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SYMMETRIC INTEGRALS DO NOT HAVE THE MARCINKIEWICZ PROPERTY

Abstract

A theorem of Marcinkiewicz asserts that the Perron integrability of a function can be deduced from the existence of a single pair of continuous major and minor functions. We show that Perron-type integrals based on various symmetric derivatives do not have this property.

1 Introduction

The classical Perron integral was defined using continuous major and minor functions. In general, to establish the integrability of a function f in this sense one must construct an infinite family of such major/minor functions. A remarkable theorem of Marcinkiewicz asserts that the integrability of f can be deduced from the existence of a single pair of continuous major and minor functions.

Since this theorem was first published in [7] in 1937 (see also [11]) it has been extended to some more general Perron-type integrals: for example the CP-integral of Burkill (in [9]), the approximate Perron integral (in [2]) and a special version of a dyadic Perron integral (in [1]). At the same time it has been observed that for some other generalizations of the Perron integral the Marcinkiewicz theorem fails to be true (see [8] for the SCP-integral of Burkill and [10] for the dyadic Perron integral): there is a nonintegrable function which nonetheless allows a pair of continuous major and minor functions in

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the sense of the integral. For more details on the history and applications of the Marcinkiewicz theorem see [3] and [4].

In this note we show that no symmetric Perron integral, ordinary or approximate, can have this property. Note that in all known cases of integrals not possessing the Marcinkiewicz property the associated derivative is computed at a point without regard to the value of the function at that point. This appears to be the factor explaining the differences in the theories.

The first study of a symmetric Perron integral appears to have been that of Ponomarev [6]. For a discussion of this integral (using continuous major and minor functions and symmetric derivates) and a variety of other symmetric integrals see [12, Ch. 9]. We shall consider as well the approximate symmetric version. Note that this integral can be given as a Perron integral or characterized using Riemann sums or the approximate symmetric variation (see [12, pp. 378–380]).

The symmetric variation and approximate symmetric variation are the main tools used and this equivalence of an approximate symmetric variational integral with the approximate symmetric Perron integral allows us to draw our conclusions. For this we need to recall some definitions. By a gauge δ on a set $E \subset \mathbb{R}$ we mean merely a positive function on E. By a density gauge Δ on a measurable set $E \subset \mathbb{R}$ we mean that Δ is a measurable subset of the plane $\mathbb{R}^2$ so that for each $x \in E$ the set $\Delta_x = \{t : t > 0,\, (x,t) \in \Delta\}$ has inner density 1 on the right at 0. The measurability assumption enters into the construction of the integral but plays no role in our deliberations otherwise.

The symmetric variation of a function F on a set E relative to a gauge δ is defined as

$$\mathrm{VS}_F(E;\delta) = \sup \sum_{i=1}^{m} |F(x_i + h_i) - F(x_i - h_i)| \tag{1}$$

where the supremum is taken over all sequences $\{[x_i - h_i, x_i + h_i]\}$ $(i = 1, 2, \ldots, m)$ of nonoverlapping intervals with centers x_i in E and $0 < h_i < \delta(x_i)$. Then the symmetric variation of a function F on a set E is defined as

$$\mathrm{VS}_F(E) = \inf \mathrm{VS}_F(E;\delta) \tag{2}$$

where the infimum is taken over all gauges δ.

In the same way the approximate symmetric variation of a function F on a set E relative to a density gauge Δ is defined as

$$\mathrm{VAS}_F(E;\Delta) = \sup \sum_{i=1}^{m} |F(x_i + h_i) - F(x_i - h_i)| \tag{3}$$

where the supremum is taken over all sequences $\{[x_i - h_i, x_i + h_i]\}$ $(i = 1, 2, \ldots, m)$ of nonoverlapping intervals with centers x_i in E and $h_i \in \Delta_{x_i}$. Then the approximate symmetric variation of a function F on a set E is defined as

$$\mathrm{VAS}_F(E) = \inf \mathrm{VAS}_F(E; \Delta) \tag{4}$$

where the infimum is taken over all density gauges Δ.

