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Katarzyna Flak and Ryszard Jerzy Pawlak, Institute of Mathematics, Łódź University, Banacha 22, 90-238 Łódź, Poland,
e-mail:rpawlak@@krysia.uni.lodz.pl

Bożena Świątek, Institute of Mathematics, Technical University of Łódź, Al. Politechniki 11, 90-924 Łódź, Poland

ON SOME METHOD FOR IMPROVING CONTINUITY, QUASI-CONTINUITY AND THE DARBOUX PROPERTY

Abstract

In the paper we give and investigate the basic properties of a method for improving continuity, quasi-continuity and the Darboux property of real functions defined on a metric Baire space. This "improvement" is carried out with the use of Blumberg sets.

A. Katafiasz in her doctoral dissertation [5] (Also see [6] and [7].) introduced the notion of an α-*improvable* discontinuous function. The main idea of this work was to examine the possibility of the *"removal" of the discontinuity* of real functions defined on some subsets of the line. The author suggested a certain method for removing the discontinuity and investigated the structure of functions for which this method is efficient as well as the successive steps of the procedure of removing of the discontinuity. This dissertation has inspired our team's investigations.

In our paper we give another method for "improving functions". In addition we show that our method is efficient for a considerably wider class of functions than A. Katafiasz's method, and that it may also be applied, in some cases, to the improving of quasi-continuity and the Darboux property. An additional merit of our method is the fact that the "improvement" is done in one step.

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We apply the classical symbols and notation. In particular, the symbol $\mathbb{R}$ denotes the set of all real numbers with the natural topology.

Throughout the paper, we consider *bounded real functions defined on some metric Baire space.*

If X is some metric space, then $I(X)$ denotes the set of all isolated points. The symbols $\overline{A}$ and A^d stand for the closure of the set A and the derived set of A (i.e. the set of all accumulation points of A), respectively. By $K(a,r) \subset X$ we denote the open ball with center at x and radius r.

Let X be a metric space with a metric ϱ, and let $W \subset X$, $x \in X$, $r > 0$. Then $\gamma(x,r,W)$ denotes the supremum of the set of all $t > 0$ for which there exists $z \in X$ such that $K(z,t) \subset K(x,r) \setminus W$. The porosity of W at x is defined to be the number $p(W,x) = 2 \cdot \lim_{r \to 0+} \frac{\gamma(x,r,W)}{r}$ ([9]) and W is porous at x if $p(W,x) > 0$.

Let $f : X \to \mathbb{R}$ and $A \subset X$. The symbol $f_{|A}$ denotes the restriction of f to A. $U_A(f) = \{x \in X :$ there exists $\lim_{x \to x_0} f_{|A}(x)\}$. (If $A = X$, then we simply write $U(f)$.) Let $x_0 \in A^d$. Then

$$L_A(f,x_0) = \{\alpha : \exists_{\{x_n\} \subset A \setminus \{x_0\}} \lim_{n\to\infty} x_n = x_0 \ \wedge \ \lim_{n\to\infty} f(x_n) = \alpha\}.$$

(If $A = X$, then we simply write $L(f,x_0)$.) If $x \in I(X)$, then we assume $L_A(f,x) = \{f(x)\}$ for an arbitrary subset $A \subset X$. If $f : \mathbb{R} \to \mathbb{R}$ and $A \subset \mathbb{R}$, then

$$L_A^+(f,x_0) = \{\alpha : \exists_{\{x_n\} \subset A \cap (x_0,+\infty)} \lim_{n\to\infty} x_n = x_0 \ \wedge \ \lim_{n\to\infty} f(x_n) = \alpha\}.$$

In an analogous way we define $L_A^-(f,x_0)$. (If $A = \mathbb{R}$, then we simply write $L^+(f,x_0)$ and $L^-(f,x_0$.) The symbol C_f denotes the set of all continuity points of f.

Let $f : \mathbb{R} \to \mathbb{R}$. A point x_0 is said to be a right-sided (left-sided) Darboux point ([3], [8]) of f provided that for each $\varepsilon > 0$ and for each number β which is strictly between $f(x_0)$ and some element of $L^+(f,x_0)$ $(L^-(f,x_0))$, there exists $z \in [x_0, x_0+\varepsilon)$ $(z \in (x_0-\varepsilon, x_0])$ such that $f(z) = \beta$. A point x_0 is a Darboux point of f provided x_0 is a right-sided and left-sided Darboux point of f.

Let $f : X \to Y$ where X and Y are metric spaces. We say that x_0 is a quasi-continuity point of f if, for any open neighborhoods U and V of x_0 and $f(x_0)$, respectively, $\operatorname{Int}(U \cap f^{-1}(V)) \neq \emptyset$. A function f is said to be quasi-continuous if each point x in the domain of f is a quasi-continuity point of f.

