RESEARCH Real Analysis Exchange
Vol. 21(2), 1995–96, pp. 430–439

Aleksandra Katafiasz, Instytut Matematyki WSP, Chodkiewicza 30, 85-064 Bydgoszcz, Poland

PROPERTIES OF THE CLASS OF IMPROVABLE FUNCTIONS

Abstract

In this paper the classes $\mathcal{A}_\alpha$ and the class $\mathcal{A}$ are characterized and compared to the classes $\mathcal{B}_1$, $\mathcal{B}_1^*$ and the class of all Darboux functions.

1 Preliminaries

The word "function" will mean a bounded real function of a real variable and D will denot a subset of $\mathbb{R}$.

Definition 1 *For each function $f : D \longrightarrow \mathbb{R}$, let*

$$C(f) = \Big\{x \in D;\ \lim_{t\to x} f(t) = f(x)\Big\};$$
$$U(f) = \Big\{x \in D;\ \lim_{t\to x} f(t) \neq f(x)\Big\};$$
$$L(f) = \Big\{x \in D; \lim_{t\to x} f(t)\ \textit{exists}\Big\}.$$

Definition 2 *A point $x_0 \in U(f)$ is called an improvable point of discontinuity of the function f.*

The following remark is easy to see.

Remark 1 *Let $f : D \to \mathbb{R}$. Then $U(f) \cap C(f) = \emptyset$ and $L(f) = U(f) \cup C(f)$.*

The following proposition is well known. (Compare to [2].)

Proposition 1 *Let $f : D \longrightarrow \mathbb{R}$. Then the set $U(f)$ is countable.*

Mathematical Reviews subject classification: Primary: 26A15
Received by the editors July 27, 1994

We define the functions $f_{(\alpha)}$ on the class of ordinal numbers.

Definition 3 *Let $f : D \to \mathbb{R}$ and let $f_{(0)}(x) = f(x)$ for each $x \in D$. For every ordinal number α, let*

$$f_{(\alpha)}(x) = \begin{cases} f(x), & \text{if } \left\{\gamma < \alpha;\ x \in U\left(f_{(\gamma)}\right)\right\} = \emptyset, \\ \lim_{t\to x} f_{(\gamma_0)}(t), & \text{if } x \in U\left(f_{(\gamma_0)}\right), \\ & \text{where } \gamma_0 = \min\left\{\gamma < \alpha;\ x \in U\left(f_{(\gamma)}\right)\right\}. \end{cases}$$

The following theorems are established in [1] where the reader should turn for pertainent definitions

Theorem 1 *Let $f : D \to \mathbb{R}$ and let $\alpha > 0$ be an ordinal number. Then*

(1,α) *for each $x \in D$, $\left\{\gamma < \alpha; x \in U\left(f_{(\gamma)}\right)\right\}$ is the empty set or has only one element,*

(2,α) *for each ordinal number γ with $\gamma < \alpha$,*

$$\left\{x \in D;\ f_{(\gamma)}(x) \neq f_{(\alpha)}(x)\right\} = \bigcup_{\gamma \le \beta < \alpha} U\left(f_{(\beta)}\right),$$

(3,α) *for each ordinal number γ with $\gamma < \alpha$, if $x \in L\left(f_{(\gamma)}\right)$, then*

$$\lim_{t\to x} f_{(\gamma)}(t) = f_{(\alpha)}(x),$$

(4,α) $\bigcup_{0\le\beta<\alpha} L\left(f_{(\beta)}\right) \subset C\left(f_{(\alpha)}\right)$.

Definition 4 *For each ordinal number α, let*

$$\mathcal{A}_\alpha = \left\{f : D \to \mathbb{R};\ C\left(f_{(\alpha)}\right) = D\right\}.$$

If a function $f : D \to \mathbb{R}$ belongs to $\mathcal{A}_\alpha \setminus \left(\bigcup_{0\le\beta<\alpha} \mathcal{A}_\beta\right)$, then it will be called an α-improvable discontinuous function.

Put $\mathcal{A} = \bigcup_{0\le\alpha<\omega_1} \mathcal{A}_\alpha$. If a function $f \in \mathcal{A}$, then it will be called an improvable function.

Definition 5 *Let $K \subset D$. Put $K^{(0)} = K$. Let*

$$K^{(1)} = K^d = \{x \in D;\ x \text{ is an accumulation point of } K \text{ in } \mathbb{R}\}$$

and $K^* = K \setminus K^d$.