2 Basic Lemmas

We shall construct a continuous function F on an interval $[a, b]$ that has an ordinary derivative off of a perfect subset of the interval. Then F' will turn out to have continuous major and minor functions in the symmetric sense and in the approximate symmetric sense and yet not be itself integrable in either sense. To see what we must accomplish we establish two lemmas.

Lemma 1 *Let F be a continuous function on an interval $[a, b]$ and let P be a subset of $[a, b]$ such that F has a derivative at every point of $[a, b] \setminus P$. Then if $VS_F(P)$ is finite the function*

$$f(x) = \begin{cases} F'(x) & \textit{if } x \in [a, b] \setminus P \\ 0 & \textit{if} \in P \end{cases}$$

has a pair of continuous minor/major functions in the symmetric Perron sense and in the approximate symmetric Perron sense.

Proof. Since $\mathrm{VS}_F(P)$ is finite there is a a gauge δ on $[a, b]$ so that $\mathrm{VS}_F(P; \delta)$ is finite. We define an interval function Ω by setting, for any closed interval I,

$$\Omega(I) = \sup \sum_{i=1}^{m} |F(x_i + h_i) - F(x_i - h_i)| \tag{5}$$

where the supremum is taken over all sequences $\{[x_i - h_i, x_i + h_i]\}$ $(i = 1, 2, \ldots, m)$ of nonoverlapping subintervals of $I \cap [a, b]$ with centers x_i in P so that $0 < h_i < \delta(x_i)$. This is just the variation $\mathrm{VS}_F(P, \delta)$ but computed using only subintervals of I.

We show that Ω is a nonnegative, continuous interval function and that the function $R(x) = \Omega([a, x])$ is continuous. Note that $R(x+h) - R(x-h) \geq 0$ for all x and that, for every $x \in P$, $h \in \Delta_x$,

$$\begin{aligned} R(x + h) - R(x - h) &= \Omega([a, x + h]) - \Omega([a, x - h]) \\ &\geq \Omega([x - h, x + h]) \geq |F(x + h) - F(x - h)|. \end{aligned}$$

From this and the fact that $F'(x) = f(x)$ on $[a,b] \setminus P$ it follows easily that $F + R$ is a major function and $F - R$ a minor function for f in the symmetric sense and also in the approximate symmetric sense. The proof is completed by showing that R is continuous.

We prove first that Ω is continuous. Fix a point $x \in [a,b]$ and let $\epsilon > 0$. This means that we must show there is an positive number η such that $\Omega(I) < \epsilon$ for every interval I with $x \in I$ and $I \subset [x-\eta, x+\eta]$.

Since F is continuous we can choose η_1 so that $|F(I)| < \frac{1}{4}\epsilon$ for every interval $I \subset [a,b]$ and $|I| < \eta_1$. Write $I_1 = [x - \frac{1}{2}\eta_1, x + \frac{1}{2}\eta_1]$. There must exist a sequence $\{[x_i - h_i, x_i + h_i]\}$ $(i = 1, 2, \dots, m)$ of nonoverlapping subintervals of I_1 with centers x_i in P so that $h_i < \delta(x_i)$ such that

$$\Omega(I_1) \geq \sum_{i=1}^{m} |F(x_i + h_i) - F(x_i - h_i)|\, \Omega(I_1) - \frac{\epsilon}{4}\,. \tag{6}$$

Choose $\eta < \frac{1}{2}\eta_1$ sufficiently small that the interval $(x-\eta, x+\eta)$ meets no more than two members of the collection $\{[x_i - h_i, x_i + h_i]\}$, say J_1 and J_2 if these exist. Note that $|F(J_1)|$ and $|F(J_2)|$ are both smaller than $\frac{1}{4}\epsilon$.

