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# WHEN IS A FAMILY OF GENERALIZED MEANS A SCALE?

**Abstract**

For a family $\{k_\alpha \,|\, \alpha \in I\}$ of real $\mathcal{C}^2$ functions defined on $U$ ($I$, $U$ – open intervals) and satisfying some mild regularity conditions, we prove that the mapping $I \ni \alpha \mapsto k_\alpha^{-1}\big(\sum_{i=1}^n w_i k_\alpha(a_i)\big)$ is a continuous bijection between $I$ and $(\min \underline{a},\, \max \underline{a})$, for every fixed non-constant sequence $\underline{a} = \big(a_i\big)_{i=1}^n$ with values in $U$ and every set, of the same cardinality, of positive weights $\underline{w} = \big(w_i\big)_{i=1}^n$. In such a situation one says that the family of functions $\{k_\alpha\}$ generates a *scale* on $U$. The precise assumptions in our result read (all indicated derivatives are with respect to $x \in U$)

(i) $k'_\alpha$ vanishes nowhere in $U$ for every $\alpha \in I$,

(ii) $I \ni \alpha \mapsto \frac{k''_\alpha(x)}{k'_\alpha(x)}$ is increasing, 1–1 on a dense subset of $U$ and onto the image $\mathbb{R}$ for every $x \in U$.

This result makes possible three things: 1) a new and extremely short proof of the classical fact that *power means* generate a scale on $(0, +\infty)$, 2) a short proof of a fact, which is in a direct relation to two results established by Kolesárová in 2001, that, for every strictly increasing convex and $\mathcal{C}^2$ function $k\colon (0,\, 1) \to (0,\, +\infty)$, the class $\{\mathfrak{M}_{k_\alpha}\}_{\alpha\in(0,\,+\infty)}$ of quasi-arithmetic means (see Introduction for the definition) generated by functions $k_\alpha$, $k_\alpha(x) = k(x^\alpha)$, $\alpha \in (0,\, +\infty)$, generates a scale on $(0, 1)$ between the geometric mean and maximum (meaning that, for every $\underline{a}$, $\underline{w}$, if $s \in \big(\prod_{i=1}^n a_i^{w_i},\, \max(\underline{a})\big)$ then there exists exactly one $\alpha$ such that $\mathfrak{M}_{k_\alpha}(\underline{a}, \underline{w}) = s$).

3) a brief proof of one of the classical results of the Italian statistics' school from the 1910-20s that the so-called *radical means* generate a scale on $(0,\, +\infty)$.

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## 1 Introduction

One of the most popular families of means encountered in the literature consists of quasi-arithmetic means (Q-A for short). That mean is defined for any continuous strictly monotone function $f\colon U \to \mathbb{R}$, $U$ – an open interval. When $\underline{a} = (a_1, \ldots, a_n)$ is a sequence of points in $U$ and $\underline{w} = (w_1, \ldots, w_n)$ is a sequence of *weights* ($w_i > 0$, $w_1 + \cdots + w_n = 1$), then the mean $\mathfrak{M} = \mathfrak{M}_f(\underline{a}, \underline{w})$ is well-defined by the equality

$$f(\mathfrak{M}) = \sum_{i=1}^{n} w_i f(a_i)\,.$$

According to [13, pp. 158–159], this family of means was dealt with for the first time in the papers [11, 15, 17], just a couple of years before the coming out of that benchmark contribution [13]. Among the names of independent, if simultaneous, contributors there is that of Kolmogorov. He had explained in [15] the *naturality* of the above construction. In fact, a very short list of his most natural postulates to be satisfied by a mean forces the existence of a continuous function governing that mean, exactly as in the definition above. The issue is also discussed in the by-now-classical encyclopaedic publications [6] and [7], and it is underlined there that one thus naturally generalizes the *Power Means*. Indeed, the latter family, containing the most popular means: arithmetic, geometric, quadratic, harmonic, is encompassed by this approach upon taking the functions

$$k_\alpha(x) = \begin{cases} x^\alpha & \text{if } \alpha \neq 0 \\ \ln x & \text{if } \alpha = 0 \end{cases},$$

for $x \in U = (0, +\infty)$, $\alpha \in I = \mathbb{R}$.

We pass now to the notion of scale in the theory of means. If a non-constant vector $\underline{a} \in U^n$ and weights $\underline{w}$ are fixed then the mapping $f \mapsto \mathfrak{M}_f(\underline{a}, \underline{w})$ takes continuous monotone functions $f\colon U \to \mathbb{R}$ to the interval $(\min \underline{a}, \max \underline{a})$. One is interested in finding such families of functions $\{k_\alpha\colon U \to \mathbb{R}\}_{\alpha \in I}$, where $I$ is an interval, that for every non-constant vector $\underline{a}$ with values in $U$ and arbitrary fixed corresponding weights $\underline{w}$, the mapping $I \ni \alpha \mapsto \mathfrak{M}_{k_\alpha}(\underline{a}, \underline{w})$ be a *bijection* onto $(\min \underline{a}, \max \underline{a})$. Every such a family of means $\mathfrak{M}_{k_\alpha}$ is called *scale on $U$*.

The problem of finding conditions, for a family of means, equivalent to its being a scale has been discussed for various families. For instance, a set of conditions pertinent for Gini means was presented in [1]. Many results concerning means may be expressed in a compact way in terms of scales. Probably the most famous is the fact that the family of power means is a scale on $(0, +\infty)$. It was proved for the first time (for arbitrary weights) in [2]. More about the underlying history, as well as another proof, was given in [6, p. 203]. In the last section of the present note we will present a new, extremely short proof of this classical fact.

