $\epsilon > 0$. For $x \in \Omega$ define

$$J_x :=]x_l, x_r[$$

with

$$x_l := \inf\{y \mid (y < x) \wedge]y, x] \subseteq \Omega\},$$

$$x_r := \sup\{y \mid (y > x) \wedge [x, y[\subseteq \Omega\}.$$

We have $J_x \subseteq \Omega$ and for $x \neq y$ either $J_x = J_y$ or $J_x \cap J_y = \emptyset$. Further, we have

$$\Omega = \bigcup_{x \in \Omega} J_x$$

$$= \bigcup_{\substack{x \in \Omega \\ |J_x| > 1}} J_x \;\cup \bigcup_{\substack{x \in \Omega \\ |J_x| \leq 1}} J_x .$$

For $x_0 \in \Omega$ with $J_{x_0} =]x_{0l}, x_{0r}[$ and $|J_{x_0}| \leq 1$ we have $J_{x_0} \subseteq x_{0l} +]0, 1[$. Clearly $x_{0l} \notin \Omega$, hence $x_{0l} + \overline{\omega} \not\subseteq \Omega$, but $J_{x_0} \subseteq (x_{0l} + \omega) \cap \Omega$. Thus $J_{x_0} \subseteq \partial_\omega \Omega$.

Consequently

$$\widetilde{\Omega} := \bigcup_{\substack{x \in \Omega \\ |J_x| \leq 1}} J_x \subseteq \partial_\omega \Omega.$$

Let moreover

$$\widehat{\Omega} := \bigcup_{\substack{x \in \Omega \\ |J_x| > 1}} J_x .$$

Since $\Omega_\omega \neq \emptyset$ there exists at least one $x \in \Omega$ where $|J_x| > 1$, so, $\widehat{\Omega}$ is not empty.

For $\xi \in \widehat{\Omega}$ there holds $J_\xi \subseteq \widehat{\Omega}$, because of $\xi \in J_x$ for some $x \in \Omega$ with $|J_x| > 1$, and since $\xi \in J_\xi \cap J_x \neq \emptyset$ there holds $J_\xi = J_x \subseteq \widehat{\Omega}$. Furthermore $\widehat{\Omega} \subseteq \Omega$ implies that $\widehat{\Omega}$ is bounded.

Now, we construct a (finite) sequence $(x_k)_k \subseteq \widehat{\Omega}$ as follows: Choose any $x_1 \in \widehat{\Omega}$. If $\widehat{\Omega} \setminus J_{x_1} \neq \emptyset$ choose any $x_2 \in \widehat{\Omega} \setminus J_{x_1}$. Then we have $J_{x_1} \neq J_{x_2}$ therefore $J_{x_1} \cap J_{x_2} = \emptyset$. If $\widehat{\Omega} \setminus (J_{x_1} \cup J_{x_2}) \neq \emptyset$ choose any $x_3 \in \widehat{\Omega} \setminus (J_{x_1} \cup J_{x_2})$. Then we have $J_{x_1} \neq J_{x_3}$ and $J_{x_2} \neq J_{x_3}$ therefore $J_{x_1} \cap J_{x_3} = \emptyset$ and $J_{x_2} \cap J_{x_3} = \emptyset$. By this we get a sequence $(J_{x_k})_k \subseteq \widehat{\Omega}$ with pairwise disjoint J_{x_k}. Since $\widehat{\Omega}$

is bounded the construction of the J_{x_k} must abort. This implies $\widehat{\Omega} = \bigcup_{i=1}^{n} I_i$ with pairwise disjoint intervals $I_i =]a_i, b_i[,\ b_i - a_i > 1$. Herewith, we get

$$\Omega = \bigcup_{i=1}^{n}]a_i, b_i[\ \cup\ \widetilde{\Omega}$$

and

$$\partial_\omega \Omega = \bigcup_{i=1}^{n} (]a_i, a_i + 1[\cup]b_i - 1, b_i[) \ \cup\ \widetilde{\Omega}.$$

Further $\overline{\Omega \setminus U} \cap \overline{\partial_\omega \Omega} = \emptyset$ implies $(\Omega \setminus U) \cap \widetilde{\Omega} = \emptyset$ hence $\Omega \setminus U \subseteq \widehat{\Omega}$.

Now, let be $I_{i_k}, k \in \{1, \ldots, s\}, i_k \in \{1, \ldots, n\}$ those ascending ordered intervals I_i for which $I_i \cap (\Omega \setminus U) \neq \emptyset$ holds. Because of $U \neq \Omega$ there exists at least one such interval. Let $\Lambda_k := I_{i_k}$, hence $\Lambda_k =]\alpha_k, \beta_k[$ with $\beta_k - \alpha_k > 1$, $k \in \{1, \ldots, s\}$ and $(\Lambda_k)_k$ pairwise disjoint, ascending ordered.