If A is a bounded subset of $\mathbb{R}$, then the symbol $[A]$ denotes the closed interval $[\inf A, \sup A]$.

H. Blumberg in paper [1] showed that, for every real function defined on $\mathbb{R}$, there exists a dense subset B of $\mathbb{R}$ such that $f_{|B}$ is continuous. If $f : X \to Y$, and B is a dense set of X such that $f_{|B}$ is continuous, then we say that B is *a Blumberg set* of f. It is known ([2]) that, for a metric space X, X is a Baire space if and only if for an arbitrary function $f : X \to \mathbb{R}$, there exists a Blumberg set of f (see also [4]).

We shall apply these facts to our considerations. First, we introduce the notion of a c-Blumberg set.

Definition 1 *Let $f : X \to \mathbb{R}$. If B is a dense set of X such that $C_f \subset B$ and $f_{|B}$ is a continuous function, then we say that B is a c-Blumberg set.*

Proposition 1 *The metric space X is a Baire space if and only if, for an arbitrary function $f : X \to \mathbb{R}$, there exists a c-Blumberg set of f.*

Proof. *Necessity.* From the Bradford-Goffman Theorem we may deduce that there exists a Blumberg set B of f. It is not hard to verify that $B' = (B \setminus \overline{C_f}) \cup C_f$ is a c-Blumberg set of f. *Sufficiency* is obvious. □

In our paper we consider only functions mapping a Baire space into the real line, and so, for each function considered in this paper, there exists (at least one) c-Blumberg set.

Let $f : X \to \mathbb{R}$ be a function and let B be a c-Blumberg set of f. By the symbol f^B we shall denote the class of all functions f_0 defined as follows.

$$f_0(x) = \begin{cases} f(x), & \text{if } x \in B, \\ \text{an arbitrary element of } L_B(f, x), & \text{if } x \notin B. \end{cases}$$

Proposition 2 *Let B be a c-Blumberg set of a function $f : X \to \mathbb{R}$. Then the class f^B contains at most one continuous function.*

Proof. Of course, if f is continuous, then $f^B = \{f\}$.

Now, we consider the case when f is a discontinuous function. Assume to the contrary that f^B contains two different continuous functions f_1, f_2. Then there exists a point $x_0 \in X$ such that $\alpha_1 = f_1(x_0) \neq f_2(x_0) = \alpha_2$. Of course, $x_0 \notin I(X)$. From the definition of the class f^B we infer that there exists a sequence $\{x_n\} \subset B$ such that $x_0 = \lim_{n\to\infty} x_n$ and $\alpha_1 = \lim_{n\to\infty} f(x_n)$. Note that $f_1(x_n) = f_2(x_n) = f(x_n)$ $(n = 1, 2, \ldots)$ and, consequently, $\lim_{n\to\infty} f_2(x_n) = \alpha_1 \neq \alpha_2$, contrary to the continuity of f_2. □

Definition 2 *Let B be a c-Blumberg set of a function $f : X \to \mathbb{R}$. Then we say that f is a B-improvable function if the family f^B contains a continuous function. (Such a function will be denoted by f^B_c.)*

According to Proposition 2, for a fixed c-Blumberg set of f, there is at most one "improvement" f_c^B of f. But, in the case of different c-Blumberg sets of f we can obtain different "improved functions".

Example 1 *Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that the image of an arbitrary interval is equal to $\mathbb{R}$. Then $C_f = \emptyset$ and, consequently, $f^{-1}(\alpha)$ is a c-Blumberg set of f for each $\alpha \in \mathbb{R}$. It is easy to see that $f_c^{f^{-1}(\alpha)}(x) = \alpha$ for $x \in \mathbb{R}$ and $\alpha \in \mathbb{R}$. So, we have obtained a continuum of "improved functions" of f.*

Proposition 3 *If $f_0 \in f^B$, then f_0 is continuous at each point of the set $U_B(f) \cup I(X)$.*

PROOF. If $x_0 \in I(X)$, then, of course, $x_0 \in C_{f_0}$.