Example 1 *Let $W = \{1/n;\ n \in \mathbb{N}\}$ and let f be the characteristic function of the set W. Then $U(f) = W$, and 0 is not an improvable point of discontinuity of f. Note that $f_{(1)}(x) = 0$ for each $x \in \mathbb{R}$, so $f \in \mathcal{A}_1$. Observe that $f_{(1)}$ is also continuous at points, which do not belong to the set $U(f)$.*

Definition 6 *For $A \subset D \subset \mathbb{R}$, let*

$$\mathcal{M}(A) = \{f : D \to \mathbb{R};\ f(A) = \{0\} \text{ and, for each } x \in D, f(x) \geq 0\}.$$

The following theorem is proved in [1].

Theorem 2 *Let A be a dense subset of D and let $f \in \mathcal{A}_\alpha$ be a function such that $C(f) = A$. Then $g = |f - f_{(\alpha)}| \in \mathcal{M}(A)$, for each $0 \leq \beta \leq \alpha$, $C\left(f_{(\beta)}\right) = C\left(g_{(\beta)}\right)$, $U\left(f_{(\beta)}\right) = U\left(g_{(\beta)}\right)$ and $g_{(\beta)} = |f_{(\beta)} - f_{(\alpha)}|$.*

In Section 3 we use the following lemma.

Lemma 1 *Let $f : \mathbb{R} \to \mathbb{R}$ and let $f \in \mathcal{A}_\alpha$ for some $\alpha < \omega_1$. Then $f \in \mathcal{B}_1$ if and only if $g = |f - f_{(\alpha)}| \in \mathcal{B}_1$.*

Proof. Let P be a perfect set. Assume that $f \in \mathcal{B}_1$. Then there exists a point $x_0 \in \mathbb{R}$ such that $x_0 \in C\left(f_{|P}\right)$. Consider two possibilities:

1. If $x_0 \in C(f)$, then $x_0 \in C(g)$; so $x_0 \in C\left(g_{|P}\right)$.
2. If $x_0 \in C\left(f_{|P}\right) \setminus C(f)$, then $\lim_{t\to x_0} f_{|P}(t)$ exists and $\lim_{t\to x_0} f_{|P}(t) = f_{|P}(x_0)$. If there existed an ordinal number $\beta_0 < \alpha$ and a sequence $(x_n)_{n=1}^{\infty} \subset \left(U\left(g_{(\beta_0)}\right) \cap P\right)$ such that $\lim_{n\to\infty} x_n = x_0$, then by Theorem 2 we would have $(x_n)_{n=1}^{\infty} \subset \left(U\left(f_{(\beta_0)}\right) \cap P\right)$, a contradiction. Thus $\lim_{t\to x_0} g_{|P}(t)$ exists and $\lim_{t\to x_0} f_{|P}(t) = g_{|P}(x_0)$.

Thus $g \in \mathcal{B}_1$.

Now, assume that $g \in \mathcal{B}_1$. We can prove that $f \in \mathcal{B}_1$ similarly. □

2 The Characterization of the Classes $\mathcal{A}_\alpha$

As we have seen in Example 1 there functions $f \in \mathcal{A}_1$ with $\mathbb{R} \setminus C(f) \neq U(f)$. Thus we can ask whether we can study continuity of the function $f_{(\alpha)}$ by considering properties of $f_{|\mathbb{R}\setminus\bigcup_{\beta<\alpha} U\left(f_{(\beta)}\right)}$. The answer is given in the following theorem.

Theorem 3 *Let $f : \mathbb{R} \to \mathbb{R}$ and let α be an ordinal number. Then $f \in \mathcal{A}_\alpha$ if and only if $f_{|\left(\mathbb{R}\setminus\bigcup_{0\leq\beta<\alpha} U\left(f_{(\beta)}\right)\right)}$ is continuous.*

Proof. First, we assume that $f \in \mathcal{A}_\alpha$. Let $x \in C(f)$. Then, of course, $f_{|\mathbb{R}\setminus\bigcup_{0\le\beta<\alpha} U\left(f_{(\beta)}\right)}$ is continuous at x. Let $x \in \mathbb{R} \setminus \left(C(f) \cup \bigcup_{0\le\beta<\alpha} U\left(f_{(\beta)}\right)\right)$. By Theorem 1 (2,$\alpha$), $\left\{x \in \mathbb{R};\ f(x) \neq f_{(\alpha)}(x)\right\} = \bigcup_{0\le\beta<\alpha} U\left(f_{(\beta)}\right)$. Hence

$$\begin{aligned} f_{(\alpha)}(x) &= \lim_{t\to x} f_{(\alpha)}(t) = \lim_{t\to x} f_{|\left(\mathbb{R}\setminus\bigcup_{0\le\beta<\alpha} U\left(f_{(\beta)}\right)\right)\ (\alpha)}(t) \\ &= \lim_{t\to x} f_{|\left(\mathbb{R}\setminus\bigcup_{0\le\beta<\alpha} U\left(f_{(\beta)}\right)\right)}(t). \end{aligned}$$

Thus $\lim_{t\to x} f_{|\left(\mathbb{R}\setminus\bigcup_{0\le\beta<\alpha} U\left(f_{(\beta)}\right)\right)}(t) = f(x)$; so $f_{|\left(\mathbb{R}\setminus\bigcup_{0\le\beta<\alpha} U\left(f_{(\beta)}\right)\right)}$ is continuous at x.

