A CHARACTERIZATION OF ORLICZ FUNCTIONS PRODUCING AN ADDITIVE PROPERTY

Abstract

It is shown that the only Luxemburg functionals that satisfy a very simply formulated property are induced by \( p \)-th-power functions, \( 0 < p < \infty \). The known result that Orlicz spaces cannot be normed analogously to \( L_p \)-spaces follows as a consequence.

1 Introduction

Let \( \Psi : [0, \infty) \rightarrow [0, \infty] \) be a nondecreasing function, \( \Psi(0) = 0 \), \( \Psi(x) \rightarrow \infty \) as \( x \rightarrow \infty \), and such that if \( 0 < a < b \), \( 0 < \Psi(a) \), \( \Psi(b) < \infty \), then \( \Psi \) is strictly increasing on \( [a, b] \) and continuous on \( [0, b] \). Such a function is called an \( O \)-function. Let \( (\Omega, \mathcal{A}, \mu) \) be a measure space. Identify real valued functions on \( \Omega \) that differ only on a set of measure zero. Let \( \mathcal{M} \) denote the corresponding set of congruence classes. It is known [2, 3] that the pair \( \{\Psi, \mu\} \) induces the Luxembourg functional on the Orlicz space

\[
\mathcal{L}_\Psi(\mu) := \left\{ f \in \mathcal{M} : \int_\Omega \Psi(\alpha |f|) \, d\mu < \infty \text{ for some } \alpha > 0 \right\}.
\]

Key Words: Orlicz function, Luxemburg norm, \( L_p \)-spaces
Mathematical Reviews subject classification: Primary: 46E30
Received by the editors August 27, 1995
*Partially supported by CONICOR and Univ. Nac. de Río Cuarto, Argentina.
Its expression is
\[
\rho_{\Psi,\mu}(f) := \inf \left\{ \lambda > 0 : \int_{\Omega} \Psi(|f|/\lambda) \, d\mu \leq 1 \right\}.
\]

A well known example is the \(L_p\)-norm (with abuse of language for \(0 < p < 1\)), induced by any \(\mu\) and \(\Psi(x) = cx^p\), \(c > 0\), for \(0 < p < \infty\), which gives \(\mathcal{L}_p(\mu) = L_p(\mu) = \{ f \in \mathcal{M} : \int_{\Omega} |f|^p \, d\mu < \infty \}, \rho_{\Psi,\mu}(f) = (c \int_{\Omega} |f|^p \, d\mu)^{1/p}, \) and induced by any \(\mu\) and a function \(\Psi\) that satisfies \(\Psi(x) = 0\) if \(0 \leq x \leq a\), \(\Psi(x) = \infty\) if \(x > a, a > 0\), for \(p = \infty\), which gives \(\mathcal{L}_{\infty}(\mu) = \{ f \in \mathcal{M} : \text{ess sup} |f| < \infty \}. \rho_{\Psi,\mu}(f) = \|f\|_{\infty} = a \text{ess sup} |f|\). The \(L_p\)-norms satisfy the following property, applicable to any functional \(\rho\) defined on a quite arbitrary real function space.

For any set \(B \in \mathcal{A}\) and \(f, g\) simple functions, if \(\rho(f \chi_B) = \rho(g \chi_B)\)
and \(\rho(f \chi_{\Omega \setminus B}) = \rho(g \chi_{\Omega \setminus B})\), then \(\rho(f) = \rho(g)\)\.

We recall that a simple function is one of form \(\sum_{i=1}^{n} c_i \chi_{A_i},\mu(A_i) < \infty\), where \(\chi_{A}\) denotes the characteristic function of the set \(A\). If \(A\) has, exactly, none, one or two disjoint sets of finite and positive measure, then the class of all simple functions can be identified with \(\{0\}, \mathbb{R}\) or \(\mathbb{R}^2\), respectively, and in these cases any homogeneous functional defined on \(\mathcal{L}_p(\mu)\), depending on \(|f|\), (e.g. a Luxemburg functional) satisfies \((*)\). We show in this paper that a rather different result follows when \(A\) has at least three disjoint sets of finite and positive measure \(\mu\). The function \(\Psi\) is said to satisfy property \(P_{\mu}\), if \(\{\Psi, \mu\}\) induces a Luxemburg functional on \(\mathcal{L}_p(\mu)\) satisfying \((*)\). We shall give a description of such functions. In all cases they yield a \(L_p\)-norm, \(0 < p \leq \infty\).

