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THE TOPOLOGICAL STRUCTURE OF ATTRACTORS FOR DIFFERENTIABLE FUNCTIONS

Abstract

Recent research has shown that there is a significant cleavage between the structure of ω-limit sets for continuous functions, and the structure of ω-limit sets for Lipschitz functions. The development of these results rested on measure theoretic considerations. In this paper we show that there is no such divergence when one considers the topological structure of these two classes of ω-limit sets. We show that an every nowhere dense compact set is homeomorphic to an ω-limit set for a differentiable function with bounded derivative.

A set E is called an ω-limit set or an attractor for a continuous function f mapping a compact interval I into itself if there exists an x in I such that $E = \omega_f(x)$ is the cluster set of the sequence $\{x, f(x), f(f(x)), \cdots\} = \{f^n(x)\}_{n=0}^{\infty}$. Recent work by Bruckner, Ceder and Smìtal in [1], [2], and [3] has shown us how the structure of $\omega_f(x)$ is affected by imposing conditions on the chaotic behavior of f. In fact, complete characterizations of ω-limit sets for continuous functions and for continuous functions of zero topological entropy can be found in [2] and [3]. Limitations imposed by smoothness are not addressed in these works, however.

In [4], we showed that there is a significant cleavage between the structure of attractors for continuous functions, and the structure of attractors for Lipschitz functions. We furnished the class $\mathcal{K}$ of nonempty closed subsets of $I = [0, 1]$ with the Hausdorff metric d, and from this complete metric space

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develop a dense G_δ subset comprised of Cantor sets that cannot be attractors for any Lipschitz function. This is in marked contrast to the continuous case since any Cantor set is an ω-limit set for some continuous function with zero topological entropy [3].

The development of our results in [4] rested exclusively on measure theoretic considerations. This then led us to consider whether or not there are also topological restrictions on the structure of attractors for Lipschitz functions. The balance of this paper is dedicated to answering this query with the development of the following result.

Theorem 1 *Every infinite nowhere dense compact set $M \subset I$ is homeomorphic to an ω-limit set of homoclinic type for a differentiable function with bounded derivative.*

Critical to the development of Theorem 1 is the idea of a set M being homoclinic with respect to a continuous function f. This is the content of our first definition.

Definition 1 *Let M be a nowhere dense compact set, and $A = \{a_0, \cdots, a_{k-1}\} \neq \emptyset$ a set of limit points of M. Assume there is a system $\{M_n^i\}_{n=0}^{\infty}$, $i = 0, 1, \cdots, k-1$, of non-empty pairwise disjoint compact subsets of M such that $M \setminus \cup_{i,n} M_n^i = A$ and $\lim_{n\to\infty} M_n^i = a_i$ for any i. Let $f : M \to M$ be a continuous map and let A be a k-cycle of f such that $f(a_i) = a_{i-1}$ for $0 < i < k$ and $f(a_0) = a_{k-1}$. If $f(M_n^i) = M_n^{i-1}$ for $0 < i < k$ and any n, $f(M_n^0) = M_{n-1}^{k-1}$ for $n > 0$, and $f(M_0^0) = a_{k-1}$, then M is called a homoclinic set (of order k) with respect to f. If M is homoclinic of order k with respect to f, then for each i, the set $M^i = \{a_i\} \cup \bigcup_{n=0}^{\infty} M_n^i$ is homoclinic of order 1 with respect to $g = f^k$, since we have $g(M_n^i) = M_{n-1}^i$ for $n > 0$ and $g(M_0^i) = g(a_i) = a_i$.*

In our effort to prove Theorem 1, we will consider the countable and uncountable cases separately. For each case, however, our general approach is the same. We first show that our general set M has a homeomorphic copy M^* that is homoclinic with respect to a function $f : M^* \to M^*$ such that the difference quotient $\lim y \to xy \in M^* \dfrac{f(y) - f(x)}{y - x}$ exists and is bounded on M^*. We then, through a series of lemmas, extend f to a differentiable function $F : I \to I$ for which $M^* = \omega_F(x)$ for an appropriate value of x in I. We begin by considering the countable case.

Our intermediate goal is to show that if M is a countable compact set, then M has a homeomorphic copy M^* that is homoclinic with respect to some continuous function $f : M^* \to M^*$ that satisfies our derivative-like condition. An important preliminary result in this direction is Proposition 2. Before developing this proposition, however, let us fix some helpful notation. Let $A \subset I$ be a countable compact set. Define a transfinite sequence $\{A_\alpha\}_{\alpha\in\Omega}$ of subsets of A as follows: $A_0 = A, A_\gamma = \bigcap\limits_{\alpha<\gamma} A_\alpha$ if γ is a limit ordinal, and A_γ is the set of limit points of $A_{\gamma-1}$ otherwise. For any such set A there is an ordinal $\beta < \Omega$ such that A_β is non-empty and finite, and $A_{\beta+1} = \emptyset$. Denote β by $T(A)$. We call β the rank of A, and A_β its set of highest order limit points.