Now assume that $f_{|\left(\mathbb{R}\setminus\bigcup_{0\le\beta<\alpha} U\left(f_{(\beta)}\right)\right)}$ is continuous. We shall show that $\mathbb{R} = C\left(f_{(\alpha)}\right)$. Of course, $C\left(f_{(\alpha)}\right) \subset \mathbb{R}$. By Theorem 1 (4,$\alpha$),

$$\bigcup_{0\le\beta<\alpha} U\left(f_{(\beta)}\right) \subset C\left(f_{(\alpha)}\right).$$

Let $x \in \mathbb{R} \setminus \bigcup_{0\le\beta<\alpha} U\left(f_{(\beta)}\right)$. Suppose that there exists a sequence $(x_n)_{n=1}^\infty$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} f_{(\alpha)}(x_n) \neq f_{(\alpha)}(x)$. We can assume that $a = \lim_{n\to\infty} f_{(\alpha)}(x_n) < f_{(\alpha)}(x)$. By Theorem 1 (2,$\alpha$), $\bigcup_{0\le\beta<\alpha} U\left(f_{(\beta)}\right) = \left\{x \in \mathbb{R};\ f_{(\alpha)}(x) \neq f(x)\right\}$ and

$$\lim_{t\to x} f_{|\left(\mathbb{R}\setminus\bigcup_{0\le\beta<\alpha} U\left(f_{(\beta)}\right)\right)\ (\alpha)}(t) = \lim_{t\to x} f_{|\left(\mathbb{R}\setminus\bigcup_{0\le\beta<\alpha} U\left(f_{(\beta)}\right)\right)}(t) = f(x) = f_{(\alpha)}(x).$$

Thus there exists $n_0 \in \mathbb{N}$ such that, for each $n > n_0$, $x_n \in \bigcup_{0\le\beta<\alpha} U\left(f_{(\beta)}\right)$. Therefore, we may assume that, for each $n \in \mathbb{N}$, $x_n \in \bigcup_{0\le\beta<\alpha} U\left(f_{(\beta)}\right)$ and $f_{(\alpha)}(x_n) < f_{(\alpha)}(x)$.

Let $\epsilon = \frac{f(x)-a}{2}$. By Theorem 1 (4,$\alpha$), $\bigcup_{0\le\beta<\alpha} U\left(f_{(\beta)}\right) \subset C\left(f_{(\alpha)}\right)$. Hence, for each $n \in \mathbb{N}$, there exists an interval (a_n, b_n) containing x_n such that, for each $z \in (a_n, b_n)$, $f_{(\alpha)}(z) < f_{(\alpha)}(x_n) + \epsilon$. Since $\bigcup_{0\le\beta<\alpha} U\left(f_{(\beta)}\right)$ is a countable set, we can choose a sequence $(z_n)_{n=1}^\infty$ such that $\lim_{n\to\infty} z_n = x$ and, for each $n \in \mathbb{N}$, $z_n \in (a_n, b_n) \cap \left(\mathbb{R} \setminus \bigcup_{0\le\beta<\alpha} U\left(f_{(\beta)}\right)\right)$. Hence, for each $n \in \mathbb{N}$, $f_{(\alpha)}(z_n) = f(z_n)$ and $f(z_n) < f_{(\alpha)}(x_n) + \epsilon$. Then

$$\limsup_{n\to\infty} f(z_n) \le \lim_{n\to\infty} f_{(\alpha)}(x_n) + \epsilon = a + \epsilon = \frac{a + f(x)}{2} < f(x)$$

and $\lim_{t\to x} f_{|\mathbb{R}\setminus\bigcup_{0\le\beta<\alpha} U\left(f_{(\alpha)}\right)}(t) \neq f(x)$, a contradiction. Hence $x \in C\left(f_{(\alpha)}\right)$. Thus $\mathbb{R} = C\left(f_{(\alpha)}\right)$ and the proof is complete. □

3 The Comparison of the Class of Improvable Functions to Other Classes of Functions

First we compare the class of improvable functions to the class of Baire 1 functions.

