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SUMS AND PRODUCTS OF QUASI-CONTINUOUS FUNCTIONS

Abstract

In this article two main results are proved. The first one is that each cliquish function $f\colon \mathbb{R}^k \to \mathbb{R}$ is the sum of two quasi-continuous functions. It is also shown that we can moreover require that the summands preserve points of continuity of f, are bounded provided that f is bounded and belong to the same class of Baire as f (if f is Borel measurable). The other main result is that each function $f\colon \mathbb{R}^k \to \mathbb{R}$ which can be written as the product of finitely many quasi-continuous functions, can be expressed as the product of two quasi-continuous functions, and we can require that the factors belong to the same class of Baire as f (if f is Borel measurable).

1 Introduction

The notion of quasi-continuity for real functions of several real variables was introduced over sixty years ago by S. Kempisty [8], as a generalization of the notion of continuity, and it has been intensively studied since then. A comprehensive survey on this topic can be found in [12].

It is easy to show that the sum of two quasi-continuous functions need not be quasi-continuous. However, it must be cliquish, as the sum of two cliquish functions is cliquish. In 1985 Z. Grande proved that each cliquish function f defined on $\mathbb{R}$ can be expressed as the sum of four quasi-continuous functions, and in the case where f is locally bounded, as the sum of three quasi-continuous functions [5]. This theorem was generalized by E. Strońska [13], [14], for cliquish functions defined on a separable, metrizable, dense in itself, Baire

space. Later J. Borsík and J. Doboš proved that each cliquish function defined on $\mathbb{R}^k$ is the sum of two simply continuous functions, each of which in turn can be written as the sum of two quasi-continuous functions [3]. The latest result in this topic is that each cliquish function defined on a separable metrizable topological space can be written as the sum of three quasi-continuous functions [1].

The story of investigating the family of functions which can be written as the product of quasi-continuous functions is much shorter. T. Natkaniec proved the following theorem [11] in 1990.

Theorem 1.1 *A function* $f\colon \mathbb{R} \to \mathbb{R}$ *can be factored into a (finite) product of quasi-continuous functions iff* f *is cliquish and*

($\star$) *each of the sets* $f^{-1}\big((-\infty,0)\big)$, $f^{-1}(0)$, *and* $f^{-1}\big((0,\infty)\big)$ *is the union of an open set and a nowhere dense set.*

However, in his representation of such functions he uses as many as eight (!) quasi-continuous functions. Lately J. Borsík proved that each real cliquish function defined on a separable metrizable topological space which fulfills condition ($\star$) can be written as the product of three quasi-continuous functions [2]. Moreover, he simplified the condition ($\star$) to the following form:

The set $f^{-1}(0)$ *is the union of an open set and a nowhere dense set.*

(It is possible if the domain is a Baire space.)

It is natural to ask which cliquish functions can be expressed as the sum (resp. the product) of <u>two</u> quasi-continuous functions. If the domain is $\mathbb{R}^k$, the answer is all (resp. all satisfying condition ($\star$)).

2   Preliminaries

First we need some notation. The real line $(-\infty, \infty)$ is denoted by $\mathbb{R}$ and the set of positive integers, by $\mathbb{N}$. Throughout this article, k is a fixed positive integer. The word *function* means a mapping from $\mathbb{R}^k$ into $\mathbb{R}$ unless otherwise explicitly stated. The words *measure*, *summable* etc. refer to Lebesgue measure and integral in $\mathbb{R}^k$. The Euclidean metric in $\mathbb{R}^k$ is denoted by ϱ. For every set $A \subset \mathbb{R}^k$ let diam A be its diameter (i.e., diam $A = \sup\{\varrho(x,y) : x,y \in A\}$), int A its interior, cl A its closure, fr A its boundary, χ_A its characteristic function, and $|A|$ its outer Lebesgue measure. A symbol like $\int_A f$ will always mean the Lebesgue integral. For any function f we write $\|f\|$ for $\sup\{|f(t)| : t \in \mathbb{R}^k\}$ (f need not be bounded) and we denote by $D(f)$ the set of points of discontinuity of f.