Let I denote any interval with $x \in I$ and $I \subset I_2 = [x-\eta, x+\eta]$. We show that $\Omega(I) < \epsilon$. For if not then, since Ω is monotone, $\Omega(I_2) \geq \epsilon$. Choose a sequence $\{[y_i - t_i, y_i + t_i]\}$ $(i = 1, 2, \dots, k)$ of nonoverlapping subintervals of I_2 with centers y_i in P so that $t_i < \delta(y_i)$ such that

$$\sum_{i=1}^{k} |F(y_i + t_i) - F(y_i - t_i)|\, \tfrac{3}{4}\epsilon\,. \tag{7}$$

Let $\{K_i\}$ denote the collection of intervals formed by combining the collections $\{[x_i - h_i, x_i + h_i]\}$ and $\{[y_i - t_i, y_i + t_i]\}$ but removing the intervals J_1, J_2 (should such exist). We must have

$$\begin{aligned}\Omega(I_1) \geq \sum_i |F(K_i)| &= \sum_{i=1}^{m} |F(x_i + h_i) - F(x_i - h_i)| - |F(J_1)| - |F(J_2)| \\ &+ \sum_{i=1}^{k} |F(y_i + t_i) - F(y_i - t_i)| \\ &> \Omega(I_1) - \tfrac{1}{4}\epsilon - \tfrac{1}{4}\epsilon - \tfrac{1}{4}\epsilon + \tfrac{3}{4}\epsilon = \Omega(I_1)\end{aligned}$$

which is impossible. It follows that Ω is continuous at x.

Now we use the continuity of Ω to show that R is continuous. Fix x and $\epsilon > 0$ and choose $\eta > 0$ so that $|F(I)| < \frac{1}{4}\epsilon$ and $\Omega(I) < \frac{1}{4}\epsilon$ if $|I| < \eta$. Choose $c < x < d$ so that $d - c < \frac{1}{2}\eta$.

There must exist a sequence $\{[x_i-h_i, x_i+h_i]\}$ $(i=1,2,\ldots,m)$ of nonoverlapping subintervals of $[a,d]$ with centers x_i in P so that $h_i<\delta(x_i)$ such that

$$\sum_{i=1}^{m} |F(x_i+h_i)-F(x_i-h_i)|\, R(d)-\tfrac{1}{4}\epsilon\,. \tag{8}$$

Let S_1 denote the sum of the terms on the left taken for $[x_i-h_i, x_i+h_i]\subset[a,c]$, let S_2 denote the term (if one exists) for which (x_i-h_i, x_i+h_i) contains c and let S_3 denote the sum of the remaining terms taken then for $[x_i-h_i, x_i+h_i]\subset[c,d]$. We have

$$R(d)-R(c)\le S_1+S_2+S_3+\tfrac{1}{4}\epsilon-R(c). \tag{9}$$

We consider the term S_2 more closely. Recall this is merely that term $[x_j-h_j, x_j+h_j]$ containing c as an interior point and we may suppose that this exists (since otherwise (10) holds trivially). In case $c\le x_j$ then, since $x_j+h_j<d$ and $d-c<\frac{1}{2}\eta$, it follows that the interval $[x_j-h_j, x_j+h_j]$ has length smaller than η and hence $|F(x_j+h_j)-F(x_j-h_j)|<\frac{1}{4}\epsilon$. Otherwise we have $c>x_j$ and we can split $[x_j-h_j, x_j+h_j]$ into

$$[x_j-h_j, x_j+h_j]=[x_j-h_j, 2x_j-c]\cup[2x_j-c, c]\cup[c, x_j+h_j]$$

so that

$$F([x_j-h_j, x_j+h_j])=F([x_j-h_j, 2x_j-c])+F([2x_j-c, c])+F([c, x_j+h_j]).$$

Note that both $|F([x_j-h_j, 2x_j-c])|$ and $|F([c, x_j+h_j])|$ are smaller than $\frac{1}{4}\epsilon$ since each interval has length smaller than η. The interval $I'=[2x_j-c, c]$ is a subinterval of $[a,c]$ and is of our required type since it must be centered in P and $c-x_j<\delta(x_j)$. Thus we obtain

$$S_1+S_2\le R(c)+\tfrac{1}{2}\epsilon. \tag{10}$$

From (8) and (10) and the fact that $S_3\le\Omega([c,d])<\frac{1}{4}\epsilon$ we have

$$\begin{aligned} R(d)-R(c) <& S_1+S_2+S_3-R(c)+\tfrac{1}{4}\epsilon \\ \le& R(c)+\tfrac{3}{4}\epsilon-R(c)+\tfrac{1}{4}\epsilon=\epsilon. \end{aligned}$$