## 2 Comparison of means

Dealing with means, we would like to know whether (a) one mean is not smaller than the other, whenever both are defined on the same interval and computed on same, but arbitrary, set of arguments. And, when (a) holds true, whether (b) the two means, evaluated on arguments, are equal only when all components in an input $\underline{a}$ are the same: $a_1 = a_2 = \cdots = a_n$. With (a) and (b) holding true, we would say that the first mean is *greater* than the second.

As long as quasi-arithmetic means are concerned, the comparability of $\mathfrak{M}_f$ and $\mathfrak{M}_g$ as such turns out to be intimately related to the convexity of the function $f \circ g^{-1}$; see items (ii) and (iii) in Proposition 2 below.

Unfortunately, however, when it comes to scales, the family of objects to handle becomes uncountable. Hence one is forced to use another tool, allowing to tell something about uncountable families of means. Its concept goes back to a seminal paper [16]. A key operator from [16] (recalled below), denoted in this paper either by $A$ or simply by bold font, is used in item (i) in our technically crucial Proposition 2.

In fact, let $U$ be an interval and $\mathcal{C}^{2\neq}(U)$ be the class of functions from $\mathcal{C}^2(U)$ with the first derivative vanishing nowhere in $U$ (if a boundary point belongs to $U$, as will happen in Section 3, then we will assume the existence of a corresponding one-sided, second derivative and nonzero, one-sided, first derivative at that point). Within this class one defines $A\colon \mathcal{C}^{2\neq}(U) \to \mathcal{C}(U)$ by the formula

$$A(f) := \frac{f''}{f'}\,.$$

However, the operator $A$ will be used so often that it is reasonable to adopt the convention that, for $a, b, c, \dots \in \mathcal{C}^{2\neq}(U)$, $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \dots$ stand for $A(a)$, $A(b)$, $A(c)$, …, respectively. Due to [16], this operator has wide applications in the comparison of means – see Proposition 2. In fact, it will enable us to compare means in huge families, not only in pairs. Precisely this kind of comparison was being advanced by Polish mathematicians in the late 1940s.

One of the most important facts was discovered by Mikusiński, who published his result, [16, (5)], in the first post-war issue of "Studia Mathematica"[1].

[1] the flagship journal of the pre-war Lvov Mathematical School, established by H. Steinhaus and S. Banach.

It is quite surprising that such a useful result has not been included in the referential book [6].

We present both necessary and sufficient conditions, for a family of functions $\{k_\alpha\}_{\alpha\in I}$ defined on a common interval $U$, to generate a scale on $U$. The key conditions in our Theorems 1 and 2 are given in terms of the operator $A$. Reiterating, it is handy to compare means with its help. We begin with

**Theorem 1.** *Let $U$ be an interval, $I=(a,b)$ – an open interval, $(k_\alpha)_{\alpha\in I}$ – a family of functions on $U$, $k_\alpha \in \mathcal{C}^{2\neq}(U)$ for all $\alpha$.*

*If $I \ni \alpha \mapsto A(k_\alpha)(x) \in \mathbb{R}$ is increasing and 1–1 on a dense subset of $U$, and is onto for all $x\in U$, then $(\mathfrak{M}_{k_\alpha})_{\alpha\in I}$ is an increasing scale on $U$.*

A proof of this theorem is given in Section 4. As a matter of fact, we will need a wider version of the above theorem. Namely, we extend the setup as follows.

In the definition of a scale (see Introduction) one may replace $\min \underline{a}$ and $\max \underline{a}$ by arbitrary bounds $L(\underline{a}, \underline{w})$ and $H(\underline{a}, \underline{w})$ respectively, with some functions $L$ and $H$.[2] Then such a modified family of means is called a *scale between $L$ and $H$*. Such generalization is very natural and is frequently used, e.g. in [6, pp. 323, 364].

Bounds in a scale, in most cases, are either quasi-arithmetic means or min, or max. In order to make the notation more homogeneous, we introduce two extra symbols $\perp$ and $\top$, and write henceforth, purely formally, $\mathfrak{M}_\perp = \min$ and $\mathfrak{M}_\top = \max$. We also adopt the convention that $A(\perp) = -\infty$ and $A(\top) = +\infty$.

**Attention.** In some papers scales may as well be decreasing. In fact, we do not lose generality if we assume that all scales are increasing, because whenever a family $\{k_\alpha\}_{\alpha\in I}$ generates a decreasing scale and $\varphi\colon J \to I$ is continuous, decreasing, 1–1 and onto, then the family $\{k_{\varphi(\alpha)}\}_{\alpha\in J}$ generates an increasing scale (see, e.g., Proposition 9 in Section 5).

**Corollary 1** (Bounded Scale)**.** *Let $U$ be an interval, $I=(a,b)$ – an open interval. Let $l,\, h \in \mathcal{C}^{2\neq}(U)\cup\{\perp, \top\}$ and $(k_\alpha)_{\alpha\in I}$ be a family of functions, $k_\alpha \in \mathcal{C}^{2\neq}(U)$ for all $\alpha\in I$.*

*If $I \ni \alpha \mapsto A(k_\alpha)(x) \in \mathbb{R}$ is increasing (decreasing) and 1–1 on a dense subset of $U$, and is onto $(A(l)(x),\, A(h)(x))$ for all $x\in U$, then $(\mathfrak{M}_{k_\alpha})_{\alpha\in I}$ is an increasing (decreasing) scale between $\mathfrak{M}_l$ and $\mathfrak{M}_h$.*

The proof is just a specification of the proof of Theorem 1.

[2] We slightly abuse the notation here, as most of the researchers active in the field of means do, e.g., in [6, p. 61].

**Remark.** *If, in the above corollary,* $l$, $h \in \mathcal{C}^{2\neq}(U)$, *then it would be enough to assume that the mapping* $\alpha \mapsto A(k_\alpha)(x)$ *be onto for almost all* $x \in U$. *(Because, then, by Corollary 3, one gets the convergence in* $L^1(U)$*.)*

The strength of Theorem 1 is visible in the following example (or exercise).