In case that \(\mu\) is \(\sigma\)-finite and \(\Psi\) is convex, this latter result can also be obtained from a classical theorem of H. F. Bohnenblust \([1, 4]\). As a consequence of that theorem, for \(\dim \mathcal{L}_p(\mu) \geq 3\) it is obtained that homogeneous functionals on \(\mathcal{L}_p(\mu)\) that satisfy \((*)\) are \(p\)-additive, \(0 < p \leq \infty\). For \(p \geq 1\) this fact implies in turn that \(\rho_{\Psi,\mu}(f)\) is a \(L_p\)-norm. However we do not follow the ideas of that theorem neither use the \(p\)-additive condition. Depending on a general measure \(\mu\), in each case our proofs directly lead to the characterization of \(\Psi\).

We consider in Section 2 a Luxemburg functional induced by a continuous \(O\)-function \(\Psi\). In Section 3 we assume that \(\Psi\) is not continuous and that \(\mu\) is in addition a \(\sigma\)-finite measure. As a consequence of Theorem 1 we get in Section 4 the known fact that the space \(\mathcal{L}_p(\mu) := \{ f \in \mathcal{M} : \int_{\Omega} \Psi(\alpha|f|) \, d\mu < \infty \) for all \(\alpha > 0\) cannot be normed analogously to \(L_p\)-spaces, \(p > 0\), whenever \(\dim \mathcal{L}_p(\mu) \geq 2\).

We say that \((\Omega, \mathcal{A}, \mu)\) is \emph{infinitely divisible} if there are measurable subsets of \(\Omega\) with positive and arbitrarily small measure. If \((\Omega, \mathcal{A}, \mu)\) is not infinitely
divisible and $A$ has at least one set of finite and positive measure, then we let
\[ r_0 = \inf \{ \mu(A), A \in A, \mu(A) > 0 \} > 0. \]

## 2 Characterization of a Continuous $O$-function $\Psi$

In this section we assume that $\Psi$ satisfies $\Psi(\mathbb{R}_+) \supseteq \mathbb{R}_+$. Hence its right inverse function $\Psi^{-1} : (0, \infty) \to \mathbb{R}_+$ exists, is continuous and satisfies $\Psi(\Psi^{-1}(x)) = x$ for all $x > 0$. We say that a Luxemburg functional induced by any measure and such a function $\Psi$ is a $L$-functional.

### Theorem 1

Assume that $A$ has at least three disjoint sets of finite and positive measure.

(a) If $(\Omega, A, \mu)$ is infinitely divisible, then $\Psi$ satisfies property $P_\mu$ if and only if $\Psi(x) \equiv cx^p$ on $[0, \infty)$, $c > 0$, $p > 0$.

(b) If $(\Omega, A, \mu)$ is not infinitely divisible, then $\Psi$ satisfies property $P_\mu$ if and only if $\Psi$ verifies $\Psi(x) = cx^p$ for any $x \in [0, \Psi^{-1}(1/r_0)]$, $p > 0$, $c > 0$.

In both cases the functional induced is a $L_p$-norm and therefore these are the only $L$-functionals that verify property $(*)$.

**Proof.** Assume that $\Psi$ satisfies property $P_\mu$. Take $F \in A$, $0 < \mu(F) < \infty$, such that there are two disjoint sets $G$ and $E$ of finite and positive measure, $G \cup E \subseteq \Omega \setminus F$, $\mu(E) \geq \mu(F)$. Such a set $F$ always exists due to the assumptions on $(\Omega, A, \mu)$. Assume first $\mu(F) = \Psi(1) = 1$. Take $h \in \mathbb{R}$, $h > 0$, such that $\rho_{\Phi, \mu}(h \chi_{F \cup G}) = \rho_{\Phi, \mu}(\chi_F) = 1$. So property $(*)$ implies $\rho_{\Phi, \mu}(h \chi_{F \cup G} + s \chi_E) = \rho_{\Phi, \mu}(\chi_F + s \chi_E) =: \delta_s$ for any $s \in [0, \infty)$. Therefore, by definition of $\rho_{\Phi, \mu}$ and the continuity of $\Psi$ (on $[0, 1]$), we get $\mu(F \cup G) \Psi(h) = 1$ on the one hand and on the other hand

