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A PROPERTY OF GREEN'S FUNCTION

For k a positive integer, let

$$M_i \subset \{0, 1, \ldots, k-1\}, \quad i = 0, 1$$

with $M_i \neq \emptyset$ (i.e. $\#M_i > 0$) and

$$\#M_0 + \#M_1 = k,$$

where $\#M_i$ is the cardinality of M_i.

Suppose that the solution $u = u(x)$ of the simple boundary value problem

$$u^{(k)} = f \quad \text{in} \quad (0,1), \tag{1}$$

$$u^{(i)}(0) = 0 \quad \text{for} \quad i \in M_0, \tag{2}$$
$$u^{(j)}(1) = 0 \quad \text{for} \quad j \in M_1$$

with f not changing sign in $(0,1)$ can be expressed uniquely in the form

$$u(x) = \int_0^x K_1(x,t) f(t) dt + \int_x^1 K_2(x,t) f(t) dt.$$

Problem: Prove that there exist positive constants c_1, c_2, and nonnegative integers α_1, α_2, β_1, β_2, γ_1, γ_2, δ_1, δ_2, such that the following estimates hold:

$$c_1 \leq \frac{K_i(x,t)}{x^{\alpha_i}(1-x)^{\beta_i} t^{\gamma_i}(1-t)^{\delta_i}} \leq c_2$$

for $0 < t < x < 1$ if $i = 1$ and for $0 < x < t < 1$ if $i = 2$, and determine the values of $\alpha_1, \ldots, \delta_2$.

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Remarks:
(i) The problem is solved in [1] under the additional assumption

$$M_0 \cap M_1 = \emptyset \quad (\text{i.e. } M_1 = \{0, 1, \ldots, k-1\} \setminus M_0).$$

(ii) The solution of this problem allows one to formulate necessary and sufficient conditions for the validity of the k–th order Hardy inequality

$$\left(\int_0^1 |u(t)|^q w_0(t) dt \right)^{1/q} \leq C \left(\int_0^1 |u^{(k)}(t)|^p w_k(t) dt \right)^{1/p}$$

for functions u satisfying condition (2).

References

[1] A. Kufner, *Higher order Hardy inequalities*, Bayreuth. Math. Schriften, **44** (1993), 105–146.