RESEARCH

Real Analysis Exchange
Vol. 37(2), 2011/2012, pp. 389–410

Antonio Boccuto,* Dipartimento di Matematica e Informatica, via Vanvitelli, 1, I-06123 Perugia, Italy. email: boccuto@yahoo.it, boccuto@dmi.unipg.it

Pratulananda Das, Department of Mathematics, Jadavpur University, Kolkata-700032, West Bengal, India. email: pratulananda@yahoo.co.in

Xenofon Dimitriou, Department of Mathematics, University of Athens, Panepistimiopolis, Athens 15784, Greece. email: xenofon11@gmail.com, dxenof@math.uoa.gr

Nikolaos Papanastassiou, Department of Mathematics, University of Athens, Panepistimiopolis, Athens 15784, Greece. email: npapanas@math.uoa.gr

IDEAL EXHAUSTIVENESS, WEAK CONVERGENCE AND WEAK COMPACTNESS IN BANACH SPACES

Abstract

Some types of compactness in the ideal context are defined and relations between ideal exhaustiveness and equicontinuity of measures are investigated. As applications, some versions of limit theorems involving ideal pointwise convergence of measure sequences and some weak compactness results related to integral functionals are presented.

1 Introduction.

Convergence with respect to ideals was introduced in [31] and further investigated in (see [6, 16, 17, 18, 19, 20, 21, 22, 30, 32]). It has been deeply studied

Mathematical Reviews subject classification: Primary: 28A20, 28B15; Secondary: 54A20
Key words: (admissible) ideal, P-ideal, ideal convergence, ideal continuous convergence, (uniform) ideal exhaustiveness, (uniform) ideal s-boundedness, (uniform) ideal countable additivity, ideal closure, ideal (weak) Fréchet compactness, ideal limit point, ideal (weak) sequential compactness, norm ideal convergence, weak ideal convergence
Received by the editors July 23, 2011
Communicated by: Luisa Di Piazza
*Corresponding Author.
Supported by Universities of Athens, Jadavpur and Perugia.

in the literature particularly in convergence of functions (see [4, 8, 13, 28, 31]) and convergence of measures and integrals (see [7, 8, 9, 10, 11, 12]). Note that, in general, ideal convergence is strictly weaker than ordinary convergence (see [30, 31]).

In this paper we firstly introduce compactness in the context of ideals and we present some comparison results about several new features of compactness in metric spaces.

Successively we focus our attention on limit theorems for measures with respect to the ideal convergence. Under suitable hypotheses it is possible to prove some versions of limit theorems even if we require the simple ideal setwise convergence of the involved measure sequence (see e.g. [9, 10, 11, 12]). Within this framework new results are established and some classical results in the literature are reproved under strictly weaker hypotheses. The main tools are the notions of ideal (α)-convergence and ideal exhaustiveness previously introduced and studied for function sequences in [13] (see also [27, 29]). Recently, some other versions of limit theorems for measures were proved in [3, 9, 10, 11, 12] with respect to this kind of convergence.

Weak convergence in measure spaces is characterized and some fundamental properties of various kinds of ideal compactness for suitable sets of measures are investigated. As an application we study the convergence in $L^\infty(\lambda)$, for λ being a regular measure.

Our thanks to the referee for his/her valuable and helpful comments and suggestions.

2 Preliminaries.

We begin with introducing the fundamental properties of ideals (see also [16, 17, 18, 22, 30, 31, 32]).

Definitions 2.1.

(a) Let $Y \neq \emptyset$ be any set. A family $\mathcal{I} \subset \mathcal{P}(Y)$ is called an *ideal* of Y iff $A \cup B \in \mathcal{I}$ whenever $A, B \in \mathcal{I}$ and for each $A \in \mathcal{I}$ and $B \subset A$ we get $B \in \mathcal{I}$.

(b) An ideal $\mathcal{I}$ of Y is said to be *non-trivial* iff $\mathcal{I} \neq \emptyset$ and $Y \notin \mathcal{I}$. A non-trivial ideal $\mathcal{I}$ is called *admissible* iff it contains all singletons.

(c) An admissible ideal $\mathcal{I}$ of $\mathbb{N}$ is said to be a *P-ideal* iff for any sequence $(A_j)_j$ in $\mathcal{I}$ there are sets $B_j \subset \mathbb{N}$, $j \in \mathbb{N}$, such that the symmetric difference $A_j \Delta B_j$ is finite for all $j \in \mathbb{N}$ and $\bigcup_{j=1}^{\infty} B_j \in \mathcal{I}$.

Several properties of P-ideals are investigated, for instance, in [26].

From now on we denote by X a Banach space, $\mathcal{I}$ an admissible ideal of $\mathbb{N}$ and $\mathcal{F} = \mathcal{F}(\mathcal{I}) = \{\mathbb{N} \setminus A : A \in \mathcal{I}\}$ its dual filter.

We now recall the norm and weak ideal convergence in the context of Banach spaces. For recent developments and studies about these concepts, see also [5, 14, 15, 34].

Definitions 2.2.

(a) A sequence $(x_n)_n$ in X is called *norm $\mathcal{I}$-convergent* to $x \in X$ iff for all $\varepsilon > 0$, $\{n \in \mathbb{N}\colon \|x_n - x\| > \varepsilon\} \in \mathcal{I}$, where $\|\cdot\|$ denotes the norm of X. We then write $\mathcal{I} - \lim_n x_n = x$.

(b) A sequence $(x_n)_n$ in X is called *norm $\mathcal{I}$-Cauchy* iff for each $\varepsilon > 0$ there exists $q \in \mathbb{N}$ such that $\{n \in \mathbb{N}\colon \|x_n - x_q\| > \varepsilon\} \in \mathcal{I}$.

(c) A sequence $(x_n)_n$ is called *weakly-$\mathcal{I}$-convergent* iff the sequence $(x^*(x_n))_n$ is $\mathcal{I}$-convergent for every $x^* \in X^*$ (the dual space of X). A sequence $(x_n)_n$ is said to be *weakly-$\mathcal{I}$-Cauchy* iff the sequence $(x^*(x_n))_n$ is $\mathcal{I}$-Cauchy for every $x^* \in X^*$.

(d) If $(x_j)_j$ is any sequence in X, then we define $\mathcal{I} - \sum_{j=1}^{\infty} x_j = \mathcal{I} - \lim_n \sum_{j=1}^{n} x_j$, provided the limit on the right hand side exists.

Note that, since X is complete, a sequence is norm (weakly-)$\mathcal{I}$-convergent iff it is norm (weakly-)$\mathcal{I}$-Cauchy (see also [16, 17, 18, 22]).

Examples 2.3. (i) If $\mathcal{I}_{fin} = \{A \subset \mathbb{N} : A \text{ finite}\}$, then $\mathcal{I}_{fin}$ is a P-ideal of $\mathbb{N}$ and $\mathcal{I}_{fin}$-convergence coincides with the ordinary convergence (see also [13, 31]).

(ii) Let $A \subset \mathbb{N}$. The *asymptotic density* of A is defined as

$$d(A) := \lim_n \frac{card(A \cap \{1, \dots, n\})}{n}$$

(provided that this limit exists), where card denotes the cardinality of the set in brackets. If $\mathcal{I}_d = \{A \subset \mathbb{N} : d(A) = 0\}$, then $\mathcal{I}_d$ is a P-ideal of $\mathbb{N}$ (see [13, 30, 31]).

We now recall another kind of convergence in the context of ideals and some fundamental properties (see [30, 31, 32]).

Definition 2.4. A sequence $(x_n)_n$ in X $\mathcal{I}^*$-*converges* to $x \in X$ iff there exists $A \in \mathcal{F}(\mathcal{I})$ with $\lim_{n \in A} x_n = x$.

Proposition 2.5. *The following results hold (see for instance* [9, 10, 11, 12, 30, 31]).

(a) If $\lim_n x_n = x$ *then* $\mathcal{I} - \lim_n x_n = x$. *Moreover, if* $X = \mathbb{R}$ *and* $(x_n)_n$ *is a monotone sequence in* $\mathbb{R}$*, then the converse is also true.*

(b) If $(x_n)_n$ *is a sequence in* $\mathbb{R}$ *with* $\mathcal{I} - \lim_n x_n = x \in \mathbb{R}$*, then there exists a subsequence* $(x_{n_q})_q$ *of* $(x_n)_n$*, such that* $\lim_q x_{n_q} = x$.

(c) The $\mathcal{I}^*$*-convergence of sequences implies always the* $\mathcal{I}$*-convergence to the same limit. Moreover, if* $(x_n)_n$ *is a sequence in* X*,* $\mathcal{I}$*-convergent to* $x \in X$*, and* $\mathcal{I}$ *is a* P*-ideal, then* $(x_n)_n$ $\mathcal{I}^*$*-converges to* x.