Theorem 4 *If $f : \mathbb{R} \to \mathbb{R}$ is an improvable function, then f is a Baire 1 function.*

Proof. By Lemma 1, we can assume that $f \in \mathcal{M}(C(f))$. Since f is an improvable function, there exists an ordinal number $\alpha < \omega_1$ such that $f \in \mathcal{A}_\alpha$.

Let P be a perfect set. Put $D(f) = \mathbb{R} \setminus C(f)$. If $P \cap C(f) \neq \emptyset$, then $C\left(f_{|P}\right) \neq \emptyset$. Thus assume that $P \cap C(f) = \emptyset$. First suppose that $P \cap U(f) \neq \emptyset$. Let $x_0 \in P \cap U(f)$. Thus for each neighborhood $U(x_0)$ of the point x_0 the set $U(x_0) \cap P$ is uncountable. Fix $U(x_0)$. Since $P \subset D(f)$ and $U(f)$ is dense in $D(f)$, for each $x \in D(f) \setminus U(f)$ there exists a sequence $(x_n)_{n=1}^{\infty} \subset U(f)$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} f(x_n) > 0$. Hence $U(x_0) \cap \bigcup_{n=1}^{\infty} \left\{x \in \mathbb{R};\ \limsup_{t\to x_0} f(t) \geq \frac{1}{n}\right\}$ is uncountable; so there exists n_0 such that $U(x_0) \cup \left\{x \in \mathbb{R};\ \limsup_{t\to x_0} f(t) \geq \frac{1}{n_0}\right\}$ is uncountable, contrary to $x_0 \in U(f)$. Thus $U(f) \cap U(x_0) = \emptyset$. Since $U(x_0) \subset \left\{x \in \mathbb{R};\ f(x) = f_{(1)}(x)\right\}$; so $U(x_0) \cap U\left(f_{(1)}\right) = \emptyset$ and by transfinite induction we can show that, for each ordinal number $\beta < \alpha$, $U(x_0) \cap U\left(f_{(\beta)}\right) = \emptyset$, contraty to $f \in \mathcal{A}_\alpha$. Thus $P \cap \bigcup_{0\leq\beta<\alpha} U\left(f_{(\beta)}\right) = \emptyset$. Hence $P \subset D(f) \setminus \bigcup_{0\leq\beta<\alpha} U\left(f_{(\beta)}\right)$ and by Theorem 3, $C\left(f_{|P}\right) \neq \emptyset$, which completes the proof. □

The following proposition shows that there exists a Baire 1 function which is not an improvable discontinuous one.

Proposition 2 *There exist a subset D of $\mathbb{R}$ and a function $f : D \to \mathbb{R}$ such that $C(f)$ is a dense subset of D and there exist no ordinal number α such that $f \in \mathcal{A}_\alpha$.*

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be the characteristic function of $[0, +\infty)$. Note that, for $x = 0$,

$$0 = \lim_{t\to x^-} f(t) \neq \lim_{t\to x^+} f(t) = 1.$$

Since $C(f) = \mathbb{R} \setminus \{0\}$; so $U(f) = \emptyset$ and, for each $x \in \mathbb{R}$, $f_{(1)}(x) = f(x)$. By Theorem 1 and by transfinite induction, we have that $f_{(\alpha)}(x) = f(x)$ for each $x \in \mathbb{R}$ and for every ordinal number α.

Denote by $(\mathcal{B}_1, \rho)$ the metric space of all bounded real Baire 1 functions defined on $\mathbb{R}$, where, for each pair of functions $f, g \in \mathcal{B}_1$, $\rho(f, g) = \sup_{x\in\mathbb{R}} |f(x) - g(x)|$. Let $f : D \to \mathbb{R}$ and let $\delta > 0$. Put $K(f, \delta) = \{g \in \mathcal{B}_1;\ \rho(f, g) < \delta\}$.

Theorem 5 *The set $\mathcal{A}$ is nowhere dense in $\mathcal{B}_1$.*

PROOF. Let $f \in \mathcal{B}_1$ and let $\delta > 0$ be a real number. Put $K = K(f, \delta)$. We shall show that there exists a ball $K_1 \subset K$ such that $K_1 \cap \mathcal{A} = \emptyset$. If $\mathcal{A} \cap K = \emptyset$, then we put $K_1 = K$. Let $g \in K \cap \mathcal{A}$. Put $\delta_1 = \rho(f, g)$ and $\sigma = \delta - \delta_1$. Since $g \in \mathcal{A}$, so $C(g)$ is residual in $\mathbb{R}$. Then there exists a perfect nowhere dense set $P \subset C(g)$. We define the function h by

$$h(x) = \begin{cases} g(x) + \frac{\sigma}{2}, & \text{if } x \in P, \\ g(x), & \text{otherwise.} \end{cases}$$