Proposition 2 *If A and B are countable compact sets with the same rank and same number of highest order limit points, then A and B are homeomorphic.*

Proof. We use transfinite induction. The result is true when $T(A) = T(B) = 0$, so we assume it is true for any $\alpha < \alpha(0)$, and let $T(A) = T(B) = \alpha(0)$. Suppose $A_{\alpha(0)} = \{a_1 < a_2 < \cdots < a_n\}$, and $B_{\alpha(0)} = \{b_1 < b_2 < \cdots < b_n\}$. Let $I_1, I_2, \cdots, I_n$ be pairwise disjoint compact intervals covering A such that I_i is a neighborhood of a_i for any i, $T(A\cap I_i) = \alpha(0)$, and $(A\cap I_i)_{\alpha(0)} = \{a_i\}$. Let $J_1, J_2, \cdots, J_n$ be an analogous covering of B. If $A \cap I_i$ is homeomorphic to $B \cap J_i$ for any i, then A is homeomorphic to B. Thus it suffices to show that our result holds when $A_{\alpha(0)} = \{a\}$ and $B_{\alpha(0)} = \{b\}$. We proceed by considering two cases.

Case 1 Suppose $\alpha(0)-1$ exists. We can write $A = \{a\} \cup \bigcup\limits_{i=0}^{\infty} A_i$ where $T(A_i) = \alpha(0) - 1$ and $|(A_i)_{\alpha(0)-1}| = 1$ for any i, with $\lim\limits_{i\to\infty} A_i = a$. Similarly, we have $B = \{b\} \cup \bigcup\limits_{j=0}^{\infty} B_j$ with $T(B_j) = \alpha(0) - 1$ and $|(B_j)_{\alpha(0)-1}| = 1$ for any j, and $\lim\limits_{j\to\infty} B_j = b$. Since A_n and B_n are homeomorphic for any n, A and B must be homeomorphic.

Case 2 Suppose $\alpha(0)$ is a limit ordinal. Using Lemma 5 of [2], we can write $A = \{a\} \cup \bigcup\limits_{i=0}^{\infty} A_i$ with $\lim\limits_{i\to\infty} A_i = a$ and $\alpha(0) > \delta(n+1) \geq \delta(n)$, where $\delta(n) = T(A_n)$ and $(A_n)_{\delta(n)} = \{a_n\}$. Similarly, we write $B = \{b\} \cup \bigcup\limits_{j=0}^{\infty} B_j$. Since $\lim\limits_{i\to\infty} T(A_i) = \lim\limits_{j\to\infty} T(B_j) = \alpha(0)$, for each k there exists $n(k)$ so that $T\left(\bigcup\limits_{i=1}^{n(k)} A_i\right) \geq T\left(\bigcup\limits_{j=0}^{k} B_j\right)$, and $\bigcup\limits_{j=0}^{k} B_j$ is homeomorphic to a subset of $\bigcup\limits_{i=0}^{n(k)} A_i$. Similarly, there exists $\overline{k} > k$ such that $\bigcup\limits_{j=0}^{\overline{k}} B_j$ is homeomorphic to a subset $B^*_{\overline{k}}$

of $\bigcup_{i=0}^{n(\bar{k})} A_i$ with $\bigcup_{i=0}^{n(k)} A_i \subseteq B^*_{\bar{k}}$. Since $n(k) \to \infty$ as $k \to \infty$ with $\bigcup_{i=0}^{n(k)} A_i \to A$ and $\bigcup_{j=0}^{k} B_j \to B$, A and B are homeomorphic. □

We are now in a position to show that any countable compact set M has a homeomorphic copy M^* that satisfies our desired conditions.

Lemma 3 *Every countable compact set $M \subseteq I$ is homeomorphic to a set $M^* \subseteq I$ that is homoclinic with respect to a function f where* $\lim y \to xy \in M^* \dfrac{f(y)-f(x)}{y-x}$ *exists and is bounded for all x in M^*.*

Proof. Suppose $T(M) = \alpha$ and $M_\alpha = \{b_0, b_1, \cdots, b_{k-1}\}$. Since any two countable compact sets with the same rank and number of highest order limit points are homeomorphic, it suffices to show that there exists an $M^* \subseteq I$ that is countable and compact for which $T(M^*) = \alpha$, M^*_α has k elements, and M^* is homoclinic with respect to a function f meeting the desired conditions. Our plan is to construct $M^* = \dot{\bigcup}_{i=0}^{k-1} M^i$ so that it is the disjoint union of k of its subsets M^i, with each of the subsets M^i congruent to all the others. In an effort to furnish M^* with a desirable homoclinic trajectory, we construct the M^i with the following considerations in mind. Our M^i are evenly spaced in I, with $j < \ell$ implying that M^j lies to the right of M^ℓ. Moreover, we set $M^i = \{a_i\} \cup \bigcup_{j=0}^{\infty} M^i_j$, so that $M^i_\alpha = \{a_i\}$ is the unique highest order limit point of M^i, and our sequence $\{M^i_j\}_{j=0}^{\infty}$ will easily accommodate a differentiable Lipschitz mapping of M^0_n to M^{k-1}_{n-1} for $n > 0$. Specifically, we begin by letting $(\max M^i - \min M^i) = \dfrac{9}{10k}$, $\min M^i = M^i_\alpha = \{a_i\}$, and $j > \ell$ imply that $\max M^j < a_\ell$ for all i, and j, ℓ, in $\{1, 2, \ldots, k-1\}$. We also take $(a_{i-1} - \max M^i) = \dfrac{1}{10(k-1)}$ for $i \in \{1, 2, \ldots, k-1\}$.