(d) Let $(x_{i,j})_{i,j}$ *be a double sequence in* X*,* $\mathcal{I}$ *be any* P*-ideal,* $\mathcal{F} = \mathcal{F}(\mathcal{I})$ *be its dual filter, and let us suppose that* $\mathcal{I} - \lim_i x_{i,j} = x_j$ *for every* $j \in \mathbb{N}$*. Then there exists* $B_0 \in \mathcal{F}$ *such that* $\lim_{h \in B_0} x_{h,j} = x_j$ *for all* $j \in \mathbb{N}$.

3 Compactness notions in the ideal context.

We now introduce the concept of sequential closure and some kinds of compactness. We will say that P-ideals have good properties, since many situations in the context of P-ideals are very similar to the corresponding classical ones.

Definitions 3.1.

(a) For $F \subset X$ and $u \in X$, we say that u is *in the* $\mathcal{I}$*-closure of* F iff there is a sequence $(x_n)_n$ of points of F such that $\mathcal{I} - \lim_n x_n = u$. We denote the $\mathcal{I}$-closure of F by $\overline{F}^I$.

(b) A point $u \in X$ is called an $\mathcal{I}$*-limit point* of $F \subset X$ iff there is a sequence $(x_n)_n$ in $F \setminus \{u\}$ such that $\mathcal{I} - \lim_n x_n = u$.

(c) A subset $F \subset X$ is said to be $\mathcal{I}$*-sequentially compact* iff every sequence $(x_n)_n$ in F contains an $\mathcal{I}$-convergent subsequence $(x_{n_k})_k$ with $\mathcal{I} - \lim_k x_{n_k} \in F$.

(d) A set $F \subset X$ is called $\mathcal{I}$*-Fréchet compact* iff every infinite subset of F has an $\mathcal{I}$-limit point.

We now prove the equivalence between ideal and classical sequential compactness.

Proposition 3.2. *Given a subset* F *of* X*, we get that* F *is* $\mathcal{I}$*-sequentially compact if and only if it is sequentially compact.*

Proof. If F is $\mathcal{I}$-sequentially compact, then from every sequence $(x_n)_n$ in F it is possible to extract a subsequence $(x_{n_k})_k$ such that the limit $\mathcal{I} - \lim_k x_{n_k}$ exists in X, say x. By Proposition 2.5 (b) there exists a sub-subsequence $(x_{n_{k_q}})_q$ of $(x_{n_k})_k$ such that $x = \lim_q x_{n_{k_q}}$. Hence F is sequentially compact. The converse implication is straightforward. □

We now give the following compactness results.

Proposition 3.3. *Any $\mathcal{I}$-closed subset of an $\mathcal{I}$-sequentially compact subset of X is $\mathcal{I}$-sequentially compact.*

PROOF. Let K be an $\mathcal{I}$-sequentially compact set and $A \subset K$ be $\mathcal{I}$-closed. Let $(x_n)_n$ be any sequence in A. Then it is also in K and it admits an $\mathcal{I}$-convergent subsequence $(y_k)_k := (x_{n_k})_k$ with $\mathcal{I} - \lim_k y_k \in K$. But since A is $\mathcal{I}$-closed, so $\mathcal{I} - \lim_k y_k$ must be in $\overline{A}^I = A$. This shows that A is $\mathcal{I}$-sequentially compact. □

Proposition 3.4. *For a P-ideal $\mathcal{I}$, any $\mathcal{I}$-sequentially compact subset $K \subset X$ is $\mathcal{I}$-closed.*

PROOF. Let K be $\mathcal{I}$-sequentially compact. Let $u \in \overline{K}^I$. Then there is a sequence $(x_n)_n$ in K, which is $\mathcal{I}$-convergent to u. Since I is a P-ideal, it is also $\mathcal{I}^*$-convergent to u, namely there is a subsequence $(y_k)_k := (x_{n_k})_k$ with $\lim_k y_k = u$. Note that consequently $\mathcal{I} - \lim_k y_k = u$. By $\mathcal{I}$-sequential compactness of K there exists a subsequence $(z_j)_j := (y_{k_j})_j$ of $(y_k)_k$ which is $\mathcal{I}$-convergent to $u_1 \in K$. But, as $(z_j)_j$ is a subsequence of $(y_k)_k$, so $\lim_j z_j = u$, and thus $\mathcal{I} - \lim_j z_j = u$. Hence we must have $u = u_1$ and so $u \in K$. Thus $K = \overline{K}^I$ and K is $\mathcal{I}$-closed. □

Corollary 3.5. *If $\mathcal{I}$ is a P-ideal, then any $\mathcal{I}$-sequentially compact subset of X is closed in X.*

Proposition 3.6. *Let $\mathcal{I}$ be a P-ideal. A subset K of X is $\mathcal{I}$-sequentially compact iff it is $\mathcal{I}$-Fréchet compact.*

PROOF. "$\Rightarrow$" Let K be $\mathcal{I}$-sequentially compact and let A be an infinite subset of K. Choose a sequence $(x_n)_n$ of distinct points from A. This sequence has an $\mathcal{I}$-convergent subsequence $(y_k)_k := (x_{n_k})_k$ with $u := \mathcal{I} - \lim_k y_k \in K$. Moreover, since $\mathcal{I}$ is a P-ideal, then $(y_k)_k$ is also $\mathcal{I}^*$-convergent, and so it has a subsequence $(z_j)_j := (y_{k_j})_j$ with $\lim_j z_j = u$. Note that $\mathcal{I} - \lim_j z_j = u$ too, and hence u is an $\mathcal{I}$-limit point of A. Thus K is $\mathcal{I}$-Fréchet compact.

"$\Leftarrow$" Conversely, let us suppose that K is $\mathcal{I}$-Fréchet compact. Let $(x_n)_n$ be a sequence in K. Note that, if there exists $x \in K$ such that $x_n = x$ for infinitely many n's, then clearly x forms a constant subsequence, which is obviously $\mathcal{I}$-convergent. So without loss of generality we can assume that $(x_n)_n$ consists of distinct points only (the proof is the same if infinitely many distinct terms are each repeated only finitely many times).

Let now $B := \{x_n : n \in \mathbb{N}\}$. Then, since B is an infinite subset of K, it has an $\mathcal{I}$-limit point $u \in K$. Consequently there exists a sequence in $B \setminus \{u\}$, say $(y_k)_k$, such that $\mathcal{I} - \lim_k y_k = u$. Since $\mathcal{I}$ is a P-ideal, so $\mathcal{I}^* - \lim_k y_k = u$ and evidently $u \in \overline{B}$, the ordinary closure of B. Since X is Hausdorff, then $u \in \overline{B_n}^I$ and so $u \in \overline{B_n}$ for every n, where $B_n := \{x_k : k \geq n\}$. Thus $u \in \bigcap_n \overline{B_n}$. We can now easily construct a subsequence $(z_k)_k$ of $(x_n)_n$ which is $\mathcal{I}$-convergent to u. □

We now recall the Eberlein-Šmulian theorem (see also [33]).

Theorem 3.7. *Let A be a subset of a normed space. Then the following are equivalent:*

a) A is relatively weakly compact;
b) A is relatively weakly sequentially compact;
c) A is relatively weakly Fréchet compact;
d) A is relatively weakly countably compact.

We now give a sufficient condition to get ideal uniform convergence of function sequences on all compact sets of a metric space (Z, d), which will be useful in the sequel (see also [2]). We recall the notions of equicontinuity and ideal exhaustiveness for sequences of functions (see also [2, 13]).

Definitions 3.8.

(a) Let (Z, d) be a metric space. We say that a sequence of functions $f_n : Z \to \mathbb{R}$, $n \in \mathbb{N}$, is *equicontinuous on* Z iff for every $z \in Z$ and $\varepsilon > 0$ there exists a $\delta > 0$ with $|f_n(y) - f_n(z)| < \varepsilon$ whenever $n \in \mathbb{N}$ and $y \in Z$ with $d(y, z) < \delta$.

(b) A function sequence $f_n : Z \to \mathbb{R}$, $n \in \mathbb{N}$, is said to be *$\mathcal{I}$-exhaustive on* Z iff for every $z \in Z$ and $\varepsilon > 0$ there are a $\delta > 0$ and a set $C \in \mathcal{I}$ with $|f_n(y) - f_n(z)| < \varepsilon$ whenever $n \in \mathbb{N} \setminus C$ and $y \in Z$ with $d(y, z) < \delta$.

Remark 3.9. It is easy to see that every equicontinuous function sequence $(f_n)_n$ is $\mathcal{I}$-exhaustive too. Note that it is an essential generalization. Indeed, let $\mathcal{I}$ be any fixed admissible ideal, choose $H \in \mathcal{I}$, $H \neq \emptyset$, and let us define $f_n : \mathbb{R} \to \mathbb{R}$ by setting $f_n(0) = 0$ for all $n \in \mathbb{N}$, $f_n(x) = 0$ for any $x \in \mathbb{R}$ and $n \in \mathbb{N} \setminus H$, $f_n(x) = 1$ for every $x \neq 0$ and $n \in H$. Obviously, for each y, $z \in \mathbb{R}$ and $n \notin H$ we get $f_n(y) - f_n(z) = 0$ and thus $(f_n)_n$ is $\mathcal{I}$-exhaustive on $\mathbb{R}$. However $(f_n)_n$ is not equicontinuous on $\mathbb{R}$, since for $z = 0$ and $\varepsilon = 1$ we have that for every $\delta > 0$ there are $\overline{y} \in \mathbb{R}$ with $0 < |\overline{y}| < \delta$ and $\overline{n} \in \mathbb{N}$ with $f_{\overline{n}}(\overline{y}) = 1 = \varepsilon$: of course, it is enough to take any $\overline{y} \neq 0$ and any $\overline{n} \in H$.