The *oscillation of a function f on a non-empty set* $A \subset \mathbb{R}^k$ will be denoted by $\omega(f, A)$ (i.e., $\omega(f, A) = \sup\{|f(x) - f(y)| : x, y \in A\}$). Similarly, the *oscillation of a function f at a point* $x \in \mathbb{R}^k$ will be denoted by $\omega(f, x)$ (i.e., $\omega(f, x) = \lim_{r \to 0^+} \omega(f, \{y \in \mathbb{R}^k : \varrho(x, y) < r\})$).

We say that a function f is *quasi-continuous* (resp. *cliquish*) at a point $x \in \mathbb{R}^k$ if for each $\varepsilon > 0$ and each open set $U \ni x$ we can find a non-empty open set $V \subset U$ such that $\omega(f, \{x\} \cup V) < \varepsilon$ (resp. $\omega(f, V) < \varepsilon$). We say that f is *quasi-continuous* (*cliquish*) if it is quasi-continuous (cliquish) at each point $x \in \mathbb{R}^k$. Cliquish functions are also known as *pointwise discontinuous*.

We will use the following well-known (and easy to prove) facts:

- A function f is quasi-continuous at a point $x \in \mathbb{R}^k$ iff there exists an open set $H \subset \mathbb{R}^k$ such that $x \in \operatorname{cl} H$ and $f|(\{x\} \cup H)$ is continuous at x.

- A function f is quasi-continuous iff for each $x \in \mathbb{R}^k$ there exists a sequence $x_1, x_2, \ldots \notin D(f)$ such that $x_n \to x$ and $f(x_n) \to f(x)$.

- The limit of a uniformly convergent sequence of quasi-continuous functions is quasi-continuous.

- Each quasi-continuous function is cliquish.

- A function f is cliquish iff $D(f)$ is of the first category.

The word *interval* (resp. *cube*) will always mean non-degenerate compact interval (resp. cube) in $\mathbb{R}^k$, i.e., Cartesian product of k non-degenerate compact intervals (resp. compact intervals of equal length) in $\mathbb{R}$. For every interval $I = [a_1, b_1] \times \cdots \times [a_k, b_k]$ we denote $I^\circ = [a_1, b_1) \times \cdots \times [a_k, b_k)$.

The proof of the following simple technical lemma is left to the reader.

Lemma 2.1 *Whenever $A \subset \mathbb{R}^k$ is closed there exists a family $\mathcal{J}$ of non-overlapping cubes such that each $x \notin A$ belongs to the interior of the union of some finite subfamily of $\mathcal{J}$ and $|J| \leq \left[\varrho(A, J)\right]^k$ for each $J \in \mathcal{J}$.*

We say that a sequence of intervals $\{I_n : n \in \mathbb{N}\}$ is *o-convergent* to a point $x \in \mathbb{R}^k$ if $x \in \bigcap_{n \in \mathbb{N}} I_n$, $\lim_{n \to \infty} \operatorname{diam} I_n = 0$, and $\limsup_{n \to \infty} (\operatorname{diam} I_n)^k / |I_n| < \infty$. We will write $I_n \overset{o}{\Rightarrow} x$. (Cf., e.g., [10].)

By an *interval function* we will mean a mapping from the family of all intervals into $\mathbb{R}$. For an arbitrary interval function F and a point $x \in \mathbb{R}^k$ we define

$$o\text{-}\limsup_{I \Rightarrow x} F(I) = \sup\left\{\limsup_{n \to \infty} F(I_n) : I_n \overset{o}{\Rightarrow} x\right\}$$

and
$$o\text{-}\liminf_{I\Rightarrow x} F(I) = \inf\Big\{\liminf_{n\to\infty} F(I_n) : I_n \overset{o}{\Rightarrow} x\Big\}.$$
If the two above limits coincide, then we denote their value by $o\text{-}\lim_{I\Rightarrow x} F(I)$.