For $k \in \{1, \ldots, s\}$ choose $N_k \in \mathbb{N}$ such that

$$\delta_k := \frac{\beta_k - (\alpha_k + 1)}{N_k} < \epsilon$$

hence $\alpha_k + 1 + N_k \cdot \delta_k = \beta_k$.

For fixed $k \in \{1, \ldots, s\}$ define now

$$D_j^k :=]\alpha_k + (j-1) \cdot \delta_k,\ \alpha_k + 1 + j \cdot \delta_k[,\ j \in \{1, \ldots, N_k\}.$$

Moreover, let be $D^k_{j_m^{(k)}}$ (with $m \in \{1, \ldots, r_k\}$ and $j_m^{(k)} \in \{1, \ldots, N_k\}$) those ascending ordered intervals D_j^k for which

$$V_j^k \cap (\Omega \setminus U) \neq \emptyset$$

where

$$V_j^k := [\alpha_k + 1 + (j-1) \cdot \delta_k,\ \alpha_k + 1 + j \cdot \delta_k[$$

holds. Such V_j^k exists. This is because since $\Lambda_k \cap (\Omega \setminus U) \neq \emptyset$, there exists some $x \in \Lambda_k \cap (\Omega \setminus U)$ and thus both $x \in \Lambda_k$ and $x \neq \partial_\omega \Omega$; therefore $x \in [\alpha_k + 1, \beta_k[\ \cap\ (\Omega \setminus U)$.

Finally, let

$$Q_1 := D^1_{j_1^{(1)}},\ \ldots\ , Q_{r_1} := D^1_{j_{r_1}^{(1)}}$$

and if $s > 1$:

$$Q_{r_1+1} := D^2_{j_1^{(2)}}, \ \dots \ , Q_{r_1+r_2} := D^2_{j_{r_2}^{(2)}},$$
$$\cdot$$
$$\cdot$$
$$\cdot$$
$$Q_{(\sum_{l=1}^{s-1} r_l)+1} := D^s_{j_1^{(s)}}, \ \dots \ , Q_{\sum_{l=1}^{s} r_l} := D^s_{j_{r_s}^{(s)}}.$$

Then the family $(Q_i)_{i\in\{1,\dots,M\}}$ with $M := \sum_{l=1}^{s} r_l$ possesses the required properties in Lemma 3. This can be seen as follows.

First, we have $\Omega \supseteq Q_i =]\gamma_i, \eta_i[, |\eta_i - \gamma_i| > 1$ with certain γ_i, η_i. Further, we get $\partial_\omega^l Q_i =]\gamma_i, \gamma_i + 1[$.

We show

$$\partial_\omega^l Q_i \cap (\Omega \setminus U) \subseteq \bigcup_{n=0}^{i-1} \overline{Q_n}, \ (Q_0 := \emptyset).$$

For $\partial_\omega^l Q_i \cap (\Omega \setminus U) = \emptyset$ there is nothing to show. But let for any $i_0 \in \{1, \dots, M\}$

$$x \in \partial_\omega^l Q_{i_0} \cap (\Omega \setminus U)$$

hence $x \in Q_{i_0} = D^k_{j_{m_0}^{(k)}}$ for some $k \in \{1, \dots, s\}$ and $m_0 \in \{1, \dots, r_k\}$. Then we get

$$x \in]\alpha_k + (j_{m_0}^{(k)} - 1) \cdot \delta_k, \ \alpha_k + 1 + j_{m_0}^{(k)} \cdot \delta_k[$$

and since $x \in \partial_\omega^l Q_{i_0}$ we get even

$$x \in]\alpha_k + (j_{m_0}^{(k)} - 1) \cdot \delta_k, \ \alpha_k + 1 + (j_{m_0}^{(k)} - 1) \cdot \delta_k[.$$

Furthermore $x \in \Omega \setminus U$ implies

$$x \notin \partial_\omega \Omega = \bigcup_{i=1}^{n} (]a_i, a_i + 1[\cup]b_i - 1, b_i[) \ \cup \ \widetilde{\Omega}$$

hence $x \notin]\alpha_k, \alpha_k + 1[$, thus $x \in [\alpha_k + 1, \ \alpha_k + 1 + (j_{m_0}^{(k)} - 1) \cdot \delta_k[$ (therefore $j_{m_0}^{(k)} > 1$). Consequently

$$x \in \bigcup_{i=1}^{j_{m_0}^{(k)}-1} [\alpha_k + 1 + (i-1) \cdot \delta_k, \ \alpha_k + 1 + i \cdot \delta_k[\ = \bigcup_{i=1}^{j_{m_0}^{(k)}-1} V_i^k.$$