The next result holds for any admissible ideal and extends [4, Proposition 5].

Proposition 3.10. *Let (Z, d) be a metric space, $f_n : Z \to \mathbb{R}$, $n \in \mathbb{N}$, be $\mathcal{I}$-exhaustive on Z and $\mathcal{I}$-pointwise convergent to a function $f : Z \to \mathbb{R}$. Then f is continuous and $(f_n)_n$ $\mathcal{I}$-converges uniformly to f on every compact subset of Z.*

PROOF. We proceed similarly as in the proof of [4, Proposition 5]. We begin with proving continuity of f. Pick $z \in Z$ and $\varepsilon > 0$, and let $\delta > 0$ and $C \in \mathcal{I}$ satisfy condition of $\mathcal{I}$-exhaustiveness. Choose arbitrarily $y \in Z$ with $d(y, z) < \delta$. Then, by $\mathcal{I}$-exhaustiveness, $|f_n(y) - f_n(z)| < \varepsilon/3$ for all $n \in \mathbb{N} \setminus C$. Moreover, thanks to $\mathcal{I}$-pointwise convergence, there is a set $C_0 \in \mathcal{I}$ with $\max\{|f_n(y) - f(y)|, |f_n(z) - f(z)|\} < \varepsilon/3$ whenever $n \in \mathbb{N} \setminus C_0$. Observe that $C \cup C_0 \neq \mathbb{N}$, since $\mathcal{I}$ is admissible. Let $n \in \mathbb{N} \setminus (C \cup C_0)$. We get:

$$|f(y) - f(z)| \le |f(y) - f_n(y)| + |f_n(y) - f_n(z)| + |f_n(z) - f(z)| < \varepsilon,$$

that is continuity of f at z. The continuity of f follows by arbitrariness of $z \in Z$.

We now prove uniform $\mathcal{I}$-convergence on compact subsets of Z. Let $K \subset Z$ be any compact set, and choose arbitrarily $\varepsilon > 0$ and $x \in K$. Since $(f_n)_n$ is $\mathcal{I}$-exhaustive on Z and f is continuous at x, in correspondence with ε and x there exist $N_x \in \mathcal{I}$ and an open ball $B_x \subset Z$ centered at x, with

$$|f_n(z) - f_n(x)| < \varepsilon/3 \quad \text{and } |f(z) - f(x)| < \varepsilon/3 \tag{1}$$

for each $n \in \mathbb{N} \setminus N_x$ and $z \in B_x$. Let us consider the family $\{B_x : x \in K\}$. Since K is compact, there is a finite subfamily $\{B_{x_1}, B_{x_2}, \ldots, B_{x_p}\}$, covering K. Since $(f_n)_n$ $\mathcal{I}$-converges pointwise to f, then in correspondence with ε and $x_1, x_2, \ldots, x_p$ there exists $N_0 \in \mathcal{I}$ with

$$|f_n(x_j) - f(x_j)| < \varepsilon/3, \quad j = 1, \ldots, p \tag{2}$$

whenever $n \in \mathbb{N} \setminus N_0$. If $N := N_0 \cup \Big(\bigcup_{j=1}^{p} N_{x_j}\Big)$, then $N \in \mathcal{I}$.

Pick arbitrarily $z \in K$: there is $j \in \{1, 2, \ldots, p\}$ with $z \in B_{x_j}$. Then, from (1) and (2), for each $n \in \mathbb{N} \setminus N$ we get

$$|f_n(z) - f(z)| \le |f_n(z) - f_n(x_j)| + |f_n(x_j) - f(x_j)| + |f(x_j) - f(z)| < \varepsilon.$$

This ends the proof. □

4 Exhaustiveness in measure spaces and applications.

We introduce the main properties of measure spaces. We begin with the notions of s-boundedness and σ-additivity. For a related literature see also [10, 11, 12, 24, 25, 23, 35, 36].

Definitions 4.1. Let G be any infinite set and $\Sigma \subset \mathcal{P}(G)$ be a σ-algebra.

(a) A finitely additive measure $\mu : \Sigma \to \mathbb{R}$ is called *$\mathcal{I} - s-$bounded* iff for every disjoint sequence $(H_n)_n$ in Σ we have $\mathcal{I} - \lim\limits_n (\|\mu\|(H_n)) = 0$, where $\|\mu\|$ denotes the variation of μ (see [25]). The finitely additive measures $\mu_j : \Sigma \to \mathbb{R}$, $j \in \mathbb{N}$, are called *uniformly $\mathcal{I} - s-$bounded* iff $\mathcal{I} - \lim\limits_n [\sup\limits_j (\|\mu_j\|(H_n))] = 0$, for every disjoint sequence $(H_n)_n$ in Σ.

(b) A finitely additive measure $\mu : \Sigma \to \mathbb{R}$ is called *$\mathcal{I}$-σ-additive* iff for every disjoint sequence $(H_n)_n$ in Σ we get: $\mathcal{I} - \lim\limits_n \Big[\|\mu\| \Big(\bigcup\limits_{l=n}^{\infty} H_l \Big) \Big] = 0$. The finitely additive measures $\mu_j : \Sigma \to \mathbb{R}$, $j \in \mathbb{N}$, are called *uniformly $\mathcal{I}$-σ-additive* iff for every disjoint sequence $(H_n)_n$ in Σ we get: $\mathcal{I} - \lim\limits_n \Big[\sup\limits_j \Big(\|\mu_j\| \Big(\bigcup\limits_{l=n}^{\infty} H_l \Big) \Big) \Big] = 0$.

We denote by $ba(\Sigma)$ the space of all real-valued finitely additive measures on Σ, and by $ca(\Sigma)$ the space consisting of all real-valued σ-additive measures on Σ endowed with the variation norm.

Observe that, if $\mathcal{I} = \mathcal{I}_{fin}$, then the given definitions coincide with the well known ones of (uniform) s-boundedness and (uniform) σ-additivity respectively.

Note that, in all next definitions and results, λ is a fixed non-negative finitely additive measure.

For $A, B \in \Sigma$ the (pseudo)-λ-distance between A and B is defined by $d_\lambda(A, B) := \lambda(A \Delta B)$, where Δ denotes the symmetric difference.

We now give the concepts of equicontinuity and ideal exhaustiveness of measures, which are close related each other, and introduce the regular measures.

Definitions 4.2. (a) A measure $\mu \in ba(\Sigma)$ is called *λ-continuous at $E \in \Sigma$* iff it is continuous at E on (Σ, d_λ). We say that μ is *λ-continuous on* Σ iff μ is λ-continuous at every $E \in \Sigma$.

Observe that μ is λ-absolutely continuous iff μ is λ-continuous at $\emptyset$.

(b) A family $A \subset ba(\Sigma)$ is called *equi-λ-continuous at $E \in \Sigma$* iff for each $\varepsilon > 0$ there is a $\delta > 0$ such that if $F \in \Sigma$ and $d_\lambda(E, F) < \delta$ we have: $|\mu(E) - \mu(F)| < \varepsilon$, for each $\mu \in A$. The family A is called *equi-λ-continuous on* Σ iff it is equi-λ-continuous at each $E \in \Sigma$.

Note that A is uniformly λ-absolutely continuous iff it is equi-λ-continuous at $\emptyset$.

(c) We say that a sequence $(\mu_n)_n$ in $ba(\Sigma)$ is *$\mathcal{I}$-exhaustive at* $E \in \Sigma$ iff for each $\varepsilon > 0$ there are a $\delta > 0$ and a set $A \in \mathcal{I}$ such that for every $F \in \Sigma$ with $d_\lambda(E, F) < \delta$ and for all $n \in \mathbb{N} \setminus A$, we have: $|\mu_n(E) - \mu_n(F)| < \varepsilon$. We say that $(\mu_n)_n$ is *$\mathcal{I}$-exhaustive on* Σ iff it is $\mathcal{I}$-exhaustive at E, for every $E \in \Sigma$.

(d) We say that a sequence $(\mu_n)_n$ in $ba(\Sigma)$ is *uniformly-$\mathcal{I}$-exhaustive on* Σ iff for each $\varepsilon > 0$ there exist a $\delta > 0$ and a set $A \in \mathcal{I}$ such that for every E, $F \in \Sigma$ with $d_\lambda(E, F) < \delta$ and for all $n \in \mathbb{N} \setminus A$, we have $|\mu_n(E) - \mu_n(F)| < \varepsilon$.

If $\mathcal{I} = \mathcal{I}_{fin}$, then the (uniformly)-$\mathcal{I}_{fin}$-exhaustive measure sequences are simply called (uniformly) exhaustive.