**Example 1.** *Let* $U = (\frac{1}{e}, +\infty)$ *and* $k_\alpha(x) = x^{\alpha x}$ *for* $\alpha \in \mathbb{R} \setminus \{0\}$.
*Find such a function* $k_0$ *that the completed family* $(k_\alpha)_{\alpha\in\mathbb{R}}$ *generates a scale on* $U$.

By the definition of the operator $A$, for $\alpha \neq 0$ there holds

$$\boldsymbol{k_\alpha}(x) = \frac{1}{x(\ln x + 1)} + \alpha(\ln x + 1)\,.$$

In view of Theorem 1 we will be done, provided $\alpha \mapsto \boldsymbol{k_\alpha}(x)$ is increasing, 1–1 and onto $\mathbb{R}$ for all $x \in U$. But

$$\mathbb{R} \setminus \{0\} \ni \alpha \mapsto \boldsymbol{k_\alpha}(x) \in \mathbb{R} \setminus \left\{ \frac{1}{x(\ln x + 1)} \right\} \quad \text{for all } x \in U\,.$$

Hence it is natural to take $k_0 = A^{-1}\left(\frac{1}{x(\ln x+1)}\right)$. Then the pattern $A^{-1}(\boldsymbol{f}) = \int e^{\int \boldsymbol{f}}$ gives automatically $k_0(x) = x \ln x$.
Therefore, an increasing scale on $(\frac{1}{e}, +\infty)$ is generated by the family

$$k_\alpha = \begin{cases} x \mapsto x^{\alpha x} & \text{if } \alpha \neq 0\,, \\ x \mapsto x \ln x & \text{if } \alpha = 0\,. \end{cases}$$

Moreover, it is now immediate to note that, in turn, the same family of functions generates a *decreasing* scale on $(0, \frac{1}{e})$.

How about a possible reversing of Theorem 1 ? This point is rather fine; the existence of a scale implies a somehow weaker set of properties than the one assumed in Theorem 1. To the best of author's knowledge, the problem of finding a set of conditions *exactly* equivalent to generating a scale is still (and, most likely, widely) open.

**Theorem 2.** *Let* $U$ *be an interval,* $I = (a, b)$ *an open interval,* $(k_\alpha)_{\alpha\in I}$, $k_\alpha \in \mathcal{C}^{2\neq}(U)$ *for all* $\alpha$.

*If* $(\mathfrak{M}_{k_\alpha})_{\alpha\in I}$ *is an increasing scale then there exists a dense subset* $X \subset U$ *such that the mapping* $I \ni \alpha \mapsto A(k_\alpha)(x) \in \mathbb{R}$ *is increasing, 1–1 and onto for all* $x \in X$.

A proof of this theorem is given in Section 4, immediately after the proof of Theorem 1.

## 3 Properties and uses of $A$

In what follows we will extensively use the operator $A$. Here we recall, after [16], some of its key properties. We also rephrase in the terms of $A$ an important result from [10].

All this will be instrumental in showing that many nontrivial families of functions do generate scales. We will also deduce about the limit properties of our quasi-arithmetic means, stating a new result (Proposition 8) inspired, to some extent, by the paper [14].

Regarding scales as such, many examples of them were furnished in [6, p. 269]. Scales were also used by the old Italian school of statisticians; see, e. g., [3, 4, 5, 12, 18, 19]. One of significant results from that last group of works will be presented, with a new and compact proof, in Proposition 9. That new approach will, we hope, show how quickly one can nowadays prove old results.

**Remark 1.** *Let $U$ be an interval and $f, g \in \mathcal{C}^{2\neq}(U)$. Then the following conditions are equivalent:*

(i) $A(f)(x) = A(g)(x)$ *for all* $x \in U$ ,

(ii) $f = \alpha g + \beta$ *for some* $\alpha, \beta \in \mathbb{R}$, $\alpha \neq 0$ ,

(iii) $\mathfrak{M}_f(\underline{a}, \underline{w}) = \mathfrak{M}_g(\underline{a}, \underline{w})$ *for all vectors* $\underline{a} \in U^n$ *and arbitrary corresponding weights* $\underline{w}$

*(see, for instance, [13, p. 66], [16]).*

Let $f \in \mathcal{C}^1(U)$ be a strictly monotone function such that $f'(x) \neq 0$ for all $x \in U$. Then there either holds $f'(x) < 0$ for all $x \in U$, or else $f'(x) > 0$ for all $x \in U$. So we define the sign $\operatorname{sgn}(f')$ of the first derivative of $f$ to be $\operatorname{sgn}(f')(x)$, where $x$ is any point in $U$. The key tool in our approach is

**Proposition 2** (Basic comparison)**.** *Let $U$ be an interval, $f, g \in \mathcal{C}^{2\neq}(U)$. Then the following conditions are equivalent:*

(i) $A(f) > A(g)$ *on a dense set in* $U$ ,

(ii) $(\operatorname{sgn} f') \cdot (f \circ g^{-1})$ *is strictly convex*,

(iii) $\mathfrak{M}_f(\underline{a}, \underline{w}) \geq \mathfrak{M}_g(\underline{a}, \underline{w})$ *for all vectors* $\underline{a} \in U^n$ *and weights* $\underline{w}$*, with both sides equal only when* $\underline{a}$ *is a constant vector.*

For the equivalence of (i) and (iii), see [16, p. 95] (this characterization of comparability of means had, in the same time, been obtained independently by S. Łojasiewicz – see footnote 2 in [16]). For the equivalence of (ii) and (iii), see, for instance, [9, p. 1053].

