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ON UNIONS OF POROUS SETS

Abstract

Some results about finite and countable unions of porous sets are established.

The collection of porous sets is a hereditary class of small sets on the real line that is smaller than the class of nowhere dense Lebesgue null sets [Z]. However, porous sets do not constitute an ideal. They generate an ideal (denoted here by $\mathcal{J}$) and they also generate a σ-ideal (denoted here by $\mathcal{I}$), the latter consisting of the σ-porous sets. Obviously, $\mathcal{I} \neq \mathcal{J}$ since the set of all rationals is σ-porous and is not expressible as a finite union of porous sets (otherwise, it would be nowhere dense). The aim of this paper is to give several sharp examples showing that the class of porous sets is not stable under finite unions. Our results are related to those obtained by Kahane [K] for H-sets, small sets investigated in Fourier analysis (cf. [BKR]). Note that each H-set is porous [Z].

Let us recall some definitions (cf. [Z]). For $E \subset \mathbb{R}$, $x \in \mathbb{R}$ and $r > 0$, we denote by $\gamma(E, x, r)$ the length of a longest interval $(a, b) \subset (x-r, x+r) \setminus E$; if there is no such interval, we put $\gamma(E, x, r) = 0$. We say that E is *porous* at x if the number

$$\mathrm{p}(E, x) = \limsup_{r \to 0^+} (\gamma(E, x, r)/r),$$

called the *porosity* of E at x, is positive. If $(x-r, x+r)$ above is replaced by $(x, x+r)$, one gets $\mathrm{p}^+(E, x)$, the *right porosity* of E at x; $\mathrm{p}^-(E, x)$ is defined

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similarly. Observe that $\mathrm{p}(E,x)>0$ iff $\mathrm{p}^+(E,x)>0$ or $\mathrm{p}^-(E,x)>0$. We say that E is a *porous* (*bilaterally porous*) [*strongly bilaterally porous*] *set* if $\mathrm{p}(E,x)>0$ ($\mathrm{p}^+(E,x)>0$ and $\mathrm{p}^-(E,x)>0$) [$\mathrm{p}^+(E,x)=\mathrm{p}^-(E,x)=1$] for each $x\in E$. Another variant is the following: E is *symmetrically porous* if $\limsup_{r\to 0^+}(s(E,x,r)/r)>0$ for each $x\in E$ where $s(E,x,r)$ stands for the length of the longest interval $(a,b)\subset(x,x+r)\setminus E$ fulfilling $(2x-b,2x-a)\cap E=\emptyset$. The notion of porosity can also be used in a metric space by replacing intervals with balls (see [Z],[V]). Note that the family of porous sets in $\mathbb{R}$ is invariant with respect to the mappings $x\mapsto ax+b$. (In short we say that it is *linear-invariant*.) The following example shows that it is not finitely additive.

Example. The sets $E_0=\{0\}$, $E_1=\{1/j : j\in\mathbb{Z}\setminus\{0\}\}$ (where $\mathbb{Z}$ stands for the set of all integers) are porous since they consist of isolated points. The union $E_0\cup E_1$ is not porous at 0 because

$$\mathrm{p}(E_0\cup E_1,0)=\lim_{n\to\infty}(1/n-1/(n+1))/(1/n)=0.$$

We will generalize the above example by constructing (for each $n\in\mathbb{N}=\{1,2,...\}$) a countable compact set that can be expressed as a union of $n+1$ porous sets but is not expressible as a union of n porous sets. We need a simple lemma.

Lemma 1 *Let E and A be subsets of $\mathbb{R}$ and $x\in\mathbb{R}$. If $0\le p(E,x)<p(A,x)\le 1$, then, for each interval (a,b) containing x, there is an interval $(c,d)\subset(a,b)$ such that $(c,d)\cap E\neq\emptyset=(c,d)\cap A$.*

PROOF. Assume $\mathrm{p}(E,x)=\alpha_1<\alpha_2=\mathrm{p}(A,x)$. Fix any β such that $\alpha_1<\beta<\alpha_2$. Then for all sufficiently small neighborhoods $(x-\varepsilon,x+\varepsilon)$ of x, there exists no interval in $(x-\varepsilon,x+\varepsilon)\setminus E$ of length $\ge\beta\varepsilon$. However, $\mathrm{p}(A,x)>\beta$ implies that for an arbitrarily small $\varepsilon>0$, there exists an interval in $(x-\varepsilon,x+\varepsilon)\setminus A$ of length $>\beta\varepsilon$. Any sufficiently short interval of the latter type can be used for (c,d). □