(e) If G is a topological space and Σ is the σ-algebra of all its Borel sets, then a measure $\lambda : \Sigma \to \mathbb{R}$ is said to be *regular on* Σ iff for all $B \in \Sigma$ and $\varepsilon > 0$ there exists a compact set $Q \subset B$ such that $\|\lambda\|(B \setminus Q) < \varepsilon$ (where the symbol $\|\cdot\|$ denotes the variation).

We now define continuous convergence of a measure sequence in the ideal context, which will be useful in the sequel (see also [13, 27, 29]).

Definition 4.3. We say that $(\mu_n)_n$ $(\mathcal{I}\alpha)$-*converges to* μ *at* $H \in \Sigma$ iff for every sequence $(H_n)_n$ in Σ with $\mathcal{I}-\lim_n d_\lambda(H_n, H) = 0$ we get $\mathcal{I}-\lim_n \mu_n(H_n) = \mu(H)$. We say that $(\mu_n)_n$ $(\mathcal{I}\alpha)$-*converges to* μ *on* Σ iff it $(\mathcal{I}\alpha)$-converges to μ at every $H \in \Sigma$.

If $\mathcal{I} = \mathcal{I}_{fin}$, then $(\mathcal{I}_{fin}\alpha)$-convergence of $(\mu_n)_n$ to μ is simply called (α)-*convergence* of $(\mu_n)_n$ to μ.

We now give a result about weak compactness and boundedness properties for subsets of countably additive measures in the context of ideal convergence, when we deal with a P-ideal. Using Proposition 3.2, Theorem 3.7, Proposition 2.5 (which yields the equivalence between (uniform) σ-additivity and (uniform) $\mathcal{I} - \sigma-$additivity) and [23, Theorem VII.13], we get:

Theorem 4.4. *Let $\mathcal{K}$ be a subset of $ca(\Sigma)$. Then the following are equivalent:*

a) $\mathcal{K}$ is relatively weakly compact.

b) $\mathcal{K}$ is relatively weakly $\mathcal{I}$-sequentially compact.

c) $\mathcal{K}$ is bounded and uniformly σ-additive.

d) $\mathcal{K}$ is bounded and uniformly $\mathcal{I} - \sigma-$additive.

e) $\mathcal{K}$ is bounded and there is a non-negative measure $\lambda \in ca(\Sigma)$ such that $\mathcal{K}$ is equi-λ-continuous.

Our next step is to prove, in the ideal setting, some convergence theorems, concerning in particular some results of existence and some good properties of the limit measure. To prove this, we deal with the tool of (uniform) ideal exhaustiveness, which allows us to assume that our involved ideal is simply a P-ideal. Some other kinds of limit theorems, like Nikodým convergence of Schur-type theorems were proved in [10] in some context different from ideal exhaustiveness, but requiring *as hypothesis* some additional properties of the limit measure and some further properties of filters (or dually of the corresponding ideals).

In general the simple $\mathcal{I}$-pointwise convergence of a sequence of σ-additive measures (even non-negative) is not sufficient to get uniform s-boundedness (and a fortiori uniform σ-additivity). Indeed, as soon as $\mathcal{I}$ is an admissible ideal different from $\mathcal{I}_{\text{fin}}$, we have the following:

Example 4.5. Let $\mathcal{I} \neq \mathcal{I}_{\text{fin}}$ be any admissible ideal, $H := \{h_1 < \ldots < h_s < h_{s+1} < \ldots\}$ be an infinite set belonging to $\mathcal{I}$ and such that $\mathbb{N} \setminus H$ is infinite. Since $\mathcal{I} \neq \mathcal{I}_{\text{fin}}$, then H does exist. For every $i \notin H$ and $E \subset \mathbb{N}$, set $m_i(E) = 0$. For any $s \in \mathbb{N}$ and $E \subset \mathbb{N}$, set $m_{h_s}(E) = 1$ if $s \in E$ and 0 otherwise. Observe that $m_0(E) := \mathcal{I} - \lim_i m_i(E) = 0$ for each $E \subset \mathbb{N}$. Moreover, it is readily seen that the m_i's are σ-additive positive bounded measures. Indeed, given $i \in \mathbb{N}$ and any disjoint sequence $(C_j)_j$ of subsets of $\mathbb{N}$, the entity $m_i(C_j)$ can be different from zero (and in this case is equal to 1) at most for one index j, because for all $s \in \mathbb{N}$ we get that $m_i(\{s\}) \neq 0$ if and only if $i = h_s$. For every $j \in \mathbb{N}$ set $C_j := \{j\}$. Then we get $1 \geq \sup_{i \in \mathbb{N}} m_i(C_j) \geq m_{h_j}(C_j) = 1$ (see also [11, Remark 2.7]).

We now prove a result about the existence of the limit measure on a whole σ-algebra under the hypotheses of its existence on a suitable subclass and uniform ideal exhaustiveness.

Lemma 4.6. *Assume that there exists a countable dense subset $\mathcal{B} = \{F_j : j \in \mathbb{N}\}$ of (Σ, d_λ). Let $(\mu_n)_n$ be a sequence in $ba(\Sigma)$ with $\mathcal{I} - \lim_n \mu_n(F_j) =: \mu(F_j)$, for all $j \in \mathbb{N}$, for some set function $\mu : \mathcal{B} \to \mathbb{R}$, and let the family $(\mu_n)_n$ be uniformly $\mathcal{I}$-exhaustive. Then μ admits an extension $\mu_0 \in ba(\Sigma)$ with*

$$\mathcal{I} - \lim_n \mu_n(E) = \mu_0(E) \text{ for all } E \in \Sigma. \tag{3}$$

Proof. Fix arbitrarily $\varepsilon > 0$. From uniform $\mathcal{I}$-exhaustiveness we can find a $\delta > 0$ and a $C \in \mathcal{I}$ such that

$$d_\lambda(E, F) < \delta \Longrightarrow |\mu_n(E) - \mu_n(F)| \leq \frac{\varepsilon}{4} \tag{4}$$

for every $E, F \in \Sigma$ and $n \in \mathbb{N} \setminus C$.

Let $E \in \Sigma$ and choose $F_j \in \mathcal{B}$ such that $d_\lambda(E, F_j) < \delta$. By virtue of the Cauchy condition (see also [16, 17, 18, 22]), in correspondence with ε and F_j there is a set $C_j^* \in \mathcal{I}$ with

$$|\mu_k(F_j) - \mu_n(F_j)| \leq \frac{\varepsilon}{2} \tag{5}$$

whenever $k, n \in \mathbb{N} \setminus C_j^*$. From (4) and (5) we get

$$\begin{aligned} |\mu_k(E) - \mu_n(E)| &\leq |\mu_k(E) - \mu_k(F_j)| + \\ &+ |\mu_k(F_j) - \mu_n(F_j)| + |\mu_n(F_j) - \mu_n(E)| \leq \\ &\leq \frac{\varepsilon}{4} + |\mu_k(F_j) - \mu_n(F_j)| + \frac{\varepsilon}{4} \leq \varepsilon \end{aligned}$$

for any $k, n \in \mathbb{N} \setminus (C \cup C_j^*)$. Thanks to the Cauchy criterion, this proves that the map μ_0 in (3) is well-defined. It is easy to check that $\mu_0 \in ba(\Sigma)$. □

We now prove the following technical lemma.

Lemma 4.7. *Let $\mathcal{I} - \lim\limits_n \mu_n(F_j) = \mu(F_j)$ for each $j \in \mathbb{N}$, where μ_n, μ, F_j's are as in Lemma 4.6. Then there exists an element $M \in \mathcal{F} = \mathcal{F}(\mathcal{I})$, such that $\lim\limits_{n \in M} \mu_n(F_j) = \mu(F_j)$ for all $j \in \mathbb{N}$, provided that $\mathcal{I}$ is a P-ideal.*

PROOF. By assumption, for each $j \in \mathbb{N}$ we get $\mathcal{I} - \lim_n \mu_n(F_j) = \mu(F_j)$. Since $\mathcal{I}$ is a P-ideal, by Proposition 2.5 (c), we have $\mathcal{I}^* - \lim\limits_n \mu_n(F_j) = \mu(F_j)$. Thus for every $j \in \mathbb{N}$ there exists $M_j' \in \mathcal{F}$ such that $\lim\limits_{n \in M_j'} \mu_n(F_j) = \mu(F_j)$. Now, as $\mathcal{I}$ is a P-ideal, in correspondence with the sequence $(M_j')_j$ there exists a sequence $(M_j)_j$ of subsets of $\mathbb{N}$ such that the set $M_j \Delta M_j'$ is finite for all $j \in \mathbb{N}$ and $\cap_j M_j \in \mathcal{F}$. Set $M := \cap_j M_j$. Note that $M \setminus M_j' \subset M_j \setminus M_j'$ is finite for every $j \in \mathbb{N}$. Now it is easy to see that $\lim\limits_{n \in M} \mu_n(F_j) = \mu(F_j)$ for all $j \in \mathbb{N}$, getting the assertion. □

This result is a sufficient condition for σ-additivity of the limit measure.