Assume that $x_0 \notin I(X)$. Let $\{x_n\}_{n=1}^{\infty} \subset X$ be an arbitrary sequence such that $x_0 = \lim_{n\to\infty} x_n$ and let $\{f_0(x_{k_n})\}_{n=1}^{\infty}$ be an arbitrary subsequence of $\{f_0(x_n)\}_{n=1}^{\infty}$. Then we may choose a subsequence $\{x_{l_{k_n}}\}_{n=1}^{\infty}$ of $\{x_{k_n}\}_{n=1}^{\infty}$ such that either $\{x_{l_{k_n}}\}_{n=1}^{\infty} \subset B$ or $\{x_{l_{k_n}}\}_{n=1}^{\infty} \subset X \setminus B$. In the first case, if $x_0 \in B$, then, according to the continuity of $f_{|B}$, we obtain $f_0(x_0) = f(x_0) = \lim_{n\to\infty} f_{|B}(x_{l_{k_n}}) = \lim_{n\to\infty} f_0(x_{l_{k_n}})$, and if $x_0 \notin B$, then ($\lim_{x\to x_0} f_{|B}(x)$ exists) we may infer that $f_0(x_0) = \lim_{x\to x_0} f_{|B}(x)$, and as a consequence, $f_0(x_0) = \lim_{n\to\infty} f_0(x_{l_{k_n}})$.

In the second case, since $f_0(x_{l_{k_n}}) \in L_B(f, x_{l_{k_n}})$, we may construct, for every n, a sequence $\{y_s^{l_{k_n}}\}_{s=1}^{\infty} \subset B$ such that $x_{l_{k_n}} = \lim_{s\to\infty} y_s^{l_{k_n}}$ and $\lim_{s\to\infty} f(y_s^{l_{k_n}}) = f_0(x_{l_{k_n}})$. So for every positive integer n, there exists a positive integer $s(n)$ such that $\varrho(y_{s(n)}^{l_{k_n}}, x_{l_{k_n}}) < \frac{1}{n}$ (where ϱ denotes the metric in the space X) and $|f(y_{s(n)}^{l_{k_n}}) - f_0(x_{l_{k_n}})| < \frac{1}{n}$. In a fashion analogous to the above, we may show that $\lim_{n\to\infty} f_0(y_{s(n)}^{l_{k_n}}) = f_0(x_0)$. Since $|f_0(x_{l_{k_n}}) - f_0(x_0)| \leq |f_0(x_{l_{k_n}}) - f_0(y_{s(n)}^{l_{k_n}})| + |f_0(y_{s(n)}^{l_{k_n}}) - f_0(x_0)|$, we deduce that $\lim_{n\to\infty} f_0(x_{l_{k_n}}) = f_0(x_0)$.

From the above considerations we infer that $\lim_{n\to\infty} f_0(x_n) = f_0(x_0)$. □

Theorem 1 *Let B be a c-Blumberg set of $f : X \to \mathbb{R}$. Then f is B-improvable function if and only if $X = U_B(f) \cup I(X)$.*

PROOF. *Necessity.* Suppose that f is a B-improvable function (so, f_c^B exists) and that there exists a point $x_0 \notin U_B(f) \cup I(X)$. Then we may choose two sequences $\{x_n\}_{n=1}^{\infty} \subset B$ and $\{y_n\}_{n=1}^{\infty} \subset B$ such that $x_n \neq x_0 \neq y_n$ $(n = 1, 2, \ldots)$ $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = x_0$ and $\lim_{n\to\infty} f(x_n) = \alpha \neq \beta = \lim_{n\to\infty} f(y_n)$.

Then $f_c^B(x_0) = \lim_{n\to\infty} f_c^B(x_n) = \lim_{n\to\infty}(x_n) = \alpha$, and on the other hand, $f_c^B(x_0) = \lim_{n\to\infty} f_c^B(y_n) = \lim_{n\to\infty} f(y_n) = \beta$, which is impossible.

According to Proposition 3, *sufficiency* is obvious. □

Proposition 4 *If x_0 is a continuity point of a function $f : X \to \mathbb{R}$, then f is also a continuity point of an arbitrary function f_0 from f^B (B is a c-Blumberg set of f).*

The easy proof is omitted.

Lemma 1 *Let $f : X \to \mathbb{R}$ be a function of Baire class one defined on the complete metric space X. Then the following conditions are equivalent:*
(i) For each c-Blumberg set B of f, f is a B-improvable function.
(ii) There exists a c-Blumberg set B of f such that f is a B-improvable function.
(iii) $L_{C_f}(f, x)$ is a singleton for each $x \in X$.

Proof. The implication $(i) \Rightarrow (ii)$ is obvious.

$(ii) \Rightarrow (iii)$. From Theorem 1 we infer that $X = U_B(f) \cup I(X)$. Of course if $x \in I(X)$, then $L_{C_f}(f, x) = \{f(x)\}$. Now, we suppose that $x \in U_B(f)$. This means that $\lim_{y\to x} f_{|B}(y)$ exists, and so, $\lim_{y\to x} f_{|C_f}(y)$ exists, also.