Hence

$$x \in \bigcup_{i=1}^{j_{m_0}^{(k)}-1} (V_i^k \cap (\Omega \setminus U))$$

thus $x \in V_{i'}^k \cap (\Omega \setminus U) \neq \emptyset$ with some $i' \in \{1, \dots, j_{m_0}^{(k)} - 1\}$ which implies $x \in D_{i'}^k$ and $i' \in \{j_1^{(k)}, \dots, j_{m_0-1}^{(k)}\}$ (in particular $m_0 > 1$). Therefore, we get

$$x \in \bigcup_{m<m_0} D_{j_m^{(k)}}^k \subseteq \bigcup_{i=1}^{i_0-1} \overline{Q_i} \ \text{ (in particular } i_0 > 1).$$

Next we show

$$\Omega \setminus U \subseteq \bigcup_{n=1}^{M} \overline{Q_n}.$$

There is

$$\Omega \setminus U \subseteq \bigcup_{k=1}^{s} \Lambda_k = \bigcup_{k=1}^{s}]\alpha_k, \beta_k[$$

and since $(\Omega \setminus U) \cap \partial_\omega \Omega = \emptyset$ we get even

$$\Omega \setminus U \subseteq \bigcup_{k=1}^{s} [\alpha_k + 1, \beta_k[= \bigcup_{k=1}^{s} \bigcup_{i=1}^{N_k} V_i^k.$$

Thus, we have

$$x \in \Omega \setminus U \Rightarrow x \in \bigcup_{k=1}^{s} \bigcup_{i=1}^{N_k} (V_i^k \cap (\Omega \setminus U)) \Rightarrow x \in V_{i_0}^k \cap (\Omega \setminus U) \neq \emptyset$$

for some $k \in \{1, \dots, s\}$ and $i_0 \in \{1, \dots, N_k\}$. Hence $x \in D_{i_0}^k$ and $i_0 \in \{j_1^{(k)}, \dots, j_{r_k}^{(k)}\}$ and we get

$$x \in \bigcup_{k=1}^{s} \bigcup_{m=1}^{r_k} D_{j_m^{(k)}}^k = \bigcup_{n=1}^{M} Q_n \subseteq \bigcup_{n=1}^{M} \overline{Q_n}.$$

Finally, for any $i_0 \in \{1, \dots, M\}$ there is $Q_{i_0} = D_{j_{m_0}^{(k)}}^k$ for some $k \in \{1, \dots, s\}$ and $m_0 \in \{1, \dots, r_k\}$ hence

$$Q_{i_0} =]\alpha_k + (j_{m_0}^{(k)} - 1) \cdot \delta_k, \ \alpha_k + 1 + j_{m_0}^{(k)} \cdot \delta_k[$$

and

$$R_{i_0} = [\alpha_k + 1 + (j^{(k)}_{m_0} - 1) \cdot \delta_k,\ \alpha_k + 1 + j^{(k)}_{m_0} \cdot \delta_k[$$

which implies length$(R_{i_0}) = \delta_k < \epsilon$. Moreover, because of $R_{i_0} = V^k_{j^{(k)}_{m_0}}$ and $V^k_{j^{(k)}_{m_0}} \cap (\Omega \setminus U) \neq \emptyset$ we get $\overline{R_{i_0}} \cap \overline{(\Omega \setminus U)} \neq \emptyset$ which completes the proof for the case $N = 1$.

Before treating the case $N > 1$ we need some additional tools.

Lemma 4. *The boundary $\partial\Omega$ is contained in $\overline{\partial_\omega \Omega}$.*

Lemma 4 was proved in [2, p. 17], for completeness we give the proof here. The proof also holds for any nonempty bounded region ω.

PROOF OF LEMMA 4. Let $z \in \partial\Omega$. Choose any $x_0 \in \omega$. Then $z \in (z-x_0)+\omega$. Since $z \in \partial\Omega$ there exists $(z_n)_n \subseteq \Omega \cap ((z-x_0)+\omega)$ and $z_n \to z$. Because of $z \in (z-x_0)+\omega$ we have $(z-x_0)+\overline{\omega} \not\subseteq \Omega$ and therefore $(z_n)_n \subseteq \partial_\omega\Omega$, hence $z \in \overline{\partial_\omega\Omega}$. □

Lemma 5. *Let U and V bounded subsets of $\mathbb{R}^N$. Then*

$$U \oplus V := \bigcup_{y \in \{t \in \mathbb{R}^N \,|\, (t+\overline{V}) \cap \overline{U} \neq \emptyset\}} (y + \overline{V})$$

is compact. If $V \neq \emptyset$ then $U \subseteq U \oplus V$ does hold.

Lemma 5 was proved in [2, p. 49], for completeness we give the proof here.