Lemma 4.8. *Let (Σ, d_λ) be separable, and $\mathcal{B} = \{F_j : j \in \mathbb{N}\}$ be as in Lemma 4.6. If $(\mu_n)_n$ is a uniformly $\mathcal{I}$-exhaustive sequence in $ca(\Sigma)$ and $\mathcal{I} - \lim\limits_n \mu_n(F_j) = \mu(F_j)$ for all $j \in \mathbb{N}$, then there are a set $L \subset \mathbb{N}$, $L \in \mathcal{F} = \mathcal{F}(\mathcal{I})$, and an extension μ_0 of μ, defined on Σ, such that $\lim\limits_{n \in L} \mu_n(E) = \mu_0(E)$ for all $E \in \Sigma$, provided that $\mathcal{I}$ is a P-ideal. Moreover, $\mu_0 \in ca(\Sigma)$.*

Proof. Let $\mathcal{F}$ be the dual filter associated with $\mathcal{I}$. By uniform $\mathcal{I}$-exhaustiveness, for all $j \in \mathbb{N}$ there are a $\delta_j > 0$ and a set $N'_j \in \mathcal{F}$ such that $|\mu_n(E) - \mu_n(F)| \leq 1/j$ for every E, $F \in \Sigma$ with $d_\lambda(E,F) < \delta_j$ and for all $n \in N'_j$. Since $\mathcal{I}$ is a P-ideal, in correspondence with the sequence $(N'_j)_j$ there is a sequence $(N_j)_j$ of subsets of $\mathbb{N}$ such that the set $N_j \Delta N'_j$ is finite for all $j \in \mathbb{N}$ and $\cap_j N_j \in \mathcal{F}$. Put $N := \cap_j N_j$ and $W_j := N \setminus N'_j$ for all j: observe that $W_j \subset N_j \setminus N'_j$ is finite for every $j \in \mathbb{N}$. So, the set N has the property that: for all $j \in \mathbb{N}$ there are a $\delta_j > 0$ and a finite set $W_j \subset N$ with $|\mu_n(E) - \mu_n(F)| \leq 1/j$ for every E, $F \in \Sigma$ with $d_\lambda(E,F) < \delta_j$ and for all $n \in N \setminus W_j$. This means that the sequence $(\mu_n)_{n\in N}$ is uniformly $\mathcal{I}_{\text{fin}}$-exhaustive.

Let now $\mathcal{B} = \{F_j : j \in \mathbb{N}\}$ be as in the hypothesis, M be as in the proof of Lemma 4.7 and set $L := M \cap N$. Note that $L \in \mathcal{F}$, the sequence $(\mu_n)_{n\in L}$ is uniformly $\mathcal{I}_{\text{fin}}$-exhaustive, and $\lim_{n\in L} \mu_n(F_j) = \mu(F_j)$ for all $j \in \mathbb{N}$, by virtue of Lemma 4.7. So, the first part of the assertion follows from this and Lemma 4.6 used with $(\mu_n)_{n\in L}$ and $\mathcal{I} = \mathcal{I}_{\text{fin}}$. From the first part and the classical Nikodým convergence theorem for measures (see [25, 23]) it follows that $\mu_0 \in$ ca(Σ). □

The following result is a characterization of weak convergence of countably additive measures and weak convergence in L^1 in terms of pointwise convergence under the hypothesis of ideal exhaustiveness, and extends [23, Theorem VII.11 and Corollary].

Theorem 4.9. *Let (Σ, d_λ) be separable. If a sequence $(\mu_n)_n$ in ca(Σ) weakly-$\mathcal{I}$-converges to $\mu \in$ ca(Σ), then for each $E \in \Sigma$, $\mu(E) = \mathcal{I} - \lim\limits_n \mu_n(E)$. Conversely, if $(\mu_n)_n$ is a sequence in ca(Σ) such that there is a set function $\mu : \Sigma \to \mathbb{R}$ such that $\mu(E) = \mathcal{I} - \lim\limits_n \mu_n(E)$ for all $E \in \Sigma$, $\mathcal{I}$ is a P-ideal and the sequence $(\mu_n)_n$ is uniformly $\mathcal{I}$-exhaustive, then $\mu \in$ ca(Σ) and $(\mu_n)_n$ weakly-$\mathcal{I}$-converges to μ in ca(Σ).*

Moreover, if $\eta \in$ ca (Σ) and $f_n : G \to \mathbb{R}$, $n \in \mathbb{N}$, is a sequence of functions in $L^1(\eta)$ with the property that the measure sequence $\mu_n(E) := \displaystyle\int_E f_n \, d\eta$, $n \in \mathbb{N}$, $E \in \Sigma$, is uniformly $\mathcal{I}$-exhaustive, then $(f_n)_n$ weakly-$\mathcal{I}$-converges to f in $L^1(\eta)$ if and only if $\displaystyle\int_E f \, d\eta = \mathcal{I} - \lim_n \mu_n(E)$ for every $E \in \Sigma$.

Proof. Since the functional $\nu \mapsto \nu(E)$ belongs to ca$(\Sigma)^*$ (the dual of $ca(\Sigma)$) for every $E \in \Sigma$, then the first part is straightforward.

Suppose now that $(\mu_n)_n$ is a sequence in ca(Σ) such that $\mu(E) = \mathcal{I} - \lim\limits_n \mu_n(E)$ exists in $\mathbb{R}$ for each $E \in \Sigma$. Then by Lemma 4.8, since the sequence

$(\mu_n)_n$ is uniformly $\mathcal{I}$-exhaustive, $\mu \in \mathrm{ca}(\Sigma)$ and there is an element $M \in \mathcal{F}(\mathcal{I})$ such that $\lim\limits_{n \in M} \mu_n(E) = \mu(E)$ for all $E \in \Sigma$. Write the set $(\mu_n)_{n \in M}$ as $(\nu_k)_{k \in \mathbb{N}}$. Then we have $\lim\limits_k \nu_k(E) = \mu(E)$ for all $E \in \Sigma$. Now, by the classical Nikodým boundedness theorem, the ν_k's are uniformly bounded, and so $\sup\limits_k \|\nu_k\|(G) < +\infty$, where the symbol $\|\cdot\|$ denotes the variation. Hence the series $\sum\limits_{k=1}^{\infty} \frac{\|\nu_k\|(\cdot)}{2^k}$ is absolutely convergent in the Banach space $\mathrm{ca}(\Sigma)$ with η as its sum. Then, by the Radon-Nikodým theorem, for each $k \in \mathbb{N}$ there is a function $f_k \in L^1(\eta)$ such that

$$\nu_k(E) = \int_E f_k \, d\eta$$

(see also [23]). Similarly, we get the existence of $f \in L^1(\eta)$ such that

$$\mu(E) = \int_E f \, d\eta. \tag{6}$$

Now, proceeding analogously as in [23, Theorem VII.11], one can show that $(f_k)_k$ weakly converges to f in $L^1(\eta)$, which in turn implies that $(\nu_k)_k$ weakly converges to μ in $\mathrm{ca}(\Sigma)$. But this implies that the sequence $(\mu_n)_n$ weakly-$\mathcal{I}^*$-converges and so weakly-$\mathcal{I}$-converges to μ in $\mathrm{ca}(\Sigma)$. Thus the first part is proved.

To get the sufficient part of the last statement of the theorem, proceeding similarly as above, we obtain the existence of an element $M \in \mathcal{F}(\mathcal{I})$ with $\lim_{n \in M} \mu_n(E) = \mu(E)$ in the usual sense for each $E \in \Sigma$, and hence the weak convergence in $L^1(\eta)$ in the ordinary sense of the subsequence $(f_n)_{n \in M}$ to f. Thus the sequence $(f_n)_{n \in \mathbb{N}}$ weakly-$\mathcal{I}$-converges to f in $L^1(\eta)$. The necessary part of the last statement is straightforward. □

Remark 4.10. Theorem 4.9 is a sufficient condition for countable additivity of the $\mathcal{I}$-limit measure of a sequence of σ-additive measures, under the hypothesis of ideal pointwise convergence and the further condition of ideal exhaustiveness. In [11, Theorems 2.5 and 2.6], σ-additivity of the limit measure was used as an hypothesis, in order to prove some limit theorems for an $\mathcal{I}$-pointwise convergent sequence of positive measures. Note that, as Example 4.5 shows, in general the ideal setwise convergence of a sequence of σ-additive measures is not enough to get uniform σ-additivity.

As a consequence of Theorem 4.9, we prove the following result, which extends [23, Theorem VII.12].