Note that h is the sum of two functions: g and k, where

$$k(x) = \begin{cases} \frac{\sigma}{2}, & \text{if } x \in P, \\ 0, & \text{otherwise.} \end{cases}$$

Thus $h \in \mathcal{B}_1$. We shall show that, for each ordinal number α with $0 \leq \alpha < \omega_1$,

(i,α) $h_{(\alpha)}(x) = \begin{cases} g_{(\alpha)}(x) + \frac{\sigma}{2}, & \text{if } x \in P, \\ g_{(\alpha)}(x), & \text{otherwise.} \end{cases}$

(ii,α) for each $x \in P$,

$$\limsup_{t \to x} h_{(\alpha)}(t) = \limsup_{t \to x} h_{|C(g)\ (\alpha)}(t) = g_{(\alpha)}(x) + \frac{\sigma}{2}$$

and

$$\liminf_{t \to x} h_{(\alpha)}(t) = \liminf_{t \to x} h_{|C(g)\ (\alpha)}(t) = g_{(\alpha)}(x),$$

(iii,α) $C\left(h_{(\alpha)}\right) = C\left(g_{(\alpha)}\right) \setminus P$ and $U\left(h_{(\alpha)}\right) = U\left(g_{(\alpha)}\right)$.

Let $\alpha = 0$. By the definition of the function h, condition (i,0) is true.

Let $x \in P$. Since $P \subset C(g)$, $P = P^d$, $P \subset (C(g))^d$ and $\mathrm{cl}\,(\mathbb{R} \setminus P) = \mathbb{R}$, we know that

$$\begin{aligned} \lim_{t \to x} h_{|P}(t) &= g(x) + \frac{\sigma}{2}, \\ \lim_{t \to x} h_{|(\mathbb{R} \setminus P)}(t) &= g(x), \\ \limsup_{t \to x} h(t) &= \limsup_{t \to x} h_{|C(g)}(t) = g(x) + \frac{\sigma}{2} \quad \text{and} \\ \liminf_{t \to x} h(t) &= \liminf_{t \to x} h_{|C(g)}(t) = g(x). \end{aligned}$$

Consequently condition (ii,0) is satisfied.

Since $\mathbb{R} \setminus P$ is an open set and since $h_{|(\mathbb{R}\setminus P)} = g_{|(\mathbb{R}\setminus P)}$, we get $C(h) \setminus P = C(g) \setminus P$ and $U(h) \setminus P = U(g) \setminus P$. By condition (ii,0), $L(h) \subset \mathbb{R} \setminus P$. Therefore $C(h) = C(h) \setminus P = C(g) \setminus P$ and $U(h) = U(h) \setminus P = U(g) \setminus P$. Since $U(g) \subset \mathbb{R} \setminus C(g) \subset \mathbb{R} \setminus P$, we have that $U(h) = U(g)$ and condition (iii,0) is true.

Now we assume that $\alpha > 0$ is an arbitrary ordinal number and, for each ordinal number $\beta (0 \le \beta < \alpha)$, conditions (i,$\beta$), (ii,$\beta$) and (iii,$\beta$) are satisfied. Let $x \in P$. Since, for each ordinal number β with $0 \le \beta < \alpha$, by (iii,β),

$$U\left(h_{(\beta)}\right) = U\left(g_{(\beta)}\right) \subset \mathbb{R} \setminus C\left(g_{(\beta)}\right) \subset \mathbb{R} \setminus C(g) \subset \mathbb{R} \setminus P,$$

we have $\left\{\beta < \alpha; x \in U\left(h_{(\beta)}\right)\right\} = \left\{\beta < \alpha; x \in U\left(g_{(\beta)}\right)\right\} = \emptyset$ and $h_{(\alpha)}(x) = h(x) = g(x) + \frac{\sigma}{2} = g_{(\alpha)}(x) + \frac{\sigma}{2}$.

Let $x \in \mathbb{R} \setminus P$. Then

$$h_{(\alpha)}(x) = \begin{cases} h(x), & \text{if } \left\{\beta < \alpha;\ x \in U\left(h_{(\beta)}\right)\right\} = \emptyset, \\ \lim_{t\to x} h_{(\beta_0)}(t), & \text{if } x \in U\left(h_{(\beta_0)}\right), \\ & \text{where } \beta_0 = \min\left\{\beta < \alpha;\ x \in U\left(h_{(\beta)}\right)\right\} \end{cases}$$

and

$$g_{(\alpha)}(x) = \begin{cases} g(x), & \text{if } \left\{\beta < \alpha;\ x \in U\left(g_{(\beta)}\right)\right\} = \emptyset, \\ \lim_{t\to x} g_{(\beta_1)}(t) & \text{if } x \in U\left(g_{(\beta_1)}\right), \\ & \text{where } \beta_1 = \min\left\{\beta < \alpha;\ x \in U\left(g_{(\beta)}\right)\right\}. \end{cases}$$