PROOF OF LEMMA 5. Let $(x_n)_n \subseteq U \oplus V$, hence $x_n = y_n + v_n$ with $y_n \in \mathbb{R}^N, v_n \in \overline{V}$ and $(y_n + \overline{V}) \cap \overline{U} \neq \emptyset$. Then $(v_n)_n$ and also $(y_n)_n$ are bounded and there exists a subsequence $(x_{n_k})_k \subseteq U \oplus V, x_{n_k} = y_{n_k} + v_{n_k}$ with $y_{n_k} \to y \in \mathbb{R}^N, v_{n_k} \to v \in \overline{V}$, thus $x_{n_k} \to x := y + v$. We show $(y+\overline{V}) \cap \overline{U} \neq \emptyset$: For all $k \in \mathbb{N}$ it exists $z_k \in (y_{n_k} + \overline{V}) \cap \overline{U}$. Hence, there exists a subsequence $(z_{k_l})_l \subseteq \overline{U}$ with $z_{k_l} \to z \in \overline{U}$. Since $y_{n_{k_l}} \to y$ we get $z_{k_l} - y_{n_{k_l}} \to z - y$ and because of $z_{k_l} - y_{n_{k_l}} \in \overline{V}$ this implies $z - y \in \overline{V}$, thus $z \in (y+\overline{V}) \cap \overline{U} \neq \emptyset$. So, we get $x \in U \oplus V$.
If $V \neq \emptyset$ choose $v_0 \in V$. Then for any $x \in U$ we have $x \in (x - v_0) + \overline{V}$. Thus $x \in ((x-v_0)+\overline{V}) \cap \overline{U} \neq \emptyset$ which implies $x \in U \oplus V$. □

Lemma 6. *Let G be any subset of Ω with*

$$\overline{G} \cap \overline{\partial_\omega\Omega} = \emptyset.$$

Then $G \oplus \omega \subseteq \Omega$ does hold.

Lemma 6 was proved in [2, p. 49], for completeness we give the proof here. The proof also holds for any nonempty bounded region ω.

PROOF OF LEMMA 6. Let G be as required. Suppose there exists $z \in G \oplus \omega$ with $z \notin \Omega$. Thus, there exists $y \in \mathbb{R}^N$ with $z \in y + \overline{\omega}$ and $(y+\overline{\omega}) \cap \overline{G} \neq \emptyset$ and since $z \notin \Omega$ we have $y + \overline{\omega} \not\subseteq \Omega$ thus $y \notin \Omega_\omega$. This implies $(y+\omega) \cap \Omega \subseteq \partial_\omega \Omega$. Because of $\overline{G} \cap \overline{\partial_\omega \Omega} = \emptyset$ we get

$$\overline{(y+\omega) \cap \Omega} \cap \overline{G} = \emptyset.$$

But we have $\overline{G} \subseteq \Omega$ because otherwise we would have $\overline{G} \cap \partial\Omega \neq \emptyset$ hence $\overline{G} \cap \overline{\partial_\omega \Omega} \neq \emptyset$ (cf. Lemma 4) in contradiction to $\overline{G} \cap \overline{\partial_\omega \Omega} = \emptyset$. Hence, since $(y+\overline{\omega}) \cap \overline{G} \neq \emptyset$, there exists $x \in (y+\overline{\omega}) \cap \overline{G}$ and $x \in \Omega$. But then, there exists also $(x_n)_n \subseteq y + \omega$ with $x_n \to x$ and $(x_n)_n \subseteq \Omega$. Consequently, we get $x \in \overline{(y+\omega) \cap \Omega}$. Since also $x \in \overline{G}$ this implies $\overline{(y+\omega) \cap \Omega} \cap \overline{G} \neq \emptyset$ in contradiction to $\overline{(y+\omega) \cap \Omega} \cap \overline{G} = \emptyset$. Thus, the assumption of the existence of $z \in G \oplus \omega$ with $z \notin \Omega$ cannot hold.

□

Now, we continue the proof of Lemma 3 for the case $N > 1$ by induction. So, let Lemma 3 be true for dimension $N-1 \in \mathbb{N}$. We prove the validity of Lemma 3 for dimension N. For this, let again $U \subseteq \Omega \subseteq \mathbb{R}^N$, $U \neq \Omega$, $\overline{\Omega \setminus U} \cap \overline{\partial_\omega \Omega} = \emptyset$ and $\epsilon > 0$ and let now

$$\Psi := \Omega \setminus U.$$

Then $\Psi \oplus \omega$ is compact (cf. Lemma 5) and nonempty. Moreover, we have $\Psi \subseteq \Psi \oplus \omega \subseteq \Omega$ (cf. Lemma 6)and since $\Psi \oplus \omega$ is compact even $\overline{\Psi} \subseteq \Psi \oplus \omega \subseteq \Omega$ does hold. Define now