Corollary 4.11. *Under the same notations as in Theorem 4.9, let (Σ, d_λ) be separable, $(\mu_n)_n$ be a uniformly $\mathcal{I}$-exhaustive family and $\mathcal{I}$ be a P-ideal. Then a sequence $(\mu_n)_n$ is weakly-$\mathcal{I}$-Cauchy in ca(Σ) if and only if it is weakly-$\mathcal{I}$-convergent in ca(Σ).*

Moreover, if $\eta \in$ ca(Σ) is positive, $f_n : G \to \mathbb{R}$, $n \in \mathbb{N}$, is a function sequence in $L^1(\eta)$ and the sequence $\mu_n(E) := \int_E f_n \, d\eta$, $n \in \mathbb{N}$, $E \in \Sigma$, is uniformly $\mathcal{I}$-exhaustive, then $(f_n)_n$ is weakly-$\mathcal{I}$-Cauchy in $L^1(\eta)$ if and only if it is weakly-$\mathcal{I}$-convergent in $L^1(\eta)$.

Proof. Let $(\mu_n)_n$ be a weakly-$\mathcal{I}$-Cauchy sequence in ca(Σ). Since each element $E \in \Sigma$ determines the member $\nu \mapsto \nu(E)$ of ca$(\Sigma)^*$, the sequence $(\mu_n(E))_n$ is $\mathcal{I}$-Cauchy in $\mathbb{R}$ for each $E \in \Sigma$. Then (see also [16, 17, 18, 22]) for every $E \in \Sigma$, $(\mu_n(E))_n$ is $\mathcal{I}$-convergent to a real number, say $\mu(E)$. So, a set function $\mu : \Sigma \to \mathbb{R}$ is defined. By Lemma 4.8, $\mu \in$ ca(Σ). Thus, the sequence $(\mu_n)_n$ is weakly-$\mathcal{I}$-convergent in ca(Σ). The converse implication of the first part is obvious.

The last assertion is a consequence of the first, by setting $\mu_n(E) := \int_E f_n \, d\eta$, $n \in \mathbb{N}$, $E \in \Sigma$, taking into account the Radon-Nikodým theorem (see also (6)) and since $L^1(\eta)$ is a closed subspace of ca(Σ) for every positive measure $\eta \in$ ca(Σ) (see also [23]). □

We now give a result on the existence of the ideal limit measure without requiring ideal exhaustiveness.

Theorem 4.12. *Let $\mathcal{A}$ be an algebra of sets generating Σ and suppose that $(\mu_n)_n$ is a uniformly σ-additive sequence with the property that $\mathcal{I} - \lim_n \mu_n(E)$ exists for each $E \in \mathcal{A}$. Then $\mathcal{I} - \lim_n \mu_n(E)$ exists for all $E \in \Sigma$.*

Proof. Let $\Lambda := \{E \in \Sigma : \mathcal{I} - \lim_n \mu_n(E) \text{ exists in } \mathbb{R}\}$. By hypothesis, $\mathcal{A} \subset \Lambda$. If we show that Λ is a monotone class, then we will get $\Lambda = \mathcal{A}$, and the result will be proved.

Let $(E_m)_m$ be a monotone sequence of elements of Λ with $\lim_m E_m = E$ in the set-theoretic sense. Since $(\mu_n)_n$ is uniformly σ-additive, then $\mu_n(E) = \lim_m \mu_n(E_m)$ uniformly in n.

Let now $\varepsilon > 0$ be given. Then an integer m can be found, such that

$$|\mu_n(E_m) - \mu_n(E)| < \varepsilon/3 \tag{7}$$

for all $n \in \mathbb{N}$. Since $\mathcal{I} - \lim_n \mu_n(E_m)$ exists, so the sequence $(\mu_n(E_m))_n$ is $\mathcal{I}$-Cauchy, and thus there exists an element $C \in \mathcal{I}$ such that

$$|\mu_p(E_m) - \mu_q(E_m)| < \varepsilon/3 \tag{8}$$

whenever $p, q \notin C$. From (7) and (8) it follows that for all $p, q \notin C$ we get:

$$\begin{aligned} &|\mu_p(E) - \mu_q(E)| \leq |\mu_p(E) - \mu_p(E_m)| + |\mu_p(E_m) - \mu_q(E_m)| + \\ + \quad &|\mu_q(E_m) - \mu_q(E)| < \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

So the sequence $(\mu_n(E))_n$ is $\mathcal{I}$-Cauchy, and thus $\mathcal{I}$-convergent. This ends the proof. □

Finally we turn to a limit theorem, which yields continuity of the limit of a setwise ideal convergent sequence of measures. Note that this result holds for any admissible ideal.

Theorem 4.13. *Assume that $\mu_n : \Sigma \to R$, $n \in \mathbb{N}$, is pointwise $\mathcal{I}$-convergent to $\mu : \Sigma \to R$. If $(\mu_n)_n$ is $\mathcal{I}$-exhaustive at H_0, then μ is λ-continuous at H_0.*

PROOF. Choose arbitrarily $\varepsilon > 0$. Then there exist $\delta > 0$ and $A_1 \in \mathcal{I}$ with

$$|\mu_n(H) - \mu_n(H_0)| < \varepsilon/3 \tag{9}$$

for any $H \in \Sigma$, with $d_\lambda(H, H_0) < \delta$ and for every $n \in \mathbb{N} \setminus A_1$.

Fix now $H \in \Sigma$, with $d_\lambda(H, H_0) < \delta$. There exists $A_2 \in \mathcal{I}$ with

$$|\mu_n(H_0) - \mu(H_0)| < \varepsilon/3, \qquad |\mu_n(H) - \mu(H)| < \varepsilon/3 \tag{10}$$

for every $n \in \mathbb{N} \setminus A_2$. Note that $A_1 \cup A_2 \neq \mathbb{N}$, because $\mathcal{I}$ is admissible. Pick $n \in \mathbb{N} \setminus (A_1 \cup A_2)$. Then by (9) and (10) we get:

$$\begin{aligned} |\mu(H) - \mu(H_0)| &\leq |\mu(H) - \mu_n(H)| + |\mu_n(H) - \mu_n(H_0)| + |\mu_n(H_0) - \mu(H_0)| \\ &< \varepsilon. \end{aligned}$$

From this the assertion follows. □

We now give some further results related with $\mathcal{I}$-exhaustiveness of measures, in connection with ideal (α)-convergence.

Theorem 4.14. *Under the same notations as above, let $H_0 \in \Sigma$, and $\mu, \mu_n : \Sigma \to \mathbb{R}$, $n \in \mathbb{N}$. If $\mathcal{I} - \lim_n \mu_n(H_0) = \mu(H_0)$ and $(\mu_n)_n$ is $\mathcal{I}$-exhaustive at H_0, then $(\mu_n)_n$ $(\mathcal{I}\alpha)$-converges to μ at H_0.*

Proof. Assume that $\mathcal{I} - \lim_n \mu_n(H_0) = \mu(H_0)$. Choose arbitrarily $\varepsilon > 0$. By $\mathcal{I}$-exhaustiveness of $(\mu_n)_n$ at H_0, there are $\delta > 0$ and $N_1 \in \mathcal{I}$ with the property that

$$|\mu_n(H) - \mu_n(H_0)| < \varepsilon/2 \tag{11}$$

whenever $n \in \mathbb{N} \setminus N_1$ and $|\lambda(H) - \lambda(H_0)| < \delta$, where λ is the measure associated with $\mathcal{I}$-exhaustiveness.

By $\mathcal{I}$-convergence of $(\mu_n(H_0))_n$ to $\mu(H_0)$ there exists an element $N_2 \in \mathcal{I}$ such that

$$|\mu_n(H_0) - \mu(H_0)| < \varepsilon/2 \tag{12}$$

for any $n \in \mathbb{N} \setminus N_2$. As $\mathcal{I} - \lim_n d_\lambda(H_n, H_0) = 0$, there is an element $N_3 \in \mathcal{I}$ with $d_\lambda(H_n, H_0) < \delta$ whenever $n \in \mathbb{N} \setminus N_3$. From this, (11) and (12) it follows that for each $n \in \mathbb{N} \setminus (N_1 \cup N_2 \cup N_3)$ we get:

$$|\mu_n(H_n) - \mu(H_0)| \le |\mu_n(H_n) - \mu_n(H_0)| + |\mu_n(H_0) - \mu(H_0)| < \varepsilon.$$

□

One can ask whether the converse of Theorem 4.14 holds. To this aim, suppose that there exists a partition of the type $\mathbb{N} = \bigcup_{k=1}^{\infty} \Delta_k$, with the property that

$$\mathcal{I} = \{A \subset \mathbb{N} : A \text{ intersects at most a finite number of } \Delta_k\text{'s}\}. \tag{13}$$

Remarks 4.15.

(a) Note that the ideal $\mathcal{I}_{\text{fin}}$ satisfies condition (13): indeed it is enough to take $\Delta_k = \{k\}$ for all $k \in \mathbb{N}$. Moreover, if for every $k \in \mathbb{N}$ the set Δ_k in (13) is infinite, then the ideal $\mathcal{I}$ associated to the Δ_k's is not a P-ideal (see also [31, Example 3.1 (g)]).