\[ \mu(F \cup G) \Psi(h/\delta_s) + \mu(E) \Psi(s/\delta_s) = \Psi(1/\delta_s) + \mu(E) \Psi(s/\delta_s) = 1, \]

whence $\Psi(h/\delta_s) = \Psi(h) \Psi(1/\delta_s)$ for any $s \geq 0$. As $s \mapsto \delta_s$ maps continuously $[0, \infty)$ onto $[1, \infty)$, we have obtained that $h$ is a multiplier for $\Psi$, i.e., $h$ is a point $m \in [0, 1]$ that satisfies

\[ \Psi(m \gamma) = \Psi(m) \Psi(\gamma) \text{ for any } \gamma \in [0, 1]. \]

Observe that the former equation implies that $\Psi(h^n) = [\Psi(h)]^n$ for $n \in \mathbb{N}$. Moreover, $h^n$ is a multiplier for $\Psi$. Since $0 < h < 1$, $0 < \Psi(h) < 1$, it follows that $h^n \downarrow 0$, $\Psi(h^n) \downarrow 0$ as $n \to \infty$ (and therefore $\Psi(x) > 0$ if $x > 0$). Take
$k_1 \in \mathbb{R}$, $k_1 > 0$, such that $\rho_{\Psi, \mu}(k_1 \chi_F + mk_1 \chi_G) = \rho_{\Psi, \mu}(\chi_F) = 1$, where $m$ is a multiplier for $\Psi$. Hence $\Psi(k_1) + \mu(G)\Psi(mk_1) = 1$. As $k_1 < 1$, it follows that

$$\Psi(k_1)[1 + \mu(G)\Psi(m)] = 1.$$ 

On the other hand, property $(\ast)$ implies $\rho_{\Psi, \mu}(k_1 \chi_F + mk_1 \chi_G + s \chi_F) = \delta_s$ for any $s \geq 0$, whence $\Psi(k_1 \gamma) = \Psi(k_1)\Psi(\gamma)$ for $\gamma \in [0, 1]$. We have just proved that if $m$ is a multiplier for $\Psi$, then $k_1 = \Psi^{-1}(1/[1 + \mu(G)\Psi(m)])$ is also a multiplier for $\Psi$. It follows that

$$m_n = m_n(\Psi, \mu) := \Psi^{-1}(1/[1 + \mu(G)\Psi(h^n)])$$

is a multiplier for $\Psi$, $n \in \mathbb{N}$. As $\Psi^{-1}$ is continuous and $\Psi^{-1}(1) = 1$, we get that $m_n \uparrow 1$ as $n \to \infty$.

The existence of the sequence $\{m_n\}$ implies that any $m \in [0, 1]$ is in the collection $\mathcal{P}$ of multipliers for $\Psi$. Indeed, observe first that $\mathcal{P}$ is closed because $\Psi$ is continuous. Hence

$$\beta_0(m) := \inf\{\beta \in \mathcal{P} : \beta > m\} \in \mathcal{P} \text{ for } m \in [0, 1),$$

and $\beta_0(m) = m$ because $m_n\beta_0(m) \in \mathcal{P}$ for all $n \in \mathbb{N}$.

For $x \in (0, 1]$ we have

$$[\Psi(xm_n) - \Psi(x)]/[x(m_n - 1)] = \Psi(x)[\Psi(m_n) - \Psi(1)]/[x(m_n - 1)]. \quad (1)$$

On the other hand, the obvious estimates below show that $\Psi$ is absolutely continuous on $[\eta, 1]$ for all $\eta \in (0, 1)$.