$$\Delta := \operatorname{dist}(\Psi \oplus \omega, \complement\Omega) > 0$$

and let

$$\delta := \min(\frac{\Delta}{2\sqrt{N}}, \frac{\epsilon}{2})$$

and

$$z_0 := \inf\{x_1 \mid (x_1, \dots, x_N) \in \Psi\},$$

$$\eta := \sup\{x_1 \mid (x_1, \dots, x_N) \in \Psi\}$$

and

$$z_k := z_0 + k \cdot \delta,\ k \in \{1, \dots, K\}$$

where $K \in \mathbb{N}$ is defined by $z_0 + (K-1)\cdot\delta \le \eta$ but $z_0 + K\cdot\delta > \eta$.
Further, let $i_1, \ldots, i_m$ be those ascending ordered $i \in \{1, \ldots, K\}$ for which

$$Z_i := ([z_{i-1}, z_i] \times \mathbb{R}^{N-1}) \cap \overline{\Psi} \neq \emptyset.$$

Because of the definition of z_0 particularly Z_1 is not empty, which implies $i_1 = 1$.

Let now $i \in \{i_1, \ldots, i_m\}$ be fixed and let P be the projection operator

$$P : \mathbb{R}^N \to \mathbb{R}^{N-1},$$

$$(x_1, \ldots, x_N) \to (x_2, \ldots, x_N)$$

and define

$$\Omega_i := P(\Omega \cap ([z_{i-1}, z_i] \times \mathbb{R}^{N-1})) \subseteq \mathbb{R}^{N-1}.$$

Finally, define $\omega' :=]0,1[^{N-1}$.

With respect to the choice of $i_1, \ldots, i_m$ we get $\Omega_i \neq \emptyset$ (note that $\overline{\Psi} \subseteq \Omega$). Further Ω_i is open: If $\xi = (\xi_2, \ldots, \xi_N) \in \Omega_i$ then there exists $x = (x_1, \xi_2, \ldots, \xi_N) \in \Omega$ with $x_1 \in [z_{i-1}, z_i]$. Since Ω is open there exists a neighborhood of $(\xi_2, \ldots, \xi_N)$ such that for any y in this neighborhood $(x_1, y) \in \Omega$ holds. Thus $y \in \Omega_i$ for any such y. In addition, obviously, Ω_i is bounded. Moreover, because of $Z_i \neq \emptyset$ there exists $x \in \overline{\Psi} \subseteq \Omega$ with $x_1 \in [z_{i-1}, z_i]$. Let now $w = (w_1, \ldots, w_N)$ any element in ω. With $t := x - w$ we get $x \in t + \omega$ thus $(t + \overline{\omega}) \cap \overline{\Psi} \neq \emptyset$. Hence $t + \overline{\omega} \subseteq \Psi \oplus \omega \subseteq \Omega$ which implies in particular $(x_1, t_2 + w_2', \ldots, t_N + w_N') \subseteq \Omega$ for all $(w_2', \ldots, w_N') \in [0,1]^{N-1}$. Thus $(t_2, \ldots, t_N) + \overline{\omega'} \subseteq \Omega_i$ and therefore $(\Omega_i)_{\omega'} \neq \emptyset$.

Define now

$$W_i := P(Z_i).$$

Then W_i is not empty and since $\overline{\Psi} \subseteq \Omega$ there is $W_i \subseteq \Omega_i$. Furthermore W_i is closed since Z_i is compact. In addition, let now

$$U_i := \Omega_i \setminus W_i \qquad (\neq \Omega_i).$$

We will apply the induction hypothesis to U_i and Ω_i. For this, it is sufficient to show

$$\overline{\Omega_i \setminus U_i} \cap \overline{\partial_{\omega'}\Omega_i} = \emptyset.$$

To do this, we assume there holds $\overline{W_i} \cap \overline{\partial_{\omega'}\Omega_i} \neq \emptyset$ and show that this leads to a contradiction. So, let be $\xi \in \overline{W_i} \cap \overline{\partial_{\omega'}\Omega_i}$, $\xi = (\xi_2, \ldots, \xi_N) \in \mathbb{R}^{N-1}$. Then there exists $(\xi^{(k)})_k \subseteq \partial_{\omega'}\Omega_i$ with $\xi^{(k)} \to \xi$. Since

$$\partial_{\omega'}\Omega_i = \bigcup_{t \notin (\Omega_i)_{\omega'}} (t+\omega') \cap \Omega_i$$

there exists $v_k \in \complement(\Omega_i)_{\omega'}$ and $w_k \in \omega'$ with $\xi^{(k)} = v_k + w_k$. Since $(w_k)_k$ and $(\xi^{(k)})_k$ are bounded this also holds for $(v_k)_k$. By the closedness of $\complement(\Omega_i)_{\omega'}$ this implies the existence of subsequences $(v_{k_l})_l$ and $(w_{k_l})_l$ with