This shows that R is continuous as required and the proof is completed. $\square$

The method used in the proof of Lemma 1 can be used in greater generality to show that a function f possesses a continuous *superadditive* major function and a continuous *subadditive* minor function in other settings where a variation can be defined. To pass to the additive case requires an argument that does not seem to work in general as it is special to the symmetric case under consideration here.

Lemma 2 *Let G be a continuous function on an interval $[a,b]$ and let P be a subset of $[a,b]$ such that G has a derivative at every point of $[a,b]\setminus P$. Then the function*

$$g(x)=\begin{cases} G'(x) & \textit{if } x\in[a,b]\setminus P\\ 0 & \textit{if } x\in P\end{cases}$$

is integrable in the approximate symmetric Perron sense on $[a,b]$ and has G as its indefinite integral if and only if $VAS_G(P)$ is zero.

Proof. This follows from material in [12, §9.3.5]. □

From Lemmas 1 and 2 we have a method for constructing a function f that has a pair of continuous minor/major functions in the approximate symmetric sense but is not in fact integrable in that sense. We shall construct a perfect set $P\subset(-1,2)$ and a continuous function F on $[-1,2]$ that satisfies the hypotheses of Lemma 1 and has $0<\mathrm{VAS}_F(P)$ and $\mathrm{VS}_F(P)\le 1$. By Lemma 1 this is enough to guarantee that the function

$$f(x)=\begin{cases} F'(x) & \text{if } x\in[a,b]\setminus P\\ 0 & \text{if } x\in P\end{cases}$$

possesses such a major/minor function pair and that, by Lemma 2, F is not its indefinite integral. For f to be integrable it must have an indefinite integral of the form $F+G$ where G is some function constant on the intervals in $[-1,2]$ contiguous to P. If we arrange that $\mathrm{VAS}_{F+G}(P)>0$ for *every* such function G then it follows from Lemma 2 that f cannot be integrable either in the symmetric sense or the approximate symmetric sense.

3 Construction of the Function F

On the unit interval $[0,1]$ define a Cantor-like set P as follows: From $L_{0,1}=[0,1]$ remove an open interval $K_{1,1}$ centered at $\frac{1}{2}$ and of length $k_1=\frac{14}{16}$. This leaves two closed intervals $L_{1,1}$ and $L_{1,2}$ each of length $\ell_1=\frac{1}{16}$. We continue inductively. Suppose that the n-th step has been completed leaving closed intervals $L_{n,1}$, $L_{n,2}$, $L_{n,3}\dots L_{n,2^n}$ each of length ℓ_n. We carry out the $n+1$-st step by removing from each interval $L_{n,j}$ $(j=1,2,\dots,2^n)$ an open interval $K_{n+1,j}$ with the same center as $L_{n,j}$ and of length $k_{n+1}=\frac{14}{16}\ell_n$. We refer to the intervals $K_{n+1,j}$ as the open intervals of rank $n+1$. The set

$$P=\bigcap_{n=1}^{\infty}\left(\bigcup_{j=1}^{2^n}L_{n,j}\right)$$

is a perfect subset of $[0,1]$ with measure zero.

Now define a continuous function F on $[-1,2]$. Put $F(x)=0$ for each x in $P\cup[-1,0)\cup(1,2]$. Let (a_n,b_n) be any of the intervals $K_{n,j}$ for a fixed n and some $j=1,2,\dots 2^{n-1}$. Put

$$F(x)=\begin{cases} 2^{-n} & \text{if } x\in[a_n+2\ell_n, a_n+\frac{1}{2}k_n-\ell_n] \\ -2^{-n} & \text{if } x\in[b_n-\frac{1}{2}k_n+\ell_n, b_n-2\ell_n] \\ 0 & \text{if } x\in(a_n,a_n+\ell_n]\cup[b_n-\ell_n,b_n). \end{cases}$$

We extend F in such a way that F is monotone on each of the intervals $(a_n+\ell_n, a_n+2\ell_n)$, $(a_n+\frac{1}{2}k_n-\ell_n, b_n-\frac{1}{2}k_n+\ell_n)$ and $(b_n-2\ell_n, b_n-\ell_n)$ and differentiable on all of (a_n,b_n). Certainly F is continuous on $[-1,2]$ since the oscillation of F on each interval $K_{n,j}$ is 2^{-n}.