We say that $x \in \mathbb{R}^k$ is an *o-Lebesgue point* of a function f if f is locally summable at x and $o\text{-}\lim_{I\Rightarrow x} \int_I |f - f(x)|/|I| = 0$. We say that f is an *o-Lebesgue function* if each $x \in \mathbb{R}^k$ is an *o*-Lebesgue point of f.

The following theorem is proved in [10].

Theorem 2.2 *Assume that $A \subset \mathbb{R}^k$ is closed, $\mathcal{J} = \big\{J_n : n \in \mathbb{N}\big\}$ is a family of non-overlapping cubes and $\big\{f_n : n \in \mathbb{N}\big\}$ is a family of summable functions such that the following conditions are satisfied:*

(i)   each $x \notin A$ belongs to the interior of the union of some finite subfamily of $\mathcal{J}$,

(ii) $|J_n| \le \big[\varrho\,(A, J_n)\big]^k$ for each $n \in \mathbb{N}$,

(iii) $f_n(x) = 0$ if $x \notin J_n$, $n \in \mathbb{N}$,

(iv) for each $x \in A$ and each $\tau > 0$ there exists an $\eta > 0$ such that for each $n \in \mathbb{N}$, if $\varrho\,(x, J_n) < \eta$, then $\int_{J_n} |f_n| \le \tau \cdot |J_n|$.

Put $f = \sum_{n=1}^{\infty} f_n$. Then the function f is locally summable and moreover, each $x \in A$ is an o-Lebesgue point of f. [10, Proposition 7.5]

3   Auxiliary Lemmas

The following lemma is a multidimensional version of Lemma 2 of [7].

Lemma 3.1 *Let a set $G \subset \mathbb{R}^k$, a function f and an $\varepsilon > 0$ be such that $\omega\,(f, x) < \varepsilon$ for each $x \in G$. Then there is a continuous function $\overline{f} \colon G \to \mathbb{R}$ such that $\big|f - \overline{f}\big| \le \varepsilon/2$ on G.*

PROOF.   For each $x \in G$ define $m(x) = \min\big\{f(x), \liminf_{t\to x} f(t)\big\}$ and $M(x) = \max\big\{f(x), \limsup_{t\to x} f(t)\big\}$. Then m is lower semicontinuous, M is upper semicontinuous, and $M(x) - m(x) = \omega\,(f, x) < \varepsilon$ for each $x \in G$. By Theorem 1.N(d) of [4], there is a continuous function $\overline{f} \colon G \to \mathbb{R}$ such that

$$f(x) - \varepsilon/2 \le M(x) - \varepsilon/2 \le \overline{f}(x) \le m(x) + \varepsilon/2 \le f(x) + \varepsilon/2$$

for each $x \in G$, whence $\big|f - \overline{f}\big| \le \varepsilon/2$ on G. $\square$

Lemma 3.2 *Assume that A is a nowhere dense closed set which contains all points of discontinuity of quasi-continuous functions h_1 and h_2. Then there exists a family of non-overlapping cubes $\mathcal{G} = \big\{G_n : n \in \mathbb{N}\big\}$ such that*

(i) each $x \notin A$ belongs to the interior of the union of some finite subfamily of $\mathcal{G}$,

(ii) $|G_n| \leq \left[\varrho\left(G_n, A\right)\right]^k$ for each $n \in \mathbb{N}$,

(iii) for $i \in \{1, 2\}$ and each $x \in A$ there exists a subfamily $\{G_{n_l} : l \in \mathbb{N}\}$ such that $x \in \operatorname{cl} \bigcup_{l \in \mathbb{N}} G_{n_l}$ and $h_i\big|\big(\{x\} \cup \bigcup_{l \in \mathbb{N}} \operatorname{int} G_{n_l}\big)$ is continuous at x.