$$v_{k_l} \to v \in \complement(\Omega_i)_{\omega'}$$

and

$$w_{k_l} \to w \in \overline{\omega'}.$$

Therefore, we get

$$\xi = v + w \in \complement(\Omega_i)_{\omega'} + \overline{\omega'}.$$

On the other hand, we have $\xi \in \overline{W_i} = W_i$ and hence, there exists $x_1 \in [z_{i-1}, z_i]$ with $x := (x_1, \xi_2, \ldots, \xi_N) \in Z_i \subseteq \overline{\Psi}$. With

$$V := (x_1, v) + \overline{\omega}$$

we get $x \in V$, therefore $V \cap \overline{\Psi} \neq \emptyset$ and thus $V \subseteq \Psi \oplus \omega$, which implies $V \subseteq \Omega$. In particular, we have

$$(x_1, v) + (\{0\} \times \overline{\omega'}) \subseteq \Omega$$

hence $v + \overline{\omega'} \subseteq \Omega_i$ which finally leads to $v \in (\Omega_i)_{\omega'}$ in contradiction to $v \in \complement(\Omega_i)_{\omega'}$.

By this the conditions for applying the induction hypothesis to (U_i, Ω_i) are fulfilled (for any $i \in \{i_1, \ldots, i_m\}$). Choose now

$$\epsilon' := \min(\frac{\Delta}{2\sqrt{N}}, \frac{\epsilon}{2}).$$

Thus, for any $i \in \{i_1, \ldots, i_m\}$, there exist by the induction hypothesis cuboids $Q_l^{(i)} \subseteq \Omega_i, Q_l^{(i)} = \prod_{j=1}^{N-1}]a_{l,j}^{(i)}, b_{l,j}^{(i)}[$, $|b_{l,j}^{(i)} - a_{l,j}^{(i)}| > 1$, $l \in \{1, \ldots, M(i)\}$ such that

$$\Omega_i \setminus U_i \subseteq \bigcup_{n=1}^{M(i)} \overline{Q_l^{(i)}}$$

and for all $l \in \{1, \ldots, M(i)\}$

$$\partial_{\omega'}^l Q_l^{(i)} \cap (\Omega_i \setminus U_i) \subseteq \bigcup_{n=0}^{l-1} \overline{Q_n^{(i)}}, \ (Q_0^{(i)} := \emptyset)$$

and (with $R_l^{(i)} := \prod_{j=1}^{N-1} [a_{l,j}^{(i)} + 1, b_{l,j}^{(i)}[$)

$$\overline{R_l^{(i)}} \cap (\overline{\Omega_i \setminus U_i}) \neq \emptyset$$

and

$$\text{length}(R_l^{(i)}) < \epsilon'.$$

For $i \in \{i_1, \ldots, i_m\}$ and $l \in \{1, \ldots, M(i)\}$ define now

$$G_l^{(i)} :=]z_{i-1} - 1, z_i[\times Q_l^{(i)}$$

and finally

$$D_1 := G_1^{(i_1)}, \ \ldots \ , D_{M(i_1)} := G_{M(i_1)}^{(i_1)}$$

and if $m > 1$

$$D_{M(i_1)+1} := G_1^{(i_2)}, \ \ldots \ , D_{M(i_1)+M(i_2)} := G_{M(i_2)}^{(i_2)}$$

$$\vdots$$

$$D_{(\sum_{j=1}^{m-1} M(i_j))+1} := G_1^{(i_m)}, \ \ldots \ , D_{\sum_{j=1}^{m} M(i_j)} := G_{M(i_m)}^{(i_m)}.$$

Then the family $(D_s)_s \subseteq \mathbb{R}^N, \ s \in \{1, \ldots, \sum_{j=1}^m M(i_j)\}$ possesses the required properties in Lemma 3. This can be seen as follows.

i) We show $D_s \subseteq \Omega$. There is $D_s =]z_{i-1} - 1, z_i[\times Q_l^{(i)}$ for some $i \in \{i_1, \ldots, i_m\}$ and $l \in \{1, \ldots, M(i)\}$. Let $x \in D_s, x = (x_1, \ldots, x_N)$, hence

$$x_1 \in]z_{i-1} - 1, z_i[$$

and

$$(x_2, \ldots, x_N) \in \prod_{j=2}^{N}]a_{l,j-1}^{(i)}, b_{l,j-1}^{(i)}[.$$

Furthermore, by the induction hypothesis we have $\overline{R_l^{(i)}} \cap \overline{W_i} \neq \emptyset$. Since $\overline{W_i} = W_i$ this implies the existence of $y := (y_1, \ldots, y_N) \in \overline{\Psi}$ such that

$$y_1 \in [z_{i-1}, z_i]$$

and

$$(y_2, \ldots, y_N) \in \overline{R_l^{(i)}}.$$

Define now $x^* \in \mathbb{R}^N,\ x^* = (x_1^*, \dots, x_N^*)$ by

$$x_j^* := \begin{cases} y_j - 1, & \text{if } x_j < y_j - 1, \\ x_j, & \text{if } y_j - 1 \le x_j \le y_j, \\ y_j, & \text{if } y_j < x_j. \end{cases}$$