We now proceed by developing one of our congruent sets M^i, as this will completely determine $M^* = \bigcup_{i=0}^{k-1} M^i$. The Lipschitz function $f : M^* \to M^*$ with respect to which M^* is homoclinic will also be defined as we develop M^i.

Case 1 Suppose $\alpha - 1$ exists. We write $M^{k-1} = \{a_{k-1}\} \cup \dot{\bigcup}_{j=0}^{\infty} M^{k-1}_j$ with

$T(M_j^{k-1}) = \alpha - 1$ for all j, $M_\alpha^{k-1} = \{a_{k-1}\} = \{0\} = \min M^{k-1}$, and

$$\overline{\text{conv}} M_j^{k-1} = \left[\frac{9}{10k2^{j+1}},\ \frac{9}{10k2^{j+1}} + \frac{9}{10k2^{2(j+1)}}\right]$$

with $\{2x : x \in M_{j+1}^{k-1}\}$ congruent to M_j^{k-1} for $j \geq 0$. Therefore, we can map $\bigcup_{i=1}^{k-1} M^i$ to $\bigcup_{i=0}^{k-2} M^i$ with $f \mid \bigcup_{i=1}^{k-1} M^i$ the identity function plus a constant. Since $|a_0 - \max M^1| = \dfrac{1}{10(k-1)}$, if $x \in \bigcup_{i=1}^{k-1} M^i$, then $\lim y \to xy \in M^* \dfrac{f(y)-f(x)}{y-x} = 1$. We can also map M^0 to M^{k-1} with $f|M^0 \in$ Lip2 so that M^0 is homoclinic with respect to f, and

$$\lim y \to xy \in M^0 \frac{f(y)-f(x)}{y-x} = \begin{cases} 2 & x \in M^0 - M_0^0 \\ 0 & x \in M_0^0 \end{cases}.$$

Case 2 Suppose α is a limit ordinal. We write $M^{k-1} = \{a_{k-1}\} \cup \bigcup_{j=0}^{\infty} M_j^{k-1}$ so that $T(M_j^{k-1}) = \alpha_j$ with $\lim_{j\to\infty} \alpha_j = \alpha$, and $p > q$ implies $\alpha_p > \alpha_q$. Again we let $M_\alpha^{k-1} = \{a_{k-1}\} = \{0\} = \min\ M^{k-1}$ with

$$\overline{\text{conv}} M_j^{k-1} = \left[\frac{9}{10k2^{j+1}}, \frac{9}{10k2^{j+1}} + \frac{9}{10k2^{2(j+1)}}\right]$$

for $j \geq$. This time, however, we take the M_j^{k-1} so that

$$\left\{4x : x \in M_{j+1}^{k-1} \bigcap \left[\frac{9}{10k2^{j+2}} + \frac{9}{10k2^{2(j+2)+1}}, \frac{9}{10k2^{j+2}} + \frac{9}{10k2^{2(j+2)}}\right] = M_{j+1}^{k-1,t}\right\}$$

is congruent to M_j^{k-1}, $j \geq 0$, and min $M_{j+1}^{k-1,t}$ is isolated from the left. Therefore, we can map $\bigcup_{i=1}^{k-1} M^i$ to $\bigcup_{i=0}^{k-2} M^i$ with $f \Big| \bigcup_{i=1}^{k-1} M^i$ the identity function plus a constant, and map $M_j^{0,t}$ to M_{j+1}^{k-1} with $f \in$ Lip 4, so that M^* is homoclinic with respect to f, and

$$\lim y \to xy \in M^* \frac{f(y)-f(x)}{y-x} = \begin{cases} 1 & x \in M^* - M^0 \\ 0 & x \in M_0^0;\ x \in M_j^0 - M_j^{0,t},\ j = 1,2,3,\cdots \\ 4 & x \in M_j^{0,t},\ j = 1,2,3,\cdots \end{cases}.$$

Moreover, by restricting the diameters of the M_j^{k-1} as we did, we have

$$\lim{}_{y} \to a_{k-1} y \in M^* \frac{f(y)-0}{y-a_{k-1}} = 2 \ .$$

□

Our goal now is to extend the function $f : M^* \to M^*$ developed in Lemma 3 to a differentiable function $F : I \to I$ for which there exists an x in I such that $M^* = \omega_F(x)$. We do this in Lemma 4, taking advantage of a result in [2] as well as a construction from [5]. From [2] we find that if for any $x \in M^*$ our extension $F : I \to I$ has the property that $F(N_x)$ is a relative neighborhood of $F(x) = f(x)$ whenever N_x is a relative neighborhood of x, then there exists some $y \in I$ such that $M^* = \omega_F(y)$. The construction found in [5] allows us to extend f to the complementary intervals of M^* in a smooth fashion without affecting the values of the difference quotient on M^*.