PROOF. First choose a family $\{I_p : p \in \mathbb{N}\}$ of non-overlapping cubes according to Lemma 2.1. For each $p \in \mathbb{N}$, h_1 and h_2 are uniformly continuous on I_p, so we can find non-overlapping cubes $G_{p,1}, \ldots, G_{p,l_p}$ such that $I_p = \bigcup_{j=1}^{l_p} G_{p,j}$ and $\omega\left(h_i, G_{p,j}\right) < \varrho\left(I_p, A\right)$ $\left(i \in \{1, 2\}, \ j \in \{1, \ldots, l_p\}\right)$. Arrange all cubes $G_{p,j}$, $p \in \mathbb{N}$, $j \in \{1, \ldots, l_p\}$, in a sequence $\{G_n : n \in \mathbb{N}\}$. Then clearly (i) and (ii) are satisfied.

To prove (iii), fix an $i \in \{1, 2\}$ and an $x \in A$. Let $l \in \mathbb{N}$. Since h_i is quasi-continuous at x, there is an open set H such that $\operatorname{diam}\left(\{x\} \cup H\right) < 1/l$ and $\omega\left(h_i, \{x\} \cup H\right) < 1/l$. Then by (i), there exists an $n_l \in \mathbb{N}$ such that $\operatorname{int} G_{n_l} \cap H \neq \emptyset$, whence $\varrho\left(G_{n_l}, A\right) < 1/l$ and

$$\omega\left(h_i, \{x\} \cup G_{n_l}\right) \leq \omega\left(h_i, G_{n_l}\right) + \omega\left(h_i, \{x\} \cup H\right) \leq 2/l.$$

Hence $x \in \operatorname{cl} \bigcup_{l \in \mathbb{N}} \operatorname{int} G_{n_l}$ and $h_i\big|\big(\{x\} \cup \bigcup_{l \in \mathbb{N}} \operatorname{int} G_{n_l}\big)$ is continuous at x, which completes the proof. $\square$

Lemma 3.3 *Suppose I is an interval, L, $\varepsilon > 0$ and $h \colon \mathbb{R}^k \to \mathbb{R}$ is continuous at each point of a non-empty perfect set $K \subset \operatorname{int} I$. Then there is a continuous function α such that $\|\alpha\| \leq L + \|h \cdot \chi_K\|$, $\alpha = 0$ outside of I, $\int_I |\alpha| < \varepsilon$, and*

$$(h + \alpha)(K) \cap (h - \alpha)(K) \supset [-L, L]. \tag{1}$$

PROOF. Clearly we may assume that $|I| < \varepsilon/\left(L + \|h \cdot \chi_K\|\right)$. Find disjoint non-empty perfect sets K_1, $K_2 \subset K$ and continuous functions φ_1, φ_2 such that $\varphi_i(K_i) = [-L, L]$ $(i \in \{1, 2\})$. Then use the Tietze extension theorem (see [4, Theorem 2.1.7]) to construct a continuous function α such that $\alpha = 0$ outside of I, $\|\alpha\| \leq L + \|h \cdot \chi_K\|$ and $\alpha = (-1)^i \cdot h + \varphi_i$ on K_i $(i \in \{1, 2\})$. It is obvious that (1) holds. $\square$

4 Sums of Quasi-Continuous Functions

Lemma 4.1 *Suppose $\overline{g}$, $\overline{\alpha}$ and g are functions which are continuous on a dense open set G, and $\overline{g}/2 + \overline{\alpha}$, $\overline{g}/2 - \overline{\alpha}$ are quasi-continuous on $A = \mathbb{R}^k \setminus G$. Then we can find an o-Lebesgue function α which is continuous on G and such that*

(i) $\alpha = 0$ *on* A,

(ii) $\|\alpha\| \leq \|g - \bar{g}\|$,

(iii) functions $g/2 + \bar{\alpha} + \alpha$ *and* $g/2 - \bar{\alpha} - \alpha$ *are quasi-continuous on* A.

