INROADS

Real Analysis Exchange
Vol. 21(1), 1995–96, pp. 345–348

Pavel Pyrih, Department of Mathematical Analysis, Charles University,
Sokolovská 83, Prague 8, CZ-18600, Czech Republic,
e-mail: pyrih@@karlin.mff.cuni.cz

SEPARATION OF FINELY CLOSED SETS BY FINELY OPEN SETS

Abstract

It is well known that the fine topology in potential theory and the density topology in real analysis are not normal. This means that there exist pairs of disjoint finely (density) closed sets which cannot be separated by disjoint finely (density) open sets. A natural question arises about which pairs can be separated. We study those pairs of disjoint finely (density) closed sets which can be separated by disjoint finely (density) open sets. The key tool is the Lusin-Menchoff property of fine (density) topology. The main result is that finely (density) closed sets are finely (density) separated iff they are F_σ-"semiseparated" (Theorem 2.1, Theorem 2.2).

1 Introduction

Let (X, ρ) be a topological space. Any topology τ finer than ρ is called a (abstract) *fine topology*. We use the terms finely open, finely closed ... with respect to the fine topology. M^f denotes the fine closure of the set M. We say that A, $B \subset X$ are *finely separated* if there are disjoint finely open sets $\mathcal{G}_A$ and $\mathcal{G}_B$ such that $A \subset \mathcal{G}_A$, $B \subset \mathcal{G}_B$.

We recall from [2, Theorem 2.1] the following proposition with its proof

Proposition 1.1 *Let τ be a fine topology on a separable Baire metric space (P, ρ). Assume that each countable subset of P is finely closed and that no countable set is finely open. If finely continuous functions on P are in Baire class one, then (P, τ) is not normal.*

Key Words: fine topology, finely separated sets, Lusin-Menchoff property
Mathematical Reviews subject classification: Primary: 54A10. Secondary: 26A03, 31C40
Received by the editors January 31, 1995

PROOF. Let A, $B \subset P$ be disjoint countable dense sets. Assuming that (P, τ) is normal, there is a finely continuous function f on P such that $f = 0$ on A and $f = 1$ on B. Since f is in Baire class one, the sets

$$\{f \leq 1/3\} \quad \text{and} \quad \{f \geq 2/3\}$$

are disjoint residual sets. But this cannot occur in any Baire space. □

Since the space (P, τ) in the proposition is not normal, there exist two disjoint finely closed sets which cannot be finely separated. The sets A and B in the above proof can be finely separated (see Theorem 2.1) .

An important tool in the study of fine topologies is the Lusin-Menchoff property. We say that the fine topology τ on (X, ρ) has the *Lusin-Menchoff property* (with respect to ρ) if for each pair of disjoint subsets F and $\mathcal{F}$ of X, F closed, $\mathcal{F}$ finely closed, there are disjoint subsets G and $\mathcal{G}$ of X, G open, $\mathcal{G}$ finely open, such that $\mathcal{F} \subset G$, $F \subset \mathcal{G}$ [2, p. 85].

We say that the fine topology has the *G_δ-insertion property* if for each finely open set $\mathcal{G}$ and each finely closed set $\mathcal{F}$ with $\mathcal{G} \subset \mathcal{F}$, there is a set G of type G_δ such that $\mathcal{G} \subset G \subset \mathcal{F}$ [2, pp. 39–40].

2 Separation of Sets

The following theorems work for both the fine topology in potential theory and the density topology in real analysis.

Theorem 2.1 *Let the fine topology have the Lusin-Menchoff property. Suppose a and b are finely closed sets. Suppose A and B are sets of type F_σ with $a \subset A$, $b \subset B$, A disjoint with b, and B disjoint with a. Then there are disjoint finely open sets α and β such that $a \subset \alpha$ and $b \subset \beta$.*

PROOF. Let $A = A_1 \cup A_2 \cup \ldots$ with $A_i \subset A_{i+1}$ and A_i closed. Let $B = B_1 \cup B_2 \cup \ldots$ with $B_i \subset B_{i+1}$ and B_i closed.

Let $n = 1$. For a and B_1 the Lusin-Menchoff property gives a disjoint open set U_1 and a finely open set $\mathcal{V}_1$ such that $a \subset U_1$ and $B_1 \subset \mathcal{V}_1$. For b and A_1 the Lusin-Menchoff property gives a disjoint open set V_1 and a finely open set $\mathcal{U}_1$ such that $b \subset V_1$ and $A_1 \subset \mathcal{U}_1$. We set $\alpha_1 = \mathcal{U}_1 \cap U_1$ and $\beta_1 = \mathcal{V}_1 \cap V_1$. α_1 and β_1 are disjoint finely open with the property that $a \cap A_1 \subset \alpha_1$ and $b \cap B_1 \subset \beta_1$.