By our assumptions, we know that $\beta_0 = \beta_1$. Since $\mathbb{R} \setminus P$ is an open set and $g_{|(\mathbb{R}\setminus P)\ (\beta_0)} = h_{|(\mathbb{R}\setminus P)\ (\beta_0)}$, we get $\lim_{t\to x} g_{(\beta_0)}(t) = \lim_{t\to x} h_{(\beta_0)}(t)$. Therefore $h_{(\alpha)}(x) = g_{(\alpha)}(x)$. So

$$h_{(\alpha)}(x) = \begin{cases} g_{(\alpha)}(x) + \frac{\sigma}{2}, & \text{if } x \in P, \\ g_{(\alpha)}(x), & \text{otherwise.} \end{cases}$$

By (i,α), as in the proof of (ii,0), we can show that, for each $x \in P$,

$$\limsup_{t\to x} h_{(\alpha)}(t) = \limsup_{t\to x} f_{|C(g)\ (\alpha)}(t) = g_{(\alpha)}(x) + \frac{\sigma}{2}$$

and

$$\liminf_{t\to x} h_{(\alpha)}(t) = \liminf_{t\to x} h_{|C(g)\ (\alpha)}(t) = g_{(\alpha)}(x).$$

Since $\mathbb{R} \setminus P$ is an open set and by (i,α), $h_{|(\mathbb{R}\setminus P)\ (\alpha)} = g_{|(\mathbb{R}\setminus P)\ (\alpha)}$. Hence $C\left(f_{(\alpha)}\right) \setminus P = C\left(g_{(\alpha)}\right) \setminus P$ and $U\left(h_{(\alpha)}\right) \setminus P = U\left(g_{(\alpha)}\right) \setminus P$. By condition (ii,$\alpha$),

$L\left(h_{(\alpha)}\right) \subset \mathbb{R} \setminus P$. Therefore $C\left(h_{(\alpha)}\right) = C\left(h_{(\alpha)}\right) \setminus P = C\left(g_{(\alpha)}\right) \setminus P$ and $U\left(h_{(\alpha)}\right) = U\left(h_{(\alpha)}\right) \setminus P = U\left(g_{(\alpha)}\right) \setminus P$. By $U\left(g_{(\alpha)}\right) \subset \mathbb{R} \setminus C\left(g_{(\alpha)}\right) \subset \mathbb{R} \setminus C(g) \subset \mathbb{R} \setminus P$, we know that $U\left(h_{(\alpha)}\right) = U\left(g_{(\alpha)}\right)$ and condition (iii,α) is true. Thus, for each α with $0 \leq \alpha < \omega_1$, conditions (i,α), (ii,α) and (iii,α) are satisfied.

We suppose that there exists an ordinal number α_0 with $0 \leq \alpha_0 < \omega_1$ such that $h \in \mathcal{A}_{\alpha_0}$. Then $C\left(h_{(\alpha_0)}\right) = \mathbb{R}$. This is impossible, since, by (iii,α_0), $C\left(h_{(\alpha_0)}\right) = C\left(g_{(\alpha_0)}\right) \setminus P \subset \mathbb{R} \setminus P \neq \mathbb{R}$. Hence $h \notin \mathcal{A}$.

Put $K_1 = K\left(h, \frac{\sigma}{6}\right)$. Let $h^* \in K_1$. Suppose that $h^* \in \mathcal{A}$. Then we may show that, for each ordinal number α with $0 \leq \alpha < \omega_1$,

(iv,α) for each $x \in C(g)$, $\left|h^*_{(\alpha)}(x) - h_{(\alpha)}(x)\right| \leq \frac{\sigma}{6}$,

(v,α) $P \subset \mathbb{R} \setminus L\left(h^*_{(\alpha)}\right)$.

Let $\alpha = 0$. By $\rho\left(h, h^*\right) < \frac{\sigma}{6}$, condition (iv,0) is obvious. Let $x \in P$. By conditions (iv,0) and (ii,0), $\liminf_{t\to x} h^*(t) \leq \liminf_{t\to x} h(t) + \frac{\sigma}{6} = g(x) + \frac{\sigma}{6}$ and $\limsup_{t\to x} h^*(t) \geq \limsup_{t\to x} h(t) - \frac{\sigma}{6} = g(x) + \frac{\sigma}{2} - \frac{\sigma}{6}$. Thus $\limsup_{t\to x} h^*(t) - \liminf_{t\to x} h^*(t) \geq \frac{\sigma}{6} > 0$ and $x \notin L(h^*)$.