PROOF. First find a family of non-overlapping cubes $\{G_n : n \in \mathbb{N}\}$ according to Lemma 3.2 (with $h_i = \bar{g}/2 + (-1)^i \cdot \bar{\alpha}$, $i \in \{1, 2\}$). Then for each $n \in \mathbb{N}$ apply Lemma 3.3 with $I = G_n$, $L = L_n = \min\{n, \|g - \bar{g}\|/2\}$, $\varepsilon = 2^{-n}$ and $h = (g - \bar{g})/2$, and find a continuous function α_n such that $\|\alpha_n\| \leq \|g - \bar{g}\|$, $\alpha_n = 0$ outside of G_n, $\int_{G_n} |\alpha_n| < 2^{-n} \cdot |G_n|$ and for $i \in \{1, 2\}$

$$\left(\frac{g - \bar{g}}{2} + (-1)^i \cdot \alpha_n\right)(\text{int } G_n) \supset [-L_n, L_n]. \tag{2}$$

Define $\alpha = \sum_{n \in \mathbb{N}} \alpha_n$. By Theorem 2.2, each $x \in A$ is an o-Lebesgue point of α and since by condition (i) of Lemma 3.2, α is continuous on G, α is an o-Lebesgue function. Clearly conditions (i) and (ii) are satisfied.

To prove (iii), take an $i \in \{1, 2\}$ and an $x \in A$. By condition (iii) of Lemma 3.2, there is a subfamily $\{G_{n_l} : l \in \mathbb{N}\}$ such that $x \in \text{cl } \bigcup_{l \in \mathbb{N}} \text{int } G_{n_l}$ and $\left(\bar{g}/2 + (-1)^i \cdot \bar{\alpha}\right)\big|\left(\{x\} \cup \bigcup_{l \in \mathbb{N}} \text{int } G_{n_l}\right)$ is continuous at x. Put

$$H = \bigcup_{l \in \mathbb{N}} \left(\text{int } G_{n_l} \cap \left(\frac{g - \bar{g}}{2} + (-1)^i \cdot \alpha\right)^{-1}\big((c - 1/l, c + 1/l)\big)\right),$$

where $c = \big(g(x) - \bar{g}(x)\big)/2$. Then H is open and by (2), $x \in \text{cl } H$. It is clear that $\big(g/2 + (-1)^i \cdot \bar{\alpha} + (-1)^i \cdot \alpha\big)\big|\left(\{x\} \cup H\right)$ is continuous at x, so (iii) holds.□

Theorem 4.2 *Given a cliquish function* f *and an* $\eta > 0$ *we can find an* o-*Lebesgue function* α *such that the functions* $f/2 + \alpha$ *and* $f/2 - \alpha$ *are quasi-continuous,* $D(\alpha) \subset D(f)$ *and* $\|\alpha\| \leq \|f\| + \eta$.

PROOF. First use Lemma 3.1 with $G = \{x \in \mathbb{R}^k : \omega(f, x) < \eta \cdot 2^{-n}\}$ to find a sequence of functions $\{f_n : n \in \mathbb{N}\}$ such that for each $n \in \mathbb{N}$: the set $A_n = \text{cl } D(f_n)$ is contained in $D(f)$, $f = f_n$ on A_n and $|f - f_n| \leq \eta \cdot 2^{-n-1}$ on $\mathbb{R}^k \setminus A_n$.