Let $n > 1$. Given $\alpha_1, \beta_1, \cdots, \alpha_{n-1}, \beta_{n-1}$ we proceed to the construction of α_n, β_n. For a and B_n the Lusin-Menchoff property gives a disjoint open set U_n and a finely open set $\mathcal{V}_n$ such that $a \subset U_n$ and $B_n \subset \mathcal{V}_n$. For b and A_n the Lusin-Menchoff property gives a disjoint open set V_n and a finely open set $\mathcal{U}_n$

such that $b \subset V_n$ and $A_n \subset \mathcal{U}_n$. We set $\alpha_n = \mathcal{U}_n \cap U_n \setminus {\beta_1}^f \setminus {\beta_2}^f \setminus \cdots \setminus {\beta_{n-1}}^f$ and $\beta_n = \mathcal{V}_n \cap V_n \setminus {\alpha_1}^f \setminus {\alpha_2}^f \setminus \cdots \setminus {\alpha_{n-1}}^f$. The sets α_n, β_n are disjoint finely open with the property that $a \cap A_n \subset \alpha_n$ and $b \cap B_n \subset \beta_n$. The sets α_i and β_j are disjoint for every i and j by construction.

Finally we set

$$\alpha = \bigcup_{n=1}^{\infty} \alpha_n \quad \text{and} \quad \beta = \bigcup_{n=1}^{\infty} \beta_n \,.$$

The sets α and β are disjoint finely open with $a \subset \alpha$ and $b \subset \beta$. □

Let $a \subset A \subset X$ and $b \subset B \subset X$ where A and B are of type F_σ, A is disjoint with b, and B is disjoint with a. In this situation we say that a and b are F_σ-"semiseparated". Theorem 2.1 says (assuming the Lusin-Menchoff property) that F_σ-"semiseparated" finely closed sets are finely separated. The following theorem shows when the converse holds.

Theorem 2.2 *Let the fine topology have the G_δ-insertion property. Suppose $\mathcal{A}$ and $\mathcal{B}$ are disjoint finely closed, $\mathcal{U}$ and $\mathcal{V}$ are disjoint finely open, $\mathcal{A} \subset \mathcal{U}$, and $\mathcal{B} \subset \mathcal{V}$. Then there exist A and B of type F_σ such that $\mathcal{A} \subset A$, $\mathcal{B} \subset B$, A is disjoint with $\mathcal{B}$, and B is disjoint with $\mathcal{A}$.*

PROOF. There exists a G_δ set $\mathcal{V}^*$ such that $\mathcal{V} \subset \mathcal{V}^* \subset \mathcal{V}^f$ by the G_δ-insertion property. We set $A = \mathbb{C} \setminus \mathcal{V}^*$. Then A is of type F_σ, $\mathcal{A} \subset A$ and A is disjoint with $\mathcal{B}$. Similarly we set $B = \mathbb{C} \setminus \mathcal{U}^*$. □

Theorem 2.2 says (assuming the G_δ-insertion property) that finely separated finely closed sets are F_σ-"semiseparated". We show that in two important cases we get the characterization of finely separated pairs of finely closed sets.

Corollary 2.3 *For the fine topology in potential theory on a σ-compact space X two finely closed sets are finely separated if and only if they are F_σ-"semiseparated".*

PROOF. Since the fine topology has the Lusin-Menchoff property [2, Corollary 10.26], Theorem 2.1 applies. Since the fine topology has the G_δ-insertion property [2, Theorem 10.30], Theorem 2.2. applies. □

Corollary 2.4 *Suppose the fine topology is the density topology on $\mathbb{R}$. Then two finely closed sets are finely separated if and only if they are F_σ-"semiseparated".*

PROOF. Since the density topology has the Lusin-Menchoff property [2, 6.A.8], Theorem 2.1 applies. Since the density topology has the G_δ-insertion property [2, 6.A.10], Theorem 2.2 applies. □

Remark 2.5 *Other material on density separated density closed sets can be found in [1, 3, 4, 5] (see [2, 6.A.5]).*

References

[1] M. Laczkovich, *Separation properties of some subclasses of Baire* 1 *functions*, Acta Math. Acad. Sci. Hungar., **26** (1975), 405–421.

[2] J. Lukeš, J. Malý and L. Zajíček, *Fine Topology Methods in Real Analysis and Potential Theory*, Lecture Notes in Mathematics 1189, Springer-Verlag, Berlin, 1986.

[3] J. Lukeš and L. Zajíček, *The insertion of G_δ sets and fine topologies*, Comment. Math. Univ. Carolinae, **18** (1977), 101–104.

[4] J. Malý, *A note on separation of sets by approximately continuous functions*, Comment. Math. Univ. Carolinae, **20** (1979), 579–588.

[5] F. D. Tall, *Normal subspaces of the density topology*, Pacific J. Math., **75** (1978), 579–588.