Let $x_i \in [\eta, 1]$, $1 \leq i \leq n + 1$, $n \in \mathbb{N}$, and $x_1 < x_2 < \cdots < x_{n+1}$. Let $\gamma_i := x_i/x_{i+1}$, $1 \leq i \leq n$. Then

$$n \sum_{i=1}^n [1 - \gamma_i] < \sum_{i=1}^n x_{i+1}[1 - \gamma_i] = \sum_{i=1}^n [x_{i+1} - x_i],$$

$$\sum_{i=1}^n [\Psi(x_{i+1}) - \Psi(x_i)] = \sum_{i=1}^n [\Psi(x_{i+1}) - \Psi(\gamma_ix_{i+1})]$$

$$= \sum_{i=1}^n \Psi(x_{i+1})[1 - \Psi(\gamma_i)] \leq \sum_{i=1}^n [1 - \Psi(\gamma_i)],$$

$$1 - \Psi(\gamma) \leq K[1 - \gamma] \text{ for some } K > 0 \text{ and any } \gamma < 1.$$ 

Therefore we get that the derivative $\Psi'(x)$ exists and is finite-valued for almost every $x$ on $[0, 1]$. Hence the left side in eq. (1) converges to $\Psi'(x)$ as $n \to \infty$ for
almost every \( x \in [0, 1] \), and it follows that the right side in eq. (1) converges to \( p\Psi(x)/x \) as \( n \to \infty \) for any \( x \in (0, 1) \), where \( p \geq 0 \). Therefore \( \Psi'(x)/\Psi(x) = p/x \) a.e. on \( (0, 1) \). As \( \Psi \) is not constant, we have \( p > 0 \). Since \( \ln \Psi \) is absolutely continuous on \([\eta, 1]\), the integration of both sides of the former equation from \( x \) to 1 gives \( \Psi(x) \equiv x^p \) on \([0, 1]\). Observe that, so far, only the restriction of \( \Psi \) on \([0,1]\) has been considered (cf. the end of Section 3).

Suppose now \( \mu(F) = r > 0, \Psi(1) \geq 0 \). The \( L \)-functional induced by \( r\Psi \) and \( \mu/r \) on \( L_\Psi(\mu) \) coincides with the \( L \)-functional induced by \( \Psi \) and \( \mu \). On the other hand, the \( L \)-functional induced by \( \tilde{\Psi}(x) := r\Psi(\Psi^{-1}(1/r)x) \) and \( \mu/r \) on \( L_\Psi(\mu) \) is \( \Psi^{-1}(1/r) \) times the \( L \)-functional induced by \( r\Psi \) and \( \mu/r \), and therefore it also satisfies property (a), with \( \tilde{\Psi}(1) = (\mu/r)(F) = 1 \). So we get \( \tilde{\Psi}(x) \equiv x^p \) on \([0, 1]\), with \( p > 0 \), whence \( \Psi(x) \equiv cx^p \) on \([0,\Psi^{-1}(1/r)]\), where \( c = 1/[r(\Psi^{-1}(1/r))^p] \).

Under the hypothesis of (a) we can take \( r \downarrow 0 \). Then the case \( \Psi^{-1}(1/r) \uparrow b, b < \infty \), leads to a contradiction, whence \( \psi^{-1}(1/r) \uparrow \infty \) and the necessary part of (a) follows. The sufficiency of (a) is obvious. The necessity of (b) follows by taking \( r \to r_0 \). (Observe that this taking of limits in \( r \) is compatible with the assumption on \( F \) at the beginning of the proof). Conversely, if \( \Psi(x) \equiv cx^p \) on \([0,\Psi^{-1}(1/r_0)]\), then \( \tilde{\Psi}(x) := r_0\Psi(\Psi^{-1}(1/r_0)x) \equiv x^p \) on \([0,1]\) and, as mentioned above, \( \Psi \) and \( \mu/r_0 \) induce, up to a multiplicative constant, the same \( L \)-functional as \( \Psi \) and \( \mu \). So, to conclude the proof, it suffices to observe that if \( \Psi(x) \equiv x^p \) on \([0, 1]\) and in addition \( \mu(C) \geq 1 \) for any measurable set \( C \) with \( \mu(C) > 0 \), then \( \{\Psi, \mu\} \) induces the standard \( L_p \)-norm on \( L_\Psi(\mu) \) (= \( L_p(\mu) \)). Indeed, if \( g \in \mathcal{M} \) and \( \int_\Omega |g| \, d\mu \leq 1 \), then \( |g| \leq 1 \) almost everywhere \( (\mu) \) on \( \Omega \), whence \( \int_\Omega \Psi(|f|/\lambda) \, d\mu \leq 1 \) is equivalent to \( \int_\Omega (|f|/\lambda)^p \, d\mu \leq 1 \). Therefore \( \rho_{\Psi, \mu}(f) = \inf\{\lambda : \int_\Omega (|f|/\lambda)^p \, d\mu = 1\} = (\int_\Omega |f|^p \, d\mu)^{1/p} \). \( \square \)