In the course of comparing means, one needs to majorate the difference between two means. If the interval $U$ is unbounded then, of course, the difference between any given two means can be unbounded (for example such is the difference between the arithmetic and geometric mean). In order to eliminate this drawback, we will henceforth suppose that the means are always defined on a compact interval. It will be with no loss of generality, because it is easy to check that a family of means defined on $U$ is a scale on $U$ if and only if those means form a scale on $D$, when treated as functions $D \to \mathbb{R}$, for every closed subinterval $D \subset U$. Indeed, if $\underline{a}$ is a vector with values in $U$, then $\underline{a}$ is also a vector with values in $D$ for some closed subinterval $D$ of $U$.

So, from now on, we have $U$ – a compact interval, $g \in \mathcal{C}^{2\neq}(U)$ increasing, and, clearly, $\boldsymbol{g} \in L^1(U)$. The following theorem is of utmost technical importance.

**Theorem 3.** *Let $U$ be a closed, bounded interval and $f, g \in C^{2\neq}(U)$. Then*

$$|\mathfrak{M}_f(\underline{a}, \underline{w}) - \mathfrak{M}_g(\underline{a}, \underline{w})| \le (\max \underline{a} - \min \underline{a}) \exp(2 \|A(f)\|_1) \sinh 2 \|A(g) - A(f)\|_1$$

*for all $\underline{a}$ and $\underline{w}$ ($\|\cdot\|_1$ is taken in the space $L^1(U)$).*

Proof. Solving a simple differential equation, in view of Remark 1, it is possible to assume, for function $f$, that

$$f(x) = \int_{\min \underline{a}}^{x} \exp\left(\int_{\min \underline{a}}^{s} \boldsymbol{f}(t)dt\right) ds, \quad x \in U. \tag{1}$$

Moreover, let us make the same simplification for $g$. Then $f(\min \underline{a}) = g(\min \underline{a}) = 0$ and both functions are positive and increasing on $(\min \underline{a}, \max \underline{a})$.

Much like in [10, pp. 215–216], we have

$$\mathfrak{M}_f(\underline{a}, \underline{w}) - \mathfrak{M}_g(\underline{a}, \underline{w}) = (f^{-1})'(\alpha) \sum_{1 \le i < j \le n} w_i w_j \Big(g(a_i) - g(a_j)\Big)\Big(\theta(z_i) - \theta(z_j)\Big),$$

where $\theta = (f \circ g^{-1})'$, for certain $\alpha \in [\min \underline{a}, \max \underline{a}]$ and $z_i \in g(U)$, $i = 1, \ldots, n$.

The vector $\underline{w}$ denotes, as usual, weights so $\sum_{1\le i<j\le n} w_i w_j < \frac{1}{2}$. Hence

$$\begin{aligned}
|\mathfrak{M}_f(\underline{a},\underline{w}) - \mathfrak{M}_g(\underline{a},\underline{w})| &= \left|(f^{-1})'(\alpha) \sum_{1\le i<j\le n} w_i w_j \Big(g(a_i)-g(a_j)\Big)\Big(\theta(z_i)-\theta(z_j)\Big)\right| \\
&\le \frac{\left\|(f^{-1})'\right\|_\infty}{2}\, g(\max\underline{a}) \sup_{z,v\in g(U)} |\theta(z)-\theta(v)|
\end{aligned}$$

Putting $\varepsilon := \|\boldsymbol{f}-\boldsymbol{g}\|_1$, we assuredly have

$$\frac{f'}{g'} = e^{\int \boldsymbol{f}-\boldsymbol{g}} \in (e^{-\varepsilon},\, e^{\varepsilon})\,.$$

Thus $\theta(z) = (f\circ g^{-1})'(z) = \frac{f'\circ g^{-1}(z)}{g'\circ g^{-1}(z)} \in (e^{-\varepsilon},\, e^{\varepsilon})$. What is more,

$$g(\max\underline{a}) = \int_{\min\underline{a}}^{\max\underline{a}} g'(s)\,ds \le \int_{\min\underline{a}}^{\max\underline{a}} e^{\varepsilon} f'(s)\,ds = e^{\varepsilon} f(\max\underline{a})\,.$$

Hence, estimating further,

$$\begin{aligned}
|\mathfrak{M}_f(\underline{a},\underline{w}) - \mathfrak{M}_g(\underline{a},\underline{w})| &\le \frac{\left\|(f^{-1})'\right\|_\infty}{2}\, g(\max\underline{a}) \sup_{z,v\in g(U)} |\theta(z)-\theta(v)| \\
&\le \frac{\left\|(f^{-1})'\right\|_\infty e^{\varepsilon}}{2} f(\max\underline{a}) \left(e^{\varepsilon}-e^{-\varepsilon}\right) \\
&= \frac{f(\max\underline{a})}{\inf f'} \cdot \frac{e^{2\varepsilon}-1}{2} \\
&\le \frac{f(\max\underline{a})}{\inf f'} \sinh 2\varepsilon\,.
\end{aligned}$$

But, by (1), we also know that

$$\begin{aligned}
f(\max\underline{a}) &= \int_{\min\underline{a}}^{\max\underline{a}} \exp\Big(\int_{\min\underline{a}}^{s} \boldsymbol{f}\Big) \\
&\le (\max\underline{a}-\min\underline{a}) \exp(\|\boldsymbol{f}\|_{L^1(\min\underline{a},\max\underline{a})}) \\
&\le (\max\underline{a}-\min\underline{a}) \exp(\|\boldsymbol{f}\|_1) \qquad (2)
\end{aligned}$$

and

$$\inf f' = \inf_{s\in U} \exp\Big(\int_{\min\underline{a}}^{s} \boldsymbol{f}\Big) \ge \exp(-\|\boldsymbol{f}\|_1). \qquad (3)$$

So, prolonging the previous chain of estimations and using (2) and (3),

$$|\mathfrak{M}_f(\underline{a},\underline{w})-\mathfrak{M}_g(\underline{a},\underline{w})| \le (\max\underline{a}-\min\underline{a})\exp(2\,\|\boldsymbol{f}\|_1)\sinh 2\,\|\boldsymbol{g}-\boldsymbol{f}\|_1 .$$

□

**Remark.** *Theorem 3 would remain true if* $\|\cdot\|_1$ *denote the standard norm in the space* $L^1(\min\underline{a},\max\underline{a})$.