Then we get

$$x_j^* - x_j = \begin{cases} y_j - 1 - x_j, & \text{if } x_j < y_j - 1, \\ 0, & \text{if } y_j - 1 \le x_j \le y_j, \\ y_j - x_j, & \text{if } y_j < x_j. \end{cases}$$

But for $x_j < y_j - 1$ there is

$$0 < y_j - 1 - x_j \le \begin{cases} z_i - 1 - (z_{i-1} - 1) = z_i - z_{i-1} = \delta \le \frac{\Delta}{2\sqrt{N}}, & \text{if } j = 1, \\ b_{l,j-1}^{(i)} - 1 - a_{l,j-1}^{(i)} < \epsilon' \le \frac{\Delta}{2\sqrt{N}}, & \text{if } j > 1, \end{cases}$$

and for $x_j > y_j$ there is

$$0 > y_j - x_j \ge \begin{cases} z_{i-1} - z_i = -\delta \ge -\frac{\Delta}{2\sqrt{N}}, & \text{if } j = 1, \\ a_{l,j-1}^{(i)} + 1 - b_{l,j-1}^{(i)} > -\epsilon' \ge -\frac{\Delta}{2\sqrt{N}}, & \text{if } j > 1. \end{cases}$$

Hence, for any j we get $|x_j^* - x_j| \le \frac{\Delta}{2\sqrt{N}}$ which implies

$$|x^* - x| \le \frac{\Delta}{2}.$$

Let now $t \in \mathbb{R}^N, t = (t_1, \dots, t_N)$ by

$$t_j := \begin{cases} -1, & \text{if } x_j < y_j - 1, \\ -1, & \text{if } y_j - 1 \le x_j \le y_j, \\ 0, & \text{if } y_j < x_j. \end{cases}$$

Then we have

$$x_j^* - (y_j + t_j) = \begin{cases} y_j - 1 - y_j + 1 = 0, & \text{if } x_j < y_j - 1, \\ x_j - y_j + 1 \in [0,1], & \text{if } y_j - 1 \le x_j \le y_j, \\ y_j - y_j = 0, & \text{if } y_j < x_j, \end{cases}$$

thus

$$x^* \in (y + t) + \overline{\omega}.$$

Obviously, we also have $y \in (y+t)+\overline{\omega}$ and since $y \in \overline{\Psi}$ we get $(y+t)+\overline{\omega} \subseteq \Psi \oplus \omega$ and therefore

$$x^* \in \Psi \oplus \omega.$$

Because of $|x^* - x| \leq \frac{\Delta}{2}$ and $\operatorname{dist}(\Psi \oplus \omega, \complement\Omega) = \Delta$ this implies $x \in \Omega$.

ii) We show

$$\partial_\omega^l D_s \cap (\Omega \setminus U) \subseteq \bigcup_{n=0}^{s-1} \overline{D_n},\ (D_0 := \emptyset)$$

for $s \in \{1, \ldots, \sum_{j=1}^m M(i_j)\}$. For this, let s be fixed, thus

$$D_s =]z_{i-1} - 1, z_i[\times Q_l^{(i)} =]z_{i-1} - 1, z_i[\times \prod_{j=1}^{N-1}]a_{l,j}^{(i)}, b_{l,j}^{(i)}[$$

for some $i \in \{i_1, \ldots, i_m\}$ and $l \in \{1, \ldots, M(i)\}$.

Let now

$$E_s := [z_{i-1}, z_i[\times \prod_{j=1}^{N-1} [a_{l,j}^{(i)} + 1, b_{l,j}^{(i)}[$$

so that

$$\partial_\omega^l D_s = D_s \setminus E_s.$$

Now, for $\partial_\omega^l D_s \cap (\Omega \setminus U) = \emptyset$ there is nothing to show.
But suppose $x \in \partial_\omega^l D_s \cap (\Omega \setminus U), x = (x_1, \ldots, x_N)$, hence $x_1 \in]z_{i-1} - 1, z_i[$.