$(iii) \Rightarrow (i)$. Assume to the contrary that, for some c-Blumberg set of f, f is not a B-improvable function. So, according to Theorem 1, there exists $x_0 \in X \setminus (U_B(f) \cup I(X))$. Then $x_0 \notin B$. It is easy to see that there exist two sequences $\{x_n\}_{n=1}^{\infty} \subset B$ and $\{y_n\}_{n=1}^{\infty} \subset B$ such that

$$\lim_{n\to\infty} x_n = x_0 = \lim_{n\to\infty} y_n, \ \lim_{n\to\infty} f(x_n) = \alpha \neq \beta = \lim_{n\to\infty} f(y_n)$$

and $x_n \neq x_0 \neq y_n$ $(n = 1, 2, \ldots)$. This means that $L_B(f, x_0)$ contains at least two distinct elements.

From the continuity of $f_{|B}$ and the density of C_f we may deduce that $\alpha, \beta \in L_{C_f}(f, x_0)$, which is impossible because $L_{C_f}(f, x_0)$ is a singleton. □

Lemma 2 *If B_1, B_2 are two c-Blumberg sets of $f : \mathbb{R} \to \mathbb{R}$ where f is a function of Baire class one and f is B_1-improvable, then f is B_2-improvable and $f_c^{B_1} = f_c^{B_2}$.*

Proof. Since f is a B_1-improvable function, according to Lemma 1, $L_{C_f}(f, x)$ is a singleton for $x \in R$ and, consequently, (according again to Lemma 1) f is a B_2-improvable function.

Now let $x_0 \in \mathbb{R}$. Then (Lemma 1) $L_{C_f}(f, x_0) = \{\alpha\}$. This means that $f_c^{B_1}(x_0) = \alpha$ and $f_c^{B_2}(x_0) = \alpha$, and so $f_c^{B_1}(x_0) = f_c^{B_2}(x_0)$. By the arbitrariness of the choice of x_0, $f_c^{B_1} = f_c^{B_2}$. □

Now, we present the basic facts from paper [5]. Let D be a subset of $\mathbb{R}$, $f : D \to \mathbb{R}$ and let

$$A(f) = \{x \in D : \lim_{t \to x} f(t) \neq f(x)\}.$$

Then $f_{(0)}(x) = f(x)$ for $x \in D$. For every ordinal number α (A. Katafiasz considers the class of all ordinal numbers.), let

$$f_{(\alpha)}(x) = \begin{cases} f(x), & \text{if } \{\gamma < \alpha : x \in A(f_{(\gamma)})\} = \emptyset; \\ \lim_{t \to x} f_{(\gamma_0)}(t) & \text{if } x \in A(f_{(\gamma_0)}), \\ & \text{where } \gamma_0 = \min\{\gamma < \alpha : x \in A(f_{(\gamma)})\}. \end{cases}$$

For every ordinal number α, $A_\alpha = \{f : D \to \mathbb{R} : C_{f_{(\alpha)}} = D\}$. If a function $f : D \to \mathbb{R}$ belongs to $A_\alpha \setminus (\bigcup_{0 \le \beta < \alpha} A_\beta)$, then it will be called an α-improvable (in the sense of Katafiasz) discontinuous function. If $f \in \bigcup_{0 \le \alpha < \omega_1} A_\alpha$, then it will be called an improvable (in the sense of Katafiasz) function.

Theorem 2 *If $f : \mathbb{R} \to \mathbb{R}$ is an α-improvable function in the sense of Katafiasz (for some ordinal number α), then, for each*[1] *c-Blumberg set B of f, f is a B improvable function.*

Proof. According to Theorem 15 of [5, p. 36], f is of Baire class one, and so, C_f is a c-Blumberg set of f.

Let $f_{(\alpha)}$ be a (continuous) function obtained by the method of Katafiasz. Now, we shall prove that

$$U_{C_f}(f) = \mathbb{R}. \tag{1}$$

Let $x_0 \in \mathbb{R}$, and let $\{b_n\}_{n=1}^{\infty} \subset C_f$ be a sequence such that $x_0 = \lim_{n\to\infty} b_n$. Theorem $1(3, \alpha)$ from paper [5] gives the equality $f(b_n) = f_{(\alpha)}(b_n)$ for $n = 1, 2, \ldots$. Of course, $\lim_{n\to\infty} f_{(\alpha)}(b_n) = f_{(\alpha)}(x_0)$ and, thus, $\lim_{n\to\infty} f(b_n) = f_{(\alpha)}(x_0)$. According to the arbitrariness of the sequence $\{b_n\}$, $x_0 \in U_{C_f}(f)$, and so, (1) is true.