Let $0<\alpha\le 1$. We say that $E\subset\mathbb{R}$ is *α–porous* if $\mathrm{p}(E,x)\ge\alpha$ for each $x\in E$. In the same way, we define E is *α–symmetrically porous.*

Theorem 1

(a) For each $n\in\mathbb{N}$, there exists a countable compact set $D_n\subset\mathbb{R}$ that is a union of $n+1$ discrete (porous) sets but cannot be expressed as a union of n porous sets.

(b) There exists a countable compact σ-porous set $D\subset\mathbb{R}$ which cannot be expressed as a union of finitely many porous sets.

(c) *Let $0 < \alpha_1 < \alpha_2 \leq 1$. Then for each $n \in \mathbb{N}$, there exists a countable compact α_1–symmetrically porous set in $\mathbb{R}$ that can be expressed as a union of $n+1$ discrete sets, but which cannot be expressed as a union of n α_2–porous sets.*

Proof. (a) Let E_0, E_1 be defined as in Example 1. For each $k \in \mathbb{N}$, if E_k is defined and it consists of isolated points, we fix an open neighborhood $U(x)$ of $x \in E_k$, disjoint from $\bigcup_{i=0}^{k} E_i \setminus \{x\}$. Then we put

$$E_{k+1} = \bigcup_{x \in E_k} \{x + 1/j : j \in \mathbb{Z} \setminus \{0\}, x + 1/j \in U(x)\}.$$

Thus we have defined inductively the sequence $\{E_k\}_{k=0}^{\infty}$. The set $D_n = \bigcup_{i=0}^{n} E_i$ will meet our needs. Indeed, it is the union of the $n+1$ discrete sets E_i. (So each E_i is porous.) We will show the second part of the assertion for $n = 2$. (The general case is similar, only involving more steps.) Suppose that $D_2 = A_1 \cup A_2$ where A_1, A_2 are porous. We may assume (by considering an appropriate translation) that $0 \in A_1$. Hence, $\mathrm{p}(A_1, 0) > 0$. Apply Lemma 1 with $E = E_1$, $A = A_1$ and $x = 0$. (Here we have $\mathrm{p}(E, x) = 0$.) Then there is an interval (c_1, d_1) disjoint from A_1 and a point $x_1 \in E_1 \cap (c_1, d_1)$. Obviously, $x_1 \in A_2$, so $\mathrm{p}(A_2, x_1) > 0$. By Lemma 1 (applied with $E = E_2$, $A = A_2$ and $(a, b) = (c_1, d_1) \cap U(x_1)$), there is an interval $(c_2, d_2) \subset (c_1, d_1) \cap U(x_1) \setminus A_2$ and a point $x_2 \in E_2 \cap (c_2, d_2)$. This is a contradiction since $x_2 \in E_2 \subset D_2$, while on the other hand, $x_2 \in (c_2, d_2)$ and (c_2, d_2) is disjoint from $A_1 \cup A_2 = D_2$.

(b) Let $D = \bigcup_{k=0}^{\infty} E_k$. By construction, D is countable and compact. Suppose that $D = \bigcup_{i=1}^{n} A_i$ for porous sets A_i. Since $D_n \subset D$, we have $D_n = \bigcup_{i=1}^{n} (A_i \cap D_n)$, where $A_i \cap D_n$ are porous $(i = 1, ..., n)$. This contradicts (a).

(c) The proof is analogous to that of (a) if one replaces the "double sequence" $\{\pm 1/j\}_{j=1}^{\infty}$ by the "double sequence" $\{\pm(1 - \alpha_1)^j\}_{j=1}^{\infty}$. $\square$

Remark. Each discrete set is porous in a strong manner. For instance, it is strongly bilaterally porous and even 1-very shell porous. (See below.)