Case 1: $x_1 \in]z_{i-1} - 1, z_{i-1}[$. Since $x \in \Omega \setminus U$ there is $x_1 \in [z_{j-1}, z_j]$ for some j with $1 \leq j < i$. Thus $x \in Z_j$ and $j \in \{i_1, \ldots, i_m\}$. Now, we have $P(x) \in \Omega_j$ and even $P(x) \in \Omega_j \setminus U_j$. But by the induction hypothesis there is $P(x) \in \overline{Q_r^{(j)}}$ for some $r \in \{1, \ldots, M(j)\}$ and because $j < i$ this implies

$$x \in [z_{j-1} - 1, z_j] \times \overline{Q_r^{(j)}} \subseteq \bigcup_{n=0}^{s-1} \overline{D_n}.$$

Case 2: $x_1 \in [z_{i-1}, z_i[$. Since $x \notin E_s$ there is

$$P(x) \notin \prod_{j=1}^{N-1} [a_{l,j}^{(i)} + 1, b_{l,j}^{(i)}[$$

hence

$$P(x) \in \partial^l_{\omega'} Q^{(i)}_l.$$

Like in Case 1 we have $P(x) \in \Omega_i \setminus U_i$ and with the induction hypothesis we get

$$P(x) \in \bigcup_{n=0}^{l-1} \overline{Q^{(i)}_n}$$

and therefore $x \in [z_{i-1}-1, z_i] \times \overline{Q^{(i)}_{n'}}$ with some $n' < l$. Thus

$$x \in \bigcup_{n=0}^{s-1} \overline{D_n}.$$

iii) We show

$$\Omega \setminus U \subseteq \bigcup_{s=1}^{\sum_{j=1}^m M(i_j)} \overline{D_s}.$$

Let $x \in \Omega \setminus U$. Then there is $x_1 \in [z_{i-1}, z_i[$ for some $i \in \{1, \ldots, K\}$, hence $Z_i \neq \emptyset$ and $i \in \{i_1, \ldots, i_m\}$. Therefore, we get $P(x) \in \Omega_i \setminus U_i$ which implies by the induction hypothesis

$$P(x) \in \bigcup_{l=1}^{M(i)} \overline{Q^{(i)}_l}.$$

Thus

$$x \in \bigcup_{l=1}^{M(i)} ([z_{i-1}-1, z_i] \times \overline{Q^{(i)}_l}) \subseteq \bigcup_{s=1}^{\sum_{j=1}^m M(i_j)} \overline{D_s}.$$

iv) We show

$$([z_{i-1}, z_i] \times \prod_{j=1}^{N-1} [a^{(i)}_{l,j}+1, b^{(i)}_{l,j}]) \cap \overline{\Omega \setminus U} \neq \emptyset$$

for any $i \in \{i_1, \ldots, i_m\}$ and $l \in \{1, \ldots, M(i)\}$. By the induction hypothesis there holds for any $i \in \{i_1, \ldots, i_m\}$ and $l \in \{1, \ldots, M(i)\}$

$$(\prod_{j=1}^{N-1} [a^{(i)}_{l,j}+1, b^{(i)}_{l,j}]) \cap \overline{\Omega_i \setminus U_i} \neq \emptyset.$$

Now, if we take for any such fixed i, l some

$$\xi \in (\prod_{j=1}^{N-1} [a_{l,j}^{(i)} + 1, b_{l,j}^{(i)}]) \cap \overline{\Omega_i \setminus U_i}$$

then there exists $x_1 \in [z_{i-1}, z_i]$ with

$$x := (x_1, \xi) \in \overline{\Omega \setminus U}$$

and therefore

$$x \in ([z_{i-1}, z_i] \times \prod_{j=1}^{N-1} [a_{l,j}^{(i)} + 1, b_{l,j}^{(i)}]) \cap \overline{\Omega \setminus U} \neq \emptyset.$$

Moreover, for any $i \in \{i_1, \ldots, i_m\}$ and $l \in \{1, \ldots, M(i)\}$ we also get by the induction hypothesis

$$\text{length}([z_{i-1}, z_i[\times \prod_{j=1}^{N-1} [a_{l,j}^{(i)} + 1, b_{l,j}^{(i)}[) \leq \max(\delta, \epsilon') \leq \max(\frac{\epsilon}{2}, \frac{\epsilon}{2}) = \frac{\epsilon}{2} < \epsilon$$

which finally completes the proof. $\square$

4 Concluding remarks

Obviously, the condition

$$\overline{\Omega \setminus U} \cap \overline{\partial_\omega \Omega} = \emptyset$$

in Theorem 1 implies $\partial_\omega \Omega \subseteq U$. This leads to the question, if Theorem 1 is even true for $U = \partial_\omega \Omega$. As mentioned before, this holds for cuboids, i.e. for the special case

$$\Omega = \prod_{i=1}^{N}]a_i, b_i[$$

with a,b $\in \mathbb{R}^N$, $|b_i - a_i| > 1$ (cf. [1, p. 192], note that in this case $\partial_\omega^l \Omega \subseteq \partial_\omega \Omega$). For general Ω this remains as an open problem.

Note, moreover, that Theorem 1 is in general *not* true for subsets U where $\partial_\omega \Omega \subseteq U$ does not hold (cf. [1, p. 180]).

The presented topic could also be extended to the case of general $k \in L^1(\omega)$ and general nonempty bounded regions ω. For this, basic concepts were given in [1]. But any result about the validity of the maximum principle in this general case remains open.

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