Lemma 4 *Every countable compact set $M \subset I$ is homeomorphic to an ω-limit set of homoclinic type for a differentiable function with bounded derivative.*

Proof. Let M^* be the homeomorphic copy of M that appears in the proof of Lemma 3, with $f : M^* \to M^*$ the function with respect to which M^* is homoclinic. We can extend f to F linearly and by using Misiurewicz' construction on $M^* - M^0$ so that for any $x \in M^* - M^0$, $F(N_x)$ is a neighborhood of $F(x)$ whenever N_x is a neighborhood of x. We proceed by considering two cases.
Case 1 Suppose $T(M) = \alpha$, and $\alpha - 1$ exists. We can then extend f to F linearly and by using Misiurewicz' construction on $M^0 - M_0^0$ so that for any $x \in M^0 - M_0^0$, $F(N_x)$ is a neighborhood of $F(x)$ whenever N_x is a neighborhood of x. It remains to show that we can extend $f|M_0^0$ to a differentiable function f so that $F(N_x)$ is a neighborhood of $a_{k-1} = 0$ whenever N_x is a neighborhood of $x \in M_0^0$. However, since the isolated points of M_0^0 are dense in M_0^0, it suffices to extend f to F so that $F(N_x)$ is a neighborhood of the origin whenever N_x is a neighborhood of an isolated point x in M_0^0. To this end, let x be an isolated point of M_0^0, with dist $(x, M_0^0 - \{x\}) = d_x$ and $\rho_x = \dfrac{d_x}{2}$. On $[x, x+\rho_x]$, let $F : [x, x+\rho_x] \to \mathbb{R}$ be defined so that $F(y) = (y-x)^2$. Let $\{x_n\}_{n=1}^{\infty}$ be an enumeration of the isolated points of M_0^0; on each x_n we perform the above construction, so that we now have F defined on $\overline{\text{conv}} M_0^0 \bigcap \left\{ \bigcup_{n=1}^{\infty} [x_n, x_n + \rho_{x_n}] \right\}$. We can now extend F to the rest of $\overline{\text{conv}} M_0^0$ using Misiurewicz' construction so that $F'(x) = 0$ for all $x \in M_0^0$.
Case 2 Suppose α is a limit ordinal. Here we extend f linearly on $M_j^{0,t}$, $j \in \{1, 2, 3, \cdots\}$, and mimic our above construction on M_0^0 and $M_j^0 - M_j^{0,t}$ for

$j \in \{1, 2, 3, \cdots\}$. We then use Misiurewicz' construction to finish our definition of F so that F is differentiable, and $F(N_x)$ is a neighborhood of $F(x)$ whenever N_x is a neighborhood of x in M_0^0 or $M_j^0 - M_j^{0,t}$ for $j \in \{1, 2, 3, \cdots\}$. $\square$

Let us now turn our attention to the uncountable case. Specifically, our goal is to show that any uncountable nowhere dense compact set is homeomorphic to a homoclinic attractor for a differentiable function with a bounded derivative. Our strategy is much like that for the countable case. We first show that every uncountable nowhere dense compact set M has a homeomorphic copy M^* that is homoclinic with respect to a function $f : M^* \to M^*$ for which $\lim {}_y \to xy \in M^* \frac{f(y) - f(x)}{y - x}$ exists and is bounded for all x in M^*. The proof of this rests on the establishment of Lemmas 5 and 6. Loosely speaking, Lemma 5 tells us that, given some rudimentary regularity conditions, a relatively thick Cantor set E can be mapped onto a relatively thin Cantor set F with a continuous function f so that $\lim {}_y \to xy \in E \frac{f(y) - f(x)}{y - x} = 0$ for every x in E. The other preliminary result we need to develop, Lemma 6, involves mapping portions of a Cantor set E onto a countable set C by a continuous function f in such a way that $\lim {}_y \to xy \in E \frac{f(y) - f(x)}{y - x} = 0$ for all x in E.

Prior to the development of Lemma 5, we need the following definition.

Definition 2 *A set E is quasi-self-similar if*

1. *there exist constants $c, r_0 > 0$ such that for any ball B with center in E, $|B| = r \leq r_0$, there exists $\psi : E \to B \cap E$ so that $r\ c \leq \frac{|\psi(x) - \psi(y)|}{x - y}$ for $x \neq y$;*

2. *there exist constants $a, b, r_0 > 0$ such that for any neighborhood N of a point in E with $|N| = r \leq r_0$, there exists $\psi : N \cap E \to E$ such that $a \leq \frac{r|\psi(x) - \psi(y)|}{|x - y|} \leq b$ for $x \neq y$.*

Generally speaking, quasi-self-similar sets are those sets that begin to look the same all over after a certain amount of magnification. It is not too difficult to develop a familiar family of sets that satisfy the conditions of this definition. If we let $K(\alpha_m)$ represent the symmetric Cantor set developed by removing the middle α_m^{th} portion from each of the $(m-1)^{th}$ stage intervals, then those sets $K(\alpha_m)$ for which $\liminf \alpha_m = p > 0$ and $\limsup \alpha_m = q < 1$ turn out to be quasi-self-similar.