## 3 Characterization of a Discontinuous \( O \)-function \( \Psi \)

Now we suppose, and only in this section, that \( \Psi \) is not \( (\Omega, \mathcal{A}, \mu) \) is in addition a \( \sigma \)-finite measure space. So we consider an \( O \)-function \( \Psi \) jumping to infinity at \( a, a > 0 \). We say that the Luxemburg functional induced by such a pair \( \{\Psi, \mu\} \) on \( L_\Psi(\mu) \) is a \( L^* \)-functional. Observe that when \( (\Omega, \mathcal{A}, \mu) \) is not infinitely divisible, a \( L^* \)-functional may coincide with a \( L \)-functional. Since \( \mu \) is \( \sigma \)-finite, the condition required to \( \mathcal{A} \) in Theorem 1 is now equivalent to \( \dim L_\Psi(\mu) \geq 3 \). We can suppose without loss of generality that \( \Psi \) is left continuous. An example of such a function is: \( \Psi(x) = 0 \) if \( 0 \leq x \leq a \), \( \Psi(x) = \infty \) if \( x > a \). It is easy to see that \( \{\Psi, \mu\} \), with this function \( \Psi \), induces the \( a \) essential sup norm on \( L_\Psi(\mu) \), which we call, as usual, a \( L_\infty \)-norm. Moreover, it is easy to show that if \( \mu(\Omega)\Psi(a) \leq 1 \), then \( \{\Psi, \mu\} \) induces a \( L_\infty \)-norm. Observe also
that if \( \mu(C) \geq 1 \) for any measurable set \( C \) of positive measure and \( \Psi \) satisfies \( \Psi(x) \equiv cx^p \) on \([0, 1]\) (e.g., \( \Psi(x) \equiv x^p \) on \([0, 1]\), \( \Psi(x) = \infty \) for \( x > 1 \)), then \( \{ \Psi, \mu \} \) induces the standard \( L_p \)-norm on \( \mathcal{L}_p(\mu) \).

Assume now that \( \{ \Psi, \mu \} \) induces a \( L^* \)-functional, where \( \mu(\Omega)\Psi(a) > 1 \). Under this condition we consider two exhaustive cases. Suppose first that there exists \( F \in \mathcal{A}, \ 0 < \mu(F)\Psi(a) < 1 \). For instance, this is the case if \( (\Omega, \mathcal{A}, \mu) \) is infinitely divisible. Assume also that there exists \( B \in \mathcal{A}, B \supseteq F, \ \mu(B) < \mu(\Omega), \ \mu(B)\Psi(a) > 1 \). (This is the case if \( \mu(\Omega) = \infty \).) Then, since \( \Psi \) is a continuous function on \([0, a]\), it follows that there exists \( b, \ 0 < b < a, \) such that \( \mu(F)\Psi(a) + \mu(B \setminus F)\Psi(b) = 1 \). Then we have \( \rho_{\Psi,\mu}(a\chi_F) = \rho_{\Psi,\mu}(a\chi_F + b\chi_{B \setminus F}) = 1 \). Take \( E \in \mathcal{A}, E \subseteq \Omega \setminus B, \ 0 < \mu(E) < \infty \). Therefore there exists \( c, \ 0 < c < a, \ \Psi(c) > 0, \) such that \( \mu(F)\Psi(a) + \mu(E)\Psi(c) \leq 1, \) whence \( \rho_{\Psi,\mu}(a\chi_F + c\chi_E) = 1 \) but \( \rho_{\Psi,\mu}(a\chi_F + b\chi_{B \setminus F} + c\chi_E) > 1 \). So property (\( * \)) does not hold.