Set $f_0 = 0$. For each $n \in \mathbb{N}$ use Lemma 4.1 with $\bar{g} = f_{n-1}$, $\bar{\alpha} = \sum_{j < n} \alpha_j$, $g = f_n$ and $G = \mathbb{R}^k \setminus \bigcup_{j \leq n} A_j$, to find an o-Lebesgue function α_n which is continuous outside of $\bigcup_{j \leq n} A_j$, $\alpha_n = 0$ on $\bigcup_{j \leq n} A_j$, $\|\alpha_n\| \leq \|f_n - f_{n-1}\|$ and such that for $i \in \{1, 2\}$ the function $f_n/2 + (-1)^i \cdot \sum_{j \leq n} \alpha_j$ is quasi-continuous.

Set $\alpha = \sum_{n=1}^{\infty} \alpha_n$. Note that this series is uniformly convergent (so α is an o-Lebesgue function) and

$$\|\alpha\| \leq \sum_{n=1}^{\infty} \|f_n - f_{n-1}\| \leq \|f\| + \eta/4 + 3 \cdot \sum_{n=2}^{\infty} \eta \cdot 2^{-n-1} = \|f\| + \eta.$$

Clearly $D(\alpha) \subset D(f)$.

Fix an $i \in \{1, 2\}$. Observe that $f/2 + (-1)^i \cdot \alpha$ is the uniform limit of the sequence $\big\{ f_n/2 + (-1)^i \cdot \sum_{j \leq n} \alpha_j : n \in \mathbb{N} \big\}$ (which consists of quasi-continuous functions), so this function is quasi-continuous. $\qquad \square$

5   Products of Quasi-Continuous Functions

We start with a lemma.

Lemma 5.1 *Assume that a function $\widetilde{g}$ is cliquish and positive on an interval I. Then we can find quasi-continuous functions $\widetilde{h}^{(1)}$, $\widetilde{h}^{(2)}$ such that*
(a) $\widetilde{g} = \widetilde{h}^{(1)} \cdot \widetilde{h}^{(2)}$ on I,
(b) for each $x \in \operatorname{fr} I$, each $i \in \{1, 2\}$ and each $s \geq 0$ there is a sequence $x_1, x_2, \dots \in \operatorname{int} I \setminus D\big(\widetilde{h}^{(i)}\big)$ such that $x_n \to x$ and $\widetilde{h}^{(i)}(x_n) \to s$.

PROOF. Define the function f by

$$f(x) = \begin{cases} \ln \widetilde{g}(x) & \text{if } x \in I, \\ 0 & \text{otherwise} \end{cases}$$

and apply Theorem 4.2 to find quasi-continuous functions $h^{(1)}$, $h^{(2)}$ such that $f = h^{(1)} + h^{(2)}$ on $\mathbb{R}^k$. Use Lemma 2.1 with $A = \mathbb{R}^k \setminus \operatorname{int} I$ to find a family $\mathcal{I} = \{I_n : n \in \mathbb{N}\}$ of non-overlapping cubes satisfying its requirements.

For each $n \in \mathbb{N}$ use Lemma 3.3 (twice) to find a continuous function α_n such that $\alpha_n = 0$ on $\operatorname{fr} I_n$ and for $i \in \{1, 2\}$

$$\big(h^{(i)} + (-1)^i \cdot \alpha_n \big)\big(\operatorname{int} I_n \setminus D(\widetilde{g}) \big) \supset [-n, n]. \tag{3}$$

Define for $i \in \{1, 2\}$ function $\widetilde{h}^{(i)}$ by

$$\widetilde{h}^{(i)}(x) = \begin{cases} \exp\big(h^{(i)}(x) + (-1)^i \cdot \alpha_n(x) \big) & \text{if } x \in I_n^\circ,\ n \in \mathbb{N}, \\ \exp\big(h^{(i)}(x) \big) & \text{otherwise.} \end{cases}$$

Then clearly (a) holds. To prove (b) fix an $x \in \operatorname{fr} I$, an $i \in \{1, 2\}$, and an $s \geq 0$.