(b) Let $\mathcal{I}$ be as in (13), and $(A_j)_j$ be any sequence of subsets of $\mathbb{N}$, with $A_j \notin \mathcal{I}$ for all $j \in \mathbb{N}$. We claim that there exists a disjoint sequence $(B_j)_j$ in $\mathcal{I}$, with $B_j \subset A_j$ for every $j \in \mathbb{N}$ and $\bigcup_{j=1}^{\infty} B_j \notin \mathcal{I}$. Indeed, first of all observe that there exists an infinite subset $P_1 \subset \mathbb{N}$ with $A_1 = \bigcup_{k \in P_1} (A_1 \cap \Delta_k)$. There is a nonempty finite set $Q_1 \subset P_1$, with the property that the set $B_1 = \bigcup_{k \in Q_1} (A_1 \cap \Delta_k)$ belongs to $\mathcal{I}$.

Let $P_2 \subset \mathbb{N}$ be such that $A_2 = \bigcup_{k \in P_2} (A_2 \cap \Delta_k)$: since $A_2 \notin \mathcal{I}$, we get that P_2 is infinite. There exists a nonempty finite set $Q_2 \subset P_2 \setminus Q_1$, such that the set $B_2 = \bigcup_{k \in Q_2} (A_2 \cap \Delta_k)$ belongs to $\mathcal{I}$. At the j-th step, we consider $P_j \subset \mathbb{N}$ with $A_j = \bigcup_{k \in P_j} (A_j \cap \Delta_k)$. There exists a nonempty finite set $Q_j \subset P_j \setminus \Big(\bigcup_{s=1}^{j-1} Q_s\Big)$, with the property that $B_j = \bigcup_{k \in Q_j} (A_j \cap \Delta_k) \in \mathcal{I}$. Since the Q_j's, by construction, are nonempty and pairwise disjoint and $\bigcup_{j=1}^{\infty} Q_j$ is infinite, we get that the B_j's are pairwise disjoint and $\bigcup_{j=1}^{\infty} B_j \notin \mathcal{I}$.

We now prove the converse of Theorem 4.14 under the hypothesis that the ideal involved satisfies (13).

Theorem 4.16. *Let $\mathcal{I}$ be as in (13), $H_0 \in \Sigma$, and μ, $\mu_n : \Sigma \to \mathbb{R}$, $n \in \mathbb{N}$. If $(\mu_n)_n$ $(\mathcal{I}\alpha)$-converges to μ at H_0, then $\mathcal{I} - \lim_n \mu_n(H_0) = \mu(H_0)$ and $(\mu_n)_n$ is $\mathcal{I}$-exhaustive at H_0.*

Proof. First of all, note that $(\mathcal{I}\alpha)$-convergence of $(\mu_n)_n$ to μ at H_0 implies that $\mathcal{I} - \lim_n \mu_n(H_0) = \mu(H_0)$: indeed, in the definition of $(\mathcal{I}\alpha)$-convergence, it is enough to consider the sequence $(H_n)_n$, defined by setting $H_n := H_0$ for each $n \in \mathbb{N}$.

Now we prove that $(\mu_n)_n$ is $\mathcal{I}$-exhaustive at H_0. For each $k \in \mathbb{N}$, set

$$V_k := \{H \in \Sigma : |\lambda(H) - \lambda(H_0)| < 1/k\} = \{H \in \Sigma : d_\lambda(H, H_0) < 1/k\}.$$

If $(\mu_n)_n$ is not $\mathcal{I}$-exhaustive at H_0, then there is $\varepsilon > 0$ such that to every $k \in \mathbb{N}$ and $A \in \mathcal{I}$ there correspond an element $H \in V_k$ and an integer $n \in \mathbb{N} \setminus A$ with $|\mu_n(H) - \mu_n(H_0)| \geq \varepsilon$. Let us consider $A = \emptyset$: so there are $n_0^k \in \mathbb{N} = \mathbb{N} \setminus A$ and $H_0^k \in V_k$ with $|\mu_{n_0^k}(H_0^k) - \mu_{n_0^k}(H_0)| \geq \varepsilon$.

We now proceed by transfinite induction. Fix $k \in \mathbb{N}$. Suppose to have chosen n_β^k and H_β^k for every $\beta < \alpha$, where α is a countable ordinal, and that $A_\alpha^k := \{n_\beta^k : \beta \leq \alpha\} \in \mathcal{I}$. Then an integer $n_{\alpha+1}^k \notin A_\alpha^k$ and a set $H_{\alpha+1}^k \in V_k$ can be found, with

$$|\mu_{n_{\alpha+1}^k}(H_{\alpha+1}^k) - \mu_{n_{\alpha+1}^k}(H_0)| \geq \varepsilon.$$

In this case, set $A^k_{\alpha+1} := A^k_\alpha \cup \{n^k_{\alpha+1}\}$. When α is a limit ordinal, if we have chosen n^k_β and H^k_β for $\beta < \alpha$, define $A^k_\alpha := \cup_{\beta<\alpha} A^k_\beta$. This procedure ends at some countable limit ordinal α_k since otherwise we would obtain a strictly increasing sequence $\{A^k_\alpha : \alpha < \omega_1\}$ of subsets of $\mathbb{N}$. Consequently, $A^k_{\alpha_k} \notin \mathcal{I}$. Set $P_k := A^k_{\alpha_k}$ for each $k \in \mathbb{N}$. By virtue of Remarks 4.15 (b) there is a disjoint sequence $(B_k)_k$ in $\mathcal{I}$ with $B_k \subset P_k$ for all $k \in \mathbb{N}$ and $\displaystyle\bigcup_{k=1}^\infty B_k \notin \mathcal{I}$.

Now for $n \in \mathbb{N}$ set $H_n := H_0$ if $n \notin \displaystyle\bigcup_{k=1}^\infty B_k$ and $H_n = H^k_\beta$ if $n \in B_k$ and $n = n^k_\beta$ (note that the integer k is uniquely determined, since the B_k's are disjoint and β is unique, because the n^k_β's are different for different choices of β: so, the H_n's are well-defined).

We get that $\mathcal{I} - \lim_n |\lambda(H_n) - \lambda(H_0)| = \mathcal{I} - \lim_n d_\lambda(H_n, H_0) = 0$. Indeed, for every neighborhood U of H_0, say $U := \{H \in \Sigma : |\lambda(H) - \lambda(H_0)| \leq \eta\}$, with $\eta > 0$, there exist $k_0 \in \mathbb{N}$ with $\{n : H_n \notin U\} \subset \displaystyle\bigcup_{k=1}^{k_0} B_k$ and $\displaystyle\bigcup_{k=1}^{k_0} B_k \in \mathcal{I}$.

Since $\mathcal{I} - \lim_n \mu_n(H_0) = \mu(H_0)$, we have

$$B := \{n \in \mathbb{N} : |\mu_n(H_0) - \mu(H_0)| \geq \varepsilon/2\} \in \mathcal{I}.$$

As $L := \{n \in \mathbb{N} : |\mu_n(H_n) - \mu_n(H_0)| \geq \varepsilon\} \supset \bigcup_{k=1}^\infty B_k$, then L does not belong to $\mathcal{I}$, and so $\{n \in \mathbb{N} : |\mu_n(H_n) - \mu(H_0)| \geq \varepsilon/2\}$ does not belong to $\mathcal{I}$, getting to a contradiction with $(\mathcal{I}\alpha)$-convergence to μ at H_0. Thus the theorem is completely proved. □

In Lemma 4.6 we dealt with some convergence results, related with weak Cauchy conditions. In this setting, as an application of ideal exhaustiveness, we will give a result involving integral-type operators, which extends [1, Lemma 2.3] to the ideal setting with respect to any admissible ideal.

Proposition 4.17. *Let (G, Σ, λ) be a measure space, assume that λ is regular and (G, d) is a complete metric space. Let $(\mu_l)_l$ be a uniformly $\mathcal{I}$-exhaustive sequence of countably additive λ-continuous measures. Suppose that there exist a set $C_0 \in \mathcal{I}$ and a positive constant M' with $\|\mu_l\|(G) \leq M'$ for all $l \in \mathbb{N} \setminus C_0$, and that the sequence $(\mu_l(A))_l$ is weakly $\mathcal{I}$-Cauchy for all $A \in \Sigma$. Let $g \in L^\infty(\lambda)$ and assume that $(s_n)_n$ is a sequence of simple functions, $\mathcal{I}$-exhaustive on G and $\mathcal{I}$-convergent pointwise to g on G.*

Then the sequence $\left(\displaystyle\int_G g\, d\mu_l\right)_l$ is $\mathcal{I}$-Cauchy.

Proof. Fix arbitrarily $\varepsilon > 0$. As the sequence $(\mu_l)_l$ is uniformly $\mathcal{I}$-exhaustive, there are a $\delta > 0$ and a set $C \in \mathcal{I}$ such that, if $\lambda(A) < \delta$ and $l \in \mathbb{N} \setminus C$, then $\|\mu_l\|(A) < \varepsilon$. Without loss of generality we can suppose that $C \supset C_0$.