If for any set \( D \) in \( \mathcal{D} = \{ D \in \mathcal{A}, D \supseteq F, \ \mu(D) < \mu(\Omega) \} \) is \( \mu(D)\Psi(a) \leq 1, \) then consider a set \( B \in \mathcal{D} \) such that \( \mu(B) = \sup\{ \mu(D), D \in \mathcal{D} \} \). Hence \( \mu(B) < \mu(\Omega) < \infty \) and \( \Omega \setminus B \) is an atom, whence \( B \) is not an atom. At this point we can assume that \( F \) satisfies at least one of the following conditions.

1. There exists \( G \in \mathcal{A} \) such that \( F \subseteq G, \ \mu(F)\Psi(a) < \mu(G)\Psi(a) < 1 \).

2. \( F \) is an atom.

In either case it follows that \( \mu(F) < \mu(B) \). We have \( \rho_{\Psi,\mu}(a\chi_F) = \rho_{\Psi,\mu}(a\chi_B) = 1 \). As \( \mu(F) + \mu(\Omega \setminus B) < \mu(\Omega) \), we get that \( F \cup (\Omega \setminus B) \in \mathcal{D}, \) i.e., \( \mu(F)\Psi(a) + \mu(\Omega \setminus B)\Psi(a) \leq 1 \). So \( \rho_{\Psi,\mu}(a\chi_F + a\chi_{\Omega \setminus B}) = 1 \) but \( \rho_{\Psi,\mu}(a\chi_B + a\chi_{\Omega \setminus B}) > 1 \), whence property (\( * \)) does not hold.

It remains to consider the case where \( (\Omega, \mathcal{A}, \mu) \) is not infinitely divisible and \( r_0\Psi(a) \geq 1 \). In this case \( \Psi^{-1} \) is well defined and continuous on the interval \((0, 1/r_0]\), whence the proof in Theorem 1(b) applies without changes. Thus, in this case \( \Psi \) satisfies property \( P_\mu \) if and only if \( \Psi(x) \equiv cx^p \) on \([0, \Psi^{-1}(1/r_0)]\). So we have the following.

**Theorem 2** If \( \dim \mathcal{L}_p(\mu) \geq 3 \), then the \( L^* \)-functional induced by \( \Psi \) and \( \mu \) satisfies property (\( * \)) if it is necessarily a \( L_p \)-norm, \( 0 < p \leq \infty \).

### 4 \( \mathcal{L}_p(\mu) \) cannot be normed analogously to \( L_p \)-spaces

Let \( \mathcal{L}_p(\mu) = \{ f \in \mathcal{M} : \int_\Omega |\Phi(a\mu)| \ d\mu < \infty \text{ for all } a > 0 \} \), where \( \Phi \) is a finite-valued convex \( O \)-function. \( \mathcal{L}_p(\mu) \) is a linear subspace of \( \mathcal{L}_\Psi(\mu) \). A natural way for trying to provide \( \mathcal{L}_p(\mu) \) with a norm, analogously to the \( L_p \)-norm, is to consider the expression \( \Phi^{-1} \int_\Omega |\Phi(a)| \ d\mu \). We refer to [6] for a historical survey of this and related questions. In this article it is proved that this
attempt is possible only if \( \Phi(x) \equiv cx^p \), in the case where \( \mu \) is the Lebesgue measure on the real line. This result was later extended [5] to the linear space \( L'_\psi(\mu) \), where \( \Psi \) is a finite-valued strictly increasing \( O \)-function and \((\Omega, \mathcal{A}, \mu)\) is such that \( \dim L'_\psi(\mu) \geq 2 \). More precisely, it is proved in that paper that if \( \varphi_{\Gamma, \psi, \mu}(f) := \Gamma(\int f \Psi(|f|) \, d\mu) \) is a homogeneous functional on \( L'_\psi(\mu) \), being \( \Gamma \) and \( \Psi \) \( O \)-functions, then \( \Psi(x) \equiv \Psi(1)x^p \), \( \Gamma(x) \equiv \Gamma(1)x^{1/p} \), \( p > 0 \). Next we prove that this result is a consequence of Theorem 1. Assume first that \( \dim L'_\psi(\mu) = 2 \). Therefore \( L'_\psi(\mu) \) can be identified with \( \mathbb{R}^2 = \{(x_1, x_2), x_1, x_2 \in \mathbb{R} \} \), and where

\[
\varphi_{\Gamma, \psi, \mu}(x_1, x_2) = \Gamma(a_1 \psi(|x_1|) + a_2 \psi(|x_2|)), a_1, a_2 > 0.
\]