In the following discussion and theorem, let X denote a metric space. Given $A \subset X$, let $A^{(1)}$ denote the derived set A' of A, i.e. the set of all accumulation points of A. Inductively, we define $A^{(n+1)} = (A^{(n)})'$ for $n \in \mathbb{N}$. (One can also go further, using ordinals.) Additionally, let $A^{(0)} = A$. If $A^{(n)} = \emptyset \neq A^{(n-1)}$, we say that the Cantor-Bendixson rank of A is n (abbreviated, $\mathrm{rk}(A) = n$). In [K] Kahane observed that, by Salinger's result [S], for any $n \in \mathbb{N}$, each countable compact set $A \subset \mathbb{R}$ with $\mathrm{rk}(A) = n$ is expressible as a union of 2^{n-1} H-sets (some of which can be empty). Since each H-set is porous (even globally porous, see [Z]), we obtain the analogous statement with H-sets replaced by porous sets. We will demonstrate a more general result which works in a metric space. Our method of proof is different from that used in [S]. Namely, we utilize shell porosity introduced by Vallin [V]. By $B_x(r)$ we mean the open ball in a metric space X, centered at x of radius r. Let cl$E$ denote the closure of $E \subset X$. For $x \in X$ and $0 < r_1 < r_2$, we put $S_x(r_1, r_2) = B_x(r_2) \setminus \mathrm{cl}B_x(r_1)$. For $E \subset X$, $x \in X$ and $R > 0$, let

$$\Lambda(E, x, R) = \sup\{h > 0 : (\exists t > 0)\ (t + h < R)\ \&\ (S_x(t, t + h) \cap E = \emptyset)\}.$$

(If no such $h > 0$ exists, put $\Lambda(E, x, R) = 0$.) A set E is called *shell porous* if $\mathrm{p}^s(E, x) > 0$ for each $x \in E$, where

$$\mathrm{p}^s(E, x) = \limsup_{R \to 0+}(\Lambda(E, x, R)/R).$$

For $X = \mathbb{R}$, shell porosity equals symmetric porosity and is stronger than bilateral porosity. Let $\alpha \in (0, 1]$. We will say that E is α-*very shell porous* if $\mathrm{p}^s_*(E, x) \geq \alpha$ for each $x \in E$, where

$$\mathrm{p}^s_*(E, x) = \liminf_{R \to 0+}(\Lambda(E, x, R)/R).$$

Clearly, α-very shell porosity implies shell porosity. But "α–very shell porous" does not imply "porous" in an arbitrary metric space. Let X be a discrete metric space and $E = X$. Then E is 1-very shell porous at each of its points, and yet E fails to be porous at any point.

Theorem 2 *For all $n \in \mathbb{N}$, each countable compact set A with $rk(A) = n$ in a metric space X is expressible as $\bigcup_{i=1}^{2^{n-1}} A_i$, where each set A_i is compact and $(1/5)$-very shell porous.*

Proof. Let $E^j = A^{(n-j)}$ for $j = 0, 1, ..., n$. Then

$$E^0 = \emptyset \neq E^1 \subset E^2 \subset ... \subset E^n = A.$$

We are going to define the family $\{A_t : t \in \bigcup_{k=0}^{n-1} \mathrm{Seq}_k\}$ of subsets of A where Seq_k denotes the set of all $0 - 1$ sequences of length k. (The empty sequence $\langle\ \rangle$ is the unique element of Seq_0.) For $t \in \mathrm{Seq}_k$ and $i \in \{0, 1\}$ let $ti \in \mathrm{Seq}_{k+1}$ stand for the extension of t with the last term equal to i. For each $k \in \{1, ..., n\}$ the sets A_t will satisfy the following conditions:

(1) A_t is closed for each $t \in \mathrm{Seq}_{k-1}$,

(2) $\bigcup_{t \in \mathrm{Seq}_{k-1}} A_t = A$,

(3) $A_t = A_{t0} \cup A_{t1}$ for all $t \in \mathrm{Seq}_{k-1}$ (if $k < n$),

(4) $\mathrm{p}^s_*(A_t, x) \geq 1/5$ for all $t \in \mathrm{Seq}_{k-1}$ and $x \in E^k$.

We will use induction with respect to k. For $k = n$, conditions (2) and (4) yield the assertion of Theorem 2.