Lemma 5 *Suppose E and F are quasi-self-similar Cantor sets with Hausdorff dimension s and t, respectively. Then there exist positive constants α and β and a bijection $\psi : E \to F$ such that $\alpha \leq \dfrac{\psi(x)-\psi(y)}{(x-y)^{s/t}} \leq \beta$ for all x and y in E, with $x > y$.*

Proof. Let $[0,a] \subseteq [0,1]$ be the convex hull of E. We first show that there exist constants $0 < c_1 \leq c_2$ such that $c_1 \leq \dfrac{\mu_s(E \cap [0,x])}{x^s} \leq c_2$ for all $x \in (0,a]$, where μ_s is Hausdorff s-dimensional measure. By the definition of μ_s-measure and the definition of quasi-self-similarity, we have that $\mu_s(B \cap E) \geq c^s r^s \mu_s(E)$ for any ball B with center in E and diameter r. This implies that $\dfrac{\mu_s(E \cap [0,r])}{r^s} \geq c^s \mu_s(E) = c_1$. We also have from the definition of quasi-self-similarity that $a^s \mu_s(N \cap E) \leq r^s \mu_s(E)$ for any neighborhood of a point in E with diameter r, so that $\dfrac{\mu_s(E \cap [0,r])}{r^s} \leq \dfrac{\mu_s(E)}{a^s} = c_2$. Now, let $[0,q]$ be the convex hull of F. As is the case with E, there exist $0 < c_3 \leq c_4$ such that $c_3 \leq \dfrac{\mu_t(F \cap [0,y])}{y^t} \leq c_4$ for $y \in (0,q]$. We now define $\psi : E \to F$ by requiring that $\mu_s(E \cap [0,x]) = \dfrac{\mu_s(E)}{\mu_t(F)} \mu_t(F \cap [0,\psi(x)])$.

Since $\dfrac{\mu_s(E \cap [0,x])}{\mu_t(F \cap [0,\psi(x)])} = \dfrac{\mu_s(E)}{\mu_t(F)}$, we have $\dfrac{\mu_t(F)}{\mu_s(E)} \cdot \dfrac{c_1}{c_4} \leq \dfrac{[\psi(x)]^t}{x^s} \leq \dfrac{\mu_t(F)}{\mu_s(E)} \cdot \dfrac{c_2}{c_3}$, or $c_5 \leq \dfrac{[\psi(x)]^t}{x^s} \leq c_6$ and $c_7 \leq \dfrac{[\psi(x)]}{x^{s/t}} \leq c_8$, for some constants $0 < c_7 \leq c_8$. Now let $\{\alpha_i\}$ be an enumeration of the open intervals complementary to E, and set $\alpha_i = (a_i, b_i)$. Then $\{b_i\}$ is dense in E, and for each b_i, if $y > b_i$ and $y \in E$, then $c_7 \leq \dfrac{\psi(y)-\psi(b_i)}{(y-b_i)^{s/t}} \leq c_8$. Thus, for each x and y in E with $x > y$, we have $c_7 \leq \dfrac{\psi(x)-\psi(y)}{(x-y)^{s/t}} \leq c_8$. We can now linearly extend ψ to $\widetilde{\psi}$ defined on all of $\overline{\text{conv}}(E)$, so that $\widetilde{\psi} : [0,a] \to [0,q]$ is a bijection with $c_7 \leq \dfrac{\widetilde{\psi}(x)-\widetilde{\psi}(y)}{(x-y)^{s/t}} \leq c_8$. □

Lemma 6 *Let E be a Cantor set and F a countable compact set. Then there exist E^* and F^* homeomorphic copies of E and F respectively, and a non-decreasing function $f : E^* \to F^*$ so that $\overline{conv}F^* = \overline{conv}F = \overline{conv}E^*$, $f(E^*) = F^*$, and for any x in E^* we have $\lim y \to xy \in E^* \dfrac{f(y)-f(x)}{y-x} = 0$.*

While a complete proof of Lemma 6 is quite lengthy, the ideas underlying it are relatively straight-forward. Proposition 2 allows us to take any set with the same rank and number of highest order limit points of F as F^*. Since all Cantor sets are homeomorphic to one another, we need only find an F^* that can be covered by a Cantor set in accordance with the conclusions of Lemma 6. In constructing F^*, our approach is to make it as spread-out as possible in order to accommodate the derivative like condition we desire.