From (1) and Theorem 1 we conclude that f is a C_f-improvable function and, consequently, according to Lemma 2, f is a B-improvable function for an arbitrary c-Blumberg set B of f.

Now, we shall show that $f_{(\alpha)} = f_c^{C_f}$. In fact, let $z_0 \in R = U_{C_f}(f)$. Then (according to Theorem $1(3, \alpha)$ from [5])

$$f_c^{C_f}(z_0) = \lim_{z \to z_0} f_{|C_f}(z) = \lim_{z \to z_0; z \in C_f} f_{(\alpha)}(z) = f_{(\alpha)}(z_0).$$

[1] Note that, according to the Proposition 1, there exists at least one c-Blumberg set of f.

From Lemma 2 we infer that $f_{(\alpha)} = f_c^B$ for an arbitrary c-Blumberg set B of f. $\square$

Note that there exists a discontinuous function $f : \mathbb{R} \to \mathbb{R}$ of Baire class one such that f is C_f-improvable, but is not improvable in the sense of Katafiasz. For example, let f be the characteristic function of the Cantor set. Since $f^{C_f} \equiv 0$, by Theorem 1 f is a C_f-improvable function, but $f_{(\alpha)} = f$ for $0 \le \alpha < \omega_1$ (cf. [5, Theorem 5, p. 10]).

The above considerations suggest the following question: How often do B-improvable functions occur in the class of Baire one functions? The answer to this question is contained in the next theorem.

Let X be a connected and complete metric space containing at least two different points, let $B_1(X)$ denote the space of all bounded functions $f : X \to \mathbb{R}$ of Baire class one with the metric of uniform convergence and let $P(X)$ denote the subset of the space $B_1(X)$ consisting of all functions $g : X \to \mathbb{R}$ satisfying the condition: there exists a c-Blumberg set B of g such that g is a B-improvable function.

Theorem 3 *In the space $B_1(X)$ the set $P(X)$ is porous at each point $f \in B_1(X)$.*

Proof. Let $f \in B_1(X)$, $\varepsilon > 0$ and let x_0 be an arbitrary point of continuity of f. Set $\alpha = f(x_0)$. Then there exists $\delta > 0$ such that $f(\overline{K(x_0, \delta)}) \subset (\alpha - \frac{\varepsilon}{4}, \alpha + \frac{\varepsilon}{4})$. Put

$$g(x) = \begin{cases} \alpha + \dfrac{\varepsilon}{2} \sin \dfrac{1}{\varrho(x_0, x)}, & \text{if } x \in K(x_0, \delta) \setminus \{x_0\}, \\ \alpha, & \text{if } x = x_0, \\ f(x), & \text{if } x \notin K(x_0, \delta), \end{cases}$$

where ϱ denotes the metric in the space X. It is not difficult to verify that $g \in B_1(X)$ and $g \in K(f, \frac{3}{4}\varepsilon)$ and, consequently, $K(g, \frac{\varepsilon}{8}) \subset K(f, \varepsilon)$.

Now, we shall show that

$$K\left(g, \frac{\varepsilon}{8}\right) \cap P(X) = \emptyset. \tag{2}$$

Let $h \in K(g, \frac{\varepsilon}{8})$ and let $\alpha_0, \beta_0 \in (0, \frac{\pi}{2})$ be numbers such that $\sin \alpha_0 = 0.2$ and $\sin \beta_0 = 0.8$. Then there exist positive integer n_0 and sequences $\{a_n\}_{n=n_0}^{\infty}$, $\{b_n\}_{n=n_0}^{\infty} \subset C_h$ such that $\varrho(x_0, a_n) \in \left(\frac{1}{\frac{\pi}{2}+2n\pi}, \frac{1}{\beta_0+2n\pi}\right)$ and $\varrho(x_0, b_n) \in \left(\frac{1}{\alpha_0+2n\pi}, \frac{1}{2n\pi}\right)$. (For simplicity we assume that $n_0 = 1$.) It is easy to see that $\lim_{n\to\infty} a_n = x_0$ and $\lim_{n\to\infty} b_n = x_0$. Without loss of generality we may assume that

$$\{a_n : n = 1, 2, \ldots\} \subset K(x_0, \delta) \supset \{b_n : n = 1, 2, \ldots\}.$$

Of course $a_n \neq x_0 \neq b_n$ $(n = 1, 2, \ldots)$. Note that

$$g(a_n) = \alpha + \frac{\varepsilon}{2} \sin \frac{1}{\varrho(x_0, a_n)} > \alpha + 0,4 \cdot \varepsilon \qquad (n = 1, 2, \ldots),$$
$$g(b_n = \alpha + \frac{\varepsilon}{2} \sin \frac{1}{\varrho(x_0, b_n)} < \alpha + 0,1 \cdot \varepsilon \qquad (n = 1, 2, \ldots).$$