By regularity of λ, there is a set $A \in \Sigma$ such that $G \setminus A$ is compact and $\lambda(A) < \delta$. By Proposition 3.10, the sequence $(s_n)_n$ $\mathcal{I}$-converges uniformly to g on $G \setminus A$, and so there exists a set $\widehat{C} \in \mathcal{I}$ with $|s_n(t) - g(t)| \le \varepsilon$ whenever $t \in G \setminus A$ and $n \notin \widehat{C}$. Let $n_0 = \min(\mathbb{N} \setminus \widehat{C})$. Since $(\mu_l(A))_l$ is weakly $\mathcal{I}$-Cauchy for all $A \in \Sigma$, there is $C^* \in \mathcal{I}$, without loss of generality $C^* \supset C$, such that for all i, $j \in \mathbb{N} \setminus C^*$ we get

$$\Big|\int_G s_{n_0} d(\mu_i - \mu_j)\Big| \le \varepsilon,$$

and hence

$$\begin{aligned}
0 \quad &\le \quad \Big|\int_G g\, d\mu_i - \int_G g\, d\mu_j\Big| \le \Big|\int_G (s_{n_0} - g)\, d\mu_i\Big| + \\
&+ \quad \Big|\int_G s_{n_0}\, d(\mu_i - \mu_j)\Big| + \Big|\int_G (s_{n_0} - g)\, d\mu_j\Big| \\
&\le \quad \Big|\int_{G\setminus A} (s_{n_0} - g)\, d\mu_i\Big| + \Big|\int_A (s_{n_0} - g)\, d\mu_i\Big| + \\
&+ \quad \Big|\int_{G\setminus A} (s_{n_0} - g)\, d\mu_j\Big| + \Big|\int_A (s_{n_0} - g)\, d\mu_j\Big| + \varepsilon \\
&\le \quad \varepsilon\|\mu_i\|(G) + \varepsilon\|\mu_j\|(G) + 6\varepsilon\|g\|_\infty + \varepsilon.
\end{aligned}$$

From this the assertion follows. □

Remark 4.18. Observe that the thesis of Proposition 4.17 can be interpreted as a weak $\mathcal{I}$-Cauchy-type condition in the space $L^\infty(\lambda)$: indeed we know from the classical literature that, by virtue of the Riesz representation theorem, the dual of $L^\infty(\lambda)$ is isomorphic to the space of all countable additive λ-continuous measures, and the integral is a functional which realizes such an isomorphism (see [1, 24, 25, 23, 35, 36]). The topics and the tools about Proposition 4.17 are related also with some properties of precompactness of sets of measures and uniform integrability (see also [1, 24, 25, 23, 35, 36]).

Open Problem: Find analogues of Theorem 4.16 for other classes of ideals or more generally for any admissible ideal $\mathcal{I}$.

References

[1] C. Abbott, E. M. Bator, R. G. Bilyeu and P. L. Lewis, *Weak precompactness, strong boundedness, and weak complete continuity,* Math. Proc. Cambridge Philos. Soc. **108** (1990), 325–335.

[2] E. Athanassiadou, A. Boccuto, X. Dimitriou and N. Papanastassiou, *Ascoli-type theorems and ideal* (α)*-convergence*, Filomat (2012), to appear.

[3] A. Aviles Lopez, B. Cascales Salinas, V. Kadets and A. Leonov, *The Schur* ℓ_1 *theorem for filters*, J. Math. Phys. Anal. Geom. **3(4)** (2007), 383–398.

[4] M. Balcerzak, K. Dems and A. Komisarski, *Statistical convergence and ideal convergence for sequences of functions*, J. Math. Anal. Appl. **328(1)** (2007), 715–729.

[5] V. K. Bhardwaj and I. Bala, *On weak statistical convergence,* Int. J. Math. Math. Sci. (2007), Art. ID 38530, 9 pp.

[6] S. Bhunia and P. Das, *Two-valued measures and summability of double sequences in asymmetric context*, Acta Math. Hung. **130(1-2)** (2011), 167–187.

[7] A. Boccuto and D. Candeloro, *Integral and ideals in Riesz spaces*, Inform. Sci. **179(17)** (2009), 2891–2902.

[8] A. Boccuto and D. Candeloro, *Defining limits by means of integrals*, Oper. Theory Adv. Appl. **201** (2010), 79–87.

[9] A. Boccuto, X. Dimitriou and N. Papanastassiou, *Basic matrix theorems for* $\mathcal{I}$*-convergence in* (ℓ)*-groups,* Math. Slovaca (2012), to appear.

[10] A. Boccuto, X. Dimitriou and N. Papanastassiou, *Schur lemma and limit theorems in lattice groups with respect to filters,* Math. Slovaca (2012), to appear.

[11] A. Boccuto, X. Dimitriou and N. Papanastassiou, *Brooks-Jewett-type theorems for the pointwise ideal convergence of measures with values in* (ℓ)*-groups,* Tatra Mt. Math. Publ. **49** (2011), 17–26.

[12] A. Boccuto, X. Dimitriou and N. Papanastassiou, *Some versions of limit and Dieudonné-type theorems with respect to filter convergence for* (ℓ)*-group-valued measures,* Cent. Eur. J. Math. **9(6)** (2011), 1298–1311.

[13] A. Boccuto, X. Dimitriou, N. Papanastassiou and W. Wilczyński, *Ideal exhaustiveness, continuity and* (α)*-convergence for lattice group-valued functions,* Int. J. Pure Appl. Math. **70(2)** (2011), 211–227.

[14] H. Çakalli, *Sequential definitions of compactness,* Appl. Math. Letters **21(6)** (2008), 594–598.

[15] J. Connor, M. Ganichev and V. Kadets, *A characterization of Banach spaces with separable duals via weak statistical convergence,* J. Math. Anal. Appl. **244** (2000), 251–261.

[16] P. Das and S. K. Ghosal, *Some further results on* I*-Cauchy results and condition* (AP), Comput. Math. Appl. **59(8)** (2010), 2597–2600.

[17] P. Das and S. K. Ghosal, *On* $\mathcal{I}$*-Cauchy nets and completeness*, Topology Appl. **157(7)** (2010), 1152–1156.

[18] P. Das and S. K. Ghosal, *When* $\mathcal{I}$*-Cauchy nets in complete uniform spaces are* $\mathcal{I}$*-convergent*, Topology Appl. **158(13)** (2011), 1529–1533.

[19] P. Das, P. Kostyrko, W. Wilczyński and P. Malík, I *and* I^**-convergence of double sequences*, Math. Slovaca **58(5)** (2008), 605–620.

[20] P. Das and P. Malík, *On extremal* I*-limit points of double sequences*, Tatra Mt. Math. Publ. **40** (2008), 91–102.

[21] P. Das and P. Malík, *On the statistical and* I*-variation of double sequences*, Real Anal. Exchange **33(2)** (2008), 351–363.

[22] K. Dems, *On* $\mathcal{I}$*-Cauchy sequences*, Real Anal. Exchange **30(1)** (2004/05), 267–276.

[23] J. Diestel, Sequences and series in Banach spaces, Springer - Verlag, 1984.

[24] J. Diestel, W. M. Ruess and W. Schachermayer, *Weak compactness in* $L^1(\mu, X)$, Proc. Amer. Math. Soc. **118(2)** (1993), 447–453.

[25] J. Diestel and J. J. Uhl, Vector Measures, **15**, Amer. Math. Soc., 1977.

[26] I. Farah, *Analytic quotients. Theory of liftings for quotients over analytic ideals on the integers,* Mem. Amer. Math. Soc. **148(702)**, (2000).

[27] V. Gregoriades and N. Papanastassiou, *The notion of exhaustiveness and Ascoli-type theorems*, Topology Appl. **155** (2008), 1111–1128.

[28] A. Komisarski, *Pointwise I-convergence and I-convergence in measure of sequences of functions*, J. Math. Anal. Appl. **340** (2008), 770–779.

[29] J. L. Kelley, General Topology, D. Van Nostrand Co., Inc., 1955.

[30] P. Kostyrko, M. Mačaj, T. Šalát and M. Sleziak, *$\mathcal{I}$-convergence and extremal $\mathcal{I}$-limit points*, Math. Slovaca **55(4)** (2005), 443–464.

[31] P. Kostyrko, T. Šalát and W. Wilczyński, *$\mathcal{I}$-convergence*, Real Anal. Exchange **26** (2000–01), 669–685.

[32] B. K. Lahiri and P. Das, *$\mathcal{I}$ and $\mathcal{I}^*$-convergence of nets*, Real. Anal. Exchange **33(2)** (2007–08), 431–442.

[33] R. Megginson, An introduction to Banach space theory, Springer - Verlag, 1998.

[34] S. Pehlivan, C. Sencimen and Z. H. Yaman, *On weak ideal convergence in normed spaces,* J. Interdiscip. Math. **13** (2010), 153–162.

[35] N. Randrianantonina and E. Saab, *Weak compactness in the space of vector-valued measures of bounded variation,* Rocky Mountain J. Math. **24(2)** (1994), 681–688.

[36] M. J. Rivera, *The De La Vallée-Poussin theorem for vector-valued measure spaces,* Rocky Mountain J. Math. **30(2)** (2000), 715–722.