One immediately gets the following

**Corollary 3.** *Let* $U$ *be a closed, bounded interval and* $f\in\mathcal{C}^{2\neq}(U)$. *Moreover, let* $(f_n)_{n\in\mathbb{N}}$ *be a sequence of functions from* $\mathcal{C}^{2\neq}(U)$ *satisfying* $\boldsymbol{f_n}\to\boldsymbol{f}$ *in* $L^1(U)$. *Then* $\mathfrak{M}_{f_n}\rightrightarrows\mathfrak{M}_f$ *uniformly with respect to* $\underline{a}$ *and* $\underline{w}$.

Heading towards the main results of the note, we state now

**Proposition 4.** *Let* $U$ *be a closed bounded interval,* $I=(a,b)$ *– an open interval,* $(k_\alpha)_{\alpha\in I}$ *– a family of functions from* $\mathcal{C}^{2\neq}(U)$.
(A) *If* $(\mathfrak{M}_{k_\alpha})_{\alpha\in I}$ *is an increasing scale then* $(A(k_\alpha))_{\alpha\in I}$ *satisfies all the conditions* (a) *through* (d) *listed below.*

(a) *if* $\alpha_i\to\alpha$ , *then* $A(k_{\alpha_i})\to A(k_\alpha)$ *on a dense subset of* $U$ *(independent of* $\alpha$ *and* $(\alpha_i)$*),*

(b) *if* $\alpha<\beta$, *then* $A(k_\alpha)<A(k_\beta)$ *on a dense subset of* $U$ *(independent of* $\alpha$ *and* $\beta$*),*

(c) *if* $\alpha\to a+$, *then* $A(k_\alpha)(x)\to-\infty$ *on a dense subset of* $U$ *(independent of the sequence* $\alpha$*),*

(d) *if* $\beta\to b-$, *then* $A(k_\beta)(x)\to+\infty$ *on a dense subset of* $U$ *(independent of the sequence* $\beta$*).*

(B) *Strengthening conditions* (a), (c) *and* (d) *to*

(e) *if* $\alpha_i\to\alpha$ , *then* $A(k_{\alpha_i})\to A(k_\alpha)$,

(f) $(\alpha\to a+\Rightarrow A(k_\alpha)(x)\to-\infty)$ *and* $(\beta\to b-\Rightarrow A(k_\beta)(x)\to+\infty)$ *for all* $x\in U$

*suffices to reverse the implication:* (b), (e), *and* (f) *imply* $(\mathfrak{M}_{k_\alpha})_{\alpha\in I}$ *being an increasing scale.*

Proof. To simplify the notation, having $\underline{a}$ and $\underline{w}$ fixed, we write shortly

$$F(\alpha)=\mathfrak{M}_{k_\alpha}(\underline{a},\,\underline{w}),$$

$F\colon I\to(\min\underline{a},\,\max\underline{a})$. And then one simply checks step by step:

(a) With no loss of generality one may consider $\alpha_i \to \alpha+$.

Suppose the converse – that there exists an open subset $V \subset U$, that $A(k_{\alpha_i}) \not\to A(k_\alpha)$ on $V$. Then there exists $m \in C^{2\neq}(V)$ such that $A(k_{\alpha_i}) < A(m) < A(k_\alpha)$. Hence for all $i$ and non-constant vector $\underline{a}$ with corresponding weights $\underline{w}$, $\mathfrak{M}_{k_{\alpha_i}} < \mathfrak{M}_m < \mathfrak{M}_{k_\alpha}$.

One need to prove that such a dense set may not depend on $\alpha$. Let

$$X_{(a)} := \Big\{x\colon \text{for an arbitrary } \alpha, \text{ if } \alpha_i \to \alpha, \text{ then } \boldsymbol{k_{\alpha_i}}(x) \to \boldsymbol{k_\alpha}(x)\Big\}.$$

Purely formally

$$X_{(a)} = \Big\{x\colon \forall\alpha \in I,\ \forall\varepsilon > 0,\ \exists\delta > 0,\ \forall\beta \in B(\alpha,\delta),\ |\boldsymbol{k_\alpha}(x) - \boldsymbol{k_\beta}(x)| < \varepsilon\Big\},$$

equivalently (using the monotonicity of the mapping $\alpha \mapsto \boldsymbol{k_\alpha}(x)$ for all $x \in U$) one obtain

$$\begin{aligned} X_{(a)} &= \Big\{x\colon \forall\alpha \in I,\ \forall\varepsilon > 0,\ \exists\delta > 0,\ |\boldsymbol{k_{\alpha-\delta}}(x) - \boldsymbol{k_{\alpha+\delta}}(x)| < \varepsilon\Big\} \\ &= \Big\{x\colon \forall\alpha \in I \cap \mathbb{Q},\ \forall\varepsilon \in \mathbb{Q}_+,\ \exists\delta > 0,\ |\boldsymbol{k_{\alpha-\delta}}(x) - \boldsymbol{k_{\alpha+\delta}}(x)| < \varepsilon\Big\} \\ &= \bigcap_{\substack{\alpha\in I\cap\mathbb{Q} \\ \varepsilon\in\mathbb{Q}_+}} \big\{x\colon \exists\delta > 0,\ |\boldsymbol{k_{\alpha-\delta}}(x) - \boldsymbol{k_{\alpha+\delta}}(x)| < \varepsilon\big\}. \end{aligned}$$

But, as $\alpha_i \to \alpha$, $\boldsymbol{k_{\alpha_i}} \to \boldsymbol{k_\alpha}$ on a dense subset of $U$. Thus

$$\{x\colon \exists\delta > 0,\ |\boldsymbol{k_{\alpha-\delta}}(x) - \boldsymbol{k_{\alpha+\delta}}(x)| < \varepsilon\}$$

is dense and open for all $\varepsilon > 0$ and $\alpha \in I$. Lastly $X_{(a)}$ is a dense $G_\delta$-set.