First suppose $s > 0$. Let $n \in \mathbb{N}$. Take an arbitrary $y_n \in \operatorname{int} I$ such that $\varrho(x, y_n) < \min\big\{ 1/n, \varrho\big(x, \bigcup_{l < |\ln s|} I_l\big) \big\}$. Then $y_n \in I_{l_n}$ for some $l_n \geq |\ln s|$ and so by (3), there is an $x_n \in I_{l_n} \setminus D(\widetilde{g})$ such that $\widetilde{h}^{(i)}(x_n) = s$. Since $\varrho(x, x_n) \leq 2/n$; so $x_n \to x$.

The case $s = 0$ immediately follows from the fact that for every $s > 0$ we are done.

Finally note that by (b), the functions $\widetilde{h}^{(1)}$ and $\widetilde{h}^{(2)}$ are quasi-continuous on $\operatorname{fr} I$. Quasi-continuity outside of $\operatorname{fr} I$ is evident. $\qquad \square$

Theorem 5.2 *For each function g the following conditions are equivalent:*
(a) There is an $l \in \mathbb{N}$ and quasi-continuous functions $h^{(1)}, \ldots, h^{(l)}$ such that
$g = h^{(1)} \cdot \ldots \cdot h^{(l)}$.

(b) There are quasi-continuous functions $h^{(1)}$ and $h^{(2)}$ with $g = h^{(1)} \cdot h^{(2)}$.

(c) Function g is cliquish and it satisfies condition $(\star)$ of Theorem 1.1.

PROOF. The proof of the implication (a) $\Rightarrow$ (c) is a repetition of the argument used in [11], while the implication (b) $\Rightarrow$ (a) is obvious; so we need only to prove (c) $\Rightarrow$ (b). Set $G_0 = \text{int}\{x \in \mathbb{R}^k : g(x) = 0\}$, $G_1 = \text{int}\{x \in \mathbb{R}^k : g(x) < 0\}$, and $G_2 = \text{int}\{x \in \mathbb{R}^k : g(x) > 0\}$. For $p \in \{0, 1, 2\}$ apply Lemma 2.1 with $A = \mathbb{R}^k \setminus G_p$ to find a family $\{I_{p,n} : n \in \mathbb{N}\}$ of non-overlapping cubes satisfying its conditions. Fix an $n \in \mathbb{N}$.

Find continuous functions $h_{0,n}^{(1)}$ and $h_{0,n}^{(2)}$ such that $h_{0,n}^{(1)} \cdot h_{0,n}^{(2)} = 0$ on $\mathbb{R}^k$, $h_{0,n}^{(1)} = h_{0,n}^{(2)} = 0$ on $\text{fr}\, I_{0,n}$, and

$$h_{0,n}^{(1)}(\text{int}\, I_{0,n}) \cap h_{0,n}^{(2)}(\text{int}\, I_{0,n}) \supset [-n, n].$$

For $p \in \{1, 2\}$ do the following. Find non-overlapping intervals $I_{p,n,1}$ and $I_{p,n,2}$ such that $I_{p,n} = I_{p,n,1} \cup I_{p,n,2}$. For $j \in \{1, 2\}$ use Lemma 5.1 with $\widetilde{g} = |g|$ and $I = I_{p,n,j}$ to get quasi-continuous functions $\widetilde{h}_{p,n,j}^{(1)}$, $\widetilde{h}_{p,n,j}^{(2)}$ satisfying its conditions.

Now for $i \in \{1, 2\}$ define the function $h^{(i)}$ by

$$h^{(i)}(x) = \begin{cases} h_{0,n}^{(i)}(x) & \text{if } x \in I_{0,n}^{\circ}, \ n \in \mathbb{N}, \\ (-1)^j \cdot \left[\text{sgn}\, f(x)\right]^i \cdot \widetilde{h}_{p,n,j}^{(i)}(x) & \text{if } x \in I_{p,n,j}^{\circ}, \ n \in \mathbb{N}, \ p, j \in \{1, 2\}, \\ (2 - i) \cdot \left(f(x) - 1\right) + 1 & \text{otherwise.} \end{cases}$$