Consequently (according to the fact that $|h(x) - g(x)| < \frac{\varepsilon}{8}$ for $x \in X$)

$$h(a_n) > \alpha + 0.275 \cdot \varepsilon, \quad h(b_n) < \alpha + 0.225 \cdot \varepsilon \quad \text{for} \quad n = 1, 2, \ldots.$$

Since $\lim_{n\to\infty} a_n = x_0 = \lim_{n\to\infty} b_n$ and $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty} \subset C_h$, the above inequality proves that $\lim_{x\to x_0} h_{|C_h}(x)$ does not exist, and so, $L_{C_h}(h, x_0)$ is not a singleton. From Lemma 1 we may infer that h is not a B-improvable function for any c-Blumberg set of h, and so, (2) is proved.

From (2) it follows that the porosity of $P(X)$ at f: $p(P(X), f) \geq \frac{1}{4}$, which means that the set $P(X)$ is porous at f. □

The results included in paper [5] and our considerations (in particular, Theorem 2) lead to the question: How large, in the space $P = P(\mathbb{R})$ (with the metric of uniform convergence), is the set K of all functions improvable in the sense of Katafiasz? The answer to this question is contained in the following assertion.

Theorem 4 *In the space P the set K is porous at each point $t \in P$.*

Proof. Let t be an arbitrary element of P and let $\eta > 0$. Fix a point z_0 of continuity of t and let $\alpha = t(z_0)$. Then there exists $\sigma > 0$ such that $t([z_0 - \sigma, z_0 + \sigma]) \subset (\alpha - \frac{\eta}{4}, \alpha + \frac{\eta}{4})$. Let C be a Cantor-like set in the segment $[\frac{\sigma}{2}, \sigma]$ and, moreover, let B be a fixed c-Blumberg set of t. Then (Lemma 2) t is a B-improvable function.

Put $\beta_1 = t_c^B(z_0 - \sigma)$ and $\beta_2 = t_c^B(z_0 + \sigma)$. Note that $\beta_1, \beta_2 \in [\alpha - \frac{\eta}{4}, \alpha + \frac{\eta}{4}]$. Let $\gamma \in (\alpha - \frac{\eta}{4}, \alpha + \frac{\eta}{4})$ be a number such that $|\gamma - \beta_2| > \frac{3}{16}\eta$. Now, we define the function $w : \mathbb{R} \to \mathbb{R}$ by

$$w(x) = \begin{cases} t(x), & \text{if } |x - z_0| \geq \sigma, \\ \beta_2, & \text{if } x - z_0 \in [\frac{\sigma}{2}, \sigma) \setminus (C \setminus \{\frac{\sigma}{2}\}), \\ \gamma, & \text{if } x - z_0 \in (\frac{\sigma}{2}, \sigma) \cap C, \\ \frac{2(\beta_2 - \beta_1)}{3\sigma} \cdot x + \frac{\sigma(\beta_1 + 2\beta_2) - 2z_0(\beta_2 - \beta_1)}{3\sigma}, & \text{if } x - z_0 \in (-\sigma, \frac{\sigma}{2}). \end{cases}$$

Since the restriction of w to any closed set possesses a continuity point, w is of Baire class one.

Let $B' = C_w$. Of course, B' is a c-Blumberg set of w. To prove the fact that w is a B'-improvable function, it is sufficient to show (Theorem 1) that

$$U_{C_w}(w) = \mathbb{R}. \tag{3}$$

Let $x \in \mathbb{R}$. We consider the following cases:

1. $|x - z_0| > \sigma$. Since t is a B-improvable function for some c-Blumberg set B of t and $C_t \subset B$, the limit $\lim_{y\to x, y\in C_t} t(y)$ exists and $\lim_{y\to x, y\in C_w} w(y) = \lim_{y\to x, y\in C_t} t(y)$.

2. $x - z_0 \in (-\sigma, \frac{\sigma}{2})$. Then, of course, the limit $\lim_{y\to x, y\in C_w} w(y)$ exists since w is linear on the interval $(z_0 - \sigma, z_0 + \frac{\sigma}{2})$.

3. $x - z_0 \in (\frac{\sigma}{2}, \sigma)$. Then, of course,

$$\lim_{y\to x, y\in C_w} w(y) = \lim_{y\to x, y\in (\frac{\sigma}{2},\sigma)\setminus C} w(y) = \beta_2.$$

4. $x - z_0 = -\sigma$. Then $\lim_{y\to x+, y\in C_w} w(y) = \beta_1$ and

$$\lim_{y\to x-, y\in C_w} w(y) = \lim_{y\to (z_0-\sigma)-, y\in C_t} t(y) = t_c^B(z_0 - \sigma) = \beta_1$$

for some c-Blumberg set B of t.