(b) if $\alpha < \beta$, we have $F(\alpha) \le F(\beta)$ and the equality holds if and only if $\underline{a}$ is constant. So by Proposition 2 we have $\boldsymbol{k_\alpha} < \boldsymbol{k_\beta}$ on a dense set.
Let

$$\begin{aligned} X_{(b)} &:= \Big\{x \in U : \forall\alpha,\ \forall\beta \neq \alpha,\ \boldsymbol{k_\alpha}(x) \neq \boldsymbol{k_\beta}(x)\Big\}, \\ E_{\alpha,\beta} &:= \Big\{x \in U\colon \boldsymbol{k_\alpha}(x) \neq \boldsymbol{k_\beta}(x)\Big\}. \end{aligned}$$

We have that if $[\alpha',\, \beta'] \subset [\alpha,\, \beta]$ then $E_{\alpha,\beta} \subset E_{\alpha',\,\beta'}$, and $E_{\alpha,\beta}$ is an open, dense set. Thus

$$X_{(b)} = \bigcap_{\substack{\alpha,\beta\in I \\ \alpha\neq\beta}} E_{\alpha,\beta} = \bigcap_{\substack{\alpha,\beta\in I\cap\mathbb{Q} \\ \alpha\neq\beta}} E_{\alpha,\beta}$$

is a dense $G_\delta$-set.

(c) The proof is completely similar to that of (d) given below.

(d) Let

$$X_{(d)} = \{x : \lim_{\beta\to b-} \boldsymbol{k_\beta}(x) \to +\infty\}.$$

If $X_{(d)}$ were not a dense set, then there would exist a closed interval $C$ such that $\operatorname{dist}(X_{(d)}, C) > 0$. Let

$$M := \sup_{x\in C} \lim_{\beta\to b-} \boldsymbol{k_\beta}(x)\,.$$

Clearly $M < +\infty$, and

$$\mathfrak{M}_{k_\beta}(\underline{v}, \underline{q}) \le \mathfrak{M}_{e^{Mx}}(\underline{v}, \underline{q}) < \max \underline{v}$$

for all $\beta$ and $\underline{v}$, $\underline{q}$ such that $\underline{v} \in C^n$ is a non-constant vector. Hence, the family $\{k_\beta\}$ would not generate a scale on $U$. So $X_{(d)}$ is a dense set.

To prove part (B) one needs to show that, under (e) and (f), $(\mathfrak{M}_{k_\alpha})_{\alpha\in I}$ is a scale on $U$. By Proposition 2 we know that $F$ is 1–1. Additionally, when arguing to this side, we know that if $\alpha \nearrow \alpha_0$ then $\boldsymbol{k_\alpha} \nearrow \boldsymbol{k_{\alpha_0}}$. So $\boldsymbol{k_\alpha} \rightrightarrows \boldsymbol{k_{\alpha_0}}$ on $[\min \underline{a}, \max \underline{a}]$. Therefore, by Corollary 3, we have $\mathfrak{M}_{k_\alpha} \rightrightarrows \mathfrak{M}_{k_{\alpha_0}}$ with respect to $\underline{a}$ and $\underline{w}$. Thus $F$ is continuous and 1–1.

To complete the proof, it is sufficient to show that

$$\lim_{\alpha\to a+} F(\alpha) = \min \underline{a}\,, \qquad \lim_{\beta\to b-} F(\beta) = \max \underline{a}\,.$$

We know that $\boldsymbol{k_\beta} \to +\infty$ on the closed interval $U$. So $\boldsymbol{k_\beta} \rightrightarrows +\infty$ on $U$. Therefore, for any $M \in \mathbb{R}$ there exists $\beta_M \in I$ such that

$$F(\beta) \ge \mathfrak{M}_{e^{Mx}}(\underline{a}, \underline{w}), \text{ for all } \beta > \beta_M.$$

Now, letting $M \to +\infty$, and knowing that $\{e^{tx} : t \neq 0\} \cup \{x\}$ generates a scale on $\mathbb{R}$ (a folk-type theorem proved in [8]; see also Remark 2 below) we get

$$F(\beta) \xrightarrow[\beta\to b-]{} \max \underline{a}\,.$$

One may similarly prove that

$$F(\alpha) \xrightarrow[\alpha\to a+]{} \min \underline{a}\,.$$

So $F$ is a continuous bijection between $I$ and $(\min \underline{a}, \max \underline{a})$. Hence $(\mathfrak{M}_{k_\alpha})_{\alpha\in I}$ is a scale on $U$. $\square$

**Remark 2.** *To prove that the family* $\{e^{tx} : t \neq 0\} \cup \{x\}$ *generates a scale on* $\mathbb{R}$ *it is enough, having data* $\underline{a}$, $\underline{w}$, *to consider the all-positive-components-vector* $\underline{v} = (e^{a_1}, \dots, e^{a_n})$. *And then to use the fact that the family of power means evaluated on* $\underline{v}$ *with weights* $\underline{w}$ *is a scale on* $\mathbb{R}_+$.