We assume that α with $0 < \alpha < \omega_1$ is an ordinal number and, for each ordinal number β with $0 \leq \beta < \alpha$, conditions (iv,β) and (v,β) are satisfied. Let $x \in C(g)$. Then, for each β with $0 \leq \beta < \alpha$, by condition (iii,β), $U\left(h_{(\beta)}\right) \subset \mathbb{R} \setminus C(g)$. Therefore $\bigcup_{0\leq\beta<\alpha} U\left(h_{(\beta)}\right) \subset \mathbb{R} \setminus C(g)$ and, by Theorem 1 (2,α), $h_{(\alpha)}(x) = h(x)$. By our assumption, we know that

$$\bigcup_{0\leq\beta<\alpha} U\left(h^*_{(\beta)}\right) \subset \bigcup_{0\leq\beta<\alpha} L\left(h^*_{(\beta)}\right) \subset \mathbb{R} \setminus P.$$

Therefore if $x \in P$, then, by Theorem 1 (2,α), $h^*_{(\alpha)}(x) = h^*(x)$. Hence, for each $x \in P$, $\left|h^*_{(\alpha)}(x) - h_{(\alpha)}(x)\right| = \left|h^*(x) - h(x)\right| \leq \frac{\sigma}{6}$. Let $x \in C(g) \setminus P$. If $\left\{\beta < \alpha;\ x \in U\left(h^*_{(\beta)}\right)\right\} = \emptyset$, then $h^*_{(\alpha)}(x) = h^*(x)$ and $\left|h^*_{(\alpha)}(x) - h_{(\alpha)}(x)\right| \leq \frac{\sigma}{6}$.

We assume that $\beta_0 = \min\left\{\beta < \alpha;\ x \in U\left(h^*_{(\beta)}\right)\right\}$. Then $h^*_{(\alpha)}(x) = \lim_{t\to x} h^*_{(\beta_0)}(t)$. Let $\epsilon > 0$ be an arbitrary real number. Thus there exists a real number $\eta > 0$ such that, for each $t \in (x-\eta, x+\eta)$ and $t \neq x$, $\left|h^*_{(\beta_0)}(t) - h^*_{(\alpha)}(x)\right| < \frac{\epsilon}{2}$. Since $x \in C(g) \setminus P = C(h)$; so there exists a real number $\eta_1 > 0$ such that, for each $t \in (x-\eta_1, x+\eta_1)$, $|h(x) - h(t)| < \frac{\epsilon}{2}$. Let $\eta_0 = \min\{\eta, \eta_1\}$. Since $h^* \in \mathcal{A}$, the set $C\left(h^*\right)$ is a residual subset of $\mathbb{R}$ and $\bigcup_{0\leq\beta<\beta_0} U\left(h^*_{(\beta)}\right) \subset \mathbb{R} \setminus C\left(h^*\right)$ is a set of the first category. Therefore

$(x-\eta_0, x) \cap \left(\mathbb{R} \setminus \bigcup_{0\le\beta<\beta_0} U\left(h^*_{(\beta)}\right)\right) \neq \emptyset$. Then, by Theorem 1 (2,$\beta_0$), there exists a point $t_0 \in (x-\eta_0, x) \cap \left\{t;\ h^*_{(\beta_0)}(t) = h^*(t)\right\}$. Hence

$$\begin{aligned}\left|h^*_{(\alpha)}(x) - h_{(\alpha)}(x)\right| &= \left|h^*_{(\alpha)}(x) - h(x)\right| \\ &\le \left|h^*_{(\alpha)}(x) - h^*_{(\beta_0)}(t_0)\right| + \left|h^*_{(\beta_0)}(t_0) - h^*(t_0)\right| \\ &\quad + |h^*(t_0) - h(t)| + |h(t) - h(x)| \\ &< \frac{\epsilon}{2} + 0 + \frac{\sigma}{6} + \frac{\epsilon}{2} = \frac{\sigma}{6} + \epsilon.\end{aligned}$$

Therefore $\left|h^*_{(\alpha)}(x) - h_{(\alpha)}(x)\right| \le \frac{\sigma}{6}$ and condition (iv,α) is satisfied.