Then it is easy to prove that for each $x \notin G_0 \cup G_1 \cup G_2$, $i \in \{1, 2\}$ and $s \in \mathbb{R}$ there is a sequence $x_1, x_2, \cdots \in \mathbb{R}^k \setminus D(h^{(i)})$ such that $x_n \to x$ and $h^{(i)}(x_n) \to s$. (Note that $G_0 \cup G_1 \cup G_2$ is dense in $\mathbb{R}^k$, by $(\star)$.) So the functions $h^{(1)}$ and $h^{(2)}$ are quasi-continuous. $\qquad\square$

6 Remarks

Remark 6.1 *If f is a function which is discontinuous on a nowhere dense set, then by Lemma 4.1, we can find quasi-continuous functions h_1, h_2 such that $\|h_i\| \leq 3\|f\|/2$ for $i \in \{1, 2\}$ and $h_1 + h_2 = f$. On the other hand, it is easy to show that $\|h_1\| + \|h_2\| \geq 3$ whenever quasi-continuous functions h_1 and h_2 are such that $h_1 + h_2 = 2\chi_{\{0\}} - 1$. However, I do not know whether the following assertion is true:*

*Whenever f is a bounded cliquish function we can find quasi-con-
tinuous functions h_1, h_2 such that $\|h_1\| + \|h_2\| \le 3\|f\|$ and $h_1 +
h_2 = f$.*

Remark 6.2 *Since each o-Lebesgue function belongs to the first class of Baire
(cf.* [9]*), the following assertion holds:*

*Let $\mathcal{A}$ be the family of all measurable functions or Baire class α.
Then for each cliquish function $f \in \mathcal{A}$ and each $\eta > 0$ we can
find quasi-continuous functions $h_1, h_2 \in \mathcal{A}$ such that $f = h_1 + h_2$,
$D(h_1) \cup D(h_2) \subset D(f)$ and $\|h_i\| \le 3\|f\|/2 + \eta$ for $i \in \{1, 2\}$.*

Though o-Lebesgue functions need not be Baire one star functions *(f is a Baire
one star function if for each non-empty perfect set P there is an interval I
with $P \cap \operatorname{int} I \ne \emptyset$ such that $f|(P \cap \operatorname{int} I)$ is continuous), the above assertion
remains true if $\mathcal{A}$ is the family of all Baire one star functions, by Lemma* 4.1.
(Cf. also [6]*.)*

Remark 6.3 *Suppose that a system of functions $\mathcal{A}$ satisfies the following pos-
tulates:*

- *$\mathcal{A}$ is a vector space containing all o-Lebesgue functions,*

- *for any function f if there are a closed set A, functions $h, h_1, h_2, \cdots \in \mathcal{A}$
  and a family $\{I_n : n \in \mathbb{N}\}$ of non-overlapping cubes satisfying require-
  ments of Lemma* 2.1 *such that $f = h$ on A and $f = h_n$ on I_n° for each
  $n \in \mathbb{N}$ then $f \in \mathcal{A}$,*

- *$\mathcal{A}$ is closed with respect to outer compositions with continuous functions.*

Then in the condition (b) of Theorem 5.2 *we can moreover require that the
functions $h^{(1)}, h^{(2)} \in \mathcal{A}$ provided that $g \in \mathcal{A}$. (The proof remains unchanged.)*

Remark 6.4 *It is easy to prove that each Baire one star function is cliquish
and satisfies condition $(\star)$ of Theorem* 1.1. *So the following assertion holds.*

*Each Baire one star function can be factored into a product of two
Baire one star quasi-continuous functions.*

(In the proof of the analogue of Lemma 5.1 *we use Lemma* 4.1 *in place of
Theorem* 4.2.*) On the other hand, not every Darboux Baire one star function
is a product of Darboux Baire one star functions* [6].

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