**Corollary 5** (strengthening of Proposition 4)**.** *Let* $U$ *be an interval,* $I = (a, b)$ *– an open interval,* $(k_\alpha)_{\alpha \in I}$ *– a family of functions,* $k_\alpha \in \mathcal{C}^{2\neq}(U)$ *for all* $\alpha$.
(A) *If* $(\mathfrak{M}_{k_\alpha})_{\alpha \in I}$ *is an increasing scale then there exists a dense set* $X \subset U$ *such that*

(a) *if* $\alpha_i \to \alpha$, *then* $A(k_{\alpha_i}) \to A(k_\alpha)$ *on* $X$,

(b) *if* $\alpha < \beta$ , *then* $A(k_\alpha) < A(k_\beta)$ *on* $X$,

(c) *if* $\alpha \to a+$, *then* $A(k_\alpha)(x) \to -\infty$ *on* $X$,

(d) *if* $\beta \to b-$, *then* $A(k_\beta)(x) \to +\infty$ *on* $X$.

(B) *Under the stronger condition*

(e) *if* $\alpha_i \to \alpha$, *then* $A(k_{\alpha_i}) \to A(k_\alpha)$,

(f) *(*$\alpha \to a+ \Rightarrow A(k_\alpha)(x) \to -\infty$*) and (*$\beta \to b- \Rightarrow A(k_\beta)(x) \to +\infty$*) for all* $x \in U$

*the entire implication of the corollary can be reversed:* (b), (e), *and* (f) *imply* $(\mathfrak{M}_{k_\alpha})_{\alpha \in I}$ *being an increasing scale.*

This corollary says that, in Proposition 4, one can have a single common subset ( $X$ ) of $U$ on which the conditions (a) through (d) hold.

Proof. The (B) parts in Proposition 4 and Corollary 5 are the same except the notion of $X$ given to the dense set in (B) in Proposition 4. We are going to show the implication (A).

We might assume that $U$ is a closed interval (compare the comment in the third paragraph below Proposition 2).

Let us denote a dense sets appearing in (a) through (d) in the Proposition 4 by $X_{(a)}, \dots, X_{(d)}$. Our aim is to prove that each of these sets is a dense $G_\delta$-set. By [the proof of] Proposition 4, $X_{(a)}$ and $X_{(b)}$ are declared to be dense $G_\delta$-sets. By definition

$$X_{(d)} = \{x \in U : \lim_{\beta \to b-} \boldsymbol{k_\beta}(x) \to +\infty\}.$$

Due to Proposition 4 we know that $X_{(d)}$ is dense. Let

$$Y_s := \{x \in U : \lim_{\beta \to b-} \boldsymbol{k_\beta}(x) > s\}.$$

Observe that $Y_s$ is dense (because $Y_s \supset X_{(d)}$). Moreover, for all $x_0 \in Y_s$ there holds $\boldsymbol{k_{\beta_0}}(x_0) > s + \delta$ for some $\beta_0 \in I$ and $\delta > 0$. Hence one may take an open neighborhood $P \ni x_0$ satisfying $\boldsymbol{k_{\beta_0}}(x) > s + \frac{1}{2}\delta$ for all $x \in P$, implying $P \subset Y_s$. So $Y_s$ is open. But the mapping $\beta \mapsto \boldsymbol{k_\beta}(x)$ is nondecreasing for all $x \in U$. Hence $X_{(d)} = \bigcap_{s=1}^{\infty} Y_s$ is a dense $G_\delta$-set. So is the set $X_{(c)}$.

Now one may take $X := X_{(a)} \cap X_{(b)} \cap X_{(c)} \cap X_{(d)}$. $X$ is clearly dense being a countable intersection of open dense sets. □

## 4 Proofs of Theorems 1 and 2

Proof of Theorem 1. Let $U$ be an interval, $I = (a, b)$ – an open interval, and $X$ be that dense subset of $U$ upon witch the mapping given in the wording of theorem is increasing and 1–1. We work with the family of functions $(k_\alpha)_{\alpha \in I}$, $k_\alpha \in \mathcal{C}^{2\neq}(U)$ for all $\alpha$.

Let us take an arbitrary $x_0 \in X$. We know that $I \ni \alpha \mapsto \boldsymbol{k_\alpha}(x_0)$ is increasing, 1–1, and onto $\mathbb{R}$. Next, let us specify the function $\Phi : \mathbb{R} \to I$ such that $\boldsymbol{k_{\Phi(\alpha)}}(x_0) = \alpha$. This function is increasing as well.

Then for $\alpha < \beta$ we have $\boldsymbol{k_{\Phi(\alpha)}} < \boldsymbol{k_{\Phi(\beta)}}$ on $X$. But, due to the fact that $I \ni \alpha \mapsto \boldsymbol{k_\alpha}(x) \in \mathbb{R}$ is onto, we have

$$\lim_{\alpha \to a} \boldsymbol{k_{\Phi(\alpha)}}(x) = -\infty \qquad \text{and} \qquad \lim_{\beta \to b} \boldsymbol{k_{\Phi(\beta)}}(x) = +\infty$$

everywhere on $U$. So one is in a position to use the part (B) of Corollary 5. Thus the family of means $(\mathfrak{M}_{k_\alpha})_{\alpha \in I}$ is an increasing scale on $U$. □

Proof of Theorem 2. Let us take $X$ from Corollary 5. Let then fix any $x_0 \in X$. Let $\{s_p\}_{p \in \mathbb{R}}$ be the reparameterized family $\{k_\alpha\}_{\alpha \in I}$, with restriction

$$s_p = k_\alpha \text{ , where } p = \boldsymbol{k_\alpha}(x_0).$$

Then we know that the mapping

$$\mathbb{R} \ni p \mapsto \boldsymbol{s_p}(x) \in \mathbb{R}$$

is 1–1 and onto for all $x \in X$, and, if $p > q$,

$$\boldsymbol{s_p}(x) > \boldsymbol{s_q}(x).$$

Moreover, due to the fact that $\boldsymbol{s_p}(x_0)$ is onto, we have for all $x_0$

$$\lim_{p \to -\infty} \boldsymbol{s_p}(x_0) = -\infty \qquad \text{and} \qquad \lim_{p \to +\infty} \boldsymbol{s_p}(x_0) = +\infty\,.$$