5. $x - z_0 = \frac{\sigma}{2}$. Then, of course,

$$\lim_{y\to x-, y\in C_w} w(y) = \beta_2 \quad \text{and} \quad \lim_{y\to x+, y\in C_w} w(y) = \beta_2.$$

6. $x - z_0 = \sigma$. Then $\lim_{y\to x-, y\in C_w} w(y) = \beta_2$ and

$$\lim_{y\to x+, y\in C_w} w(y) = \lim_{y\to (z_0+\sigma)+, y\in C_t} t(y) = t_c^B(z_0 + \sigma) = \beta_2$$

for some c-Blumberg set B of t. The proof of (3) is complete.

Of course, $w \in K(t, \frac{\eta}{2})$ and, consequently, $K(w, \frac{\eta}{16}) \subset K(t, \eta)$. Now, we shall show that

$$K\left(w, \frac{\eta}{16}\right) \cap K = \emptyset \tag{4}$$

Let $h \in K(w, \frac{\eta}{16})$. Then $\gamma - \frac{\eta}{16} < h(x) < \gamma + \frac{\eta}{16}$ for each x such that $x - z_0 \in (\frac{\sigma}{2}, \sigma) \cap C$, $\beta_2 - \frac{\eta}{16} < h(x) < \beta_2 + \frac{\eta}{16}$ for each x such that $x - z_0 \in [\frac{\sigma}{2}, \sigma] \setminus (C \setminus \{\frac{\sigma}{2}\})$. Note that

$$h \text{ possesses no limit at any point } x \text{ such that } x - z_0 \in \left(\frac{\sigma}{2}, \sigma\right) \cap C. \tag{5}$$

Indeed, if x_0 is an element such that $x_0 - z_0 \in (\frac{\sigma}{2}, \sigma) \cap C$, then there exist two sequences $\{a_n\}_{n=1}^{\infty} \subset (\frac{\sigma}{2}, \sigma) \setminus C$ and $\{b_n\}_{n=1}^{\infty} \subset (\frac{\sigma}{2}, \sigma) \cap C$ such that $\lim_{n\to\infty}(z_0 + a_n) = x_0 = \lim_{n\to\infty}(z_0 + b_n)$ and

$$h(z_0 + b_n) \in \left(\gamma - \frac{\eta}{16}, \gamma + \frac{\eta}{16}\right),$$
$$h(z_0 + a_n) \in \left(\beta_2 - \frac{\eta}{16}, \beta_2 + \frac{\eta}{16}\right).$$

From the assumptions on γ we infer that

$$\left[\gamma - \frac{\eta}{16}, \gamma + \frac{\eta}{16}\right] \cap \left[\beta_2 - \frac{\eta}{16}, \beta_2 + \frac{\eta}{16}\right] = \emptyset.$$

This finishes the proof of (5).

By the transfinite induction (applying (5)), we may prove that

$$h_{(\alpha)}(x) = h(x) \text{ for } x \in \left(z_0 + \frac{\sigma}{2}, z_0 + \sigma\right) \text{ and } 0 \leq \alpha < \omega_1. \tag{6}$$

According to (5) and (6), we infer that $h_{(\alpha)}$ possesses no limit at any point x such that $x - z_0 \in (\frac{\sigma}{2}, \sigma) \cap C$, which finishes the proof of (4). From (4) we deduce $p(K, t) \geq \frac{1}{8}$, which means that K is porous at t. □

Our method for improving the continuity of a fixed function is not always effective but, in some cases, we may "improve" properties "near" the continuity, for example quasi-continuity or the Darboux property. The following theorem is connected with this problem.

Theorem 5 *Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that C_f is a dense set. If $x_0 \in \mathbb{R}$ is a point such that there exists $\delta > 0$ such that $(x_0, x_0 + \delta) \subset U_{C_f}(f)$ and $[L^+_{C_f}(f, x_0)] \subset \overline{L^+_{C_f}(f, x_0)}$, then f^{C_f} contains a function for which x_0 is a right-sided Darboux point and x_0 is a quasi-continuity point.*