We show first that the symmetric variation of F on P does not exceed 1, i.e. that $\mathrm{VS}_F(P)\le 1$. Let $\{[x_i-h_i,x_i+h_i]\}$ $(i=1,2,\dots m)$ be any sequence of nonoverlapping subintervals of $[-1,2]$ with centers x_i in P. Notice that if one of the values $|F(x_i-h_i)|$ or $|F(x_i+h_i)|$ is not zero then the other one must be zero and if

$$2^{-n-1}<|F(x_i+h_i)-F(x_i-h_i)|\le 2^{-n} \tag{11}$$

then $h_i>\ell_n$. This implies that if (11) holds then at least one of the intervals $L_{n,j}$ must be covered by (x_i-h_i,x_i+h_i).

Introduce a singular measure μ on P in such a way that $\mu(P\cap L_{n,j})=2^{-n}$. It follows that $\mu(P)=1$. For each n denote by $\sum_i^{(n)}$ a summation taken over indices i for which the inequality (11) holds. Let $\mathcal{J}_n$ denote the set of indices j for which $L_{n,j}$ is covered by the collection $\{(x_i-h_i,x_i+h_i)\}$. Then we obtain

$$\begin{aligned}\sum_{i=1}^{m}|F(x_i+h_i)-F(x_i-h_i)| &\le \sum_n \sum\nolimits_i^{(n)} |F(x_i+h_i)-F(x_i-h_i)| \\ &\le \sum_n\sum_{j\in\mathcal{J}_n} 2^{-n} = \sum_n\sum_{j\in\mathcal{J}_n}\mu(P\cap L_{n,j}) \\ &\le \mu(P)=1.\end{aligned}$$

From this we see that $\mathrm{VS}_F(P)\le 1$.

Now let G be any function on $[-1,2]$ that is constant on each of the intervals $K_{n,j}$. We prove that

$$\mathrm{VAS}_{F+G}(P)\ge \tfrac{1}{4}. \tag{12}$$

Let Δ be any density gauge on $[-1,2]$. There is a gauge δ on P so that

$$|\Delta_x\cap(0,t)|>\tfrac{11}{14}t \tag{13}$$

for each $0 < t < \delta(x)$.

We extend δ to all of the interval $[-1,2]$ by setting $\delta(x) = \operatorname{dist}(x, P)$ for $x \in [-1,2] \setminus P$. By applying the covering theorem [12, Theorem 3.13, p. 72] we can obtain a partition $\{[x_i - h_i, x_i + h_i]\}$ of an interval that covers P, with centers x_i in $[-1,2]$ and so that $h_i < \delta(x_i)$. We then remove from this partition all intervals that are not centered in P and obtain a sequence $\{[x_i - h_i, x_i + h_i]\}$ $(i = 1, 2, \ldots, m)$ of nonoverlapping subintervals of $[-1,2]$ with centers x_i in P so that $h_i < \delta(x_i)$ and so that

$$P \subset \bigcup_{i=1}^{m} [x_i - h_i, x_i + h_i]. \tag{14}$$

We shall pass to a collection $\{[x_i - h_i^*, x_i + h_i^*]\}$ of smaller intervals with the same centers so that $h_i^* \in \Delta_{x_i}$. For each index i find a minimal n so that for some j

$$L_{n-1,j} \subset [x_i - h_i, x_i + h_i].$$

Then $h_i < l_{n-2} < k_{n-2}$ so that $[x_i - h_i, x_i + h_i]$ has non-void intersection with only one interval L_{n-2,j_1}, namely with the one for which $L_{n-2,j_1} \supset L_{n-1,j}$. Hence

$$\mu\left(P \cap [x_i - h_i, x_i + h_i]\right) \leq \mu\left(P \cap L_{n-2,j_1}\right) = 2^{-n-2} = 4(2^{-n}) \tag{15}$$

