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A MONOTONE C^1 FUNCTION AND A RIEMANN INTEGRABLE FUNCTION WHOSE COMPOSITION IS NOT RIEMANN INTEGRABLE

Abstract

In this paper the author constructs a C^1 function G and a Riemann integrable function H, and shows that the composition $H \circ G$ is not Riemann integrable.

If G is just required to be continuous, we can choose the function defined on the interval $[0,1]$ as $G(x) = 1$ if x belongs to a Cantor set of positive measure C, and

$$G(x) = 1 - \frac{1}{2}(b-a) + \left|x - \frac{1}{2}(a+b)\right|$$

if x belongs to some interval (a,b) contiguous to $[0,1]$. In this case $H(x)$ would be 0 on $[0,1)$ and $H(1) = 1$. This construction can be seen in [1]. With some modifications we can get G to be a C^∞ function. But, if we also want G to be monotone, we need different arguments.

Let P be a Cantor perfect subset of $[0,1]$ which contains 0, 1 and has positive measure. Let $\{(a_n, b_n)\}$, $n = 1, 2, \ldots$ be the intervals contiguous to P. Let $g : [0,1] \mapsto [0,1]$ defined as follows:

$$g(x) = \begin{cases} 0 & \text{if } x \in P \\ \frac{1}{2^n} & \text{if } x = \frac{a_n+b_n}{2},\ n = 1, 2, \ldots \\ \text{linearly} & \text{on } [a_n, \frac{a_n+b_n}{2}] \text{ and on } [\frac{a_n+b_n}{2}, b_n] \end{cases}$$

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Clearly g is continuous on $[0,1]$. Let $G : [0,1] \mapsto \mathbb{R}$, $G(x) = \int_0^x g(t)dt$ Then $G(0) = 0$, $G(1) = \sum_{n=1}^{\infty}(b_n - a_n)/2^{n+1} = \alpha$. It follows that G is strictly increasing and of class C^1. Also $G(P) \cup (\cup_{n=1}^{\infty} G((a_n, b_n))) = [0, \alpha]$. But $G((a_n, b_n)) = (G(a_n), G(b_n)) = \int_{a_n}^{b_n} g(t)dt = (b_n - a_n)/2^{n+1}$. Therefore $G(P)$ is a perfect subset of $[0, \alpha]$ of measure 0. Let $H : [0, \alpha] \mapsto \{0, 1\}$,

$$H(x) = \begin{cases} 1 & \text{if } x \in G(P) \\ 0 & \text{if } x \notin G(P) \end{cases}$$

Clearly G and H are Riemann integrable, but

$$H \circ G(x) = \begin{cases} 1 & \text{if } x \in P \\ 0 & \text{if } x \notin P \end{cases}$$

is not Riemann integrable.

References

[1] B. R. Gelbaum and J. M. Olmsted, *Counterexamples in analysis*, Holden–Day, Inc. San Francisco, 1964.