First, we need an auxiliary construction. Consider a set $C \subset X$ of the form $C = \bigcup_{x \in D} B_x(r^x)$ where $D \subset A$ and the balls $B_x(r^x)$, $x \in D$, are pairwise disjoint. We define two operations $C \mapsto C_0$ and $C \mapsto C_1$ as follows. For each $x \in D$, let $\{r^x_n\}_{n=1}^\infty$ be a decreasing sequence of reals such that $r^x_1 = r^x$ and $\lim_{n\to\infty}(r^x_{n+1}/r^x_n) = 1/2$. Since A is countable, we may assume the numbers r^x_n (for $x \in D$ and $n > 1$) have been chosen so that there are no points in A which lie exactly a distance r^x_n away from x. Put

$$C_0 = D \cup \bigcup_{x\in D} \bigcup_{j=1}^\infty S_x(r^x_{2j-1}, r^x_{2j}) \cup (X \setminus C), \quad C_1 = D \cup \bigcup_{x\in D} \bigcup_{j=1}^\infty S_x(r^x_{2j}, r^x_{2j+1}).$$

Now, we will start defining sets A_t. Let $k = 1$ and $A_{\langle\,\rangle} = A$. Then conditions (1) to (4) are obviously satisfied. Assume that $1 \leq k < n$ and that the sets A_t, for $t \in \mathrm{Seq}_{k-1}$, are defined. The set $E^{k+1} \setminus E^k$ is discrete, so we can pick pairwise disjoint balls $B_x(r^x)$, $x \in E^{k+1} \setminus E^k$. It is easy to see these balls are disjoint from E^k. Additionally, we can ensure that the collection of points in X which lie exactly a distance r^x away from x does not meet A. Now, for any fixed $t \in \mathrm{Seq}_{k-1}$, we put

$$A_{ti} = A_t \cap (\left(\bigcup_{x \in (E^{k+1}\setminus E^k)\cap A_t} B_x(r^x) \right)_i \cup E^k),$$

where $i = 0, 1$. (Here we have used the operations $C \mapsto C_i$, $i = 0, 1$, defined above.) It is not hard to check that (by the compactness of A and the choice of numbers r^x) the sets A_{ti} $(i = 0, 1)$ are closed. The equality $A_{t0} \cup A_{t1} = A_t$ is also clear. This and (2) applied to k give (2) for $k + 1$. From (4) applied to A_t $(t \in \mathrm{Seq}_{k-1})$ and from $A_{ti} \subset A_t$ it follows that $\mathrm{p}^s_*(A_{ti}, x) \geq 1/5$ for $x \in E_k$ $(i = 0, 1)$. It suffices to prove that

$$(4') \quad \mathrm{p}^s_*(A_{ti}, x) \geq 1/5 \text{ for each } x \in E^{k+1} \setminus E^k (i = 0, 1).$$

To this end, pick $x \in E^{k+1} \setminus E^k$. If $x \notin A_{ti}$, then (4') holds because A_{ti} is closed. If $x \in A_{ti}$, then by the construction we have

$$\bigcup_{j=1}^\infty S_x(r^x_{2j}, r^x_{2j+1}) \cap A_{t0} = \bigcup_{j=1}^\infty S_x(r^x_{2j-1}, r^x_{2j}) \cap A_{t1} = \emptyset.$$

A simple calculation now shows (4') holds. □

Corollary 1 *For each $n \in \mathbb{N}$, there exists a countable compact set $F_n \subset \mathbb{R}$ that is a union of $n+1$ compact (bilaterally) [symmetrically] porous sets but cannot be expressed as a union of n compact (bilaterally) [symmetrically] porous sets.*

Proof. Consider the set D_n from Theorem 1(a). Since $\mathrm{rk}(D_n) = n+1$, Theorem 2 implies that $D_n = \bigcup_{i=1}^{2^n} A_i$, where A_i $(i = 1, ..., 2^n)$ are compact porous sets. At first consider all subsets of D_n that are unions of $n+1$ distinct sets A_i. If at least one of these subsets cannot be expressed as a union of n compact porous sets, we pick it as F_n. Otherwise, consider all subsets of D_n that are unions of $n+2$ distinct sets A_i. (In fact, those unions can be reduced to unions of $n+1$ compact porous sets.) and repeat the above procedure. After fewer than 2^n such steps, we find a subset F_n of D_n fulfilling the assertion. (Otherwise, in the last step, $\bigcup_{i=1}^{2^n} A_i = D_n$ would be expressible as a union of n compact porous sets which contradicts Theorem 1(a).) The proof for bilateral (or symmetric) porosity is analogous. □