OUTLINE OF THE PROOF OF LEMMA 6. Let $[p, q] = \overline{\text{conv}}F$, and suppose p is a limit point of order α_p and q is a limit point of order α_q. Then there exist neighborhoods P and Q of p and q respectively such that $T(P \cap F) = \alpha_p$, $T(Q \cap F) = \alpha_q$, $(P \cap F)_{\alpha_p} = \{p\}$, $(Q \cap F)_{\alpha_q} = \{q\}$, and $P \cap F$, $Q \cap F$ and their complements are all both closed and open. In constructing F^* we will take $\overline{\text{conv}}F^* = [p, q]$, with p a limit point of order α_p and q a limit point of order α_q. A homeomorphic copy of $P \cap F$ will be placed in $[p, p + \frac{q-p}{5}]$, and a homeomorphic copy of $Q \cap F$ will be placed in $[q - \frac{q-p}{5}, q]$. A homeomorphic copy of $F^t = F - \{(P \cap F) \cup (Q \cap F)\}$ will be placed in $[p + \frac{q-p}{4}, q - \frac{q-p}{4}]$. Let $T(F^t) = \alpha_t$, and suppose $F^t_{\alpha_t} = \{a_0, a_1, \cdots, a_{k-1}\}$. We continue our outline in several parts.

1. We construct $(P \cap F)^*$ a homeomorphic copy of $P \cap F$.

Case 1 Suppose $\alpha_p - 1$ exists. We assume that $\dfrac{q-p}{5} = 1$ and $\left[p, p + \dfrac{q-p}{5}\right] = [0, 1]$. We take $\left\{\dfrac{1}{2^n} : n = 1, 2, 3, \cdots\right\}$ to be limit points of order $\alpha_p - 1$ of a sequence of points contained in $\left[\dfrac{1}{2^n}, \dfrac{1}{2^n} + \dfrac{1}{2^{n+2}}\right]$ for each n. In defining these particular sequences, we iteratively decompose the intervals $\left[\dfrac{1}{2^n}, \dfrac{1}{2^n} + \dfrac{1}{2^{n+2}}\right]$ into closed subintervals with lengths of negative powers of two, making each point a distance $2^{-(n+2+j)}$ to the right of $\dfrac{1}{2^n}$ a limit point of an appropriate lower order. For example, if $\alpha_p - 2 > 0$ exists, for each n, we let $\left\{\dfrac{1}{2^n} + \dfrac{1}{2^{n+2+j}} : j = 1, 2, 3, \cdots\right\}$ be limit points of order $\alpha_p - 2$. If $\alpha_p - 2 > 0$ does not exist, we let $\left\{\dfrac{1}{2^n} + \dfrac{1}{2^{n+2+j}} : j = 1, 2, 3, \cdots\right\}$ be limit points of increasing order as j increases so that the limit of their orders is $\alpha_p - 1$.

Case 2 Suppose now that α_p is a limit ordinal. We again assume $\dfrac{q-p}{5} = 1$

and that $\left[p, p+\dfrac{q-p}{5}\right] = [0,1]$. In this instance we take $\left\{\dfrac{1}{2^n} : n = 1,2,3,\cdots\right\}$ to be limit points of increasing order as n increases so that the limit of their orders is α_p. We then continue the construction described in the previous case.

2. We construct $(Q \cap F)^*$ a homeomorphic copy of $Q \cap F$. We mimic our construction of $(P \cap F)^*$ so that points accumulate to the right rather than to the left.

3. We construct $(F^t)^*$ a homeomorphic copy of F^t. We will let $(F^t)^* = \bigcup_{i=0}^{k-1} F^i$, with all the F^i congruent to one another, and so that $T(F^i) = \alpha_t$ and $F^i_{\alpha_t} = \{b_i\}$ for all $i \in \{0,1,\cdots,k-1\}$. Moreover, we let $|\max F^i - \min F^i| = \dfrac{9}{10k} \cdot \dfrac{q-p}{2}$, $\min F^i = b_i$, and $j > \ell$ imply $\max F^j < b_\ell$ for $i,j,\ell \in \{0,1,\cdots,k-1\}$, with $|a_{i-1} - \max F^i| = \dfrac{1}{10(k-1)} \cdot \dfrac{q-p}{2}$. We now construct each F^i as we constructed $(P \cap F)^*$.