Proof. Let g be a function belonging to f^{C_f} such that $g(x_0) \in L^+_{C_f}(f, x_0)$. We shall show that

$$x_0 \text{ is a right-sided Darboux point of } g. \tag{7}$$

Let $\alpha \in L^+(g, x_0) \setminus \{g(x_0)\}$. It is not hard to verify that $\alpha \in L_{C_f}(f, x_0)$. Assume, for instance, that $g(x_0) < \alpha$, and let β be an arbitrary number such that $\beta \in (g(x_0), \alpha)$; moreover, let $\gamma > 0$. To prove (7), it is sufficient to show that there exists $y_0 \in [x_0, x_0 + \gamma)$ such that $g(y_0) = \beta$. Note that there exist β_1 and β_2 such that

$$\beta_1 \in L^+_{C_f}(f, x_0) \cap (g(x_0), \beta) \wedge \beta_2 \in L^+_{C_f}(f, x_0) \cap (\beta, \alpha). \tag{8}$$

Indeed, $\alpha \in L^+_{C_f}(f, x_0)$, $(g(x_0), \beta) \subset [L^+_{C_f}(f, x_0)] \supset (\beta, \alpha)$. Since $L^+_{C_f}(f, x_0)$ is dense in $[L^+_{C_f}(f, x_0)]$, the proof of (8) is finished.

From (8) we infer that there exists a nondegenerate segment $[x_1, x_2]$ such that $[x_1, x_2] \subset (x_0, x_0 + \min(\gamma, \delta))$, $f(x_1) \in (g(x_0), \beta)$, $f(x_2) \in (\beta, \alpha)$ and $x_1, x_2 \in C_f$. Assume, for instance, that $x_1 < x_2$. Let

$$y_0 = \sup\big\{x > x_1 : f([x_1, x) \cap C_f) \subset (-\infty, \beta)\big\}.$$

(Of course, $\{x > x_1 : f([x_1, x) \cap C_f) \subset (-\infty, \beta)\} \neq \emptyset$.) Note that

$$f([x_1, y_0) \cap C_f) \subset (-\infty, \beta) \text{ and } y_0 < x_2.$$

By the above, $f(x) < \beta$ for any $x \in [x_1, y_0) \cap C_f$ and, consequently,

$$\lim_{x \to y_0-, x \in C_f} f(x) \leq \beta. \tag{9}$$

(Such a limit exists because $x_0 < y_0 < x_0 + \delta$, which means that $y_0 \in U_{C_f}(f)$.) Now, we shall show that

$$L_{C_f}(f, y_0) = \{\beta\}. \tag{10}$$

Indeed, from the definition of y_0 we deduce that:

$$\begin{aligned} &\text{for each } n = 1, 2, \ldots, \text{ there exists } y_n \in \big[y_0, y_0 + \frac{1}{n}\big) \cap C_f \\ &\text{such that } \ f(y_n) \geq \beta. \end{aligned} \tag{11}$$

Consider the following cases:

1^o There exists n_0 such that $f(y) < \beta$ for each $y \in (y_0, y_0 + \frac{1}{n_0}) \cap C_f$. By (11) and the assumptions of this case, $f(y_0) \geq \beta$ and $y_0 \in C_f$. According to (9), $\lim_{y \to y_0, y \in C_f} f(y) = \beta$, which means that $L_{C_f}(f, y_0) = \{\beta\}$.

2^o For every n, there exists $y_n \in (y_0, y_0 + \frac{1}{n}) \cap C_f$ such that $f(y_n) \geq \beta$. Since $y_0 \in U_{C_f}(f)$, $\lim_{n\to\infty} f(y_n)$ exists and $\lim_{n\to\infty} f(y_n) \geq \beta$. From (9) we infer that $\lim_{y \to y_0} f(y) = \beta$. The proof of (10) is finished.

If $y_0 \in C_f$, then $g(y_0) = f(y_0) = \beta$. If $y_0 \notin C_f$, then ($g \in f^{C_f}$) $g(y_0) \in L_{C_f}(f, y_0) = \{\beta\}$ and, consequently, $g(y_0) = \beta$. The proof of the fact that x_0 is a right-sided Darboux point of g is completed.

Now, we shall show that x_0 is a quasi-continuity point of g. Let V be an open neighborhood of $g(x_0)$ and let U be an open neighborhood of x_0. Since $g(x_0) \in L^+_{C_f}(f, x_0)$, there exists a sequence $\{q_n\} \subset C_f$ such that $\lim_{n\to\infty} q_n = x_0$ and $\lim_{n\to\infty} f(q_n) = g(x_0)$. According to Proposition 4, $\{q_n\} \subset C_g$ and $\lim_{n\to\infty} g(q_n) = g(x_0)$. Then there exists n_0 such that $q_{n_0} \in U$ and, moreover, there exists an open neighborhood $G \subset U$ of q_{n_0} such that $g(G) \subset V$. This completes the proof of the quasi-continuity of g at x_0. □

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