The next application of Theorem 1 deals with ideals $\mathbb{J}$ being nice generators of a given linear-invariant σ-ideal $\mathbb{I}$ of subsets of $\mathbb{R}$ containing all singletons. According to [BS] (see also [B]), a *nice generator* $\mathbb{J}$ should be an ideal generating $\mathbb{I}$, be linear-invariant, contain all singletons, differ from the σ-ideal $\mathbb{I}$ on each nonempty open set, fail to be a σ-ideal and, moreover, it should satisfy the following Świątkowski condition:

($*$) for each $E \subset \mathbb{R}$, from $E \cap U \in \mathbb{J}$ for any open bounded $U \subset \mathbb{R}$ it follows that $E \in \mathbb{J}$.

Additionally, $\mathbb{I}$ and $\mathbb{J}$ in [BS] are required to be *Borel-supported*, which means each set from $\mathbb{I}$ (or $\mathbb{J}$) is contained in a Borel set from the same family. It is easily seen that the ideal of nowhere dense sets is a nice generator of the σ-ideal of meager sets in $\mathbb{R}$. In [BS], nice generators of the σ-ideal of Lebesgue null sets are constructed. (There are many of them.) We show that the family $\mathcal{J}$ of finite unions of porous sets is not a nice generator of the σ-ideal $\mathcal{I}$ of σ-porous sets, which answers a question from [BS].

Corollary 2 *The ideal $\mathcal{J}$ of finite unions of porous sets does not satisfy ($*$).*

Proof. We transform the sets D_n $(n \in \mathbb{N})$ from Theorem 1(a) into sets $D_n^* \subset (n, n+1)$ by suitable mappings $x \mapsto ax+b$ $(a \neq 0)$. Then we put $E = \bigcup_{n=1}^{\infty} D_n^*$. By Theorem 1(a), we have $E \cap U \in \mathcal{J}$ for each open bounded $U \subset \mathbb{R}$. However, $E \notin \mathcal{J}$. Indeed, suppose that $E = \bigcup_{i=1}^{n} E_i$ where E_i are porous sets. Then $E \cap (n, n+1) = D_n^*$ is expressible as the union of n porous sets $E_i \cap (n, n+1)$ $(i = 1, ..., n)$, which contradicts Theorem 1(a). □

Theorem 3 *The ideal $\mathcal{J}^*$ of sets that can be covered by G_δ σ-porous sets in $\mathbb{R}$ is a nice generator of the σ-ideal $\mathcal{I}$ of σ-porous sets.*

Proof. First, recall that each σ-porous set is contained in a $G_{\delta\sigma}$ σ-porous set [FH]. Thus, $\mathcal{J}^*$ generates $\mathcal{I}$. By considering the rationals we see that $\mathcal{J}^*$ is not a σ-ideal and that $\mathcal{J}^*$ differs from $\mathcal{I}$ on each nonempty open set. The remaining conditions, except for $(*)$, are obvious. (Recall that a finite union of G_δ sets is G_δ.) To show $(*)$, take $E \subset \mathbb{R}$ and assume $E \cap U \in \mathcal{J}^*$ for any open bounded $U \subset \mathbb{R}$. Then, for each $n \in \mathbb{Z}$, the set $E \cap (2n, 2n+3/2)$ is contained in a G_δ σ-porous set $F_n \subset (2n, 2n+3/2)$. It easily follows that $F = \bigcup_{n\in\mathbb{Z}} F_n$ is a G_δ σ-porous set. Similarly, for each $n \in \mathbb{Z}$, the set $E \cap (2n-1, 2n+1/2)$ is contained in a G_δ σ-porous set $H_n \subset (2n-1, 2n+1/2)$, and $H = \bigcup_{n\in\mathbb{Z}} H_n$ is a G_δ σ-porous set. Finally, E is contained in the G_δ σ-porous set $F \cup H$, and therefore $E \in \mathcal{J}^*$. □

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