4. We set $F^* = (P \cap F)^* \mathbin{\dot\cup} (F^t)^* \mathbin{\dot\cup} (Q \cap F)^*$.

5. We construct E^* a homeomorphic copy of E.

We first define a homeomorphic copy F' of F^*. We begin with $(P \cap F)^*$ and our assumption that $\dfrac{q-p}{5} = 1$ with $\left[p, p + \dfrac{q-p}{5}\right] = [0,1]$. We start our copy by moving each of the points $\left\{\dfrac{1}{2^n} : n = 1,2,3,\cdots\right\}$ to $\left\{\sqrt{\dfrac{1}{2^n}} : n = 1,2,3,\cdots\right\}$, and for each n, we move the sequence $\left\{\dfrac{1}{2^n} + \dfrac{1}{2^{n+2+j}} : j = 1,2,3,\cdots\right\}$ to $\left\{\sqrt{\dfrac{1}{2^n}} + \dfrac{1}{2^n}\sqrt{\dfrac{1}{2^{j+2}}} : j = 1,2,3,\cdots\right\}$. By continuing this process, we arrive at a copy of $(P \cap F)^*$ contained in $\left[p, p + \dfrac{q-p}{5}\right]$. We now do the same with each of the sets F^i, $i \in \{0,1,\ldots,k-1\}$, comprising $(F^t)^*$, with $\min(F^t)^*$ left fixed. For $(Q \cap F)^*$ we use the same iterative process, but apply instead the mirror of the square root function with the point q left fixed. Now, let $\{c_n\}$ be an enumeration of F^*, with $\{d_n\}$ an enumeration of F' such that c_n is mapped to d_n by the homeomorphism just defined. We now use F' to construct a Cantor set E^*. We consider two cases.

Case 1 Suppose $d_n \in F'$ is a limit point. Then let $e_n = d_n$.

Case 2 Suppose $d_n \in F'$ is isolated, where $\min\{\text{dist}(d_n, x) : x \in F' - \{d_n\}\} = \mathcal{N}$. Then let e_n be a copy of the middle thirds Cantor set contained in $[d_n, d_n + \mathcal{N}/2^{10}]$.

Let $E^* = \bigcup_{n=0}^{\infty} e_n$. We can now define $f : E^* \to F^*$ by $x \mapsto c_n$ for all $x \in e_n$. $\square$

With Lemmas 5 and 6, we are in a position to prove that every uncountable nowhere dense compact set M is homeomorphic to a set M^* which is homoclinic with respect to a function f for which $\lim y \to xy \in M^* \frac{f(y)-f(x)}{y-x}$ exists and is bounded for all $x \in M^*$. Our strategy is to break the set M into appropriate concentric annuli, making homeomorphic copies of each annulus as we go along. It is in mapping one annulus onto another as we develop our homoclinic trajectory that we use Lemmas 5 and 6. The idea is to use the maximal perfect part P^* of $M^* = P^* \dot{\cup} C^*$ to cover both itself and the at most countable remainder C^* of M^* in the homoclinic trajectory. Specifically, we have the following proposition.

Proposition 7 *Every uncountable nowhere dense set $M \subseteq I$ is homeomorphic to a set $M^* \subseteq [-1,1]$ that is homoclinic with respect to a function f, where*

$$\lim y \to xy \in M^* \frac{f(y)-f(x)}{y-x} = \begin{cases} 2 & x = 0 \\ 0 & x \in M^* - \{0\}\ . \end{cases}$$

PROOF. Let $M = P \dot{\cup} C$, where P is a Cantor set and C is countable, and take $a \in P \subseteq M$ to be an accumulation point in P from both the left and the right. We construct sets $\{M_n\}_{n=0}^{\infty}$ by taking concentric annuli about a so that each M_n is both open and closed in M, $M_n^\ell = \{x \in M_n : x < a\}$ and $M_n^r = \{x \in M_n : x > a\}$ are both uncountable, and $m > n$ implies $\max M_n^\ell < \min M_m^\ell$, $\min M_n^r > \max M_m^r$. We put a homeomorphic copy M_0^* of M_0 in $\left[-\frac{3}{4}, -\frac{1}{2}\right] \cup \left[\frac{1}{2}, \frac{1}{2} + \frac{1}{2^2}\right]$, a homeomorphic copy M_1^* of M_1 in $\left[-\frac{5}{16}, -\frac{1}{4}\right] \cup \left[\frac{1}{4}, \frac{1}{4} + \frac{1}{4^2}\right]$, and in general a homeomorphic copy M_{n-1}^* of M_{n-1} in $\left[-\frac{2^n+1}{2^{2n}}, -\frac{1}{2^n}\right] \cup \left[\frac{1}{2^n}, \frac{1}{2^n} + \frac{1}{2^{2n}}\right]$. We send the point $a \in P \subseteq M$ to the origin, and use Lemmas 5 and 6 to make appropriate copies M_n^*. Our restriction of the diameters of the M_n^* gives us $\lim y \to 0y \in M^* \frac{f(y)}{y} = 2$, since $\lim_{n\to\infty} \frac{\left(\frac{1}{2^{n-1}} + \frac{1}{2^{2n-2}}\right)}{\frac{1}{2^n}} = 2$ and $\lim_{n\to\infty} \frac{\frac{1}{2^n}}{\frac{1}{2^{n+1}} + \frac{1}{2^{2n+2}}} = 2$ as well. We now construct our sets M_n^*. We begin with M_0^*. We write $M_0 = P_0 \dot{\cup} C_0$, where P_0 is a Cantor set and C_0 is countable. We let $M_0^* = P_0^* \dot{\cup} C_0^*$, where P_0^* is a closed and open portion of $K\left(\frac{1}{3}\right)$, the middle thirds Cantor set, and C_0^*