Assume that \( \varphi_{\Gamma, \psi, \mu}(x_1, x_2) \) is a homogeneous functional. For any \( x > 0 \) we have

\[
\varphi_{\Gamma, \psi, \mu}(x, 0) = \Gamma(a_1 \psi(x)) = x \varphi_{\Gamma, \psi, \mu}(1, 0) = x \Gamma(a_1 \psi(1)).
\]

As \( \varphi_{a, \Gamma, \psi, \mu} := a \varphi_{\Gamma, \psi, \mu} \) is also a homogeneous functional for all \( a > 0 \), we can suppose without loss of generality that \( \Gamma(a_1 \psi(1)) = 1 \). Under this assumption, \( a_1 \psi = \Gamma^{-1} \). Define on \( \mathbb{R}^3 \) the homogeneous functional

\[
\varphi'(x_1, x_2, x_3) = \varphi_{\Gamma, \psi, \mu}(\varphi_{\Gamma, \psi, \mu}(x_1, x_2), x_3)
\]

\[= \Gamma(a_1 \psi(\Gamma(a_1 \psi(|x_1|) + a_2 \psi(|x_2|)))) + a_2 \psi(|x_3|)) = \Gamma(a_1 \psi(|x_1|) + a_2 \psi(|x_2|)) + a_2 \psi(|x_3|).
\]

This functional is of the form \( \varphi_{\Gamma, \psi, \mu'} \), where \( \dim L'_\psi(\mu') = 3 \). This fact shows that it suffices to consider a homogeneous functional \( \varphi_{\Gamma, \psi, \mu} \) defined on \( L'_\psi(\mu) \), \( \dim L'_\psi(\mu) \geq 3 \). After dividing \( \Gamma \) by \( \Gamma(1) \) we can suppose \( \Gamma(1) = 1 \). For any \( f \in L'_\psi(\mu) \) we have that \( \varphi_{\Gamma, \psi, \mu}(f) = 1 \) if and only if \( \rho_{\psi, \mu}(f) = \inf\{\lambda : \int \psi(|f|/\lambda) \, d\mu \leq 1 \} = 1 \), and since these two functionals are homogeneous, it follows that \( \varphi_{\Gamma, \psi, \mu}(f) = \rho_{\psi, \mu}(f) \) for all \( f \in L'_\psi(\mu) \). Note that \( \dim L'_\psi(\mu) \geq 3 \) implies that there exist three measurable sets of finite and positive measure, since \( \Psi \) is strictly increasing. As the simple functions belong to \( L'_\psi(\mu) \) and \( \varphi_{\Gamma, \psi, \mu} \) satisfies property \( (\ast) \), we get that Theorem 1 implies that \( \varphi_{\Gamma, \psi, \mu}(f) = (c \int f^p \, d\mu)^{1/p} \) for all \( f \in L'_\psi(\mu) \), and also \( \Psi(x) \equiv cx^p \) in the case where \( (\Omega, \mathcal{A}, \mu) \) is infinitely divisible, and \( \Psi(x) \equiv cx^p \) on \([0, \Psi^{-1}(1/r_0)] \) in the case where \( (\Omega, \mathcal{A}, \mu) \) is not infinitely divisible, \( c > 0 \), \( p > 0 \). For \( \epsilon > 0 \) define

\[
\Gamma^*(x) = \Gamma(x/\epsilon)/\Gamma(1/\epsilon).
\]

Then applying in the latter case the same conclusion to the homogeneous functional \( \varphi_{\Gamma, \psi, \epsilon \mu} = [1/\Gamma(1/\epsilon)] \varphi_{\Gamma, \psi, \mu} \), and taking \( \epsilon \downarrow 0 \), it follows that also in this case \( \Psi(x) = cx^p \) for any \( x \in [0, \infty) \). Therefore

\[
\Gamma(c \int f^p \, d\mu) = (c \int f^p \, d\mu)^{1/p} \quad \text{for all} \quad f \in L'_\psi(\mu),
\]

whence it is easy to show that \( \Gamma(x) \equiv x^{1/p} \).
References