So $p \mapsto \boldsymbol{s_p}(x)$ is increasing, 1–1, and onto $\mathbb{R}$ for all $x \in X$. □

## 5 Applications

**Proposition 6** (power means do generate a scale)**.** *Let* $U = \mathbb{R}_+$ *and* $(k_\alpha)_{\alpha\in\mathbb{R}}$*, given by*

$$k_\alpha(x) = \begin{cases} x^\alpha & \alpha \neq 0 \\ \ln x & \alpha = 0 \end{cases},$$

*be the family of power functions. Then the family* $(k_\alpha)$ *generates a scale on* $\mathbb{R}_+$*.*

PROOF. We compute $\boldsymbol{k_\alpha}$,

$$\boldsymbol{k_\alpha}(x) \;=\; \frac{\alpha - 1}{x}\,,$$

and see that the mapping $\alpha \mapsto \boldsymbol{k_\alpha}(x)$ is increasing, 1–1 and onto for all $x \in \mathbb{R}_+$. So the assumptions in Theorem 1 hold, implying that the family $(k_\alpha)$ generates an increasing scale on $\mathbb{R}_+$. □

Before giving our second application we reproduce a 10 years old' result.

**Proposition 7** ([14])**.** *Let* $k\colon [0,1] \to \mathbb{R}$ *be a continuous monotone function. Writing* $k_\alpha(x) := k(x^\alpha)$ *for any* $\alpha > 0$*, there hold:*

(i) *if there exists the one side, nonzero derivative* $k'(0+)$ *then*

$$\lim_{\alpha\to+\infty} \mathfrak{M}_{k_\alpha} \;=\; \max\,,$$

(ii) *if there exists the one side, nonzero derivative* $k'(1-)$ *then*

$$\lim_{\alpha\to 0+} \mathfrak{M}_{k_\alpha} \;=\; \mathfrak{M}_{\ln x}\,.$$

We prove a somehow similar, yet not so close, result.

**Proposition 8.** *Let* $k \in \mathcal{C}^{2\neq}[0,1] \to (0, +\infty)$ *and* $k_\alpha(x) := k(x^\alpha)$*,* $\alpha \in (0, +\infty)$*. Then*

$$\lim_{\alpha\to 0+} \mathfrak{M}_{k_\alpha} = \mathfrak{M}_{\ln x} \qquad \textit{and} \qquad \lim_{\alpha\to+\infty} \mathfrak{M}_{k_\alpha} = \max\,. \tag{4}$$

*If, in addition,* $k$ *is convex,*[3] *then* $(k_\alpha)_{\alpha\in(0,+\infty)}$ *generates a scale between the geometric mean and* max.

[3] in this situation we can just assume that $k \in \mathcal{C}^2[0,1]$ and $k$ is strictly monotone, instead of assuming $k \in \mathcal{C}^{2\neq}[0,1]$

PROOF. We have to prove that the mapping $(0, +\infty) \ni \alpha \mapsto \boldsymbol{k_\alpha}(x) \in \mathbb{R}$ is 1–1 and onto for all $x \in (0, 1)$. Let us fix an arbitrary $x \in (0, 1)$. Then we have

$$\boldsymbol{k_\alpha}(x) \;=\; \alpha x^{\alpha-1}\boldsymbol{k}(x^\alpha) + \frac{\alpha-1}{x}\,.$$

When $\alpha \to 0+$, then

$$\boldsymbol{k_0}(x) := \lim_{\alpha\to 0+} \boldsymbol{k_\alpha}(x) \;=\; \frac{-1}{x}\,.$$

In turn, when $\alpha \to +\infty$, there holds

$$\boldsymbol{k_\alpha}(x) \;=\; \underbrace{\alpha x^{\alpha-1}\frac{k''(0)}{k'(0)}}_{>-\infty} + \frac{\alpha-1}{x} \to +\infty\,.$$

The proof of formulas (4) is now completed.

When, additionally, $g$ is convex, then $\boldsymbol{k} \geq 0$ and, by Corollary 1, the family $\{k_\alpha\}_{\alpha\in\mathbb{R}_+}$ generates a scale on $(0,1)$ between the geometric mean and max. □

Now we would like to present one classical result of the Italian school of statisticians from 1910-20s. That result has been reported in [6, p. 269]. We now give it a new short proof based on Corollary 1.

**Proposition 9** (Radical Means)**.** *Let $U = \mathbb{R}_+$ and $(k_\alpha)_{\alpha\in\mathbb{R}_+}$, $k_\alpha(x) = \alpha^{1/x}$ for $\alpha \neq 1$, completed by $k_1(x) = 1/x$, be the family of radical functions. Then this family generates a decreasing scale on $\mathbb{R}_+$.*

PROOF. The proof appears to be quite close to the proof of Proposition 6. Indeed, we quickly compute

$$\boldsymbol{k_\alpha}(x) = -\frac{2x + \ln\alpha}{x^2}\,,$$

finding that the mapping $\alpha \mapsto \boldsymbol{k_\alpha}(x)$ is decreasing, 1–1 and onto for all $x \in \mathbb{R}_+$. So the assumptions in Corollary 1 hold, and hence the family $(k_\alpha)_{\alpha\in\mathbb{R}_+}$ generates a decreasing scale on $\mathbb{R}_+$. □

**Open problem.** How to unify Theorem 1 and Theorem 2 so as to get a set of conditions that would simultaneously be necessary and sufficient?

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