is arranged so that Lemma 6 is satisfied in each complementary interval of P_0^*. For M_1^*, we let $M_1^* = P_1^* \dot{\cup} C_1^*$ where P_1^* is a union of a closed and open portion of $K\left(\frac{1}{4}\right)$ as well as closed and open portions of $K\left(\frac{1}{4}\right)$ that will map onto C_0^* as in Lemma 6. We arrange C_1^* as in Lemma 6 as well. In the general case of M_{n+1}^*, we let $M_{n+1}^* = P_{n+1}^* \dot{\cup} C_{n+1}^*$ so that P_{n+1}^* is a union of a closed and open portion of $K\left(\frac{1}{n+4}\right)$ as well as closed and open portions of $K\left(\frac{1}{n+4}\right)$ that will map onto C_n^* as in Lemma 6. We arrange C_{n+1}^* as in Lemma 6. From Lemmas 5 and 6, then, we can map P_{n+1}^* onto M_n^* so that $\lim {}^{y} \to xy \in P_{n+1}^* \dfrac{f(x) - f(y)}{x - y} = 0$ for every $x \in P_{n+1}^*$. We complete our map of M_{n+1}^* to M_n^*, for $n > 0$, by retracting the points of C_{n+1}^* into the endpoints of the complementary intervals of the Cantor set P_{n+1}^*. By mapping the origin to itself and M_0^* to the origin, we have a map which is homoclinic with respect to M^* that satisfies our derivative-like condition. □

We now need only extend our function $f : M^* \to M^*$ developed in Proposition 7 to a function $F : [-1, 1] \to [-1, 1]$ for which there exists an x in $[-1, 1]$ with $\omega_F(x) = M^*$. This is the content of Lemma 8.

Lemma 8 *Every uncountable nowhere dense compact set $M \subseteq I$ is homeomorphic to an ω-limit set of homoclinic type for a differentiable function with bounded derivative.*

Proof. Let M^* be the homeomorphic copy of M that appears in the proof of Lemma 7, with $f : M^* \to M^*$ the function with respect to which M^* is homoclinic. We describe our construction for $0 \leq x \leq 1$, since the extension of f to $-1 \leq x \leq 0$ is completely analogous. We may assume that for every $n \geq 0$, f maps P_{n+1}^* onto M_n^* so that $f|P_{n+1}^*$ is non-decreasing. We want to now extend $f|M_{n+1}^*$ to all of $\overline{\text{conv}} M_{n+1}^*$ so that the extension F is differentiable, $F'(x) = 0$ for all $x \in M_{n+1}^*$ and if N_x is a neighborhood of $x \in M_{n+1}^*$, then $F(N_x)$ is a neighborhood of $F(x) = f(x)$. Let $M_{n+1}^* = P_{n+1}^* \dot{\cup} C_{n+1}^*$ as before, with $\{(a_m, b_m)\}$ an enumeration of the complementary intervals of P_{n+1}^* in $\overline{\text{conv}} M_{n+1}^*$. If $C_{n+1}^* \cap (a_j, b_j) \neq \emptyset$ for some $j \in \mathbb{N}$, we then extend the retraction of this set to $f(a_j)$ and $f(b_j)$ by using that part of the construction in Lemma 4 that applies to intervals upon which f is constant (1).
We can use Misiurewicz' construction on the rest of (a_j, b_j). Now, let Q_{n+1} be a maximal closed and open portion of M_{n+1}^* that maps onto some $c_n \in C_n^*$, and let $\{(a_j, b_j)\}$ be an enumeration of those complementary intervals of Q_{n+1} that contain no points of C_{n+1}^*. Let $c_j = \dfrac{b_j + a_j}{2}$ and $\rho_j = \dfrac{b_j - a_j}{4}$. On

$[c_j - \rho_j, c_j + \rho_j]$, let $F : [c_j - \rho_j, c_j + \rho_j] \to \mathbb{R}$ be defined so that

$$F(y) = \begin{cases} (y - c_j)^2 & y \geq c_j \\ -(y - c_j)^2 & y < c_j\ . \end{cases} \tag{2}$$

We can now extend F to the rest of $\overline{\text{conv}} Q_{n+1}$ using Misiurewicz' construction. It remains to extend f to those complementary intervals $\{(a_\ell, b_\ell)\}$ of M^*_{n+1} that contain no points of C^*_{n+1} and for which $f(a_\ell) < f(b_\ell)$. We do this with Misiurewicz' construction so that $\max\{F(x) : x \in [a_\ell, b_\ell]\} = F(b_\ell) = f(b_\ell)$, $\min\{F(x) : x \in [a_\ell, b_\ell]\} = F(a_\ell) = f(a_\ell)$, and F is non-decreasing on $[a_\ell, b_\ell]$. In the same way we extend f to the intervals $\{(\max M^*_{n+1}, \min M^*_n)\}$. Finally, we use the same ideas found in (1) and (2) above to extend f to $\overline{\text{conv}} M^*_0$. Moreover, from our development of M^* in the proof of Proposition 7, we can take $F'(x) < 4 + \varepsilon$ for all $x \in [-1, 1]$, and $\varepsilon > 0$. □

Theorem 1 now follows as an immediate consequence of Lemmas 4 and 8.

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