Let $x \in P$. Then, by (iv,α) and (ii,α),

$$\begin{aligned}\liminf_{t\to x} h^*_{(\alpha)}(t) \le& \liminf_{t\to x} h^*_{|C(g)\ (\alpha)}(t) \le \liminf_{t\to x} h_{|C(g)\ (\alpha)}(t) + \frac{\sigma}{6} \\ =& g_{(\alpha)}(x) + \frac{\sigma}{6}\end{aligned}$$

and

$$\begin{aligned}\limsup_{t\to x} h^*_{(\alpha)}(t) \ge& \limsup_{t\to x} h^*_{|C(g)\ (\alpha)}(t) \ge \limsup_{t\to x} h_{|C(g)\ (\alpha)}(t) - \frac{\sigma}{6} \\ =& g_{(\alpha)}(x) + \frac{\sigma}{2} - \frac{\sigma}{6}.\end{aligned}$$

Therefore $\limsup_{t\to x} h^*_{(\alpha)}(t) - \liminf_{t\to x} h^*_{(\alpha)}(t) \ge \frac{\sigma}{6} > 0$ and $x \notin L\left(h^*_{(\alpha)}\right)$; so, for each ordinal number $\alpha (0 \le \alpha < \omega_1)$, conditions (iv,$\alpha$) and (v,$\alpha$) are satisfied. By our assumptions, there exists an ordinal number α_0 with $0 \le \alpha < \omega_1$ such that $h^* \in \mathcal{A}_{\alpha_0}$. Then $\mathbb{R} = C\left(h^*_{(\alpha_0)}\right) \subset L\left(h^*_{(\alpha_0)}\right)$. By (v,$\alpha_0$), $P \subset \mathbb{R} \setminus L\left(h^*_{(\alpha_0)}\right) = \emptyset$, a contradiction. Thus $h^* \notin \mathcal{A}$ and $K_1 \cap \mathcal{A} = \emptyset$.

Now we show that $K_1 \subset K$. Let $h^* \in K_1$. Assume that $x \in P$. Then

$$\begin{aligned}|h^*(x) - f(x)| \le& |h^*(x) - h(x)| + |h(x) - f(x)| \\ =& |h^*(x) - h(x)| + |g(x) + \frac{\sigma}{2} - f(x)| \\ \le& |h^*(x) - h(x)| + |g(x) - f(x)| + \frac{\sigma}{2} < \frac{2\sigma}{3} + \delta_1.\end{aligned}$$

Let $x \in \mathbb{R} \setminus P$. Then

$$\begin{aligned}|h^*(x) - f(x)| \leq& |h^*(x) - h(x)| + |h(x) - f(x)| \\ =& |h^*(x) - h(x)| + |g(x) - f(x)| \\ \leq& |h^*(x) - h(x)| + |g(x) - f(x)| < \frac{\sigma}{6} + \delta_1.\end{aligned}$$

Therefore $\rho(f, h^*) \leq \frac{2\sigma}{3} + \delta_1 < \delta$ and $h^* \in K$ and the proof is complete. □

We will say that $f : \mathbb{R} \to \mathbb{R}$ belongs to $\mathcal{B}_1^*$ if for every perfect set P there exists an open interval (a, b) such that $f_{|(a,b)\cap P}$ is a continuous function.

Proposition 3 *The class $\mathcal{A}_1$ is not contained in $\mathcal{B}_1^*$.*

PROOF. Let $P \subset [0, 1]$ be the Cantor set. We define the function $f : \mathbb{R} \to \mathbb{R}$ as follows. If x is the end of some contiguous interval of P, then $f(x)$ is equal to the length of this interval, otherwise $f(x) = 0$. Note that $f \in \mathcal{A}_1 \setminus \mathcal{A}_0$. Let (a, b) be a non-empty interval such that $(a, b) \cap P \neq \emptyset$. Then there exists a point $x_0 \in (a, b)$ such that x_0 is the end of some contiguous interval of P; so $f(x_0) > 0$ and there exists a real number $\eta > 0$ such that either $(x_0 - \eta, x_0) \subset \{x \in (a, b);\ f(x) = 0\}$ or $(x_0, x_0 + \eta) \subset \{x \in (a, b);\ f(x) = 0\}$. Thus either $\lim_{t \to x_0^-} f(t) = 0$ or $\lim_{t \to x_0^+} f(t) = 0$. Hence for every open interval (a, b) such that $(a, b) \cap P \neq \emptyset$, the function $f_{|(a,b)\cap P}$ is not continuous on the set P. Thus $f \notin \mathcal{B}_1^*$, so $\mathcal{A}_1 \not\subset \mathcal{B}_1^*$ and the proof is complete. □

It is easy to see that $L(f) = C(f)$ for every Darboux function. We have thus the following theorem.

Theorem 6 *There is no Darboux discontinuous function, which is improvable.*

References

[1] A. Katafiasz, *Improvable functions*, Real Analysis Exch., **21** (1995–96).

[2] W. H. Young, *La sym etrie de structure des fonctions des variables r eelles*, Bulletin des Sciences Math ematiques, **52** (2) (1928), 265–280.