We have $L_{n-1,j} = L_{n,2j-1} \cup K_{n,j} \cup L_{n,2j}$. The point x_i belongs either to $L_{n,2j-1}$ or to $L_{n,2j}$. In the first case $K_{n,j} \subset [x_i, x_i + h_i]$ and in the second case $K_{n,j} \subset [x_i - h_i, x_i]$. In any case we have $k_n \leq h_i$.

Let

$$T_1 = (3\ell_n, \tfrac{1}{2}k_n - \ell_n) \quad \text{and } T_2 = (\tfrac{1}{2}k_n + 2\ell_n, k_n - 2\ell_n) \tag{16}$$

and compute that

$$|T_1| = |T_2| = \tfrac{1}{2}k_n - 4\ell_n = \tfrac{3}{14}k_n. \tag{17}$$

Since $k_n \leq h_i < \delta(x_i)$ we can apply the requirement (13) using $t = k_n$ to obtain points h_i' and h_i'' with

$$h_i' \in \Delta_x \cap T_1 \quad \text{and} \quad h_i'' \in \Delta_x \cap T_2. \tag{18}$$

We wish to estimate

$$K_1 = F(x_i + h_i') + G(x_i + h_i') - F(x_i - h_i') - G(x_i - h_i')$$

and

$$K_2 = F(x_i + h_i'') + G(x_i + h_i'') - F(x_i - h_i'') - G(x_i - h_i'').$$

Note that the left hand endpoint of $L_{n-1,j}$ is the right hand endpoint $b_{n'}$ of some interval $(a_{n'}, b_{n'})$ of rank $n' < n$ and the right hand endpoint of $L_{n-1,j}$ is the left hand endpoint $a_{n''}$ of some interval $(a_{n''}, b_{n''})$ of rank $n'' < n$.

Let us denote the constant values taken by the function G in the intervals $(a_n, b_n) = K_{n,j}$, $(a_{n'}, b_{n'}) = K_{n',j'}$ and $(a_{n''}, b_{n''}) = K_{n'',j''}$ by g_n, $g_{n'}$ and $g_{n''}$ respectively.

We consider two cases: (i) $x_i \in L_{n,2j-1}$ and (ii) $x_i \in L_{n,2j}$. In case (i) from (16) and (18) we obtain

$$\begin{aligned} b_{n'} - \ell_{n'} < & b_{n'} - \tfrac{1}{2}k_n \le b_{n'} - h'_i < x_i - h'_i < b_{n'} \\ b_{n'} - \ell_{n'} < & b_{n'} - k_n \le b_{n'} - h''_i < x_i - h''_i < b_{n'} \\ a_n + 2\ell_n \le & x_i + 3\ell_n \le x_i + h'_i < a_n + \tfrac{1}{2}k_n - \ell_n \end{aligned}$$

and

$$b_n - \tfrac{1}{2}k_n + \ell_n = a_n + \tfrac{1}{2}k_n + \ell_n \le x_i + h''_i < b_n - 2\ell_n.$$

Then we get from the definition of F that $F(x_i - h'_i) = F(x_i - h''_i) = 0$, $F(x_i + h'_i) = 2^{-n}$ and $F(x_i + h''_i) = -2^{-n}$. Hence we obtain $K_1 = 2^{-n} + g_n - g_{n'}$ and $K_2 = -2^{-n} + g_n - g_{n'}$.

If $g_n \ge g_{n'}$ then we choose $h^*_i = h'_i$ while if $g_n < g_{n'}$ we choose $h^*_i = h''_i$. This gives then, in the case (i) that we are considering, a choice of point $h^*_i \in \Delta_{x_i}$ so that

$$|F(x_i + h^*_i) + G(x_i + h^*_i) - F(x_i - h^*_i) - G(x_i - h^*_i)| \ge 2^{-n}. \tag{19}$$

Consider now the case (ii) where $x_i \in L_{n,2j}$. We employ a similar argument. In this case $F(x_i - h'_i) = -2^{-n}$, $F(x_i - h''_i) = 2^{-n}$ and $F(x_i + h'_i) = F(x_i + h''_i) = 0$. Hence we obtain $K_1 = 2^{-n} + g_{n''} - g_n$ and $K_2 = -2^{-n} + g_{n''} - g_n$. We set $h^*_i = h'_i$ if $g_{n''} \ge g_n$ and $h^*_i = h''_i$ if $g_{n''} < g_n$ and again we get (19).

In this way we obtain a new collection $\{[x_i - h^*_i, x_i + h^*_i]\}$ of nonoverlapping subintervals of $[-1, 2]$ with centers x_i in P so that $h^*_i \in \Delta_{x_i}$, and, because of (15) and (19), with the property that

$$\begin{aligned} |F(x_i + h^*_i) + G(x_i + h^*_i) \ & - \ F(x_i - h^*_i) - G(x_i - h^*_i)| \\ & \ge \ \tfrac{1}{4}\mu\left(P \cap [x_i - h_i, x_i + h_i]\right). \end{aligned} \tag{20}$$

From this we obtain from (14) by summing that

$$\begin{aligned} \sum_{i=1}^{m} |F(x_i + h^*_i) + G(x_i + h^*_i) \ & - \ F(x_i - h^*_i) - G(x_i - h^*_i)| \\ & \ge \ \tfrac{1}{4}\mu\left(P\right) = \tfrac{1}{4}. \end{aligned} \tag{21}$$

As the collection of intervals was chosen relative to an arbitrary density gauge Δ we have proved (12).

The existence of the function F proves the following theorem.

Theorem 3 *There is a measurable function f that is not integrable in the symmetric Perron or approximate symmetric Perron sense on an interval $[a, b]$ and yet f possesses a pair of continuous major/minor functions in both senses.*

We remark that the major and minor functions for this particular example can be constructed directly without appealing to the general lemmas of the preceding section.

As before, let μ be a singular measure on P such that $\mu(P \cap L_{n,j}) = 2^{-n}$ and $\mu(P) = 1$. Define a function on the interval $[-1, 2]$ by

$$R(x) = \mu\left(P \cap [-1, x]\right).$$

Clearly R is a continuous and monotone function on $[-1, 2]$ that is constant on each $K_{n,j}$ and on $[-1, 0]$ and $[1, 2]$. We prove that for each $x \in P$ and any $(x - h, x + h) \subset [-1, 2]$ that

$$|F(x + h) - F(x - h)| \leq R(x + h) - R(x - h). \tag{22}$$

As we have already noted, if one of the values $|F(x - h)|$ or $|F(x + h)|$ in (22) is not zero then the other one must be zero and in this case an integer n can be chosen so that

$$2^{-n-1} \leq |F(x + h) - F(x - h)| \leq 2^{-n} \tag{23}$$

and $h > \ell_n$. This means that where (23) holds at least one of the intervals $L_{n,j}$ for some j must be covered by $(x - h, x + h)$. Fix this j. Then, from (23), we have

$$|F(x + h) - F(x - h)| \leq 2^{-n} = \mu\left(P \cap L_{n,j}\right) \leq R(x + h) - R(x - h)$$

thus establishing (22). As before then $F + R$ is a continuous major function for f and $F - R$ is a continuous minor function.

We note, in conclusion, that this same example supplies a function that is Denjoy-Khintchine integrable (see [5]) but not integrable in a symmetric sense. Clearly F as constructed is ACG on $[-1, 2]$; it follows that the corresponding function f (as defined in Lemma 1) is Denjoy-Khintchine integrable on $[-1, 2]$ and so we have proved the following theorem.

Theorem 4 *There is a function f that is Denjoy-Khintchine integrable on an interval but not integrable there in either the symmetric Perron or approximate symmetric Perron sense.*

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